

Strategic Health Workforce Planning

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Abstract

Analysts predict impending shortages in the health care workforce, yet wages for health care workers already account for over half of U.S. health expenditures. It is thus increasingly important to adequately plan to meet health workforce demand at reasonable cost. Using infinite linear programming methodology, we propose an infinite-horizon model for health workforce planning in a large health system for a single worker class, e.g. nurses. We give a series of common-sense conditions any system of this kind should satisfy, and use them to prove the optimality of a natural lookahead policy. We then use real-world data to examine how such policies perform in more complex systems with additional detail.

1 Introduction

Health workforce planning plays a key role in the United States and worldwide. Analysts project that by 2020 the U.S. will experience a shortage of up to 100,000 physicians, up to one million nurses and up to 250,000 public health professionals [55]. Adequate staffing of medical units has been shown to have a direct impact in the quality of patient care [42], and also accounts for a considerable fraction of health care costs, with wages for health care workers representing 56% of the \$2.6 trillion spent on health care in the United States in 2010 [35].

As the U.S. population continues to age [54] and demand for health care continues to grow, different sectors of the population will compete for constrained and costly health care resources. It thus becomes increasingly important to understand how the health care needs of the population are linked to long-term workforce management plans of doctors, nurses and other medical personnel. The challenge is to ensure that sufficient resources are available in the future to meet the growing health care needs of the population, while accounting for the costs associated with meeting these needs. These workforce levels should meet the demand for resources in the present and be positioned to meet demand for the foreseeable future [49], an essentially infinite horizon. Furthermore,

workforce plans should account for lags implied by training new members of the workforce, attrition stemming from retirements, firings and resignations, and also the adequate supervision of workers at different levels of the workforce hierarchy by their superiors.

Given the significant costs and the impact on health care outcomes associated with workforce decisions, an understanding of the role of the planning horizon and the long-term consequences of health workforce plans becomes essential to stakeholders in large health care systems. We therefore propose to study the planning of workforce training, promotion and hiring within such systems, with the main goal of designing a natural policy for decision makers to implement, and concurrently determining common-sense conditions under which this policy is in fact optimal. Governments, regulatory bodies, professional associations, representatives from the private sector and senior health system executives may use the results presented in this paper to gain a deeper understanding on where incentives should be placed to best meet the health workforce needs of the population.

Our focus is on decisions at a health care policy or public policy level (i.e. not on individual hiring and firing decisions), and thus our model includes a few stylized simplifications. For instance, the problem scale we are interested in has workforces numbering in the thousands or the tens of thousands, e.g. state or provincial health systems, large hospital conglomerates, or the U.S. Veterans Administration. We therefore model the workforce as a continuous flow and allow fractional quantities in our solutions. Similarly, we assume centralized control of the system, which may only be realistic in some cases. Nevertheless, even for those systems in which this is not entirely the case, the conditions we list can help decision makers with limited control in monitoring the system's behavior.

1.1 Our Contribution

We consider the following to be our main contributions:

- i)* We propose a discounted, infinite linear programming model for strategic workforce planning, which includes training, promotion and hiring decision for a class of health workers within a hierarchical system. The model takes as input a demand forecast, workforce payroll, training and hiring costs, workforce hierarchy parameters and a discount factor.
- ii)* We give a series of common-sense conditions any system of this kind should satisfy under our assumptions, demonstrate the pathological behavior that can occur when they are not satisfied and derive useful structural properties of the optimal solutions from the conditions. Though based on our assumptions, these conditions may help guide decision making in more complex systems with additional detail.
- iii)* We prove that a natural lookahead policy is optimal for our model, and use duality theory to demonstrate how sensitivity analysis can be performed. The optimality result is particularly useful because lookahead policies mimic how more complex models may be managed in practice.
- iv)* We provide computational examples, based on real-world nursing workforce data, demonstrating the efficacy of our proposed policy in more complex models with additional constraints and practical details.

The remainder of the paper is organized as follows: This section closes with a literature review. Section 2 formulates our model and states the conditions we assume. Section 3 uses the conditions

to show some structural properties of optimal solutions, proves the optimality of our proposed policy and discusses duality and sensitivity. Section 4 discusses experiments that test our policy on more complex models, and Section 5 concludes outlining future research avenues. The Appendix contains technical proofs and some additional modeling information.

1.2 Literature Review

Workforce planning models are not new to the operations research and management science literature, with work stretching back several decades, such as [1, 9, 32, 40, 45, 50]. Workforce management models have been developed to manage workforce in call centers [26], military personnel [28], medical school budgets [16, 39], as well as to address cross-training and flexibility of the workforce [43, 56]. [11, 22, 53] provide overviews of workforce/manpower planning models, while [15] discuss the need of a greater interface between operations and human resource management models and the complexities associated with those models. Recent work continues to address workforce issues in operational or tactical time frames, e.g. [10]; this focus on shorter horizons extends also to health care and emergency workforce planning [14, 21, 25, 60]. The long-term workforce capacity planning models [6, 27, 52] are related to our work, yet they concentrate only on the recruitment and retention of personnel without incorporating some of the other decisions required to manage health care personnel. On the other hand, models such as [8, 59] concentrate on skill acquisition and on-the job learning, focusing on a shorter time scale. The results in [57, 58] and the recent survey [53] particularly highlight the need to research long-term health workforce planning, among other areas.

The work in [36, 37, 38] developed an integrated workforce planning model of the registered nursing workforce of British Columbia. The model was solved over a 20-year planning horizon, providing guidance on the number of nurses to train, promote and recruit to achieve specified workforce levels. However, as the model was solved over a finite horizon, one of the key challenges was understanding the impact of the length of the planning horizon and terminal conditions on the resulting solution.

Infinite-horizon optimization has been widely applied to various operational problems; the dynamic programming paradigm in particular is extensively used [44, 61]. The last two or three decades have also seen the direct study of infinite mathematical programming models and specifically infinite linear programs for operations management applications. Some problems studied in the literature include inventory routing [2, 3], joint replenishment [4, 5], production planning [19, 47], and equipment replacement [34]. A general reference for infinite linear programming is [7]. Our models operate in countable dimensions; recent results for this topic include [19, 29, 30, 46, 47, 48, 51]. Workforce management possesses differences with other resource management problems that deal mostly with products [27]. To our knowledge, although dynamic programming has been applied to model strategic workforce management, e.g. [27, 45], infinite linear programming has not yet been considered in the literature to address this topic, and this methodology allows for additional research insights such as sensitivity analysis, duality and the modeling of workforce as a network flow.

For a recent overview of optimization in health care, we refer the reader to [12].

2 Model Formulation and Assumptions

We consider an infinite-horizon, discounted workforce planning model with the following characteristics. There is a deterministic demand forecast given as an infinite sequence; each period, the population of workers at the lowest level of the system, e.g. junior nurses, must be at least equal to that period's demand. The system has a fixed number of levels above this first level; worker population at each higher level must be at least a fixed fraction of the same period's population one level below, to ensure adequate supervision. Between one period and the next, a fixed fraction of each level's population leaves the system, accounting for retirements, firings and resignations. New workers may be added to any level directly via hiring, or indirectly through student admission and training at the first level, and promotion at higher levels; there is no down-sizing, i.e. mass firing to reduce workforce levels. Student populations take one period to train before entering at the first workforce level; similarly, only workers who have been in a level for at least one period may be promoted. We discuss how to extend our results to models with longer training in the subsequent sections.

The model is defined by the following parameters.

- $n \geq 2$: Number of workforce hierarchy levels.
- $h_k > 0$: Per-period variable payroll costs for level $k = 1, \dots, n$.
- $c_k > 0$: Variable training ($k = 0$) or hiring ($k \geq 1$) costs for level $k = 0, \dots, n$.
- $c_{k,k+1} \geq 0$: Variable promotion cost from level $k = 1, \dots, n-1$ to $k+1$. Workers may only be promoted once they have worked at a particular level for at least one period.
- $\gamma \in (0, 1)$: Discount factor, adjusted to account for cost increases. That is, if $\bar{\gamma}$ is the nominal discount rate and $\alpha > 1$ is the cost growth rate, then $\gamma = \alpha\bar{\gamma}$; this is the reciprocal of the “health care inflation.”
- $d_t > 0$: Forecasted level-1 workforce demand for period $t = 1, \dots$.
- $q_{k,k+1} \in (0, 1)$: Minimum fraction of level- k workers needed at level $k+1$, for $k = 1, \dots, n-1$.
- $p_k \in (0, 1)$: Per-period *retention* rate of workers that stay in the system at level $k = 0, \dots, n$ from one period to the next. The *attrition* rate $1 - p_k$ is the fraction of workers at level k expected to leave the system from one period to the next; this includes firing, retirement and quitting.
- s_k^0 : Students ($k = 0$) or workers in level $k = 1, \dots, n$ at the start of the current period, before attrition.

The model's decision variables are:

- s_k^t : Students ($k = 0$) or workers in level $k = 1, \dots, n$ at end of period $t = 1, \dots$.
- x_k^t : Students admitted ($k = 0$) or workers hired at level $k = 1, \dots, n$ in period $t = 1, \dots$.
- $x_{k,k+1}^t$: Workers promoted from level $k = 1, \dots, n-1$ to $k+1$.

Our strategic workforce planning problem then has the following formulation.

$$\inf C(s, x) = \sum_{t=1}^{\infty} \gamma^{t-1} \left(\sum_{k=0}^n c_k x_k^t + \sum_{k=1}^{n-1} c_{k,k+1} x_{k,k+1}^t + \sum_{k=1}^n h_k s_k^t \right) \quad (1a)$$

$$\text{s.t. } s_1^t \geq d_t, \quad \forall t = 1, \dots \quad (1b)$$

$$s_{k+1}^t - q_{k,k+1} s_k^t \geq 0, \quad \forall k = 1, \dots, n-1, \quad \forall t = 1, \dots \quad (1c)$$

$$p_1 s_1^{t-1} - s_1^t + p_0 x_0^{t-1} - x_{12}^t + x_1^t = 0, \quad \forall t = 1, \dots \quad (1d)$$

$$p_k s_k^{t-1} - s_k^t + x_{k-1,k}^t - x_{k,k+1}^t + x_k^t = 0, \quad \forall k = 2, \dots, n-1, \quad \forall t = 1, \dots \quad (1e)$$

$$p_n s_n^{t-1} - s_n^t + x_{n-1,n}^t + x_n^t = 0, \quad \forall t = 1, \dots \quad (1f)$$

$$p_k s_k^{t-1} - x_{k,k+1}^t \geq 0, \quad \forall k = 1, \dots, n-1, \quad \forall t = 1, \dots \quad (1g)$$

$$s^t, x^t \geq 0, \quad \forall t = 1, \dots, \quad (1h)$$

where $s_0^t = x_0^t$ for $t = 0, \dots$. We take the feasible region to be the subset of solutions for which the objective is well defined and finite [47]. In the model, the objective (1a) minimizes discounted cost over the infinite horizon. The demand satisfaction constraint (1b) ensures enough level-1 workers are present to satisfy projected demand each period, while (1c) ensures the minimum required fraction of level- $(k+1)$ workers are present to supervise level- k workers. The flow balance constraints (1d–1f) track workers present at each level from one period to the next, and (1g) limits the promoted workers from level k to $k+1$ to those present in level k for at least one period. The domain constraints (1h) ensure non-negativity of worker levels, hires, promotions and student admissions.

We next list several conditions the model should satisfy. These conditions are common in many real world settings or are reasonable approximations, and are necessary for most of our subsequent results. Many are also necessary to avoid pathological behavior. We begin with technical assumptions.

Assumption 2.1 (Technical assumptions).

i) Finite total demand: Total discounted demand converges.

$$\sum_{t=1}^{\infty} \gamma^{t-1} d_t < \infty \quad (2a)$$

ii) Growing demand: The sequence (d_t) is non-decreasing.

$$d_t \leq d_{t+1}, \quad \forall t = 1, \dots \quad (2b)$$

The former assumption is necessary to have a finite objective and thus a feasible problem. The latter reflects most contemporary health care systems in which demand is expected to grow for the foreseeable future, and ensures that training and promotion will be perpetually necessary within the system. In more general cases, even if demand is only expected to be eventually non-decreasing, our conclusions can be applied starting at the period where non-decreasing growth begins, with a finite model accounting for the system in preceding periods.

The first non-technical assumption concerns the relative costs of payroll, promotion and hiring.

Assumption 2.2 (Promotion is preferable). Even when factoring attrition, payroll costs and discounting, promotion is cheaper than hiring.

$$\frac{c_0}{\gamma p_0} \leq c_1, \quad \frac{c_k + h_k}{\gamma p_k} + c_{k,k+1} \leq c_{k+1}, \quad \forall k = 1, \dots, n-1 \quad (2c)$$

If this assumption does not hold at some level in the hierarchy, there is no incentive to train and promote from within beyond that point. This condition should be satisfied by many workforce systems, both in health care and in other industries.

