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VECTOR MEASURES APPLIED TO OPTIMAL CONTROL FOR A CLASS OF EVOLUTION EQUATIONS ON BANACH SPACES

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Abstract. In this paper we consider a class of nonlinear evolution equations on infinite dimensional Banach spaces driven by vector measures. We prove existence and uniqueness of solutions and continuous dependence of solutions on the control measures. Using these results we prove existence of optimal controls for Bolza problems. Based on this result we present necessary conditions of optimality.

1. Introduction

It is well known that vector measures cover probability measures, signed measures, Borel measures, Radon measures, Dirac measures etc. Therefore it has broader application in many different fields including the area of impulsive control systems [3, 4]. A dynamic system may be subject to natural impulsive forces or it may be controlled by impulsive forces or both. In a recent paper [6] we considered a class of nonlinear evolution equations on Banach spaces driven by finitely additive vector measures and developed several results on optimal control theory. There we considered nonatomicity of the dominating measure. Here in this paper we relax this condition by some alternate assumptions thereby improving the scope of application to a broader class of problems.

In Section 2, after some preliminaries, we introduce a class of evolution equations defined on infinite dimensional Banach spaces and controlled by finitely additive vector measures (covering discrete measures given by a sum of vector valued Dirac measures) having bounded variation. In Section 3, we prove existence and uniqueness of mild solutions [1]. In Section 4, we present a result on weak to strong continuity of solutions with respect to vector measures. This

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result is then used to prove existence of optimal controls. Following this, in Section 5, we develop necessary conditions of optimality. The paper is concluded with reference to some open questions.

2. Preliminaries

Let $I \equiv [0,T]$ be a closed bounded interval and let $\mathcal{A}_I$ denote a field of subsets of the set I and Z be a Banach space with the dual Z^*. Let $B_\infty(I,Z)$ denote the space of bounded measurable functions defined on I and taking values in the Banach space Z. Furnished with sup-norm topology, it is a Banach space. A vector measure γ is a set function defined on $\mathcal{A}_I$ and taking values in the Banach space Z^*. It is said to be finitely additive if $\gamma(\emptyset) = 0$ and for any finite family $\{E_i\}$ of disjoint $\mathcal{A}_I$ measurable subsets

$$\gamma(\cup E_i) = \sum \gamma(E_i).$$

Let $\Sigma_I = \sigma(\mathcal{A}_I)$ denote the sigma field (sigma algebra) of subsets of the set I generated by $\mathcal{A}_I$. In this case the set function $\gamma : \Sigma_I \longrightarrow Z^*$ is said to be countably additive if the above equality holds for any countable family of disjoint Σ_I measurable subsets of the set I. For convenience of the reader we recall the definition of the variation norm. Let D be a $\mathcal{A}_I$ measurable subset of the set I and let Π denote any finite disjoint $\mathcal{A}_I$ measurable partition of the set D. The total variation of γ on D, denoted by $|\gamma|(D)$, is given by

$$|\gamma|(D) \equiv \sup_{\Pi} \sum_{\sigma \in \Pi} \|\gamma(\sigma)\|_{Z^*},$$

where the summation is taken over the elements of the partition Π and the supremum is taken with respect to the class of all such finite partitions. The norm of the measure γ is then given by $\|\gamma\| \equiv |\gamma|(I)$. We denote by $\mathcal{M}_{bfa}(\mathcal{A}_I, Z^*)$ the class of finitely additive Z^*-valued vector measures having bounded total variation. Endowed with the total variation norm, $\mathcal{M}_{bfa}(\mathcal{A}_I, Z^*)$ is a Banach space. Let $\mathcal{M}_{ca}(\Sigma_I, Z^*)$ denote the class of countably additive Z^* valued vector measures having bounded total variation. This is a closed linear subspace of $\mathcal{M}_{bfa}(\mathcal{A}_I, Z^*)$ and hence a Banach space. A continuous linear functional $\hat{\ell}$ on $B_\infty(I,Z)$ has the representation through an element $\gamma \in \mathcal{M}_{bfa}(\mathcal{A}_I, Z^*)$ giving

$$\hat{\ell}(f) = \int_I \langle f(t), \gamma(dt) \rangle_{Z,Z^*}$$

for $f \in B_\infty(I,Z)$. This is easily proved for simple functions $\mathcal{S}(I,Z)$ and then, for the general case, one uses the fact that simple functions are dense in $B_\infty(I,Z)$. In fact one can prove the following result.

Lemma 2.1. *The topological dual of the Banach space $B_\infty(I,Z)$ denoted by $(B_\infty(I,Z))^*$ is isometrically isomorphic to the space of finitely additive vector measures $\mathcal{M}_{bfa}(\mathcal{A}_I, Z^*)$ expressed symbolically by*

$$(B_\infty(I,Z))^* \cong \mathcal{M}_{bfa}(\mathcal{A}_I, Z^*).$$

Proof. In principle the proof follows from similar arguments as given in the proof of theorem 3.1 in [5, Theorem 3.1, p. 70]. □

In case when Z^* has the Radon-Nikodym property (RNP) with respect to a finitely additive nonnegative measure $\nu \in \mathcal{M}^+_{bfa}(\mathcal{A}_I)$ and the measure γ is ν-continuous then it is given by $\gamma(dt) = h(t)\nu(dt)$ where $h \in L_1(\nu, Z^*)$. In this case the functional $\hat{\ell}$ has the representation

$$\hat{\ell}(f) = \int_I \langle f(t), \gamma(dt)\rangle = \int_I \langle f(t), h(t)\rangle_{Z,Z^*} \nu(dt).$$

We can use these measures as perturbation or as controls for dynamic systems defined on infinite dimensional Banach space X. The system we consider here is given by the following evolution equation,

$$dx(t) = Ax(t)dt + F(t, x(t))dt + G(t, x(t))\gamma(dt),\ t \in I,\ x(0) = x_0, \tag{2.1}$$

where A is the infinitesimal generator of a C_0 semigroup [1] $\{S(t), t \geq 0\} \subset \mathcal{L}(X)$, and the functions $F : I \times X \longrightarrow X$ and $G : I \times X \longrightarrow \mathcal{L}(Y, X)$ are Borel measurable maps and $\gamma \in \mathcal{M}_{bfa}(\mathcal{A}_I, Y)$. Using the semigroup and variation of constants formula (Duhamel's formula) this differential equation can be transformed into an integral equation on the Banach space X as follows:

$$\begin{aligned} x(t) = S(t)x_0 &+ \int_0^t S(t-s)F(s, x(s))ds \\ &+ \int_0^t S(t-s)G(s, x(s))\gamma(ds),\ t \in I. \end{aligned} \tag{2.2}$$

Throughout this paper, by a solution of the evolution equation (2.1) we mean the mild solution [1] given by the solution (if one exists) of the integral equation (2.2).

3. Existence and uniqueness of solutions

Here we consider the system (2.2) driven by finitely additive Y valued vector measures. Let $\mathcal{M}_{ad} \subset \mathcal{M}_{bfa}(\mathcal{A}_I, Y)$ be a nonempty bounded set denoting the set of admissible control measures. Later, we state more precise characterization of this set. Let $B_\infty(I, X)$ denote the Banach space of bounded Borel measurable functions defined on I and taking values from the Banach space X.

We introduce the following basic assumptions.

(A1) The linear operator A is the infinitesimal generator of a C_0-semigroup $\{S(t), t \geq 0\}$ of bounded linear operators in X satisfying, for some finite positive number M, $\sup\{\|S(t)\|_{\mathcal{L}(X)},\ t \in [0, T]\} \leq M$.

(A2) $F : I \times X \longrightarrow X$ is Borel measurable and there exists a constant $K > 0$ such that

(1) $\|F(t, x)\|_X \leq K(1 + \|x\|_X)$, $x \in X$, $t \in I$,

(2) $\|F(t, x_1) - F(t, x_2)\|_X \leq K\|x_1 - x_2\|$, $x_1, x_2 \in X$, $t \in I$.

(A3) $G : I \times X \longrightarrow \mathcal{L}(Y, X)$ is Borel measurable and there exists a constant $L > 0$ such that

(1) $\|G(t, x)\|_{\mathcal{L}(Y,X)} \leq L(1 + \|x\|_X)$, $x \in X$, $t \in I$,

(2) $\|G(t, x_1) - G(t, x_2)\|_{\mathcal{L}(Y,X)} \leq L\|x - y\|_{\mathcal{L}(Y,X)}$, $x_1, x_2 \in X$, $t \in I$.

Theorem 3.1. *Consider the evolution equation* (2.1) *with the control measure* $\gamma \in \mathcal{M}_{bfa}(\mathcal{A}_I, Y)$ *satisfying* $\gamma(\{0\}) = 0$*, and suppose the assumptions* (A1), (A2), (A3) *hold. Then, for every* $x_0 \in X$*, the system* (2.1) *has a unique mild solution* $x \in B_\infty(I, X)$.