The next assumption is slightly more specific to the health care industry, but still common in other industries.

Assumption 2.3 (Non-increasing retention). The hierarchy does not tend to become top-heavy:

$$p_k \geq p_{k+1}, \quad \forall k = 1, \dots, n-1 \quad (2d)$$

This assumption is natural in health care hierarchies such as nursing, where higher-level workers are usually older, since older workers tend to retire or leave the system for other reasons at a higher rate. The assumption is more problematic, for example, in industries where tenure guarantees at an intermediate level imply an unnaturally high attrition at lower levels.

For some of our results, it is necessary to further strengthen the previous assumption.

Assumption 2.3' (Equal retention). Retention and attrition are equal at all hierarchy levels:

$$p_k = p_{k+1}, \quad \forall k = 1, \dots, n-1 \quad (2d')$$

Though it appears restrictive, in many real-world systems the top and bottom retention rates in fact only differ by a few percentage points [36, 37, 38].

Assumption 2.4 (Non-decreasing payroll). Salaries increase within the hierarchy, even when accounting for attrition:

$$\frac{h_k}{1 - \gamma p_k} \leq \frac{h_{k+1}}{1 - \gamma p_{k+1}}, \quad \forall k = 1, \dots, n-1 \quad (2e)$$

As the next example shows, this condition prevents undesirable behavior.

Example 1 (Down-sizing by promotion). Consider a two-level system which is drastically over-staffed. Let $d_t = \varepsilon$ for all t , where $\varepsilon > 0$ is a small positive number, and let $s_1^0 \gg \varepsilon$. If (2e) is not satisfied, it may be optimal because of (2d) to promote all but ε workers to level 2, effectively down-sizing the workforce by promoting most of it, and achieving lower costs in the process. Such behavior could lead to detrimental side effects, such as poor morale in the remaining workforce.

Assumption 2.5 (Moderate demand growth). Demand does not grow too quickly:

$$\frac{d_{t+1}}{d_t} \leq \frac{p_{\min}}{q_{\max}}, \quad \forall t = 1, \dots, \quad (2f)$$

where $p_{\min} = \min_k p_k$ and $q_{\max} = \max_k q_{k,k+1}$.

Intuitively, the assumption ensures enough worker population at each level to promote to the next level as demand grows; it is easily satisfied in most systems. For example, if $n = 2$, $p_1 = p_2 = 0.8$ and $q_{12} = 0.25$, (2f) requires the demand growth to be no more than 320% per period, a condition met in virtually any system. Furthermore, as the next example shows, when this assumption is not met, the planning horizon necessary to compute an optimal solution may be arbitrarily long.

Example 2 (Excessive demand growth). We consider a two-level system that experiences excessive demand growth for a given number of periods, and constant demand thereafter. To simplify the numbers in the example, we set $p_1 = p_2 = q_{12} = 1$. For a fixed $m \geq 2$ let

$$d_1^m = 1, \quad d_t^m = \begin{cases} 2^{t-1} - 1, & t = 2, \dots, m \\ 2^{m-1} - 1, & t = m+1, \dots \end{cases},$$

and $s_0^0 = 0$, $s_1^0 = s_2^0 = 1$; note that $d_{t+1}/d_t > p_1/q_{12} = 1$ for $t = 2, \dots, m-1$. Table 1 details the first demand values in the sequence, for $m = 3, 4, 5$. The table also lists a solution that satisfies demand without any hiring, which can be made optimal by choosing large enough hiring costs. Although projected demand for the first three periods is identical in all cases, the optimal number of students admitted in the first period changes with m ; for general m , we get $x_0^1 = (2^{m-2} - 1)/2^{m-3}$. In other words, the current period's decision may depend on a horizon of arbitrary length m .

t	1	2	3	4	5	...
d_t^3	1	1	3	3	3	...
x_0^t	1	3	0	0	0	...
s_1^t	1	3/2	3	3	3	...
s_2^t	1	3/2	3	3	3	...
d_t^4	1	1	3	7	7	...
x_0^t	3/2	7/2	7	0	0	...
s_1^t	1	3/2	7/2	7	7	...
s_2^t	1	3/2	7/2	7	7	...
d_t^5	1	1	3	7	15	...
x_0^t	7/4	15/4	15/2	15	0	...
s_1^t	1	15/8	15/4	15/2	15	...
s_2^t	1	15/8	15/4	15/2	15	...

Table 1: Sample demand sequences and solutions with no hiring for $m = 3, 4, 5$ in Example 2.

As Example 2 suggests, the condition (2f) can be relaxed; we include the best possible condition of this kind we could derive in the Appendix (see the proof of Claim A.3). However, (2f) is much simpler to state and suffices for any practical situation.

3 Optimal System Behavior

We begin our characterization of optimal solutions of (1) by outlining structural properties satisfied in models that meet our assumptions. We include only simple proofs here and relegate any complex proof to the Appendix.

Lemma 3.1 (No unnecessary hiring). *Suppose the model parameters satisfy Assumptions 2.1 and 2.2. There is an optimal solution of (1) in which no hiring takes place when promotion is possible:*

$$x_1^t = 0, \quad \forall t = 2, \dots \quad (3a)$$

$$(p_k s_k^{t-1} - x_{k,k+1}^t) x_{k+1}^t = 0, \quad \forall k = 1, \dots, n-1, \quad \forall t = 1, \dots \quad (3b)$$

Proof. If a solution does not satisfy either condition, a simple substitution produces another solution with equal or lesser objective that does satisfy the conditions. \square

Lemma 3.2 (No excess training or promotion). *Suppose Assumptions 2.1, 2.2, 2.4 and 2.5 hold. Furthermore, suppose either Assumption 2.3 holds and $n = 2$, or Assumption 2.3' holds. Then there is an optimal solution of (1) in which no excess promotion or student admittance occurs:*

$$(s_1^t - d_t) x_0^{t-1} = 0, \quad \forall t = 2, \dots \quad (4a)$$

$$(s_{k+1}^t - q_{k,k+1} s_k^t) x_{k,k+1}^t = 0, \quad \forall k = 1, \dots, n-1, \quad \forall t = 1, \dots \quad (4b)$$

Like the preceding lemma, Lemma 3.2 follows from applying a substitution or perturbation to any solution that does not satisfy it. However, unlike in the hiring case, a perturbation in promotion has ripple effects in higher levels of the hierarchy and in later periods that render it much more complex.

With these two structural properties in place, we are able to characterize optimal solutions of (1). Consider the two-period restriction of (1) given by

$$\min \sum_{k=0}^n c_k (x_k^1 + \gamma x_k^2) + \sum_{k=1}^{n-1} c_{k,k+1} (x_{k,k+1}^1 + \gamma x_{k,k+1}^2) + \sum_{k=1}^n h_k (s_k^1 + \gamma s_k^2) \quad (5a)$$

$$\text{s.t. } s_1^t \geq d_t, \quad \forall t = 1, 2 \quad (5b)$$

$$s_{k+1}^t - q_{k,k+1} s_k^t \geq 0, \quad \forall k = 1, \dots, n-1, \quad \forall t = 1, 2 \quad (5c)$$

$$p_1 s_1^{t-1} - s_1^t + p_0 x_0^{t-1} - x_{12}^t + x_1^t = 0, \quad \forall t = 1, 2 \quad (5d)$$

$$p_k s_k^{t-1} - s_k^t + x_{k-1,k}^t - x_{k,k+1}^t + x_k^t = 0, \quad \forall k = 2, \dots, n-1, \quad \forall t = 1, 2 \quad (5e)$$

$$p_n s_n^{t-1} - s_n^t + x_{n-1,n}^t + x_n^t = 0, \quad \forall t = 1, 2 \quad (5f)$$

$$p_k s_k^{t-1} - x_{k,k+1}^t \geq 0, \quad \forall k = 1, \dots, n-1, \quad \forall t = 1, 2 \quad (5g)$$

$$s^t, x^t \geq 0, \quad \forall t = 1, 2, \quad (5h)$$

A *one-period lookahead policy* constructs a solution to (1) by iteratively solving (5), fixing the variables for $t = 1$, stepping one period forward by relabeling $t \leftarrow t + 1$ for all variables and parameters, and repeating the process. In practice, this corresponds to a decision maker planning the current period's promotion, training and hiring based on current demand and the next period's forecasted demand, while ignoring demand for subsequent periods.

Theorem 3.3 (Optimality of one-period lookahead policy). *Suppose Assumptions 2.1, 2.2, 2.4 and 2.5 are satisfied. Suppose either Assumption 2.3 holds and $n = 2$, or Assumption 2.3' holds. Then one-period lookahead policies are optimal.*

Corollary 3.4 (Increased training time). *Suppose students require $L \geq 1$ periods to train instead of one, with all other system characteristics remaining the same. Under the conditions of Theorem 3.3, L -period lookahead policies are optimal, where an L -period lookahead is defined analogously to a one-period lookahead but with L additional periods instead of one.*

Proof. The proof of Theorem 3.3 still applies; we are simply relabeling level-0 variables. \square

These results indicate that good workforce planning decisions can be made using a minimal amount of forecasted information, which strengthens the robustness of the resulting solution since forecasts of more distant demand naturally tend to be less reliable. This also places our result within the context of *solution* and *forecast horizons*; see, e.g., [18] for formal definitions and discussion. Moreover, lookahead policies mimic how such large workforce systems might be managed in practice, suggesting that these policies could be useful in more complex settings. We explore this idea experimentally in Section 4, where we study a model with a four-year training program, i.e. $L = 4$.

Another important question related to (1) is duality. A dual satisfying the typical complementary relationships can shed additional light on the structure of optimal solutions to (1). Furthermore, optimal dual prices may also be useful as indicators of the model's sensitivity to parameters such as demand. However, the infinite horizon implies significant technical complications and gives rise to pathologies not encountered in the finite case.

Extending the typical LP dual construction to (1) yields

$$\sup D(\mu, \lambda, \eta) = \sum_{t=1}^{\infty} d_t \mu_1^t - p_0 s_0^0 \lambda_1^1 - \sum_{k=1}^{n-1} p_k s_k^0 (\lambda_k^1 + \eta_{k,k+1}^1) - p_n s_n^0 \lambda_n^1 \quad (6a)$$

$$\text{s.t. } \mu_k^t - q_{k,k+1} \mu_{k+1}^t - \lambda_k^t + p_k \lambda_k^{t+1} + p_k \eta_{k,k+1}^{t+1} \leq \gamma^{t-1} h_k, \quad \forall k = 1, \dots, n-1, \quad \forall t = 1, \dots \quad (6b)$$

$$\mu_n^t - \lambda_n^t + p_n \lambda_n^{t+1} \leq \gamma^{t-1} h_n, \quad \forall t = 1, \dots \quad (6c)$$

$$p_0 \lambda_1^t \leq \gamma^{t-2} c_0, \quad \forall t = 2, \dots \quad (6d)$$

$$\lambda_k^t \leq \gamma^{t-1} c_k, \quad \forall k = 1, \dots, n, \quad \forall t = 1, \dots \quad (6e)$$

$$-\lambda_k^t + \lambda_{k+1}^t - \eta_{k,k+1}^t \leq \gamma^{t-1} c_{k,k+1}, \quad \forall k = 1, \dots, n-1, \quad \forall t = 1, \dots \quad (6f)$$

$$\mu^t, \eta^t \geq 0, \quad \lambda^t \text{ unrestricted}, \quad \forall t = 1, \dots, \quad (6g)$$

where we similarly define the feasible region as a subset of the points for which the objective is well defined and finite. However, this model does not satisfy strong or even weak duality with (1).

Example 3 (No weak duality; adapted from [51]). Suppose $s_n^0 > 0$ and let $M > 0$. Define $\hat{\lambda}_n^t = -M/p_n^t$, $\forall t = 1, \dots$, and set all other variables to zero. The solution is feasible for (6), and its objective function value is positive and goes to infinity as $M \rightarrow \infty$. However, (1) is clearly feasible and bounded below by zero.

The following result addresses this problem.

Theorem 3.5. *Suppose we can change the equality constraints (1d–1f) to greater-than-or-equal constraints (and thus impose $\lambda^t \geq 0$) for all but a finite number of indices t without affecting optimality in (1). Let (\hat{s}, \hat{x}) and $(\hat{\mu}, \hat{\lambda}, \hat{\eta})$ be feasible for (1) and (6) respectively.*

i) *Weak duality:* $D(\hat{\mu}, \hat{\lambda}, \hat{\eta}) \leq C(\hat{s}, \hat{x})$.

ii) *Strong duality:* Both solutions are optimal and $D(\hat{\mu}, \hat{\lambda}, \hat{\eta}) = C(\hat{s}, \hat{x})$ if and only if complementary slackness holds (in the usual sense) and transversality [47, 48, 51] holds:

$$\liminf_{t \rightarrow \infty} p_0 \hat{\lambda}_1^{t+1} \hat{x}_0^t + \sum_{k=1}^{n-1} p_k (\hat{\lambda}_k^{t+1} + \hat{\eta}_k^{t+1}) \hat{s}_k^t + p_n \hat{\lambda}_n^{t+1} \hat{s}_n^t = 0. \quad (7)$$

Proof. If all constraints eventually become greater-than-or-equal, then the off-diagonal constraint matrix of (1) in inequality form is eventually non-negative, implying that [47, Assumption 3.1] holds, and thus the results follow from [47, Theorems 3.3 and 3.7]. \square

Corollary 3.6. *The conditions of Theorem 3.5 apply, and therefore weak and strong duality hold, if demand is eventually non-decreasing.*

The results in [47] imply we can use optimal solutions of (6) as shadow prices to perform sensitivity analysis on (1).

Example 4 (Sensitivity analysis). Consider a two-level system in which the incoming worker populations in period 1 require some promotion from level 1 to level 2, with enough level-1 workers remaining after promotion to meet demand in period 1 but not later. Based on these initial conditions and Assumptions 2.1 through 2.5, Lemmas 3.1 and 3.2 imply the following structure to the optimal solution:

$$\begin{aligned} x_1^t = x_2^t = 0, \quad x_0^t, x_{12}^t > 0, \quad x_{12}^t < p_1 s_1^{t-1}, \quad s_2^t = q_{12} s_1^t, \quad \forall t = 1, \dots \\ s_1^1 > d_1; \quad s_1^t = d_t, \quad \forall t = 2, \dots \end{aligned}$$

The solution for (6) that satisfies complementary slackness and transversality is:

$$\begin{aligned} \mu_1^1 &= 0 \\ \mu_1^t &= \frac{\gamma^{t-2} c_0}{p_0} (1 - \gamma p_1 + q_{12} (1 - \gamma p_2)) + \gamma^{t-1} q_{12} c_{12} (1 - \gamma p_2) \\ &\quad + \gamma^{t-1} (h_1 + q_{12} h_2), \quad \forall t = 2, \dots \\ \mu_2^1 &= \frac{1}{1 + q_{12}} \left(\frac{c_0}{p_0} (p_1 - p_2) + c_{12} (1 - \gamma p_2) + (h_2 - h_1) \right) \\ \mu_2^t &= \gamma^{t-2} (1 - \gamma p_2) \left(\frac{c_0}{p_0} + \gamma c_{12} \right) + \gamma^{t-1} h_2, \quad \forall t = 2, \dots \\ \lambda_1^1 &= \frac{1}{1 + q_{12}} \left(\frac{c_0}{p_0} (p_1 + q_{12} p_2) - q_{12} c_{12} (1 - \gamma p_2) - (h_1 + q_{12} h_2) \right) \\ \lambda_1^t &= \frac{\gamma^{t-2} c_0}{p_0}, \quad \forall t = 2, \dots \\ \lambda_2^1 &= \frac{1}{1 + q_{12}} \left(\frac{c_0}{p_0} (p_1 + q_{12} p_2) + c_{12} (1 + \gamma q_{12} p_2) - (h_1 + q_{12} h_2) \right) \\ \lambda_2^t &= \gamma^{t-2} \left(\frac{c_0}{p_0} + \gamma c_{12} \right), \quad \forall t = 2, \dots \\ \eta_{12}^t &= 0, \quad \forall t = 1, \dots \end{aligned}$$

It can be verified that this solution is dual feasible provided the assumptions hold. Suppose in particular that demand grows based on a rate $1 < \beta < 1/\gamma$, so that $d_t = \beta^{t-1} d_1$. It follows that

$$\sum_{t=1}^{\infty} d_t \mu_1^t = d_1 \left[\frac{c_0}{\gamma p_0} (1 - \gamma p_1 + q_{12} (1 - \gamma p_2)) + q_{12} c_{12} (1 - \gamma p_2) + h_1 + q_{12} h_2 \right] \frac{\beta \gamma}{1 - \beta \gamma}.$$

This expression indicates how the optimal cost would change if either d_1 or β vary slightly from their given values.

4 Computational Examples

To evaluate the efficacy of our proposed models and policies in a more complex setting, we performed computational experiments based on the British Columbia nursing workforce described in [36, 37, 38]. Health care human resource data is more readily available from Canadian provinces because of their centralized control of health care. However, our ongoing research also involves collecting comparable data from various U.S. health systems.

We extended model 1 to differentiate by worker age (as it affects attrition rates) and set the length of student training to four periods (years in this case); we include further description of the model in the Appendix. We began by solving the problem over a 25-year planning horizon (a *full information* model) and used the solution to the first 20 years as our benchmark. We then compared the results to a solution obtained by implementing a four-year lookahead policy.

Figure 1 outlines the structure of the model and its parameters. Students are admitted into the training program, where they take four years to train before entering the workforce. The probabilities of students continuing their education depend on the school year of the student (with greater attrition in the first year of the program). After graduation, students enter the first workforce level as direct care nurses. In level 1, the number of workers has to meet current demand. This demand is met by workers that have not retired or been promoted, graduates from the training program and workers hired externally. Level 2 consists of nurse managers, a supervisory position to the first level; nurse managers are either hired externally or promoted from the first workforce level. Level-1 workers must have worked for at least one year before being promoted into the second workforce level. In both levels, retention rates depend on the age of the workers. The average retention in level 2 is slightly higher than the average retention in level 1, which would violate Assumption 2.3 if the averages applied to all age groups. However, since the parameter is age-dependent in this model, the actual retention in each level depends on the age distribution of the worker population. This slight difference did not impact our results, further supporting the robustness of our findings. We set the discount factor to $\gamma = 0.95$.

4.1 Scenarios

We tested the model in nine scenarios. Among the nine scenarios, the baseline scenario represents the estimated demand in British Columbia, Canada, starting in 2007; we calculated demand by extrapolating the population growth between 1996 and 2006 [13]. Scenarios 1 through 4 evaluate the impact of different demand growth rates. Scenario 5 evaluates the impact of limiting the growth of the training program. Scenarios 6, 7 and 8 evaluate the performance of the lookahead policy in extreme conditions where demand has a peak, hiring growth is limited, and costs are varied. The parameter characteristics and descriptions of the scenarios are summarized in the following list.

Baseline Scenario Fixed demand growth rate of 1.25% per year. Projected demand growth in British Columbia, Canada.

Scenario 1 Fixed demand growth rate of 0.01% per year. Very low demand growth.

Scenario 2 Fixed demand growth rate of 2.5% per year. High demand growth.

Scenario 3 Linearly accelerating demand growth from 0% per year to 2.5% per year over 25 years.

Scenario 4 Linearly decelerating demand growth from 2.5% per year to 0% per year over 25 years.

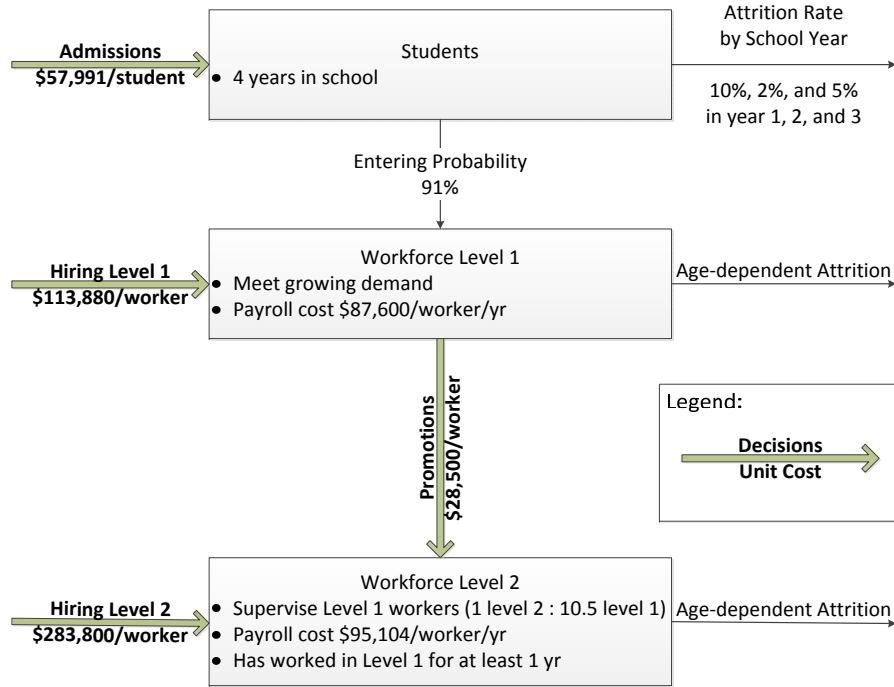


Figure 1: Flow chart of model used in computational examples.

Scenario 5 Fixed demand growth rate of 1.25% per year and student population growth limited to no more than 1% per year. Major restrictions in training growth.

Scenario 6 Fixed demand growth rate of 1.25% per year in years 1 through 9 and 11 through 25, demand doubled in year 10. Level-1 hiring growth limited to no more than 50% per year. This scenario simulated a sudden jump of demand, which might be due to a drastic change in roles and scope of practice of the workforce. We assumed that drastic changes in the number of workers hired could not be made without incurring very large recruitment costs.

Scenario 7 Fixed demand growth rate of 1.25% per year in years 1 through 9 and 11 through 25, demand doubled in year 10. Level-1 hiring growth limited to no more than 50% per year, and zero student admission cost. In addition to the jump in demand and limited hiring growth, we eliminated the admission cost to increase the incentive to admit students in advance and thus potentially undermine the four-year lookahead model.

Scenario 8 Fixed demand growth rate of 1.25% per year in years 1 through 9 and 11 through 25, demand doubled in year 10. Level-1 hiring growth limited to no more than 50% per year, and zero level-1 payroll cost. In addition to the jump in demand and limited hiring growth, we eliminated the level-1 payroll cost to increase the incentive to admit students in advance and thus potentially undermine the four-year lookahead model.

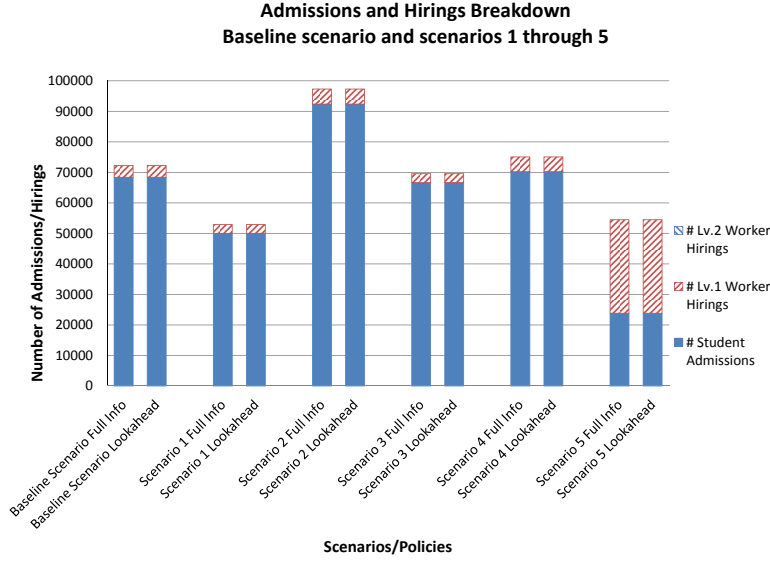


Figure 2: Breakdown of the total number of admissions and hirings in baseline scenario and scenarios 1 through 5 over the course of 20 years.

4.2 Results and Discussion

We compared the solutions obtained using the full information model and the lookahead model. Figure 2 shows results for the baseline scenario and scenarios 1 through 5. In these scenarios, we obtained the same solutions using the full information and the lookahead models. The lookahead model was robust in these scenarios, even if Assumption 2.3 was slightly violated by our system’s parameters. Even in Scenario 5, where education growth was drastically limited, the full information model did not differ from the lookahead policy because training students a year in advance incurred extra payroll costs, making early training more expensive than hiring. Though education growth was limited, hires served as a back-up action in Scenario 5 and made the lookahead and the full information methods operate identically.

Figure 3 shows results for scenarios 6, 7, and 8; in this case, the lookahead policy resulted in slightly higher total costs. Compared to the full information solution, the percentage differences in total cost were only 0.026%, 0.129%, and 0.014% respectively. The lookahead model resulted in more admissions, more level-2 hirings, and fewer level-1 hirings than the full information model. Since level-1 hiring was limited, fewer level-1 workers were hired and more students were trained as an alternative. Level-2 workers were hired when the model reached a point where promotions could not meet the level-2 workforce demand due to the insufficient number of level-1 workers. The lookahead model failed to anticipate future changes in demand, not training sufficient students nor hiring sufficient level-1 workers in advance.