Proof. For the proof we use the Banach fixed point theorem. For any given $x_0 \in X$ and $\gamma \in \mathcal{M}_{bfa}(\mathcal{A}_I, Y)$, we use the semigroup $S(t), t \geq 0$, and define the operator Γ on $B_\infty(I, X)$ as follows,

$$(\Gamma x)(t) \equiv S(t)x_0 + \int_0^t S(t-s)F(s, x(s))ds + \int_0^t S(t-s)G(s, x(s))\gamma(ds), \ t \in I. \tag{3.1}$$

Under the given assumptions, we show that Γ maps $B_\infty(I, X)$ to itself. Since $\{S(t), t \geq 0\}$ is a C_0-semigroup on X and I is a finite interval there exists a finite positive number M such that $\sup\{\|S(t)\|_{\mathcal{L}(X)}, \ t \in I\} \leq M$. Computing the norm of $(\Gamma x)(t)$ and using the assumptions (A2) and (A3) it follows from triangle inequality that for each $t \in I$,

$$\|(\Gamma x)(t)\|_X \leq M\|x_0\|_X + MK\int_0^t (1 + \|x(s)\|_X)ds + ML\int_0^t (1 + \|x\|_X)|\gamma|(ds), \tag{3.2}$$

where $|\gamma|(\cdot)$ denotes the finitely additive nonnegative measure induced by the variation of the vector measure γ. Consider the measure μ on $\mathcal{A}_I$ given by

$$\mu(\sigma) \equiv \int_\sigma M[Kds + L|\gamma|(ds)], \ \sigma \in \mathcal{A}_I.$$

Since $|\gamma|$ is a finitely additive nonnegative measure it is clear that μ is a finitely additive bounded strictly positive measure. Define the constant

$$C \equiv [M\|x_0\|_X + MKT + ML|\gamma|(I)].$$

Since γ is a finitely additive measure having bounded total variation, the constant C is positive and finite. Using these parameters we can rewrite the inequality (3.2) in the following compact form

$$\|(\Gamma x)(t)\|_X \leq C + \int_0^t \|x(s)\|_X \mu(ds), \ t \in I. \tag{3.3}$$

Hence

$$\sup\{\|(\Gamma x)(t)\|_X, \ t \in I\} = \|\Gamma x\|_{B_\infty(I,X)} \leq C + \mu(I)\|x\|_{B_\infty(I,X)} < \infty. \tag{3.4}$$

This shows that the operator Γ maps $B_\infty(I,X)$ into itself. We prove that it has a unique fixed point in $B_\infty(I,X)$. For any pair of elements $x_1, x_2 \in B_\infty(I,X)$, it follows from equation (3.1) that

$$[(\Gamma x_1)(t) - (\Gamma x_2)(t)] = \int_0^t S(t-s)[F(s,x_1(s)) - F(s,x_2(s))]ds$$
$$+ \int_0^t S(t-s)[G(s,x_1(s)) - G(s,x_2(s))]\gamma(ds),\ t \in I. \tag{3.5}$$

Using the assumptions (A2) and (A3) and triangle inequality applied to the above expression we obtain the following inequality,

$$\|(\Gamma x_1)(t) - (\Gamma x_2)(t)\|_X \le \int_0^t \|x_1(s) - x_2(s)\|_X \mu(ds),\ t \in I. \tag{3.6}$$

Define the function β as follows

$$\beta(t) \equiv \int_0^t \mu(ds),\ t \in I.$$

Using this function, the expression (3.6) can be rewritten as

$$\|(\Gamma x_1)(t) - (\Gamma x_2)(t)\|_X \ \le \int_0^t \|x_1(s) - x_2(s)\|_X d\beta(s),\ t \in I. \tag{3.7}$$

It is clear from the definition of the measure μ, that β is a nonnegative, monotone increasing function of bounded variation on I with $\beta(0) = 0$. Thus β is differentiable almost everywhere with the derivative being measurable and Lebesgue integrable with $\dot\beta(t) > 0,$ for a.e $t \in I$. Hence

$$0 < \int_0^t \dot\beta(s)ds \le \beta(t+),\ t \in I,$$

and

$$0 < \int_0^T \dot\beta(s)ds \le \beta(T) - \beta(0) = \beta(T) < \infty.$$

Define the set Λ as follows:

$$\Lambda \equiv \Big\{ f \in L_1^+(I) : f(t) > \dot\beta(t) \text{ for a.e } t \in I \Big\}.$$

Clearly, the set $\Lambda \neq \emptyset$ and so we can choose an element $f_o \in \Lambda$ such that

$$\mathcal{I}_t(\dot\beta) \equiv \int_0^t \dot\beta(s)ds \le \beta(t+) \le \int_0^t f_o(s)ds,\ \forall\ t \in [0,T] \tag{3.8}$$

with $\beta(T+) = \beta(T)$, and the function $t \longrightarrow \beta_o(t) \equiv \int_0^t f_o(s)ds,\ t \in I$, is continuous and differentiable almost everywhere on I. It is clear from this construction that

$$\beta(t+) \le \beta_o(t),\ t \in I;\ \text{and } \dot\beta(t) < \dot\beta_o(t) \text{ for a.e. } t \in I$$

and β_o is continuous while β is only right continuous. It is the continuity property of β_o which is critical in the integration by parts used in the process of iteration as follows. Using the function β_o, dominating the function β, in the inequality (3.7) we obtain

$$\begin{aligned} \|(\Gamma x_1)(t) - (\Gamma x_2)(t)\|_X &\leq \int_0^t \|x_1(s) - x_2(s)\|_X d\beta(s) \\ &\leq \int_0^t \|x_1(s) - x_2(s)\|_X d\beta_o(s),\ t \in I. \end{aligned} \tag{3.9}$$

For any pair $x_1, x_2 \in B_\infty(I, X)$ and $t \in I$, define

$$\rho_t(x_1, x_2) \equiv \sup\{\|x_1(s) - x_2(s)\|_X,\ 0 \leq s \leq t\}$$

and note that $\rho_T(x_1, x_2) = \|x_1 - x_2\|_{B_\infty(I,X)}$. Using this notation it is not difficult to verify that we can rewrite the inequality (3.9) as follows:

$$\rho_t(\Gamma x_1, \Gamma x_2) \leq \int_0^t \rho_s(x_1, x_2)\ d\beta_o(s),\ t \in I. \tag{3.10}$$

Considering the second iteration of the operator Γ (i.e., $\Gamma^2 \equiv \Gamma \circ \Gamma$), it follows from the above expression, and the fact that $t \longrightarrow \rho_t(x, y)$ is a nondecreasing function of $t \geq 0$, that, for each $t \in I$, we have

$$\begin{aligned} \rho_t(\Gamma^2(x_1), \Gamma^2(x_2)) &\leq \int_0^t \rho_s(\Gamma(x_1), \Gamma(x_2)) d\beta_o(s) \\ &\leq \int_0^t \left(\int_0^s \rho_\theta(x_1, x_2) d\beta_o(\theta) \right) d\beta_o(s). \end{aligned}$$

Hence it follows from the above inequality that

$$\rho_t(\Gamma^2(x_1), \Gamma^2(x_2) \leq \int_0^t \rho_s(x_1, x_2)\beta_o(s) d\beta_o(s),\ t \in I \tag{3.11}$$

and consequently it follows from integration by parts that

$$\rho_t(\Gamma^2(x_1), \Gamma^2(x_2)) \leq \rho_t(x_1, x_2)\big(\beta_o^2(t)/2\big),\ t \in I. \tag{3.12}$$

Continuing this process of iteration m times we arrive at the following inequality

$$\rho_t(\Gamma^m(x_1), \Gamma^m(x_2)) \leq \rho_t(x_1, x_2)\big(\beta_o^m(t)/m!\big),\ t \in I. \tag{3.13}$$

Thus, for $t = T$, we have

$$\|\Gamma^m(x_1) - \Gamma^m(x_2)\|_{B_\infty(I,X)} \leq \alpha_m \|x_1 - x_2\|_{B_\infty(I,X)}, \tag{3.14}$$

where $\alpha_m = ((\beta_o(T))^m/m!)$. Since $\beta_o(T)$ is finite it is clear that for $m \in N$ sufficiently large, $\alpha_m < 1$ and hence the m-th iterate of the operator Γ is a contraction. Thus it follows from Banach fixed point theorem that Γ^m has a unique fixed point $x^o \in B_\infty(I, X)$. Using this fact one can easily verify that x^o is also the unique fixed point of the operator Γ itself. This proves the existence of a unique solution of the integral equation (2.2) and hence a unique mild solution of the equation (2.1) in the Banach space $B_\infty(I, X)$. □

4. Existence of optimal controls

In this section we introduce the admissible set of controls and state the optimal control problem. Let $\mathcal{M}_{ad} \subset \mathcal{M}_{bfa}(\mathcal{A}_I, Y)$ denote the class of admissible control measures. Consider the system (2.1) with the admissible controls $\mathcal{M}_{ad}$ and the objective functional $J(\gamma)$ given by

$$J(\gamma) \equiv \int_0^T \ell(t, x(t))\nu(dt) + \Phi(x(T)), \tag{4.1}$$

where ℓ and Φ are real valued Borel measurable functions from $I \times X$ and X to $R_0 = [0, \infty) \cup \{\infty\}$ respectively, and $x \in B_\infty(I, X)$ is the unique solution of the equation (2.1) corresponding to the control measure $\gamma \in \mathcal{M}_{ad}$ and $\nu \in \mathcal{M}^+_{bfa}(\mathcal{A}_I)$ is a measure associated with the set $\mathcal{M}_{ad}$ to be discussed shortly. The objective is to find a control $\gamma_o \in \mathcal{M}_{ad}$ that minimizes the above cost functional. In order to solve this problem we must characterize the admissible set $\mathcal{M}_{ad}$ in more details. This is given in the following Lemma.