Overall, the lookahead policy demonstrated strong robustness in the nine scenarios modeled. In the most extreme scenarios, where demand had a sudden jump and hirings or admissions were

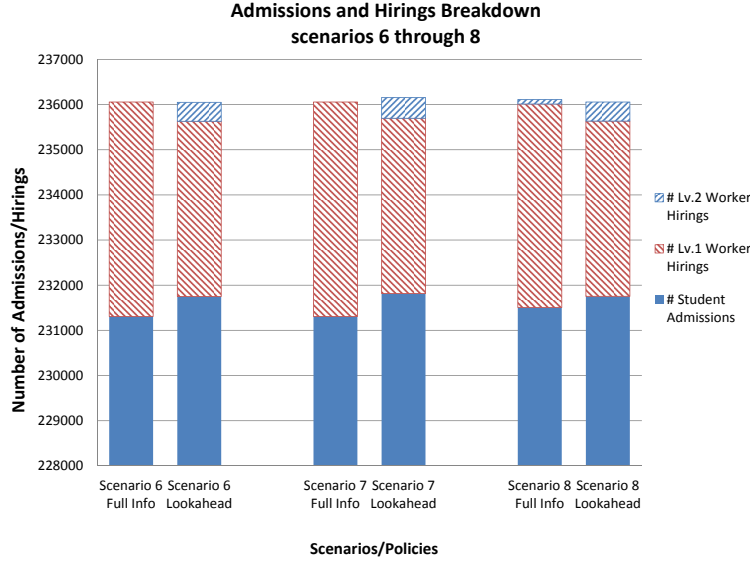


Figure 3: Breakdown of the total number of admissions and hirings in scenarios 6 through 8 over the course of 20 years.

limited, the lookahead policy and the full information policy still showed very little difference, particularly in total cost.

5 Conclusions

This paper contributes a new modeling framework for strategic health workforce planning. Through infinite-horizon optimization, we are able to model the long-term implications of training, hiring and promotion decisions made within a health care system. Our approach enables us to understand the planning horizon length necessary to obtain optimal decisions. We derive common-sense system conditions that should hold in any situation and also imply the optimality of simple lookahead workforce management policies. Using real-world data from British Columbia, we further demonstrate how lookahead policies perform well in a variety of scenarios, even in more complex systems. These results are particularly useful, as they mirror workforce management policies implemented in practice.

Given that long-term workforce planning should be an important component of a well-functioning health care system, this kind of model can be used to obtain qualitative checks on whether a particular health workforce system is actually behaving optimally, and what conditions it must meet to do so. Though we have assumed exact knowledge of demand and retention rates throughout this paper, even within a deterministic model it is possible to explore the consequences of unforeseen outcomes via sensitivity analysis of the optimal cost.

A next step in our work is to directly model and optimize the system’s uncertainty, specifically in demand growth and retention rates [41]. It is important to understand whether the conditions

we develop in this work and their structural consequences (or appropriate modifications) still hold in more general settings. For example, it is possible that under uncertain demand growth condition (4a) of Lemma 3.2 does not hold in level 0 – we may need to train in excess of forecasted demand – but it may be that a similar property holds which accounts for the risks of under-training and over-training. The more nuanced analysis required in this case may give insight into the impact of uncertainty on health workforce costs and management decisions; for example, [20, 31] investigate similar questions for short-term nurse staffing.

Because this work is applied to guide strategic health workforce decisions, we can formulate more realistic models by incorporating other elements. For instance, (1) could be expanded to include a variety of health care providers and changes in scopes of practice. As a first step, the impact of multiple worker types can be modeled indirectly in (1) through scaling or modification of the demand forecast. This approach has the advantage of allowing for non-linear interactions between multiple health care providers and demand, if, for example, different worker types cannot serve patient demand in the same fashion. Assuming that the interaction of all worker types with demand is linear, multiple worker types can be incorporated in models similar to (1), by differentiating across both type and level, where each worker type includes its own hierarchy with its own supervision constraints (1c) and dynamics, but the multiple types serve patient demand jointly.

Furthermore, clinical inactivity has been a well documented phenomenon among health care providers [17, 23, 24, 33], and therefore policy makers may be interested in understanding the role of such inactivity in workforce planning. As before, one possibility is to incorporate expected inactivity in the demand calculations. A more complex option is to incorporate additional states representing the number of health care providers that are inactive each period. While this second option entails an expansion of the model, by following this option it could be possible to study the impact of adding incentives to bring inactive health care professionals back to the workforce.

By providing an initial understanding of this infinite-horizon model, our goal is to move a step forward in the field of strategic health workforce planning, and to motivate others to continue doing research in this important application.

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A Proof of Lemma 3.2

All the arguments below apply to solutions that satisfy Lemma 3.1.

A.1 Proof when $n = 2$

A.1.1 Proof of (4b)

Assume a feasible solution is given for which (4b) is violated in some period. We start from the earliest such period, relabeling it as period 1 without loss of generality, and make the following changes:

$$\Delta x_{1,2}^t = \begin{cases} -\epsilon, & t = 1 \\ \frac{p_2 + p_1 q_{1,2}}{1 + q_{1,2}} \epsilon, & t = 2 \\ \frac{(p_1 + p_2 q_{1,2})^{t-3}}{(1 + q_{1,2})^{t-1}} (p_1 - p_2)^2 q_{1,2} \epsilon, & t = 3, \dots \end{cases},$$

$$\Delta s_1^t = \begin{cases} \epsilon, & t = 1 \\ \frac{(p_1 + p_2 q_{1,2})^{t-2}}{(1 + q_{1,2})^{t-1}} (p_1 - p_2) \epsilon, & t = 2, \dots \end{cases},$$

$$\Delta s_2^t = \begin{cases} -\epsilon, & t = 1 \\ \frac{(p_1 + p_2 q_{1,2})^{t-2}}{(1 + q_{1,2})^{t-1}} (p_1 - p_2) q_{1,2} \epsilon, & t = 2, \dots \end{cases}.$$

The resulting solution is feasible for small positive ϵ . Furthermore, we achieve an objective improvement

$$\begin{aligned} \Delta C &= c_{1,2} \left(-1 + \gamma \frac{p_2 + p_1 q_{1,2}}{1 + q_{1,2}} + \sum_{t=3}^{\infty} \frac{(p_1 + p_2 q_{1,2})^{t-3}}{(1 + q_{1,2})^{t-1}} q_{1,2} (p_1 - p_2)^2 \gamma^{t-1} \right) \epsilon \\ &\quad + h_1 \left(1 + \sum_{t=2}^{\infty} \frac{(p_1 + p_2 q_{1,2})^{t-2}}{(1 + q_{1,2})^{t-1}} (p_1 - p_2) \gamma^{t-1} \right) \epsilon \\ &\quad + h_2 \left(-1 + q_{1,2} \sum_{t=2}^{\infty} \frac{(p_1 + p_2 q_{1,2})^{t-2}}{(1 + q_{1,2})^{t-1}} (p_1 - p_2) \gamma^{t-1} \right) \epsilon \\ &= c_{1,2} \frac{(1 + q_{1,2})(1 - p_1 \gamma)(p_2 \gamma - 1)}{1 + q_{1,2} - (p_1 + p_2 q_{1,2}) \gamma} \epsilon \\ &\quad + \frac{(1 + q_{1,2})((1 - \gamma p_2)h_1 - (1 - \gamma p_1)h_2)}{1 + q_{1,2} - (p_1 + p_2 q_{1,2}) \gamma} \epsilon \\ &< 0, \end{aligned}$$

where the last inequality follows by Assumption 2.4.

The rationale behind the construction is to choose training and promotion perturbations so that staff at the two levels increase proportionally in later periods, which implies feasibility; on the other hand, the cost decrease exceeds the increase when discounts and monotonic payrolls are applied, which leads to the lower objective.

A.1.2 Proof of (4a)

Since (4b) can be achieved without resorting to (4a), we consider the solutions where (4b) is satisfied while (4a) is not. Again we rename the earliest such period to be period 1. We have $x_0^0 > 0$.

Case 1: $x_{1,2}^2 < p_1 s_1^1$.

Construct a new feasible solution with the formulas below:

$$\Delta x_0^t = \begin{cases} -\frac{\epsilon}{p_0}, & t = 0 \\ \frac{p_1}{p_0} \epsilon, & t = 1 \\ 0, & t = 2, \dots \end{cases},$$

$$\Delta s_1^t = \begin{cases} -\epsilon, & t = 1 \\ 0, & t = 2, \dots \end{cases}.$$

The resulting objective improvement is

$$\begin{aligned} \Delta C &= c_0 \left(-\frac{1}{p_0 \gamma} + \frac{p_1}{p_0} \right) \epsilon - h_1 \epsilon \\ &< 0. \end{aligned}$$

Case 2: $x_{1,2}^2 = p_1 s_1^1$.

We first note that at most one of $x_{1,2}^{t+1} = p_1 s_1^t$ and $s_1^{t+1} = d_{t+1}$ can be true provided (4b) for any $t \geq 1$. Assuming both equalities hold for some t , we then have $s_2^{t+1} \geq p_2 s_2^t + x_{1,2}^{t+1} = p_2 s_2^t + p_1 s_1^t$. Since $x_{1,2}^{t+1} > 0$ implies $s_2^{t+1} = q_{1,2} s_1^{t+1}$ by (4b), we further have $d_{t+1} = s_1^{t+1} \geq \frac{p_2 s_2^t + p_1 s_1^t}{q_{1,2}} \geq \frac{p_2 q_{1,2} d_t + p_1 d_t}{q_{1,2}} = \left(p_2 + \frac{p_1}{q_{1,2}}\right) d_t$, but this contradicts Assumption 2.5.

Let i be the smallest possible period with $x_{1,2}^i < p_1 s_1^{i-1}$. From the above observation we have $s_1^t > d_t, t \leq i-1$. Thus we can perturb as follows to obtain a new feasible solution:

$$\begin{aligned} \Delta x_0^t &= \begin{cases} -\frac{\epsilon}{p_0}, & t = 0 \\ -\frac{p_1(p_1 + p_2 q_{1,2})^{t-1}}{p_0 q_{1,2}^t} \epsilon, & t = 1, \dots, i-2 \\ \frac{p_1(p_1 + p_2 q_{1,2})^{i-2}}{p_0 q_{1,2}^{i-2}} \epsilon, & t = i-1 \\ 0, & t = i, \dots \end{cases}, \\ \Delta x_{1,2}^t &= \begin{cases} -p_1 \epsilon, & t = 2 \\ -\frac{p_1^2(p_1 + p_2 q_{1,2})^{t-3}}{q_{1,2}^{t-2}} \epsilon, & t = 3, \dots, i-1 \\ \frac{p_1 p_2(p_1 + p_2 q_{1,2})^{i-3}}{q_{1,2}^{i-3}} \epsilon, & t = i \\ 0, & t = i+1, \dots \end{cases}, \\ \Delta s_1^t &= \begin{cases} -\epsilon, & t = 1 \\ -\frac{p_1(p_1 + p_2 q_{1,2})^{t-2}}{q_{1,2}^{t-1}} \epsilon, & t = 2, \dots, i-1 \\ 0, & t = i, \dots \end{cases}, \\ \Delta s_2^t &= \begin{cases} -\frac{p_1(p_1 + p_2 q_{1,2})^{t-2}}{q_{1,2}^{t-2}} \epsilon, & t = 2, \dots, i-1 \\ 0, & t = i, \dots \end{cases}. \end{aligned}$$

The corresponding objective improvement is

$$\begin{aligned} \Delta C &= c_0 \left(-\frac{1}{p_0 \gamma} - \sum_{t=1}^{i-2} \frac{p_1(p_1 + p_2 q_{1,2})^{t-1} \gamma^{t-1}}{p_0 q_{1,2}^t} + \frac{p_1(p_1 + p_2 q_{1,2})^{i-2} \gamma^{i-2}}{p_0 q_{1,2}^{i-2}} \right) \epsilon \\ &\quad + h_1 \left(-1 - \sum_{t=2}^{i-1} \frac{p_1(p_1 + p_2 q_{1,2})^{t-2} \gamma^{t-1}}{q_{1,2}^{t-1}} \right) \epsilon \\ &\quad + h_2 \left(-\sum_{t=2}^{i-1} \frac{p_1(p_1 + p_2 q_{1,2})^{t-2} \gamma^{t-1}}{q_{1,2}^{t-2}} \right) \epsilon \\ &\quad + c_{1,2} \left(-p_1 \gamma - \sum_{t=3}^{i-1} \frac{p_1^2(p_1 + p_2 q_{1,2})^{t-3} \gamma^{t-1}}{q_{1,2}^{t-2}} + \frac{p_1 p_2(p_1 + p_2 q_{1,2})^{i-3} \gamma^{i-1}}{q_{1,2}^{i-3}} \right) \epsilon \\ &< c_0 \left(-\frac{p_1 p_2^{i-2} \gamma^{i-2}}{p_0} - \sum_{t=1}^{i-2} \frac{p_1^2(p_1 + p_2 q_{1,2})^{t-1} p_2^{i-2-t} \gamma^{i-2}}{p_0 q_{1,2}^t} + \frac{p_1(p_1 + p_2 q_{1,2})^{i-2} \gamma^{i-2}}{p_0 q_{1,2}^{i-2}} \right) \epsilon \end{aligned}$$

$$\begin{aligned}
& + c_{1,2} \left(-p_1 p_2^{i-2} \gamma^{i-1} - \sum_{t=3}^{i-1} \frac{p_1^2 (p_1 + p_2 q_{1,2})^{t-3} p_2^{i-t} \gamma^{i-1}}{q_{1,2}^{t-2}} + \frac{p_1 p_2 (p_1 + p_2 q_{1,2})^{i-3} \gamma^{i-1}}{q_{1,2}^{i-3}} \right) \epsilon \\
& = 0.
\end{aligned}$$

A.2 Problem structure when $n \geq 3$

Given a solution that violates Lemma 3.2, our goal is to construct a new solution that is both feasible and incurs a lower total cost. While the big picture appears similar to the proof when $n = 2$, things are much more complicated here: the effect of Δs_k^t may not end at level $k+1$; instead it can force $x_{k+1,k+2}^{t+2}$ and thus s_{k+2}^{t+2} to change, which will propagate to higher levels; even worse, lower levels may also be influenced since there may be multiple violated levels and the perturbation may not start from level 1. Therefore, it is unlikely that we can rely on one-time substitutions as before.

Instead, our strategy is to construct a perturbation period by period. To develop such a dynamic approach we first introduce three sets of new variables:

- $r_k^t = \frac{s_k^t}{p^t}, \quad \forall k = 0, \dots, n, \quad \forall t = 1, \dots,$
- $z_k^t = \frac{x_k^t}{p^t}, \quad \forall k = 0, \dots, n, \quad \forall t = 1, \dots,$
- $z_{k,k+1}^t = \frac{x_{k,k+1}^t}{p^t}, \quad \forall k = 1, \dots, n-1, \quad \forall t = 1, \dots$

The original problem can be reformulated as follows.

$$\inf W(r, z) = \sum_{t=1}^{\infty} \gamma^{t-1} p^t \left(\sum_{k=0}^n c_k z_k^t + \sum_{k=1}^{n-1} c_{k,k+1} z_{k,k+1}^t + \sum_{k=1}^n h_k r_k^t \right) \quad (8a)$$

$$\text{s.t. } r_1^t \geq d_t / p^t \quad \forall t = 1, \dots \quad (8b)$$

$$r_{k+1}^t \geq q_{k,k+1} r_k^t, \quad \forall k = 1, \dots, n-1, \quad \forall t = 1, \dots \quad (8c)$$

$$r_1^{t-1} - r_1^t + z_0^{t-1} - z_{1,2}^t + z_1^t = 0, \quad \forall t = 1, \dots \quad (8d)$$

$$r_k^{t-1} - r_k^t + z_{k-1,k}^t - z_{k,k+1}^t + z_k^t = 0, \quad \forall k = 2, \dots, n-1, \quad \forall t = 1, \dots \quad (8e)$$

$$r_n^{t-1} - r_n^t + z_{n-1,n}^t + z_n^t = 0, \quad \forall t = 1, \dots \quad (8f)$$

$$z_{k,k+1}^t \leq r_k^{t-1}, \quad \forall k = 1, \dots, n-1, \quad \forall t = 1, \dots \quad (8g)$$

$$r^t, z^t \geq 0, \quad \forall t = 1, \dots, \quad (8h)$$

where $r_0^t = z_0^t$ for $t = 0, \dots$. The constraints above can be divided into three sets: demand/ratio constraints (8b–8c), promotion bounds (8g), and network flow constraints (including flow conservation (8d–8f) and nonnegativity (8h)). Graphically, if we consider the r variables as flows between successive periods and the z variables as flows between successive levels, a feasible solution can be represented by an infinite time-space network. The equivalence of the reformulated problem and the original problem stems from a one-to-one correspondence between their solutions. Therefore, any result obtained from one version applies to the other as well.

Next we identify four structural characteristics of our problem(s). Claim A.1 describes the cost of certain structures and will help justify the superiority of a perturbed solution; Claim A.2 is a dominance property and will enable us to consider a relatively small set of solutions for perturbation; Claims A.3 and A.4 analyze necessary conditions for feasibility and will shed light on how to perturb.

Claim A.1. *For the reformulated problem, any flow circulating counterclockwise (either in a cycle, on a doubly-infinite path, or on a one-way infinite path) incurs negative cost.*

Proof. $z_0^t, \forall t = 1, \dots$ can be reduced to a common super source node representing level 0 in the network. Define a *basic unit* in the grid-like network as either case below:



The corresponding total costs per unit counterclockwise flow are

$$\begin{aligned} \gamma^{t-1}p^t(h_k - h_{k+1}) + p^t c_{k,k+1}(p\gamma^t - \gamma^{t-1}) &< 0, \\ -\gamma^{t-1}h_1p^t + (-\gamma^{t-1}c_0p^t + \gamma^t c_0p^{t+1}) &< 0, \end{aligned}$$

respectively. We will refer to the two types of basic units as *basic square* and *basic triangle*, respectively. Any cycle can be decomposed into a finite number of basic squares and/or triangles; any doubly-infinite path can be decomposed into a countable number of basic squares; and any one-way infinite path can be decomposed into a countable number of basic squares and/or triangles. Since counterclockwise flows around both basic units incur negative costs, the same is true for arbitrary cycles/infinite paths. \square

Claim A.2. *For $t \geq 1$, let ℓ_t and ℓ_{t+1} be levels such that $s_{\ell_{t+1}}^t > q_{\ell_t, \ell_{t+1}} s_{\ell_t}^t$, $x_{\ell_t, \ell_{t+1}}^t > 0$, and $s_{\ell_{t+1}+1}^{t+1} > q_{\ell_{t+1}, \ell_{t+1}+1} s_{\ell_{t+1}}^{t+1}$. Assuming ℓ_t and ℓ_{t+1} exist, for any k with $\min\{\ell_t, \ell_{t+1}\} \leq k \leq \max\{\ell_t, \ell_{t+1}\}$ there exists some $t' \leq t$ such that $x_{k, k+1}^{t'} > 0$. A solution cannot be optimal if both $s_{\ell_{t+1}}^{t+1} = q_{\ell_t, \ell_{t+1}} s_{\ell_t}^{t+1}$ and $x_{k, k+1}^{t+1} = 0$ hold for some k with $\min\{\ell_t, \ell_{t+1}\} \leq k \leq \max\{\ell_t, \ell_{t+1}\}$.*

Proof. Clearly $\ell_t \neq \ell_{t+1}$. Consider level i , the largest such k if $\ell_t > \ell_{t+1}$, or the smallest such k if $\ell_t < \ell_{t+1}$.

Case 1: $i = \ell_t$.

Let $\Delta z_{\ell_t, \ell_{t+1}}^{t+1} = -\Delta z_{\ell_t, \ell_{t+1}}^t = \epsilon$. Since $x_{\ell_t, \ell_{t+1}}^{t+1} = 0$, we know $x_{\ell_{t+1}, \ell_{t+1}+2}^{t+1} < p s_{\ell_{t+1}}^t$ and thus feasibility is not violated. By Claim A.1 this corresponds to a counterclockwise flow around a basic square and incurs less total cost.

Case 2: $i \neq \ell_t$.

We first have

$$\begin{aligned} s_{i+1}^{t+1} &= p s_{i+1}^t - x_{i+1, i+2}^{t+1} \leq p s_{i+1}^t, \\ s_i^{t+1} &\geq p s_i^t + x_{i-1, i}^{t+1} \geq p s_i^t. \end{aligned}$$

If $\ell_t > \ell_{t+1}$, then $x_{i+1,i+2}^{t+1} > 0$ by definition of i , and hence $s_{i+1}^{t+1} < ps_{i+1}^t$, which together with $s_{i+1}^{t+1} \geq q_{i,i+1}s_i^{t+1}$ indicates that $s_{i+1}^t > q_{i,i+1}s_i^t$. Similarly, if $\ell_t < \ell_{t+1}$, then $x_{i-1,i}^{t+1} > 0$ and again $s_i^{t+1} > ps_i^t$ indicates that $s_{i+1}^t > q_{i,i+1}s_i^t$. If $x_{i,i+1}^t = 0$, then $s_{i+1}^t > q_{i,i+1}s_i^t$ further indicates that $s_{i+1}^{t-1} > q_{i,i+1}s_i^{t-1}$. Recursively utilizing this fact for $t-1, t-2, \dots$, finally we can find a period $t_0 \geq t'$ ($t_0 = t$ if $x_{i,i+1}^t > 0$) where $s_{i+1}^{t_0} > q_{i,i+1}s_i^{t_0}$ and $x_{i,i+1}^{t_0} > 0$. Now construct a new solution by letting $\Delta z_{i,i+1}^{t+1} = -\Delta z_{i,i+1}^{t_0} = \epsilon$; it is feasible due to slack and zero promotions at level i in periods $t_0 + 1, \dots, t$, and its lower cost is guaranteed by Claim A.1. \square

Claim A.3. For $t \geq 1$, For $t \geq 1$, let g be a level such that $x_{g,g+1}^{t+1} = 0$, $g \leq n$. If $s_g^{t+1} = q_{g-1,g}s_{g-1}^{t+1} = \dots = q_{1,g}s_1^{t+1}$, where $q_{k,\ell} = q_{k,k+1}q_{k+1,k+2} \dots q_{\ell-1,\ell}$, then at most one of $x_{k,k+1}^{t+1} = ps_k^t$ and $s_1^{t+1} = d_{t+1}$ can be true for each $k \leq g-1$.

Proof. Assume both equalities hold. Adding together equations (1e) for levels $k+1, \dots, g$ in period $t+1$, plugging in $x_{k,k+1}^{t+1} = ps_k^t$ and $s_g^{t+1} = q_{g-1,g}s_{g-1}^{t+1} = \dots = q_{1,g}s_1^{t+1}$, we have

$$s_1^{t+1} \sum_{i=k+1}^g q_{1,i} \geq ps_1^t \sum_{i=k}^g q_{1,i}. \quad (9)$$

Note that

$$\begin{aligned} \frac{\sum_{i=k}^g q_{1,i}}{\sum_{i=k+1}^g q_{1,i}} &= \frac{q_{1,k}}{\sum_{i=k+1}^g q_{1,i}} + 1 \\ &= \frac{1}{\sum_{i=k+1}^g q_{k,i}} + 1 \\ &\geq \frac{1}{\sum_{i=1}^{g-k} q_{\max}^i} + 1 \\ &= \frac{1 + \sum_{i=1}^{g-k} q_{\max}^i}{q_{\max}(1 + \sum_{i=1}^{g-k-1} q_{\max}^i)}. \end{aligned}$$

Combined with (9), this results in

$$s_1^{t+1} \geq \frac{1 + \sum_{i=1}^{g-k} q_{\max}^i}{q_{\max}(1 + \sum_{i=1}^{g-k-1} q_{\max}^i)} ps_1^t > \frac{p}{q_{\max}} s_1^t \geq \frac{p}{q_{\max}} d_t \geq d_{t+1}$$

by Assumption 2.5. We have arrived at a contradiction. \square

Claim A.4. For $t \geq 1$, let k be an arbitrary level, and $g = \min\{i : x_{i,i+1}^{t+1} = 0, k+1 \leq i \leq n\}$. If $x_{k,k+1}^{t+1} = ps_k^t$ and $s_g^{t+1} = q_{g-1,g}s_{g-1}^{t+1} = \dots = q_{\ell+1,g}s_{\ell+1}^{t+1}$, $0 \leq \ell \leq k-1$, then $x_{\ell,\ell+1}^{t+1} > 0$.

Proof. The result follows directly by Lemma 3.1 if $x_{\ell,\ell+1}^{t+1} > 0$. Consider when $x_{\ell,\ell+1}^{t+1} = 0$. $\ell = k-1$ is trivial since $x_{k-1,k}^{t+1} = s_k^{t+1} > 0$. For $\ell \leq k-2$, we show that $x_{\ell,\ell+1}^{t+1} > pq_{\ell+1,k}s_{\ell+1}^t$ by induction.