Lemma 4.1 (Brooks and Dinculeanu)**.** *Let the Banach space Y and its dual Y^* satisfy the Radon-Nikodym property* (RNP)*. Then a set $\mathcal{M}_{ad} \subset \mathcal{M}_{bfa}(\mathcal{A}_I, Y)$ is relatively weakly compact if and only if it satisfies the following conditions*:

(1) *the set $\mathcal{M}_{ad}$ is bounded* (*in variation norm*),

(2) *there exists a nonnegative finitely additive measure $\nu \in \mathcal{M}^+_{bfa}(\mathcal{A}_I)$ such that the set $\mathcal{M}_{ad}$ is uniformly ν continuous,that is, for every $E \in \mathcal{A}_I$,*

$$\lim_{\nu(E)\to 0} |\gamma|(E) = 0$$

uniformly in $\gamma \in \mathcal{M}_{ad}$,

(3) *for each set $E \in \mathcal{A}_I$, the set $\{\gamma(E), \gamma \in \mathcal{M}_{ad}\}$ is a relatively weakly compact subset of Y.*

Proof. See Diestel [8, Corollary 6, p. 106]. This is a generalization of a celebrated theorem due to Bartle-Dunford-Schwartz [8, Theorem 5, p. 105] from countably additive vector measures to finitely additive vector measures. □

According to this result for any $\gamma \in \mathcal{M}_{ad}$ there exists a unique $g_\gamma \in L_1(\nu, Y)$ such that for every $\sigma \in \mathcal{A}_I$, $\gamma(\sigma) = \int_\sigma g_\gamma(s)\nu(ds)$.

Under the assumptions of Theorem 3.1, we show that the solution set is a bounded subset of the Banach space $B_\infty(I, X)$.

Corollary 4.2. *Consider the system* (2.1)*, suppose the assumptions of Theorem 3.1 hold, and that the admissible set $\mathcal{M}_{ad}$ is a bounded subset of $\mathcal{M}_{bfa}(\mathcal{A}_I, Y)$. Then the solution set*

$$\mathcal{S} \equiv \{x \in B_\infty(I, X) : x = x(\gamma) \text{ for some } \gamma \in \mathcal{M}_{ad}\} \tag{4.2}$$

is a bounded subset of $B_\infty(I, X)$.

Proof. It follows from Theorem 3.1 that, for each $\gamma \in \mathcal{M}_{ad}$, the evolution equation (2.1) has a unique mild solution $x(\gamma) \in B_\infty(I, X)$. Thus $x(\gamma)$ satisfies the following integral equation

$$x(\gamma)(t) = S(t)x_0 + \int_0^t S(t-s)F(s, x(\gamma)(s))ds + \int_0^t \int_U S(t-s)G(s, x(\gamma)(s))\gamma(ds), \ t \in I. \tag{4.3}$$

By taking the norm on either side and using the assumptions (A1), (A2) and (A3) it follows from triangle inequality that

$$\|x(\gamma)(t)\| \leq C(\gamma) + \int_0^t \|x(\gamma)(s)\| \mu_\gamma(ds), \tag{4.4}$$

where the measure μ_γ is given by

$$\mu_\gamma(\sigma) \equiv \int_\sigma \{MKds + ML|\gamma|(ds)\}, \ \sigma \in \mathcal{A}_I,$$

and the constant $C(\gamma)$ is given by

$$C(\gamma) \equiv M(\|x_0\|_X + KT + L|\gamma|(I)).$$

Applying generalized Gronwall inequality [2, Lemma 5, p. 268] to (4.4) we arrive at the following inequality

$$\|x(\gamma)\|_{B_\infty(I,X)} \equiv \sup\{\|x(\gamma)(t)\|_X, \ t \in I\} \leq C(\gamma)\Big(1 + \mu_\gamma(I) \exp \mu_\gamma(I)\Big). \tag{4.5}$$

Since the admissible set $\mathcal{M}_{ad}$ is a bounded subset of $\mathcal{M}_{bfa}(\mathcal{A}_I, Y)$ it is clear that

$$\sup\{\mu_\gamma(I), \gamma \in \mathcal{M}_{ad}\} = \Big(MKT + ML \sup\{|\gamma|(I) = \|\gamma\|, \gamma \in \mathcal{M}_{ad}\}\Big) < \infty$$

and similarly the constant

$$\sup\{C(\gamma) \equiv M(\|x_0\|_X + KT + L|\gamma|(I)), \gamma \in \mathcal{M}_{ad}\} < \infty.$$

Thus it follows from (4.5) that there exists a constant $0 < b < \infty$ such that

$$\sup\{\|x(\gamma)\|_{B_\infty(I,X)}, \gamma \in \mathcal{M}_{ad}\} \leq b < \infty.$$

This proves that the solution set $\mathcal{S}$ is a bounded subset of $B_\infty(I, X)$. □

Before we prove the existence of optimal control we need the following important result on continuity of the control to solution map $\gamma \to x(\gamma)$. This is presented in the following theorem.

Theorem 4.3. *Consider the system* (2.1) *with the operator* A *being the generator of a compact* C_0*-semigroup* $\{S(t), t > 0\}$ *on* X*, and the assumptions of Theorem 3.1 and Corollary 4.2 hold. Suppose the set of admissible controls* $\mathcal{M}_{ad}$ *is weakly closed satisfying the assumptions of Lemma 4.1 and that the measure* ν *is nonatomic. Further suppose there exists a measure* $\mu^* \in \mathcal{M}^+_{bfa}(\mathcal{A}_I)$ *such that* $|\gamma|(\sigma) \leq \mu^*(\sigma)$ *for each* $\sigma \in \mathcal{A}_I$ *uniformly with respect to* $\gamma \in \mathcal{M}_{ad}$*. Then the map* $\gamma \longrightarrow x(\gamma)$ *from* $\mathcal{M}_{ad}$ *to* $B_\infty(I, X)$ *is continuous with respect to the relative weak topology on* $\mathcal{M}_{ad}$ *and the norm topology on* $B_\infty(I, X)$*.*

Proof. Let $\{\gamma^n, \gamma^o\} \in \mathcal{M}_{ad}$ and suppose $\gamma^n \xrightarrow{w} \gamma^o$. Let $x^n \equiv x(\gamma^n)$ and $x^o \equiv x(\gamma^o)$ denote the unique mild solutions of equation (2.1) corresponding to the same initial state, $x(\gamma^n)(0) = x(\gamma^o)(0) = x_0$ and driving measures γ^n and γ^o respectively. Clearly, this means that $\{x^n, x^o\}$ satisfy the following integral equations:

$$\begin{aligned} x^n(t) = S(t)x_0 &+ \int_0^t S(t-s)F(s, x^n(s))ds \\ &+ \int_0^t S(t-s)G(s, x^n(s))\gamma^n(ds), \ t \in I, \end{aligned} \tag{4.6}$$

$$\begin{aligned} x^o(t) = S(t)x_0 &+ \int_0^t S(t-s)F(s, x^o(s))ds \\ &+ \int_0^t S(t-s)G(s, x^o(s))\gamma^o(ds), \ t \in I, \end{aligned} \tag{4.7}$$

where $\{x^n(t) \equiv x(\gamma^n)(t), x^o(t) \equiv x(\gamma^o)(t), \ t \in I\}$. Subtracting the expression (4.7) from (4.6) term by term and suitably rearranging terms we obtain the following identity

$$\begin{aligned} x^n(t) - x^o(t) = &\int_0^t S(t-s)[F(s, x^n(s)) - F(s, x^o(s))]ds \\ &+ \int_0^t S(t-s)[G(s, x^n(s)) - G(s, x^o(s))]\gamma^n(ds) \\ &+ \int_0^t S(t-s)G(s, x^o(s))\big(\gamma^n - \gamma^o\big)(ds), \ t \in I. \end{aligned} \tag{4.8}$$

We denote the last term of the above expression by e_n giving

$$e_n(t) \equiv \int_0^t S(t-s)G(s, x^o(s))\big(\gamma^n - \gamma^o\big)(ds), \ t \in I. \tag{4.9}$$

By virtue of Lemma 4.1, this can be written in the equivalent form

$$e_n(t) \equiv \int_0^t S(t-s)G(s, x^o(s))\big(g^n(s) - g^o(s)\big)\nu(ds), \ t \in I, \tag{4.10}$$

where $\gamma^n(ds) = g^n(s)\nu(ds)$ and $\gamma^o(ds) = g^o(s)\nu(ds)$ with $\{g^n, g^o\} \in L_1(\nu, Y)$ being the corresponding Radon-Nikodym derivatives. Computing the X norm of either side of the expression (4.8) and using the assumptions (A1), (A2) and (A3) and triangle inequality we obtain the following inequality

$$\|x^n(t)-x^o(t)\|_X \le \int_0^t MK\|x^n(s)-x^o(s)\|_X ds$$
$$+\int_0^t ML\|x^n(s)-x^o(s)\|_X|\gamma^n|(ds)+\|e_n(t)\|_X,\ t\in I. \tag{4.11}$$

Since by assumption the set $\mathcal{M}_{ad}$ is set wise dominated by a finitely additive bounded positive measure μ^* it follows from the above inequality that

$$\|x^n(t)-x^o(t)\|_X \le MK\int_0^t \|x^n(s)-x^o(s)\|_{R^n} ds$$
$$+ML\int_0^t \|x^n(s)-x^o(s)\|_X\mu^*(ds)+\|e_n(t)\|_X,\ t\in I. \tag{4.12}$$

Define the measure ϱ by

$$\varrho(E)\equiv MK\int_E ds+ML\int_E \mu^*(ds), E\subset I, E\in \mathcal{A}_I.$$

Since μ^* is a finitely additive bounded positive measure, it follows from the above expression that ϱ is also a finitely additive bounded positive measure. Using this measure we can rewrite the inequality (4.12) as follows:

$$\|x^n(t)-x^o(t)\|_X \le \int_0^t \|x^n(s)-x^o(s)\|_X\varrho(ds)+\|e_n(t)\|_X,\ t\in I. \tag{4.13}$$

Defining $\varphi_n(t)\ \equiv\ \|x^n(t)-x^o(t)\|_X, t\in I$, again it follows from generalized Gronwall inequality [2, Lemma 5, p. 268] that

$$\varphi_n(t)\ \le\ \|e_n(t)\|_X+\int_0^t \exp\Big\{\int_s^t \varrho(d\theta)\Big\}\|e_n(s)\|_X\varrho(ds),$$
$$\le\ \|e_n(t)\|+\ \exp(\varrho(I))\int_0^t \|e_n(s)\|_X\varrho(ds),\ t\in I. \tag{4.14}$$

We show that $e_n(t)$, given by the expression (4.10), converges to zero strongly in X uniformly on I. Here we use the compactness of the semigroup $S(t), t>0$, nonatomicity of ν and the weak convergence of γ^n to γ^o. For any $t\in I$, and any $\varepsilon>0$ satisfying $t\ge\varepsilon$, we can decompose the expression (4.10) into two parts as

$$e_n(t)=e_n^{(1)}(t)+e_n^{(2)}(t),\ t\in I,$$

where the first term is given by

$$e_n^{(1)}(t)\equiv S(\varepsilon)\Big(\int_0^{t-\varepsilon} S(t-\varepsilon-s)G(s,x^o(s))\big(\gamma^n-\gamma^o\big)(ds)\Big),\ t\in I_\varepsilon\equiv[\varepsilon,T].$$

The second term is given by

$$e_n^{(2)}(t)=\int_{t-\varepsilon}^t S(t-s)G(s,x^o(s))(\gamma^n-\gamma^o)(ds)$$

$$= \int_{t-\varepsilon}^{t} S(t-s)G(s,x^o(s))(g^n(s)-g^o(s))\nu(ds),\ t \in I_\varepsilon,$$

where $\{g^n, g^o\}$ are the Radon-Nikodym derivatives of the measures $\{\gamma^n, \gamma^o\}$ with respect to the scalar measure ν. Referring to the first term, it follows from weak convergence of γ^n to γ^o that the integral within the round bracket weakly converges to zero. Since by assumption the semigroup is compact, the operator $S(\varepsilon)$ is compact and hence the first term converges strongly to zero uniformly with respect to $t \in I_\varepsilon$. In other words,

$$\lim_{n\to\infty} \sup\{\|e_n^{(1)}(t)\|_X,\ t \in I_\varepsilon\} = 0$$

for every $\varepsilon > 0$. By Lemma 4.1(2) the set of admissible controls $\mathcal{M}_{ad}$ is uniformly ν-continuous, and by assumption γ^n weakly converges to γ^o, and ν is nonatomic. Thus the second term converges to zero as $\varepsilon \downarrow 0$ uniformly on I. Hence $\|e_n\|_{B_\infty(I,X)} \overset{s}{\longrightarrow} 0$. Thus, by the Lebesgue bounded convergence theorem, the expression on the right hand side of the inequality (4.14) converges to zero uniformly with respect to $t \in I$. Hence $\varphi_n(t) \longrightarrow 0$ uniformly in $t \in I$. In other words, $x^n \longrightarrow x^o$ in the norm topology of $B_\infty(I,X)$. This proves the continuity of the map $\gamma \longrightarrow x(\gamma)$ in the sense as stated in the theorem. □

Remark 4.4. In the preceding theorem we assumed that the measure $\nu \in \mathcal{M}_{bfa}^+(\mathcal{A}_I)$, which plays a crucial role in the characterization of weak compactness of the set of admissible control measures $\mathcal{M}_{ad}$, is nonatomic. This assumption can be relaxed (as stated in the following Proposition) by requiring that the operator valued function $G_o(t) \equiv G(t, x^o(t)),\ t \in I$, takes values in the Banach space of compact operators $\mathcal{K}(Y,X) \subset \mathcal{L}(Y,X)$.

Proposition 4.5. *Suppose the assumptions of Theorem 4.3 hold without the nonatomicity condition for the measure $\nu \in \mathcal{M}_{bfa}^+(\mathcal{A}_I)$ with respect to which $\mathcal{M}_{ad}$ is uniformly absolutely continuous; and suppose $G_o(\cdot)$ is an operator valued function with values in $\mathcal{K}(Y,X)$. Then the control to solution map $\gamma \longrightarrow x(\gamma)$ is continuous with respect to the weak topology on $\mathcal{M}_{ad}$ and strong* (*norm*) *topology on $B_\infty(I,X)$.*

Proof. (Outline) Using the Lebesgue decomposition theorem we can write $\nu = \nu_c + \nu_a$ where ν_c is absolutely continuous with respect to Lebesgue measure $\lambda(dt) = dt$ and ν_a is the purely atomic part. It suffices to verify that e_n given by (4.10) converges to zero strongly in $B_\infty(I,X)$. We can rewrite this expression as

$$\begin{aligned} e_n(t) &\equiv \int_0^t S(t-s)G_o(s)\big(g^n(s)-g^o(s)\big)\nu(ds),\ t \in I \\ &= \int_0^t S(t-s)G_o(s)(\big(g^n(s)-g^o(s)\big)\nu_c(ds) \\ &\quad + \int_0^t S(t-s)G_o(s)(\big(g^n(s)-g^o(s)\big)\nu_a(ds). \end{aligned} \tag{4.15}$$

By Theorem 4.3, the continuous part converges to zero strongly in $B_\infty(I,X)$. So we must verify that this is also true for the atomic part. Since our control measures have bounded total variation there can be at most countably many atoms which may be denoted by the set of singletons $\{\theta_i\} \subset I$. Hence the atomic part is given by the sum

$$\begin{aligned}&\int_0^t S(t-s)G_o(s)\big(g^n(s)-g^o(s)\big)\nu_a(ds)\\ &= \sum_{\theta_i\le t} S(t-\theta_i)G_o(\theta_i)(g^n(\theta_i)-g^o(\theta_i))\nu_a(\{\theta_i\}),\ t\in I.\end{aligned} \tag{4.16}$$

Since g^n converges weakly to g^o in $L_1(\nu,Y)$, and $\nu=\nu_c+\nu_a$, it is clear that, at the atoms $\{\theta_i\}$, $g^n(\theta_i)$ converges weakly to $g^o(\theta_i)$ in the Banach space Y. Thus it follows from compactness of the operators $G_o(\theta_i)\in\mathcal{K}(Y,X)$ that, along a subsequence if necessary, the sum in (4.16) converges strongly in X for all $t\in I$ (including the atoms $\{\theta_i\}$). Hence it follows from the expression (4.15) that e_n converges to zero in X uniformly on I. This completes the outline of our proof. □

Next we consider the question of existence of optimal controls. This is presented in the following theorem.

Theorem 4.6. *Consider the system* (1)*, suppose that the assumptions of Theorem 4.3, Proposition 4.5 hold, and that the set of admissible control measures $\mathcal{M}_{ad}$ is a weakly compact subset of $\mathcal{M}_{bfa}(\mathcal{A}_I,Y)$ and the objective functional is given by*

$$J(\gamma)\equiv\int_I \ell(t,x(t))\nu(dt)+\Phi(x(T)), \tag{4.17}$$

where $x(t)\equiv x(\gamma)(t),\ t\in I$, is the mild solution of the evolution equation (2.1) *corresponding to the control measure $\gamma\in\mathcal{M}_{ad}$. Suppose the functions ℓ and Φ satisfy the following assumptions*:

(1) *$\ell: I\times X\longrightarrow R$ is nonnegative, Borel measurable in all the arguments, and lower semicontinuous in the second argument $x\in X$ and ν-integrable on bounded subsets of $I\times X$, the function $\Phi: X\longrightarrow R$ is nonnegative and lower semicontinuous satisfying*

(a1) *$\ell(t,x)\le\alpha_1(t)+\alpha_2\|x\|_X^{p_1}$ for some $p_1\in[1,\infty)$, $\alpha_1\in L_1^+(\nu)$ and $\alpha_2>0$,*

(a2) *$\Phi(x)\le\alpha_3+\alpha_4\|x\|_X^{p_2}$ for some $p_2\in[1,\infty)$, and $\alpha_3,\alpha_4>0$.*

Then, there exists an optimal control measure at which J attains its minimum.

Proof. Since $\mathcal{M}_{ad}$ is weakly compact, it suffices to prove that the map $\gamma\longrightarrow J(\gamma)$ is weakly lower semicontinuous on $\mathcal{M}_{ad}$. Let $\gamma^n\xrightarrow{w}\gamma^o$ in $\mathcal{M}_{ad}$. It follows from Theorem 4.3 and Proposition 4.5 that, (along a subsequence if necessary), $x(\gamma^n)\xrightarrow{s}x(\gamma^o)$ in the Banach space $B_\infty(I,X)$. Thus it follows from lower semicontinuity of ℓ and Φ in $x\in X$ that

$$\ell(t,x^o(t))\le\underline{\lim}\,\ell(t,x^n(t)), \tag{4.18}$$

$$\Phi(x^o(T)) \leq \underline{\lim}\, \Phi(x^n(T)) \tag{4.19}$$

for almost all $t \in I$. By Corollary 4.2, the solution set $\mathcal{S}$ is a bounded subset of $B_\infty(I, X)$ and, since ℓ is ν-integrable on bounded subsets of $I \times X$, both sides of the first inequality are ν-integrable. Hence it follows from the inequality (4.18) that

$$\int_I \ell(t, x^o(t))\ \nu(dt) \leq \int_I \underline{\lim}\, \ell(t, x^n(t))\ \nu(dt). \tag{4.20}$$

Since ℓ is nonnegative and ν-integrable, it follows from Fatou's Lemma and the above inequality that

$$\int_I \ell(t, x^o(t))\ \nu(dt) \leq \underline{\lim} \int_I \ell(t, x^n(t))\ \nu(dt). \tag{4.21}$$

Summing (4.19) and (4.21) we conclude that $J(\gamma^o) \leq \underline{\lim}\, J(\gamma^n)$. This proves that J is weakly lower semicontinuous on $\mathcal{M}_{ad}$ and since $\mathcal{M}_{ad}$ is weakly compact we conclude that there exists a $\gamma^o \in \mathcal{M}_{ad}$ at which J attains its minimum. This completes the proof. □

Remark 4.7. Theorem 4.6 can be easily extended to cover objective functionals that include control cost in terms of total variation norm such as the following functional

$$J_e(\gamma) \equiv \int_I \ell(t, x(t))\nu(dt) + \Phi(x(T)) + \|\gamma\|_v,$$

where $\|\gamma\|_v$ denotes the total variation norm of the measure γ. Since the norm in any Banach space is weakly lower semi continuous, under the given assumptions of Theorem 4.6, the functional J_e is weakly lower semi continuous on $\mathcal{M}_{ad}$. Hence there exists a control in $\mathcal{M}_{ad}$ that minimizes the functional J_e under the dynamic constraint of the evolution equation (2.1).