Base case: Here $\ell = k-2$. $s_{k-1}^{t+1} = ps_{k-1}^t + x_{k-2,k-1}^{t+1} - s_k^{t+1} = ps_{k-1}^t + x_{k-2,k-1}^{t+1} - q_{k-1,k}s_{k-1}^{t+1}$ implies $s_{k-1}^{t+1} = \frac{ps_{k-1}^t + x_{k-2,k-1}^{t+1}}{1 + q_{k-1,k}}$. $q_{k-1,g}s_{k-1}^{t+1} = s_g^{t+1} = ps_g^t + x_{g-1,g}^{t+1} > ps_g^t \geq pq_{k-1,g}s_{k-1}^t$ implies $s_{k-1}^{t+1} > ps_{k-1}^t$. Therefore, $\frac{ps_{k-1}^t + x_{k-2,k-1}^{t+1}}{1 + q_{k-1,k}} > ps_{k-1}^t$, and thus $x_{k-2,k-1}^{t+1} > pq_{k-1,k}s_{k-1}^t > 0$.

Induction: Assume that the claim holds for ℓ , $1 \leq \ell \leq k-2$, i.e. $x_{\ell,\ell+1}^{t+1} > pq_{\ell+1,k}s_{\ell+1}^t$, then

$$\begin{aligned} s_\ell^{t+1} &= ps_\ell^t + x_{\ell-1,\ell}^{t+1} - x_{\ell,\ell+1}^{t+1} \\ &< ps_\ell^t + x_{\ell-1,\ell}^{t+1} - pq_{\ell+1,k}s_{\ell+1}^t \\ &< ps_\ell^t + x_{\ell-1,\ell}^{t+1} - pq_{\ell,k}s_\ell^t. \end{aligned}$$

On the other hand, $q_{\ell,g}s_\ell^{t+1} = s_g^{t+1} > ps_g^t \geq pq_{\ell,g}s_\ell^t$ implies $s_\ell^{t+1} > ps_\ell^t$. Therefore, $x_{\ell-1,\ell}^{t+1} > pq_{\ell,k}s_\ell^t > 0$, i.e. the claim holds for $\ell-1$ as well. \square

A.3 A perturbation procedure for $n \geq 3$

(4a) is a special case of (4b) if we define $s_0^t = d_t$, $x_{0,1}^t = px_0^{t-1}$, $q_{0,1} = 1$; the only difference is that $x_{0,1}^t$ have no upper bound. Pick the earliest period where (4b) is violated, as before we rewrite it as period 1 and redefine successive periods as $2, 3, \dots$. Let m be any violated level in period 1. Below is the key notation we will use:

- j_t : a level that has ever seen staff reduction and has full promotions in period t ; mathematically this means $x_{j_t,j_t+1}^t = ps_{j_t}^{t-1}$ and $\Delta s_{j_t}^{t'} < 0$ for some $t' < t$.
- ℓ_t : a level with ratio slack and for which there is a j_t where all levels in between have positive promotions in period t , i.e. $s_{\ell_t+1}^t > q_{\ell_t,\ell_t+1}s_{\ell_t}^t$ and $x_{k,k+1}^t > 0$, $\forall k = \ell_t, \dots, j_t$.
- $\bar{\ell}_t$: an ℓ_t where the ratio relationship would be violated if not perturbed in period $t+1$, i.e. $s_{\bar{\ell}_t+1}^{t+1} + p\Delta s_{\bar{\ell}_t+1}^t < q_{\bar{\ell}_t,\bar{\ell}_t+1}(s_{\bar{\ell}_t}^{t+1} + p\Delta s_{\bar{\ell}_t}^t)$ and $\bar{\ell}_t \in L_t$. An implicit constraint is $s_{\bar{\ell}_t+1}^{t+1} = q_{\bar{\ell}_t,\bar{\ell}_t+1}s_{\bar{\ell}_t}^{t+1}$.
- J_t, L_t, \bar{L}_t : the set of all j_t , the set of all ℓ_t , and the set of all $\bar{\ell}_t$, respectively. $\bar{L}_t \subseteq L_t$.
- $\max j_t$: the largest element in J_t , i.e. $\max\{j : j \in J_t\}$, with other maxima and minima defined analogously.

Suppose we have perturbed periods $1, \dots, t-1$ and the current perturbed solution satisfies constraints in these periods. Clearly all j_t should be perturbed to guarantee feasibility. By Claims A.3 and A.4 there must exist an ℓ_t for each j_t , and thus it is a candidate for the perturbation in period t to stop. The perturbation in period $t-1$ also causes infeasibility at $\bar{\ell}_{t-1}$ in period t , which constitutes an additional source for further perturbation. Obviously we can set $J_1 = L_1 = \{m\}$, $\bar{L}_0 = \emptyset$. Procedure 1 illustrates how to identify J_t, L_t and \bar{L}_{t-1} when $t \geq 2$.

We are now ready to construct perturbing operations. Since Lemma 3.1 and Claim A.2 have identified several non-optimal cases, we only consider solutions that satisfy the conditions therein.

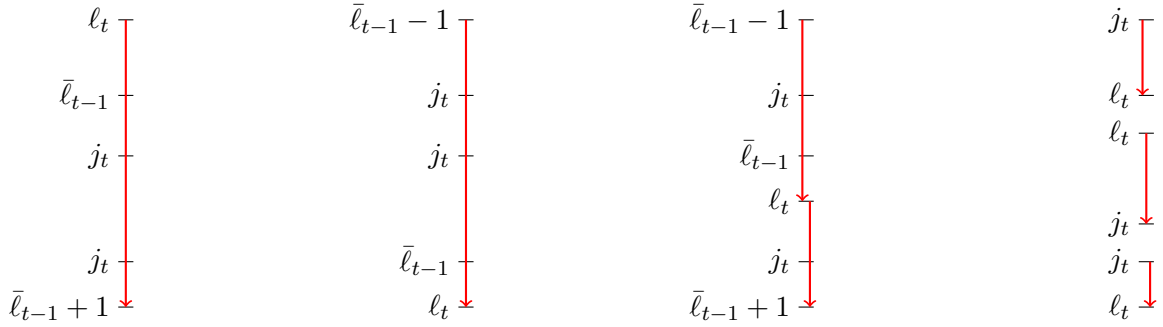
Perturbation for period 1 is trivial: $\Delta r_{m+1}^1 = -\Delta r_m^1 = \Delta z_{m,m+1}^1 = -\epsilon$. For an arbitrary $t \geq 2$ before the possible end period, $\Delta z_{k,k+1}^t$ consists of two parts: a change due to full promotions at j_t , and a change due to tight ratio relationships at $\bar{\ell}_{t-1}$. The calculation of Δz consequently depends on the locations of j_t, ℓ_t and $\bar{\ell}_{t-1}$. If there is an $\bar{\ell}_{t-1} \notin L_t$, then the levels between any pair of j_t and $\bar{\ell}_{t-1}$ must have been reached at some point before t , thus by Claim A.2 we know $x_{k,k+1}^t > 0$, $\forall k = \bar{\ell}_{t-1}, \dots, \ell_t, \forall \ell_t \in L_t$. Otherwise $\bar{L}_{t-1} = \emptyset$. Hence although there may be multiple j_t, ℓ_t and $\bar{\ell}_{t-1}$, and a large number of possible locations, it suffices to check the following four cases:

Procedure 1 J_t , L_t , and \bar{L}_{t-1} when $t \geq 2$

```

1:  $J_t = L_t = \bar{L}_{t-1} = \emptyset, L'_{t-1} = \{\ell_{t-1} : \Delta x_{\ell_{t-1}, \ell_{t-1}+1}^{t-1} < 0\}$ 
2: for  $\ell_{t-1} \in L_{t-1}$  do
3:   if  $s_{\ell_{t-1}+1}^t + p\Delta s_{\ell_{t-1}+1}^{t-1} < q_{\ell_{t-1}, \ell_{t-1}+1}(s_{\ell_{t-1}}^t + p\Delta s_{\ell_{t-1}}^{t-1})$  then
4:      $\bar{L}_{t-1} = \bar{L}_{t-1} \cup \{\ell_{t-1}\}$ 
5:   end if
6: end for
7: for  $k = \min\{i : i \in \bigcup_{t' \leq t-1} J_{t'} \cup L'_{t'}\}$  to  $\max\{i : i \in \bigcup_{t' \leq t-1} J_{t'} \cup L'_{t'}\} + 1$  do
8:   if  $x_{k, k+1}^t = ps_k^{t-1}$  and  $\Delta s_k^{t'} < 0$  for some  $t' < t$  then
9:      $J_t = J_t \cup \{k\}$ 
10:     $g = \min\{i : x_{i, i+1}^{t+1} = 0, k+1 \leq i \leq n\}$ 
11:    for  $i \leq g-1$  do
12:      if  $s_{i+1}^t > q_{i, i+1}s_i^t$  and  $x_{k, k+1}^t > 0, \forall k = i, \dots, g-1$  then
13:         $L_t = L_t \cup \{i\}$ 
14:      end if
15:    end for
16:  end if
17: end for

```



In the above graph, promotions are positive at levels connected by the vertical lines, a gap between levels indicates zero promotions, and the direction of the arrows is consistent with the perturbation flows. Claims A.5 through A.8 validate the operations we use in each case.

Claim A.5. *Assuming a perturbed solution is feasible for periods $1, \dots, t-1$, if there exists an $\ell_t \in L_t$ with $\ell_t \geq \max\{\max j_t, \max \bar{\ell}_{t-1} + 1\}$ and $x_{k, k+1}^t > 0$ for k with $\min\{\min j_t, \min \bar{\ell}_{t-1} + 1\} \leq k \leq \ell_t$, then a solution also feasible for period t can be obtained by sequentially applying equations*

$$\Delta z_{k, k+1}^t = \min\{\Delta z_{k-1, k}^t + \Delta r_k^{t-1} - \Delta r_{k-1}^t q_{k-1, k}, \Delta r_k^{t-1}\}, \quad (10a)$$

$$\Delta r_k^t = \Delta z_{k-1, k}^t + \Delta r_k^{t-1} - \Delta z_{k, k+1}^t, \quad (10b)$$

to $k = k_b, \dots, k_e$, and finally letting

$$\Delta r_{k_e+1}^t = \Delta z_{k_e, k_e+1}^t + \Delta r_{k_e+1}^{t-1}, \quad (10c)$$

where $k_b = \min\{\min j_t, \min \bar{\ell}_{t-1} + 1\}$, $k_e = \min\{\ell_t : \ell_t \geq \max\{\max j_t, \max \bar{\ell}_{t-1} + 1\}\}$.

Proof. We can choose the smallest such ℓ_t as a common level for the perturbation driven by all j_t and $\bar{\ell}_{t-1}$ to stop in period t . To fix potential infeasibility caused by full promotions or tight ratio relationships, it is reasonable to decrease the promotions at these levels and update employment accordingly. The level to start such operations should of course be $\min\{\min j_t, \min \bar{\ell}_{t-1} + 1\}$. The decreasing effect will finally be conveyed to the ℓ_t we choose.

We consider level k to determine $\Delta z_{k,k+1}^t$. Since Δr_k^{t-1} , Δr_{k-1}^t , and $\Delta z_{k-1,k}^t$ (in particular, $\Delta z_{k_b-1,k_b}^t = 0$) are all known as we reach node (k, t) , we can solve

$$\begin{cases} \Delta z_{k-1,k}^t + \Delta r_k^{t-1} = \Delta z_{k,k+1}^t + \Delta r_k^t \\ \Delta r_k^t \geq \Delta r_{k-1}^t q_{k-1,k} \\ \Delta z_{k,k+1}^t \leq \Delta r_k^{t-1} \end{cases},$$

and choose the largest possible $\Delta z_{k,k+1}^t$ (so that $|\Delta z_{k,k+1}^t|$ is as small as possible), which yields (10a). The three constraints above represent flow conservation, ratio relationships, and promotion bounds respectively. Δr_k^t can then be determined via flow conservation, i.e. (10b). (10c) is a result of stopping perturbation at k_e . \square

Claim A.6. *Assuming a perturbed solution is feasible for periods $1, \dots, t-1$, if there exists an $\ell_t \in L_t$ with $\ell_t \leq \min\{\min j_t, \min \bar{\ell}_{t-1} - 1\}$ and $x_{k,k+1}^t > 0$ for k with $\ell_t \leq k \leq \max\{\max j_t, \max \bar{\ell}_{t-1} - 1\}$, then a solution also feasible for period t can be obtained by sequentially applying equations*

$$\Delta z_{k,k+1}^t = \min\{\Delta z_{k+1,k+2}^t + \Delta r_{k+2}^t / q_{k+1,k+2} - \Delta r_{k+1}^{t-1}, \Delta r_k^{t-1}\}, \quad (11a)$$

$$\Delta r_{k+1}^t = \Delta z_{k,k+1}^t + \Delta r_{k+1}^{t-1} - \Delta z_{k+1,k+2}^t, \quad (11b)$$

to $k = k_b, \dots, k_e$, and finally letting

$$\Delta r_{k_e}^t = \Delta r_{k_e}^{t-1} - \Delta z_{k_e,k_e+1}^t, \quad (11c)$$

where $k_b = \max\{\max j_t, \max \bar{\ell}_{t-1} - 1\}$, $k_e = \max\{\ell_t : \ell_t \leq \min\{\min j_t, \min \bar{\ell}_{t-1} - 1\}\}$.