Remark 4.8. It follows from weak lower semicontinuity of J that the set of optimal controls

$$\mathcal{O}_p \equiv \left\{ \gamma \in \mathcal{M}_{ad} : J(\gamma) = \inf\{J(\varrho), \varrho \in \mathcal{M}_{ad}\} \right\}$$

is a weakly closed subset of $\mathcal{M}_{ad}$ and hence a weakly compact subset of $\mathcal{M}_{ad}$.

Remark 4.9. In Theorem 4.3, we assumed that the admissible set of control measures $\mathcal{M}_{ad}$ is set wise dominated by a measure $\mu^* \in \mathcal{M}^+_{bfa}(\mathcal{A}_I)$. In fact, in view of recent developments in the theory of integration with respect to submeasures due to Gould [11], it suffices if μ^* is only a submeasure of finite variation. This is because bounded measurable functions are integrable in the sense of Gould [10, Theorem 2.4]. The function $\eta : \mathcal{A}_I \longrightarrow [0, \infty]$, given by

$$\eta(E) \equiv \sup\{|\gamma|(E), \gamma \in \mathcal{M}_{ad}\} \text{ for } E \in \mathcal{A}_I,$$

is a nonnegative finitely subadditve set function having bounded total variation. It is easy to check that this is a submeasure having bounded total variation. Since by Corollary 4.2 the solution set is bounded, the integrands in the expression (4.12) related to μ^* (third term on the righthand side) are uniformly bounded. Thus the measure μ^* in Theorem 4.3 can be replaced by the submeasure η.

5. Necessary conditions of optimality

In this section we present necessary conditions of optimality whereby one can determine optimal policies.

Theorem 5.1. *Let X be a separable reflexive Banach space and suppose the assumptions of Theorem 4.3 and Proposition 4.5 hold and the set $\mathcal{M}_{ad}$ is a weakly compact and convex subset of $\mathcal{M}_{bfa}(\mathcal{A}_I, Y)$. Further, suppose the pair $\{F, G\}$ is once Gâteaux differentiable in the state variable with the Gâteaux derivatives being continuous and bounded, and the functions $\{\ell, \Phi\}$ appearing in the objective functional* (4.17) *are once continuously Gâteaux differentiable with respect to the state variable satisfying $\ell_x(\cdot, x^o(\cdot)) \in L_1(\nu, X^*)$ and $\Phi_x(\cdot) \in X^*$. Then, in order for the control state pair $\{\gamma^o, x^o\} \in \mathcal{M}_{ad} \times B_\infty(I, X)$ to be optimal, it is necessary that there exists a $\psi \in B_\infty(I, X^*)$ such that the triple $\{\gamma^o, x^o, \psi\}$ satisfies the following system of evolution equations, and the inequality*:

$$dx^o(t) = Ax^o dt + F(t, x^o(t))dt + G(t, x^o(t))\gamma^o(dt), x(0) = x_0, \tag{5.1}$$

$$\begin{aligned} -d\psi(t) = {} & A^*\psi dt + (DF)^*(t, x^o(t))\psi(t)dt + (D\mathcal{G})^*(t, x^o(t); \psi(t))\ \gamma^o(dt) \\ & + \ell_x(t, x^o(t))\ \nu(dt),\ \ \psi(T) = \Phi_x(x^o(T)), \end{aligned} \tag{5.2}$$

$$\int_I \langle \psi(t), G(t, x^o(t))(\gamma - \gamma^o)(dt)\rangle_{X^*, X} \geq 0\ \forall\ \gamma \in \mathcal{M}_{ad}. \tag{5.3}$$

Proof. Let $\gamma^o \in \mathcal{M}_{ad}$ denote the optimal control and $\gamma \in \mathcal{M}_{ad}$ any arbitrary element, and $\varepsilon > 0$. By convexity of $\mathcal{M}_{ad}$, it is clear that $\gamma^\varepsilon \equiv \gamma^o + \varepsilon(\gamma - \gamma^o) \in \mathcal{M}_{ad}$ for all $\varepsilon \in [0, 1]$. Then, by optimality of γ^o, it is evident that

$$J(\gamma^\varepsilon) \geq J(\gamma^o)\ \forall \gamma \in \mathcal{M}_{ad},\ \text{and}\ \varepsilon \in [0, 1].$$

Hence

$$(1/\varepsilon)(J(\gamma^\varepsilon) - J(\gamma^o)) \geq 0\ \forall \gamma \in \mathcal{M}_{ad}\ \ \text{and}\ \varepsilon \in (0, 1]. \tag{5.4}$$

Let $\{x^\varepsilon, x^o\} \subset B_\infty(I, X)$ denote the mild solutions of the state equation (2.1) corresponding to the control measures $\{\gamma^\varepsilon, \gamma^o\}$ respectively. In other words, $\{x^\varepsilon, x^o\}$ satisfy the following integral equations

$$\begin{aligned} x^\varepsilon(t) = {} & S(t)x_0 + \int_0^t S(t-s)F(s, x^\varepsilon(s))ds \\ & + \int_0^t S(t-s)G(s, x^\varepsilon(s))\gamma^\varepsilon(ds),\ \ t \in I, \end{aligned} \tag{5.5}$$

$$x^o(t) = S(t)x_0 + \int_0^t S(t-s)F(s,x^o(s))ds$$
$$+ \int_0^t S(t-s)G(s,x^o(s))\gamma^o(ds),\ t \in I. \tag{5.6}$$

It is evident that, as $\varepsilon \to 0$, $\gamma^\varepsilon \xrightarrow{w} \gamma^o$. In fact it follows from the construction of γ^ε that this convergence also holds in the total variation norm. In any case, it follows from Theorem 4.3 and Proposition 4.5 that $x^\varepsilon \xrightarrow{s} x^o$ in $B_\infty(I,X)$. Subtracting the equation (5.6) from the equation (5.5) term by term and computing the difference quotient $(1/\varepsilon)(x^\varepsilon(t) - x^o(t))$ and letting $\varepsilon \downarrow 0$, and denoting the limit by y, if one exists, we have

$$y(t) = \int_0^t S(t-s)DF(s,x^o(s))y(s)ds$$
$$+ \int_0^t S(t-s)DG(s,x^o(s);y(s))\gamma^o(ds)$$
$$+ \int_0^t S(t-s)G(s,x^o(s))(\gamma(ds) - \gamma^o(ds)),\ t \in I. \tag{5.7}$$

Hence one can easily verify that y satisfies the following evolution equation in the mild sense

$$dy(t) = Aydt + DF(t,x^o(t))y(t)dt + DG(t,x^o(t);y(t))\gamma^o(dt)$$
$$+ G(t,x^o(t))(\gamma - \gamma^o)(dt),\ y(0) = 0,\ t \in I, \tag{5.8}$$

where $\eta \longrightarrow DG(t,x^o(t);\eta)$ is a bounded linear map from X to $\mathcal{L}(Y,X)$. This is a linear differential equation in y and can be written compactly as

$$dy = Aydt + B(t)y(t)dt + C(t;y(t))\gamma^o(dt) + \vartheta_\gamma(dt),\ y(0) = 0,\ t \in I, \tag{5.9}$$

where $B(t) \equiv DF(t,x^o(t)), C(t;y(t)) \equiv DG(t,x^o(t);y(t)),\ t \in I$, and the measure ϑ_γ is given by

$$\vartheta_\gamma(E) \equiv \int_E G(t,x^o(t))(\gamma - \gamma^o)(dt),\ E \in \mathcal{A}_I.$$

Since, under the given assumptions, both F and G are continuously Gâteaux differentiable in the state variable with the Gâteaux derivatives being bounded on bounded sets, and $x^o \in B_\infty(I,X)$, it is clear that both B and C are bounded operator valued functions with values in $\mathcal{L}(X)$, and $\mathcal{L}(Y,X)$ respectively. Since $\{\gamma,\gamma^o\} \in \mathcal{M}_{ad}$ and the members of the set $\mathcal{M}_{ad}$ are finitely additive having bounded variation, it is clear from the above expression that ϑ_γ is finitely additive having bounded variation and hence $\vartheta_\gamma \in M_{bfa}(\mathcal{A}_I,X)$. Using Banach fixed point theorem one can easily verify that, for each $\gamma \in \mathcal{M}_{ad}$, equation (5.9) has a unique mild solution $y \in B_\infty(I,X)$ given by the solution of the linear integral equation (5.7). Thus the map,

$$\vartheta_\gamma \longrightarrow y, \tag{5.10}$$

from $\mathcal{M}_{bfa}(\mathcal{A}_I, X)$ to $B_\infty(I, X)$, is a continuous linear map and hence bounded. On the other hand, computing the difference quotient (5.4) and letting $\varepsilon \downarrow 0$, we obtain the Gâteaux differential (the directional derivative) of J at γ^o in the direction $\gamma - \gamma^o$ as follows:

$$dJ(\gamma^o; \gamma - \gamma^o) = \lim_{\varepsilon \downarrow 0} (1/\varepsilon)(J(\gamma^\varepsilon) - J(\gamma^o))$$

$$= \int_I \langle \ell_x(t, x^o(t)), y(t) \rangle_{X^*,X} \nu(dt) + \langle \Phi_x(x^o(T)), y(T) \rangle_{X^*,X}. \tag{5.11}$$