Proof. We can choose the largest such ℓ_t as a common level for the perturbation driven by all j_t and $\bar{\ell}_{t-1}$ to stop in period t . Again we fix potential infeasibility by decreasing promotions. But unlike the previous claim, we operate in a top-down fashion since the perturbation is expected to end at the $\ell_t \in L_t$ we choose.

We consider level $k+1$ to determine $\Delta z_{k,k+1}^t$. Since Δr_{k+1}^{t-1} , Δr_k^{t-1} , Δr_{k+2}^t , and $\Delta z_{k+1,k+2}^t$ (in particular, $\Delta z_{k_b+1,k_b+2}^t = 0$) are all known as we reach node $(k+1, t)$, we can solve

$$\begin{cases} \Delta z_{k,k+1}^t + \Delta r_{k+1}^{t-1} = \Delta z_{k+1,k+2}^t + \Delta r_{k+1}^t \\ \Delta r_{k+1}^t \leq \Delta r_{k+2}^t / q_{k+1,k+2} \\ \Delta z_{k,k+1}^t \leq \Delta r_k^{t-1} \end{cases},$$

and choose the largest possible $\Delta z_{k,k+1}^t$, which yields (11a). Δr can then be determined via flow conservation, i.e. (11b) and (11c). \square

Claim A.7. *Assuming a perturbed solution is feasible for periods $1, \dots, t-1$, if there exists an $\ell_t \in L_t$ with $\min\{\min j_t, \min \bar{\ell}_{t-1} + 1\} \leq \ell_t \leq \max\{\max j_t, \max \bar{\ell}_{t-1} - 1\}$ and $x_{k,k+1}^t > 0$ for k with $\min\{\min j_t, \min \bar{\ell}_{t-1} + 1\} \leq k \leq \max\{\max j_t, \max \bar{\ell}_{t-1} - 1\}$, then a solution also feasible for period t can be obtained by sequentially applying equations (10) to levels $\min\{\min j_t, \min \bar{\ell}_{t-1} + 1\} \leq k \leq \ell_t - 1$, equations (11) to levels $\ell_t + 1 \leq k \leq \max\{\max j_t, \max \bar{\ell}_{t-1} - 1\}$, and finally letting*

$$\Delta z_{\ell_t, \ell_t+1}^t = \min\{\Delta r_{\ell_t}^{t-1} + \Delta z_{\ell_t-1, \ell_t}^t - \Delta r_{\ell_t-1}^t q_{\ell_t-1, \ell_t}, \Delta z_{\ell_t+1, \ell_t+2}^t + \Delta r_{\ell_t+2}^t / q_{\ell_t+1, \ell_t+2} - \Delta r_{\ell_t+1}^{t-1}, \Delta r_{\ell_t}^t\} \quad (12a)$$

$$\Delta r_{\ell_t}^t = \Delta z_{\ell_t-1, \ell_t}^t + \Delta r_{\ell_t}^{t-1} - \Delta z_{\ell_t, \ell_t+1}^t \quad (12b)$$

$$\Delta r_{\ell_t+1}^t = \Delta z_{\ell_t, \ell_t+1}^t + \Delta r_{\ell_t+1}^{t-1} - \Delta z_{\ell_t+1, \ell_t+2}^t. \quad (12c)$$

Proof. This is a hybrid of the previous two claims. By a similar analysis the perturbation should only cover levels from $\min\{\min j_t, \min \bar{\ell}_{t-1} + 1\}$ to $\max\{\max j_t, \max \bar{\ell}_{t-1} - 1\}$. We can treat any such ℓ_t as a breakpoint above which Claim A.6 applies and below which Claim A.5 applies. It only remains to perturb level ℓ_t itself. To guarantee feasibility we can choose the minimum of the values provided by Claims A.5 and A.6 to determine $\Delta z_{\ell_t, \ell_t+1}^t$, and $\Delta r_{\ell_t+1}^t$ and $\Delta r_{\ell_t}^t$ can then be calculated by flow conservation. This yields (12). \square

Claim A.8. *Assuming a perturbed solution is feasible for periods $1, \dots, t-1$, if there are zero promotions between successive levels ℓ_t, \dots, j_t or j_t, \dots, ℓ_t , then a solution also feasible for period t can be obtained by applying (10) to each succession ℓ_t, \dots, j_t or (11) to each succession j_t, \dots, ℓ_t , as long as the j_t values partition J_t .*

Proof. By Claim A.2 there is no $\bar{\ell}_{t-1}$ and so we only consider the impact of full promotions at j_t . The applicability of the claims follows immediately from the fact that each succession here is an instance of Claim A.5 or A.6.

We need to ensure, though, that each j_t is included in exactly one succession. Starting from $\max j_t$, if there exists some ℓ_t with $\ell_t \geq \max j_t$ and $x_{k,k+1}^t > 0$ for k with $\max j_t \leq k \leq \ell_t$, there may be a lower j_t that satisfies this condition as well; hence we can decrease promotions from the lowest such j_t to any such ℓ_t , and use the formulas from Claim A.5 to determine $\Delta z_{k,k+1}^t$ and Δr_k^t . Otherwise, by Claims A.3 and A.4 there must exist some ℓ_t with $\ell_t < \max j_t$ and $x_{k,k+1}^t > 0$ for k with $\ell_t \leq k \leq \max j_t$, and so we can use the formulas from Claim A.6 to determine $\Delta z_{k,k+1}^t$ and Δr_{k+1}^t until reaching the highest such ℓ_t . After either case is done, we can move downwards to the next j_t that has not been visited, and apply the same argument again. This process goes on until reaching $\min j_t$. \square

We now state our perturbation procedure as Procedure 2. Note that if in some period we find $x_{k+1,k+2}^t < ps_{k+1}^{t-1}$ for each perturbed level k (including when $J_t = \bar{L}_{t-1} = \emptyset$), the procedure can end in this period, and from then on the perturbed solution will remain the same as the initial solution; otherwise, the procedure will iterate forward infinitely but converge to a new feasible solution. Claim A.9 justifies the lower cost of the final perturbed solution.

Claim A.9. *Procedure 2 modifies the given solution by adding to it a series of negative cost cycles or infinite paths in the time-space network.*

Proof. Except for the possible end period, all the perturbations are initiated by decreasing the z values. Pick any node (k, t) with negative Δr flows in the perturbation network. Each time we conduct an operation as in Claims A.5 to A.8, by flow conservation Δr_k^t is passed to either Δr_k^{t-1} or $\Delta z_{k-1, k}^t$, resulting in a left arc and a downward arc, respectively. As this propagates, two cases may occur.

Case 1: We reach some ℓ_t ($t \leq t$) where the flow turns right, then follows a right-down-right pattern, and finally turns upwards at the perturbation's end period. This constitutes a counterclockwise cycle as described in Claim A.1.

Case 2: The flow may continue shifting in a right-down-right pattern perpetually, which constitutes a counterclockwise infinite path as described in Claim A.1. \square

Procedure 2 Perturbation when $n \geq 3$ and $p_k = p_{k+1}, \forall k = 1, \dots, n-1$

```

1:  $\Delta r_{m+1}^1 = -\Delta r_m^1 = \Delta z_{m, m+1}^1 = -\epsilon, J_1 = L_1 = \{m\}, \bar{L}_0 = \emptyset, t = 2, ep = 0$ 
2: while  $ep = 0$  do
3:   if  $x_{k+1, k+2}^t < ps_{k+1}^{t-1}$  for every perturbed level  $k$  then
4:      $ep = 1$ 
5:   else
6:     Run Procedure 1 to identify  $J_t, L_t$ , and  $\bar{L}_{t-1}$ 
7:     if Claim A.5 is applicable then
8:       Perturb according to Claim A.5
9:     else if Claim A.6 is applicable then
10:      Perturb according to Claim A.6
11:    else if Claim A.7 is applicable then
12:      Perturb according to Claim A.7
13:    else
14:      Perturb according to Claim A.8
15:    end if
16:    Update  $\Delta x, \Delta s, x, s$ 
17:     $t = t + 1$ 
18:  end if
19: end while
20: for perturbed levels  $k$  in increasing order do
21:    $\Delta z_{k, k+1}^t = \Delta r_k^{t-1} + \Delta z_{k-1, k}^t$ 
22:   Update  $\Delta x, \Delta s, x, s$ 
23: end for
```

A.4 A technical note on the perturbation amount ϵ

We have constructed perturbation operations that are feasible for small enough ϵ . To obtain a valid perturbed solution, however, we need to guarantee that $\epsilon > 0$. Because the perturbation may range over infinitely many periods, it could be that the required ϵ eventually converges to zero. We next argue why this is not the case.

If the perturbation ends in some period, it essentially works in finite dimensions and thus $\epsilon > 0$. On the other hand, if the procedure iterates infinitely, ϵ depends on the x and s values. In particular, the values of the training variables, promotion variables, and slack between consecutive levels matter since we are decreasing them. To eliminate the possibility of ϵ converging to zero, it suffices to bound those values from below wherever they are perturbed.

When $n = 2$, the proof of (4b) reduces the promotion in period 1 and increases it in later periods. The increments depend on the reduction in period 1 and so depend on how much we can decrease there, which clearly is positive. In the proof of (4a), ϵ depends on $s_1^{t+1} - d_{t+1}$ if $x_{1,2}^{t+1} = p_1 s_1^t$, and $p_1 s_1^t - x_{1,2}^{t+1}$ if $s_1^{t+1} = d_{t+1}$. In the former case, $s_1^{t+1} - d_{t+1} = \frac{p_2 s_2^t + p_1 s_1^t}{q_{1,2}} - d_{t+1} \geq (p_{\min} + \frac{p_{\min}}{q_{\max}})d_t - d_{t+1} \geq p_{\min}d_t$, which is bounded away from zero since $d_t \geq d_1 > 0$. In the latter case, the perturbation ends in period $t + 1$ so the perturbation is finite.

Now consider when $n \geq 3$. Recall that in all the cases considered for perturbation, we perturb levels between j_t and ℓ_t , or $\bar{\ell}_{t-1}$ and ℓ_t , or both. For any level k with $\ell_t \leq k \leq j_t$, the proof of Claim A.4 actually provides a lower bound independent of t , i.e. $x_{k,k+1}^{t+1} > pq_{k+1,j_t} s_{k+1}^t \geq pq_{1,j_t} d_1$. Furthermore, if we redefine $g = \min \{i : x_{i,i+1}^{t+1} \leq C, j_t + 1 \leq i \leq n\}$, where $C > 0$ can be any constant less than $\min_{t'=1,\dots,t} \{ps_i^{t'}\}$ for each i (such as $C = pq_{1,n}d_1$), the bound still holds, and hence the x variables at levels k with $j_t \leq k \leq \ell_t$ are also bounded below by a constant independent of t .

The same trick can be applied to any level k between ℓ_t and $\bar{\ell}_{t-1}$, i.e. the lower bounds on the x variables in Claim A.2 can be strengthened from zero to the same constant C above. Finally, Claim A.3 is still correct for $x_{g,g+1}^{t+1} \leq C$ as long as $C < pq_{\max}^{n-1}d_1$. This enables us to use a g that still satisfies the properties in both claims as the level starting from which a search of ℓ_t is conducted in Procedure 1.

Since the proofs of the claims only utilize the linear relationships between s and x , we can obtain similar bounds for the slack between consecutive s variables. All the lower bounds depend only on d_1 and fixed parameters like n , p and q . It follows that the ϵ in the infinite case is indeed positive.

B Proof of Theorem 3.3

We will construct a one-period lookahead policy based on Lemmas 3.1 and 3.2, and then demonstrate that the resulting solution is unique.

B.1 A one-period lookahead policy

The notation we use is summarized below:

- $B_{i,i+1}^t$: upper promotion bound for level i in period t .
- I : a list of levels where promotion bounds would be violated if hiring were not allowed.