By optimality of γ^o, it follows from (5.4) that

$$dJ(\gamma^o; \gamma - \gamma^o) \geq 0 \ \forall \ \gamma \in \mathcal{M}_{ad}. \tag{5.12}$$

By assumption, $\ell_x(\cdot, x^o(\cdot)) \in L_1(\nu, X^*)$ and $\Phi_x(x^o(T)) \in X^*$. Combining this with the fact that equation (5.9) has a unique mild solution $y \in B_\infty(I, X)$, we conclude that the functional L, given by

$$L(y) \equiv \int_I \langle \ell_x(t, x^o(t)), y(t) \rangle_{X^*,X} \nu(dt) + \langle \Phi_x(x^o(T)), y(T) \rangle_{X^*,X}, \tag{5.13}$$

is a well defined bounded linear functional on the Banach space $B_\infty(I, X)$. Thus $y \longrightarrow L(y)$ is a continuous linear functional on $B_\infty(I, X)$ and hence it follows from (5.10) that the composition map

$$\vartheta_\gamma \longrightarrow y \longrightarrow L(y) \equiv \tilde{L}(\vartheta_\gamma) \tag{5.14}$$

is a continuous linear functional on the Banach space $\mathcal{M}_{bfa}(\mathcal{A}_I, X)$. Hence there exists a $\psi \in (\mathcal{M}_{bfa}(\mathcal{A}_I, X))^* \equiv \mathcal{M}^*_{bfa}(\mathcal{A}_I, X)$ such that

$$\tilde{L}(\vartheta_\gamma) = \langle\langle \psi, \vartheta_\gamma \rangle\rangle_{\mathcal{M}^*_{bfa}(\mathcal{A}_I,X),\mathcal{M}_{bfa}(\mathcal{A}_I,X)} \equiv \int_I \langle \psi(t), \vartheta_\gamma(dt) \rangle_{X^*,X}, \tag{5.15}$$

where $\mathcal{M}^*_{bfa}(\mathcal{A}_I, X)$ denotes the topological dual of the space $\mathcal{M}_{bfa}(\mathcal{A}_I, X)$. Since by our assumption X is reflexive, under the canonical embedding of a Banach space into its bidual, we have

$$B_\infty(I, X^*) \hookrightarrow \mathcal{M}^*_{bfa}(\mathcal{A}_I, X) = \mathcal{M}^*_{bfa}(\mathcal{A}_I, X^{**}).$$

It is known that a reflexive Banach space X is separable if and only if X^* is separable. Thus our assumption implies that X^* is also separable. Hence, by Pettis measurability theorem as seen in Dunford and Schwartz [9, Theorem III.6.11, p. 149] the elements of $B_\infty(I, X^*)$ are also strongly measurable functions with values in X^*. So the duality pairing in (5.15) is also well defined for $\psi \in B_\infty(I, X^*)$. Later we show that actually ψ does belong to this smaller space. Using the expression for ϑ_γ in the equation (5.15) we obtain

$$\tilde{L}(\vartheta_\gamma) = \int_I \langle \psi(t), G(t, x^o(t))(\gamma - \gamma^o)(dt) \rangle_{X^*,X}. \tag{5.16}$$

It follows from (5.11)–(5.16) that

$$\int_I \langle \psi(t), G(t, x^o(t))(\gamma - \gamma^o)(dt) \rangle_{X^*,X} \geq 0 \ \forall \ \gamma \in \mathcal{M}_{ad}. \tag{5.17}$$

Thus we have proved the necessary condition (5.3). It remains to prove the necessary condition given by (5.2). Using the variational equation (5.8) or equivalently (5.9) in the above expression and integrating by parts, which can be justified by use of Yosida approximation of the operator A as indicated in [2, Theorem 4.1, p. 15], and using Fubini's theorem one can derive the following identity,

$$\begin{aligned}\tilde{L}(\vartheta_\gamma) = \langle \psi(T), y(T)\rangle_{X^*,X} - \int_0^T \langle y(t), d\psi(t) + A^*\psi(t)dt\rangle_{X,X^*} \\ - \int_0^T \langle y(t), (DF)^*(t, x^o(t))\psi(t)\rangle_{X,X^*} dt \\ - \int_0^T \langle y(t), (D\mathcal{G})^*(t, x^o(t); \psi(t))\gamma^o(dt)\rangle_{X,X^*},\end{aligned} \tag{5.18}$$

where the operator valued function $D\mathcal{G}(t, x^o(t); \cdot)$ is uniquely determined by the following duality pairings,

$$\begin{aligned}\langle DG(t, x^o(t); \eta)\xi, x^*\rangle_{X,X^*} &= \langle \xi, (DG)^*(t, x^o(t); \eta)x^*\rangle_{Y,Y^*} \\ &\equiv \langle \xi, D\mathcal{G}(t, x^o(t); x^*)\eta\rangle_{Y,Y^*}\end{aligned}$$

for all $t \in I$, $\eta \in X, \xi \in Y$ and $x^* \in X^*$. It is clear from the above pairings that for each $t \in I$, $(DG)^*(t, x^o(t); \cdot) \in \mathcal{L}(X, \mathcal{L}(X^*, Y^*))$ while $(D\mathcal{G})(t, x^o(t); \cdot) \in \mathcal{L}(X^*, \mathcal{L}(X, Y^*))$. The dual of this operator, denoted by $(D\mathcal{G})^*$, has the property $(D\mathcal{G})^*(t, x^o(t); \cdot) \in \mathcal{L}(X^*, \mathcal{L}(Y^{**}, X^*))$. As the elements of the set of admissible controls $\mathcal{M}_{ad}$ are Y-valued, it is clear that the restriction of $(D\mathcal{G})^*$ to Y gives $(D\mathcal{G})^*(t, x^o(t); \cdot) \in \mathcal{L}(X^*, \mathcal{L}(Y, X^*))$. Hence the pairings in the expression (5.18) are all well defined. Since the identity (5.14), expressed by $\tilde{L}(\vartheta_\gamma) = L(y)$, must hold it follows from the expression (5.18) that ψ must satisfy the following equations:

$$\psi(T) = \Phi_x(x^o(T)), \tag{5.19}$$

$$\begin{aligned}d\psi + A^*\psi dt + (DF)^*(t, x^o(t))\psi(t)dt + (D\mathcal{G})^*(t, x^o(t); \psi(t))\gamma^o(dt) \\ = - \ell_x(t, x^o(t))\nu(dt), \ t \in I.\end{aligned} \tag{5.20}$$

This is precisely the necessary condition given by the equation (5.2). The equation (5.1) is the given dynamic system with x^o being the solution corresponding to the optimal control measure γ^o and hence nothing to prove. To complete the proof, it remains to show that the adjoint variable ψ, whose existence was guaranteed by the duality pairing (see the equation (5.15)), is actually given by the mild solution of the evolution equation (5.20) with the terminal condition given (5.19). The equation (5.20), or equivalently (5.2), is a backward linear evolution equation on the Banach space X^* called the adjoint evolution equation. This equation can be written in the compact form as follows,

$$\begin{aligned}-d\psi = A^*\psi(t)dt + B^*(t)\psi(t)dt + \mathcal{C}^*(t; \psi(t))\gamma^o(dt) + \Lambda_\nu(dt), \\ \psi(T) = \Phi_x(x^o(T)) \equiv \psi_o(T), \ t \in I,\end{aligned} \tag{5.21}$$

where A^* is the conjugate (adjoint) of the semigroup generator A, $B^*(t) \equiv (DF)^*(t, x^o(t))$, $\mathcal{C}^*(t; \psi(t)) \equiv (D\mathcal{G})^*(t, x^o(t); \psi(t))$ for all $t \in I$, and

$$\Lambda_\nu(\sigma) = \int_\sigma \ell_x(t, x^o(t))\nu(dt) \text{ for each } \sigma \in \mathcal{A}_I.$$

Since by our assumption $\ell_x(\cdot, x^o(\cdot)) \in L_1(\nu, X^*)$, Λ_ν is a bounded finitely additive (in this case countably additive) X^* valued ν-continuous vector measure. The mild solution of equation (5.21) is given by the solution of the following linear integral equation on the Banach space X^*

$$\psi(t) = S^*(T-t)\Phi_x(x^o(T)) + \int_t^T S^*(s-t)B^*(s)\psi(s)ds$$
$$+ \int_t^T S^*(s-t)\mathcal{C}^*(s; \psi(s))\gamma^o(ds) + \int_t^T S^*(s-t)\Lambda_\nu(ds),\ t \in I. \tag{5.22}$$