Claim B.1. *Consider a subsystem consisting of levels from j to $k + 1$, $0 \leq j \leq k \leq n - 1$ in period t . Assume $s_i^{t-1}, i = j, \dots, k + 1$ and s_j^t are known. If $p_{j+1}s_{j+1}^{t-1} < q_{j,j+1}s_j^t$ and $p_{i+1}s_{i+1}^{t-1} \leq q_{i,i+1}p_i s_i^{t-1}, i = j + 1, \dots, k$, then the unique solution to the following equations provides a solution*

that satisfies promotion bounds at levels in I :

$$p_{j+1}s_{j+1}^{t-1} + x_{j,j+1}^t - x_{j+1,j+2}^t + x_{j+1}^t = q_{j,j+1}(s_j^t - x_{j,j+1}^t) \quad (13a)$$

$$p_{i+1}s_{i+1}^{t-1} + x_{i,i+1}^t - x_{i+1,i+2}^t + x_{i+1}^t = q_{i,i+1}(p_i s_i^{t-1} + x_{i-1,i}^t - x_{i,i+1}^t + x_i^t), \quad (13b)$$

$$i = j + 1, \dots, k - 1$$

$$p_{k+1}s_{k+1}^{t-1} + x_{k,k+1}^t + x_{k+1}^t = q_{k,k+1}(p_k s_k^{t-1} + x_{k-1,k}^t - x_{k,k+1}^t + x_k^t), \quad (13c)$$

where $x_i^t = 0$ if $i \notin I$ and $x_{i,i+1}^t = B_{i,i+1}^t$ if $i \in I$.

Proof. Clearly any solution to the above linear system satisfies promotion bounds at any $i \in I$. Nonnegativity of the x variables is guaranteed by the deficiency of staff at level j and the tight ratio relationships (with respect to retention from period $t - 1$) at levels j, \dots, k . For a specific solution x , a policy can be obtained by letting

$$s_i^t = \begin{cases} s_i^t - x_{i,i+1}^t, & i = j \\ p_i s_i^{t-1} + x_{i-1,i}^t - x_{i,i+1}^t + x_i^t, & i = j + 1, \dots, k \\ p_i s_i^{t-1} + x_{i-1,i}^t + x_i^t, & i = k + 1 \end{cases} \quad (14)$$

Let the coefficient matrix be such that column ℓ ($\ell = 1, \dots, k - j + 1$) records the coefficients of $x_{j+\ell-1,j+\ell}^t$ if $j + \ell - 1 \notin I$ and $x_{j+\ell}^t$ otherwise. The elements are

$$a_{m\ell} = \begin{cases} -q_{j+\ell-1,j+\ell}, & m = \ell + 1 \\ 1 \text{ if } j + \ell - 1 \in I \text{ and } 1 + q_{j+\ell,j+\ell+1} \text{ otherwise}, & m = \ell \\ 0 \text{ if } j + \ell - 1 \in I \text{ and } -1 \text{ otherwise}, & m = \ell - 1 \\ 0, & \text{otherwise} \end{cases}.$$

Define D_m ($m = 1, \dots, k - j + 1$) as the determinant of the submatrix composed of the first m rows and the first m columns. For $m \geq 2$, we have the recursion

$$D_m = \begin{cases} D_{m-1}, & j + m - 1 \in I \\ (1 + q_{m+j-1,m+j})D_{m-1} - q_{m+j-1,m+j}D_{m-2}, & \text{otherwise} \end{cases}.$$

By induction we know $D_{k-j+1} \geq D_{k-j} \geq \dots \geq D_1 > 0$, and hence the solution is unique. \square

We now construct a feasible one-period lookahead policy by solving subproblems composed of levels $1, \dots, k + 1$ sequentially until $k = n - 1$. During each loop, we first check if the resulting solution is feasible without promotion at level k , if yes then we are done. Otherwise, we try to get a solution which uses only promotions, i.e. solve (13) with $I = \emptyset$. If this happens to be feasible, then we update the s^t and x^t values and exit; otherwise we calculate a feasible solution by allowing hiring, i.e. solve (13) with $I \neq \emptyset$. For each k , we keep iterating these steps for subproblems composed of levels $j, \dots, k + 1$ so that we can stop at the highest j and the lower levels are not affected. When determining promotion and hiring, we force the ratio constraints to be tight so that we use the smallest possible x^t . In other words, we promote and hire only if necessary. A formal statement is described in Procedure 3.

We end this section with two comments. First, the procedure is applicable to both $n = 2$ and $n \geq 3$. Second, once i enters I at some iteration, it will be there forever: The first time i enters I ,

Procedure 3 A one-period lookahead policy

```
1:  $s_i^t = d_t$  if  $i = 0$  and  $p_i s_i^{t-1}$  if  $i \geq 1$ ,  $B_{0,1}^t = 0$  if  $t = 1$  and  $\infty$  if  $t \geq 2$ ,  $B_{i,i+1}^t = p_i s_i^{t-1}$ ,  $i = 1, \dots, n-1$ 
2:  $k = 0$ 
3: while  $k \leq n-1$  do
4:    $j = k$ 
5:   while  $j \geq 0$  and  $s_{j+1}^t < q_{j,j+1} s_j^t$  do
6:      $I = \emptyset$ 
7:      $B_{j,j+1}^t = B_{j,j+1}^t - x_{j,j+1}^t$ 
8:     Solve (13)
9:      $I = \{i : x_{i,i+1}^t > B_{i,i+1}^t, j \leq i \leq k\}$ 
10:    if  $I \neq \emptyset$  then
11:      while  $I$  is changed do
12:        Solve (13)
13:        Update  $I$ 
14:      end while
15:    end if
16:    Update  $s$  with (14)
17:     $j = j - 1$ 
18:  end while
19:   $k = k + 1$ 
20: end while
```

$x_{i,i+1}^t$ must be decreased (from infeasibility) to full promotion and so s_i^t must be increased in the next iteration (which is true since the only possibility for s_i^t not to be increased is then to decrease x_i^t or $x_{i-1,i}^t$, but this would induce infeasibility between s_i^t and lower levels). To further satisfy the ratio relationships at levels i to k , $s_{\ell+1}^t$ ($i \leq \ell \leq k$) cannot be decreased either, which in turn forces the promotions at these levels to be full if they were. This implies the procedure terminates.

B.2 Optimality of the one-period lookahead policy

Claim B.2. *Recursively applying Procedure 3 yields the unique solution that satisfies Lemmas 3.1 and 3.2.*

Proof. Clearly the solution satisfies the lemmas. Suppose there are multiple feasible solutions for which the lemmas hold. We compare an arbitrary one of them, say (u, y) , with (s, x) obtained from Procedure 3. By Lemma 3.1 $y_{k+1}^t > 0$ only if $y_{k,k+1}^t = p_k u_k^{t-1}$. Start from the earliest period, say t , where there is a difference between x^t and y^t . Pick the lowest different level, say i . We have $s_k^t = u_k^t, \forall k \leq i-1$, and $s^{t'} = u^{t'}, \forall t' \leq t-1$.

We first note that $y_{i,i+1}^t < x_{i,i+1}^t$ or $y_i^t < x_i^t$ cannot be true; otherwise we should be able to obtain a smaller x^t as Procedure 3 finishes since a feasible solution must satisfy the ratio and bound constraints at every level.

We next show that $y_{i,i+1}^t > x_{i,i+1}^t$ or $y_i^t > x_i^t$ cannot be true, either. Since (u, y) is feasible, by (1e) $y_{i,i+1}^t > x_{i,i+1}^t$ or $y_i^t > x_i^t$ implies $u_k^t > s_k^t$ for some $k \geq i$, so there is over promotion/hiring and the lemmas must be violated somewhere in (u, y) . \square

C Extended model used in the computational examples

We modified (1) for our computational examples as follows. Let:

- $T \geq 1$: Length of planning horizon.
- $L \geq 1$: Length of the training program.
- a : Age of the student or worker, $a_l \leq a \leq a_u$.
- $p_{0,i} \in (0, 1)$: Per-period rate of continuing education for students in school year $i = 1, \dots, L-1$, or per-period rate of graduating and going to the workforce for students in school year $i = L$.
- $p_{k,a} \in (0, 1)$: Per-period retention rate of workers of age $a = a_l, \dots, a_u$ that stay in the system at level $k = 0, \dots, n$ from one period to the next.
- $m_{k,a}$: The age distribution of students ($k = 0$) or workers ($k = 1, \dots, n$) of age $a = a_l, \dots, a_u$.
- $s_{0,i,a}^t$: Students of age $a = a_l, \dots, a_u$ in school year $i = 1, \dots, L$ at end of period $t = 1, \dots, T$.
- $s_{k,a}^t$: Workers of age $a = a_l, \dots, a_u$ in level $k = 1, \dots, n$ at end of period $t = 1, \dots, T$.

Our modified problem has the following formulation.

$$\begin{aligned}
\min \quad & C(s, x) = \sum_{t=1}^T \gamma^{t-1} \left(\sum_{k=0}^n c_k x_k^t + \sum_{k=1}^{n-1} c_{k,k+1} x_{k,k+1}^t + \sum_{k=1}^n h_k \sum_{a=a_l}^{a_u} s_{k,a}^t \right) \\
\text{s.t.} \quad & \sum_{a=a_l}^{a_u} s_{1,a}^t \geq d_t, \quad \forall t = 1, \dots, T \\
& \sum_{a=a_l}^{a_u} s_{k+1,a}^t - q_{k,k+1} \sum_{a=a_l}^{a_u} s_{k,a}^t \geq 0, \quad \forall k = 1, \dots, n-1, \quad \forall t = 1, \dots, T \\
& s_{0,1,a}^t - m_{0,a} x_0^t = 0, \quad \forall a = a_l, \dots, a_u, \quad \forall t = 1, \dots, T \\
& s_{0,i,a}^t - p_{0,i-1} s_{0,i-1,a-1}^{t-1} = 0, \quad \forall i = 2, \dots, L \quad \forall a = a_l + 1, \dots, a_u - 1, \quad \forall t = 1, \dots, T \\
& s_{0,i,a_u}^t - p_{0,i-1} (s_{0,i-1,a_u}^{t-1} + s_{0,i-1,a_u-1}^{t-1}) = 0, \quad \forall i = 2, \dots, L, \quad \forall t = 1, \dots, T \\
& s_{1,a}^t - p_{1,a-1} s_{1,a-1}^{t-1} - m_{1,a} x_1^t - p_{0,L} s_{0,L,a-1}^{t-1} + m_{2,a} x_{1,2}^t = 0, \\
& \quad \forall a = a_l + 1, \dots, a_u - 1, \quad \forall t = 1, \dots, T \\
& s_{1,a_u}^t - p_{1,a_u} s_{1,a_u}^{t-1} - p_{1,a_u-1} s_{1,a_u-1}^{t-1} - m_{1,a_u} x_1^t - p_{0,L} (s_{0,L,a_u}^{t-1} + s_{0,L,a_u-1}^{t-1}) \\
& \quad + m_{2,a} x_{1,2}^t = 0, \quad \forall t = 1, \dots, T \\
& p_{k,a-1} s_{k,a-1}^{t-1} - s_{k,a}^t + m_{k,a} x_{k-1,k}^t - m_{k+1,a} x_{k,k+1}^t + m_{k,a} x_k^t = 0, \quad \forall k = 2 \dots, n-1, \\
& \quad \forall a = a_l + 1, \dots, a_u - 1, \quad \forall t = 1, \dots, T \\
& p_{k,a_u-1} s_{k,a_u-1}^{t-1} + p_{k,a_u} s_{k,a_u}^{t-1} - s_{k,a_u}^t + m_{k,a_u} x_{k-1,k}^t - m_{k+1,a_u} x_{k,k+1}^t + m_{k,a_u} x_k^t = 0, \\
& \quad \forall k = 2 \dots, n-1, \quad \forall t = 1, \dots, T \\
& p_{n,a-1} s_{n,a-1}^{t-1} + m_{n,a} (x_{n-1,n}^t + x_n^t) - s_{n,a}^t = 0, \quad \forall a = a_l + 1, \dots, a_u - 1, \\
& \quad \forall t = 1, \dots, T
\end{aligned}$$

$$\begin{aligned}
& p_{n,a_u-1} s_{n,a_u-1}^{t-1} + p_{n,a_u} s_{n,a_u}^{t-1} + m_{n,a_u} (x_{n-1,n}^t + x_n^t) - s_{n,a_u}^t = 0, \quad \forall t = 1, \dots, T \\
& p_{k,a-1} s_{k,a-1}^{t-1} - m_{k+1,a} x_{k,k+1}^t \geq 0, \quad \forall k = 1, \dots, n-1, \\
& \quad \forall a = a_l + 1, \dots, a_u, \quad \forall t = 1, \dots, T \\
& s_{0,i,a}^t = 0, \quad \forall i = 2, \dots, L \quad \forall a = a_l, \dots, a_l + i - 2, \quad \forall t = 1, \dots, T \\
& s_{k,a}^t = 0, \quad \forall k = 2, \dots, n-1, \quad \forall a = a_l, \dots, a_l + k + L - 2, \quad \forall t = 1, \dots, T \\
& x^t, s^t \geq 0, \quad \forall t = 1, \dots, T
\end{aligned}$$