Existence and uniqueness of solution of this equation can be proved by use of successive approximation technique. By reversal of the flow of time $\tau \longrightarrow T-\tau$ and defining $\varphi(\tau) = \psi(T-\tau)$, the above backward integral equation can be written as an equivalent forward integral equation

$$\varphi(\tau) = S^*(\tau)\varphi(0) + \int_0^\tau S^*(\tau-\theta)B^*(T-\theta)\varphi(\theta)d\theta$$
$$+ \int_0^\tau S^*(\tau-\theta)\Lambda_{\nu,T}(d\theta) + \int_0^\tau S^*(\tau-\theta)\mathcal{C}^*(T-\theta; \varphi(\theta))\gamma_T^o(d\theta),\ \tau \in I, \tag{5.23}$$

where, $\varphi(0) = \Phi_x(x^o(T)) \equiv \psi_o(T)$ and both $\Lambda_{\nu,T}$ and γ_T^o are the time reversed measures corresponding to the measures Λ_ν and γ^o respectively in the sense that for any $\sigma = [t_1, t_2]$, $t_1, t_2 \in [0, T] \equiv I$, $\Lambda_{\nu,T}(\sigma) = \Lambda_\nu([T-t_2, T-t_1])$ and similarly $\gamma_T^o(\sigma) = \gamma^o([T-t_2, T-t_1])$. For convenience of notation let us introduce the X^*-valued function h as follows

$$h(\tau) \equiv S^*(\tau)\varphi(0) + \int_0^\tau S^*(\tau-\theta)\Lambda_{\nu,T}(d\theta),\ \tau \in I, \tag{5.24}$$

and rewrite the integral equation (5.23) in the following compact form

$$\varphi(\tau) = h(\tau) + \int_0^\tau S^*(\tau-\theta)B^*(T-\theta)\varphi(\theta)d\theta$$
$$+ \int_0^\tau S^*(\tau-\theta)\mathcal{C}^*(T-\theta; \varphi(\theta))\gamma_T^o(d\theta),\ \tau \in I. \tag{5.25}$$

We prove that this equation has a unique solution. Since $\ell_x(\cdot, x^o(\cdot)) \in L_1(\nu, X^*)$ and $\Phi_x(x^o(T)) \in X^*$ and ν is a bounded positive measure, it follows from triangle inequality applied to (5.24) that

$$\sup\{\|h(\tau)\|_{X^*}, \tau \in I\} \le M\|\Phi_x(x^o(T))\|_{X^*} + M\|\ell_x\|_{L_1(\nu, X^*)} < \infty.$$

Hence $h \in B_\infty(I, X^*)$. We use successive approximation technique. Define $\varphi_0 \equiv h$ and construct the sequence $\{\varphi_n\}$ as follows:

$$\varphi_1(t) = h(t) + \int_0^t S^*(t-\theta)B^*(T-\theta)\varphi_0(\theta)d\theta + \int_0^t S^*(t-\theta)\mathcal{C}^*(T-\theta;\varphi_0(\theta))\gamma_T^o(d\theta),\ t \in I$$

$$\varphi_{n+1}(t) = h(t) + \int_0^t S^*(t-\theta)B^*(T-\theta)\varphi_n(\theta)d\theta + \int_0^t S^*(t-\theta)\mathcal{C}^*(T-\theta;\varphi_n(\theta))\gamma_T^o(d\theta),\ t \in I, \tag{5.26}$$

for all nonnegative integers $n \in N_0 \equiv \{n \geq 0\}$. Clearly, since the operator $\mathcal{C}^*$ is linear in the second argument, for all $n \geq 1$ we have

$$\varphi_{n+1}(t) - \varphi_n(t) = \int_0^t S^*(t-\theta)B^*(T-\theta)[\varphi_n(\theta) - \varphi_{n-1}(\theta)]d\theta + \int_0^t S^*(t-\theta)\mathcal{C}^*(T-\theta;[\varphi_n(\theta)-\varphi_{n-1}(\theta)])\gamma_T^o(d\theta),\ t \in I. \tag{5.27}$$

By assumption the Gâteaux derivatives of F and G are bounded and hence there exist nonnegative constants $\{b, c\}$ such that

$$\|\varphi_1(t) - \varphi_0(t)\|_{X^*} \leq Mb\int_0^t \|h(\theta)\|_{X^*}d\theta + Mc\int_0^t \|h(\theta)\|_{X^*}|\gamma_T^o|(d\theta), \tag{5.28}$$

where $|\gamma_T^o|(\cdot)$ denotes (as usual) the measure induced by its variation (on any set in $\mathcal{A}_I$). Define the measure

$$\mu^o(\sigma) \equiv \int_\sigma \{Mbd\theta + Mc|\gamma_T^o|(d\theta)\},\ \sigma \in \mathcal{A}_I.$$

Clearly $\mu^o \in \mathcal{M}_{bfa}^+(\mathcal{A}_I)$ and hence the function $\beta(t) \equiv \int_0^t \mu^o(ds),\ t \in I$, is a nonnegative monotone increasing function of bounded variation. Thus by similar arguments as seen in the proof of Theorem 3.1, there exists a continuous nonnegative monotone increasing function β_o of bounded variation dominating the function β in the sense that

$$\beta(t+) \leq \beta_o(t) \text{ for all } t \in I, \text{ and } \dot{\beta}(t) < \dot{\beta}_o(t), \text{ for a.e } t \in I.$$

Based on these facts, the inequality (5.28) can be written as

$$\|\varphi_1(t) - \varphi_0(t)\|_{X^*} \leq \int_0^t \|h(\theta)\|_{X^*}d\beta(\theta) \leq \int_0^t \|h(\theta)\|_{X^*}d\beta_o(\theta),\ t \in I. \tag{5.29}$$

Similarly one can verify that

$$\|\varphi_2(t) - \varphi_1(t)\|_{X^*} \leq \int_0^t \|\varphi_1(\theta) - \varphi_0(\theta)\|_{X^*}d\beta(\theta)$$

$$\leq \int_0^t \|\varphi_1(\theta) - \varphi_0(\theta)\|_{X^*} d\beta_o(\theta) \tag{5.30}$$

Using the inequality (5.29) in the expression (5.30) and using the fact that $h \in B_\infty(I, X^*)$ we find that

$$\|\varphi_2(t) - \varphi_1(t)\|_{X^*} \leq \|h\|_{B_\infty(I,X^*)}(\beta_o(t))^2/2, \ \forall\, t \in I. \tag{5.31}$$

Using the expression (5.27) and following similar steps we find that, for all $n \geq 1$,

$$\|\varphi_n(t) - \varphi_{n-1}(t)\|_{X^*} \leq \|h\|_{B_\infty(I,X^*)}\big((\beta_o(t))^n/n!\big), \ \forall\, t \in I. \tag{5.32}$$

Hence, for any $p \geq 1$, one can easily verify that

$$\begin{aligned} &\|\varphi_{n+p}(t) - \varphi_n(t)\|_{X^*} \\ &\leq \|h\|_{B_\infty(I,X^*)} \exp\{\beta_o(t)\}\{(\beta_o(t))^{n+1}/(n+1)!\}, \ t \in I. \end{aligned} \tag{5.33}$$

Since β_o is a nonnegative monotone increasing function of bounded variation, it follows from the above estimate that

$$\begin{aligned} &\|\varphi_{n+p} - \varphi_n\|_{B_\infty(I,X^*)} \\ &\leq \|h\|_{B_\infty(I,X^*)} \exp\{\beta_o(T)\}\{(\beta_o(T))^{n+1}/(n+1)!\}. \end{aligned} \tag{5.34}$$

Hence for any $p \in N$, letting $n \to \infty$ we arrive at the conclusion that $\{\varphi_n\}$ is a Cauchy sequence in the Banach space $B_\infty(I, X^*)$. Thus there exists a unique $\varphi^o \in B_\infty(I, X^*)$ such that $\varphi_n \xrightarrow{s} \varphi^o$ in $B_\infty(I, X^*)$. Letting $n \to \infty$ in the expression (5.26) we conclude that φ^o satisfies the integral equation (5.25). By reversal of time once again, the function $\psi(t) \equiv \varphi^o(T - t), \ t \in I$, satisfies the backward integral (5.22). This proves the existence of a unique mild solution of the adjoint evolution equation (5.2) in the Banach space $B_\infty(I, X^*)$. This completes the proof of all the necessary conditions as stated. □

6. A convergence theorem

Using the necessary conditions of optimality one can develop a computational algorithm which can be used to obtain the optimal control. The question whether or not the algorithm converges is important for applications. Here we present a theorem that ensures convergence. Recall that $\mathcal{M}_{ad} \subset \mathcal{M}_{bfa}(\mathcal{A}_I, Y)$ is weakly compact. For simplicity, and only for the following theorem, we assume that Y is also a reflexive Banach space.

Theorem 6.1. *Consider the necessary conditions of optimality given by theorem 5.1 and suppose Y is a reflexive Banach space. Then there exists a sequence of control measures $\gamma^n \in \mathcal{M}_{ad}$ along which the cost functional $J(\gamma^n)$ converges (at least) to a local minimum.*

Proof. In step 1, we choose an arbitrary element, say $\gamma^1 \in \mathcal{M}_{ad}$, and solve the state equation (5.1) giving $x^1 \in B_\infty(I, X)$. In step 2, we replace the pair $\{\gamma^o, x^o\}$ with the pair $\{\gamma^1, x^1\}$ in the adjoint equation (5.2) and solve it for

$\psi^1 \in B_\infty(I, X^*)$. At this stage we have the triple $\{\gamma^1, x^1, \psi^1\}$. We use this triple in the inequality (5.3) replacing the triple $\{\gamma^o, x^o, \psi\}$. This gives us the inequality

$$\int_I \langle \psi^1(t), G(t, x^1(t))(\gamma - \gamma^1)(dt)\rangle_{X^*,X} \geq 0, \ \forall\, \gamma \in \mathcal{M}_{ad}. \tag{6.1}$$

If this inequality holds, γ^1 is optimal. Instead of verifying this we proceed to determine an element γ^2 on the basis of the triple $\{\gamma^1, x^1, \psi^1\}$. We rewrite the expression on the left of the above inequality as follows:

$$\begin{aligned} &\int_I \langle \psi^1(t), G(t, x^1(t))(\gamma - \gamma^1)(dt)\rangle_{X^*,X} \\ &= \int_I \langle G^*(t, x^1(t))\psi^1(t), (\gamma - \gamma^1)(dt)\rangle_{Y^*,Y}. \end{aligned} \tag{6.2}$$

At this stage we need the duality map. Let $\Xi \equiv B_\infty(I, Y^*)$, and let $\Xi^* \equiv \mathcal{M}_{bfa}(\mathcal{A}_I, Y^{**})$ denote its topological dual. Since Y is a reflexive Banach space $\mathcal{M}_{bfa}(\mathcal{A}_I, Y^{**}) = \mathcal{M}_{bfa}(\mathcal{A}_I, Y)$. Define the duality map,

$$D(\xi) \equiv \{\eta^* \in \Xi^* : \langle\langle \xi, \eta^* \rangle\rangle = \|\xi\|^2_\Xi = \|\eta\|^2_{\Xi^*}\}. \tag{6.3}$$

Note that, by virtue of Hahn-Banach theorem the set valued function $D(\xi)$ is nonempty. Letting $\xi^1(t) \equiv G^*(t, x^1(t))\psi^1(t), t \in I$, and using the duality map and the above expression we can rewrite the Gâteaux differential of J as follows:

$$\begin{aligned} dJ(\gamma^1; \gamma - \gamma^1) &= \int_I \langle G^*(t, x^1(t))\psi^1(t), (\gamma - \gamma^1)(dt)\rangle_{Y^*,Y^{**}} \\ &= \langle\langle \xi^1, \gamma - \gamma^1 \rangle\rangle_{\Xi,\Xi^*}. \end{aligned} \tag{6.4}$$

Taking any element $\eta^1 \in D(\xi^1)$, and $\varepsilon > 0$ sufficiently small, we construct $\gamma^2 \equiv \gamma^1 - \varepsilon\eta^1$ so that it is an element of the set $\mathcal{M}_{ad}$. Using γ^2 in place of γ in the above expression we obtain

$$dJ(\gamma^1; \gamma^2 - \gamma^1) = \langle\langle \xi^1, \gamma^2 - \gamma^1 \rangle\rangle_{\Xi,\Xi^*} = -\varepsilon\|\xi^1\|^2 = -\varepsilon\|\eta^1\|^2_{\Xi^*}. \tag{6.5}$$

Using Lagrange formula we can express the cost functional J at γ^2 in terms of its value at γ^1 giving

$$\begin{aligned} J(\gamma^2) &= J(\gamma^1) + dJ(\gamma^1; \gamma^2 - \gamma^1) + o(\varepsilon) \\ &= J(\gamma^1) - \varepsilon\|\xi^1\|^2_\Xi + o(\varepsilon) = J(\gamma^1) - \varepsilon\|\eta^1\|^2_{\Xi^*} + o(\varepsilon). \end{aligned} \tag{6.6}$$

Hence for $\varepsilon > 0$ sufficiently small, we have $J(\gamma^2) < J(\gamma^1)$. Using this γ^2 and returning to step 1, and repeating the process, we construct a sequence of controls $\{\gamma^n\}$ satisfying

$$J(\gamma^1) > J(\gamma^2) > \cdots > J(\gamma^n) > \cdots.$$

Thus we have a sequence of controls along which the cost functional J decreases monotonically. Under the assumptions of Theorem 4.6, $J(\gamma) \geq 0$. Hence there exists a nonnegative number m_0 such that $J(\gamma^n) \longrightarrow m_0$. This completes the proof. □

Remark 6.2. In the above convergence theorem we assumed Y to be reflexive. It should be interesting to relax this assumption.

7. Purely impulsive controls

Purely impulsive controls, or purely atomic measures as controls used extensively in applications [7], are given by a series of Dirac measures. Let $\{\alpha_i\}$ be a sequence of nonnegative real numbers and define the scalar valued measure $\nu(dt) \equiv \sum_{i\geq 1} \alpha_i \delta_{t_i}(dt)$. Clearly the total variation norm of ν is given by $|\nu| = \sum \alpha_i$ provided the series is convergent. We assume that $\alpha \equiv \{\alpha_i\} \in \ell_1$. Let Y_0 be a weakly compact convex subset of Y and let $\mathcal{M}_\delta$ denote the following set of discrete vector measures:

$$\mathcal{M}_\delta \equiv \Big\{\gamma \in \mathcal{M}_{bfa}(\Sigma_I, Y) : \\ \gamma(dt) = \sum_{i\geq 1} \alpha_i y_i \delta_{t_i}(dt) : \alpha_i \geq 0, (\alpha_i) \in \ell_1 \text{ and } y_i \in Y_0\Big\},$$

where $t_i \in I_0 \equiv \{t_i, i \in \mathbb{N}, 0 < t_1 < t_2 < \cdots < t_\kappa < \cdots < T\}$. It is clear that the total variation norm of any element of this family is given by $\|\gamma\| = \sum_{i\geq 1} \alpha_i \|y_i\|_Y$ provided that $\{\|y_i\|_Y\}_{i\geq 1} \in \ell_\infty$. Since by assumption Y_0 is weakly compact (so bounded) this is automatically satisfied. The reader can easily verify that the family of measures $\mathcal{M}_\delta$ satisfies all the conditions of Lemma 4.1. Thus by Lemma 4.1, the set $\mathcal{M}_\delta$ is a weakly compact subset of $\mathcal{M}_{bfa}(\mathcal{A}_I, Y)$. Let $I_0 \equiv \{t_i, i \in \mathbb{N}\}$. In this case the system (2.1) can be represented by the following pair of equations:

$$dx(t) = Ax(t)dt + F(t, x(t))dt, x(0) = x_0,\ t \in I \setminus I_0, \tag{7.1}$$

$$\Delta x(t_i) = G(t_i, x(t_i))\gamma(\{t_i\}),\ t_i \in I_0, \tag{7.2}$$

where $\Delta x(t_i)$ denotes the jump at time t_i. The cost functional is given by

$$J(\gamma) \equiv \int_0^T \ell(t, x(t))\nu(dt) + \Phi(x(T)) \\ = \sum \alpha_i \ell(t_i, x(t_i)) + \Phi(x(T)). \tag{7.3}$$

The necessary conditions of optimality corresponding to the admissible set $\mathcal{M}_\delta$ is given by the following corollary.

Corollary 7.1. *Consider the system* (7.1)-(7.2) *with the cost functional* (7.3). *In order that a control measure* $\gamma^o (\equiv \sum \alpha_i y_i^o \delta_{t_i}(dt) \in \mathcal{M}_\delta)$ *and the corresponding solution* $x^o \in B_\infty(I, X)$ (*of equations* (7.1)-(7.2)) *be optimal, it is necessary* *that there exists a* $\psi^o \in B_\infty(I, X^*)$ *such that the triple* $\{\gamma^o, x^o, \psi^o\}$ *satisfies the following equations and inequalities*:

$$dx^o = Ax^o dt + F(t, x^o(t))dt, x(0) = x_0,\ t \in I \setminus I_0, \tag{7.4}$$

$$\Delta x^o(t_i) = \alpha_i G(t_i, x^o(t_i))y_i^o,\ i \in N,\ t_i \in I_0, \tag{7.5}$$

$$-d\psi^o = A^*\psi^o dt + DF^*(t, x^o(t))\psi^o(t)dt, \tag{7.6}$$

$$\psi^o(T) = \Phi_x(x^o(T)),\ t \in I \setminus I_0,$$

$$-\Delta\psi^o(t_i) = \alpha_i(D\mathcal{G})(t_i, x^o(t_i); \psi^o(t_i))y_i^o + \alpha_i \ell_x(t_i, x^o(t_i)),\ i \in N, \tag{7.7}$$

and

$$\sum_i \alpha_i \langle \psi^o(t_i), G(t_i, x^o(t_i))y_i \rangle_{X^*,X}$$

$$\geq \sum_i \alpha_i \langle \psi^o(t_i), G(t_i, x^o(t_i))y_i^o \rangle_{X^*,X},\ \forall\ y_i \in Y_0. \tag{7.8}$$

Proof. Proof readily follows from Theorem 5.1 by choosing $\mathcal{M}_\delta$ as the set of admissible controls. □

Remark 7.2. It follows from the expression (7.8) that

$$\sum_i \alpha_i \langle \psi^o(t_i), G(t_i, x^o(t_i))y_i \rangle_{X^*,X}$$

$$= \sum_i \alpha_i \langle G^*(t_i, x^o(t_i))\psi^o(t_i), y_i \rangle_{Y^*,Y} \equiv \sum_i L_i(y_i).$$

It is clear that, for each index i, $G^*(t_i, x^o(t_i))\psi^o(t_i) \in Y^*$. Hence each L_i is a continuous linear functional on Y. Since by assumption Y_0 is a weakly compact subset of the Banach space Y, each L_i attains its minimum on Y_0. Thus the existence of $\{y_i^o\}$ minimizing the linear functional is obvious.

Remark 7.3. In the field of vector measures, the measure ν appearing in Lemma 4.1 is known as the Rybakov control [8] for the set $\mathcal{M}_{ad}$. This measure plays an important role in the characterization of weak compactness of the set $\mathcal{M}_{ad}$. Practical implication is that, in the case of discrete measures, this determines the temporal positions (distribution) of the impulses and the number of such impulses in the set $\mathcal{M}_\delta$.

An Open Problem. Given a weakly compact set $\mathcal{M}_{ad} \subset \mathcal{M}_{bfa}(\mathcal{A}_I, Y)$, does there exist a $\mu^* \in \mathcal{M}_{bfa}^+(\mathcal{A}_I)$ that dominates the family $\mathcal{M}_{ad}$ set wise. The author is not aware of any such result. However we have seen that one can construct a dominating submeasure.

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