Injectivity radius of manifolds with a Lie structure at infinity

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Abstract

Using Lie groupoids, we prove that the injectivity radius of a manifold with a Lie structure at infinity is positive.

Introduction

Manifolds with a Lie structure at infinity were introduced by Ammann, Lauter and Nistor in [1], forming a class of non-compact complete Riemannian manifolds of infinite volume. In the same article, they conjectured that the injectivity radius of a (connected) manifold with Lie structure at infinity is positive. In this paper, we give a proof of this conjecture using the associated groupoid given by [5] and [6]. Together with the results from [1], this implies that manifolds with a Lie structure at infinity are of bounded geometry. In particular, the hypothesis of injectivity radius in [2] is now automatically satisfied, as well as in [3], where positivity of the injectivity radius is used to obtain uniform parabolic Schauder estimates. Bounded geometry also yields uniform elliptic Schauder estimates, see [4] for a recent application in this direction.

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1 Preliminaries

Following [1] and [8], we recall some definitions and facts.

Definition 1.1. A *groupoid* is a small category G in which every morphism is invertible.

The objects of the category are also called *units*, and the set of units is denoted by G^0. The set of morphisms is denoted by G^1. The range and domain maps are denoted respectively $r, d : G^1 \to G^0$. The multiplication operator μ is defined on the set of composable pairs of morphisms by:

$$\mu : G^2 = G^1 \times_{G^0} G^1 = \{(g,h) : d(g) = r(h)\} \to G^1$$

The inversion operation is a bijection $\iota : g \mapsto g^{-1}$ of G^1. The identity morphisms give an inclusion $u : x \mapsto \mathrm{id}_x$ of G^0 into G^1.

Definition 1.2. An *almost differentiable groupoid* $G = (G^0, G^1, d, r, \mu, u, \iota)$ is a groupoid such that G^0 and G^1 are manifolds with corners ([7]), the structural maps d, r, μ, u, ι are differentiable, and the domain map d is a submersion.

Consequently, for an almost differentiable groupoid, ι is a diffeomorphism, $r = d \circ \iota$ is a submersion and each fiber $G_x = d^{-1}(x) \subset G^1$ is a smooth manifold whose dimension n is constant on each connected component of G^0.
Following the convention in [5, p. 578], we require G^0 and $d^{-1}(x)$ to be Hausdorff (for all $x \in G^0$), but not necessarily G^1 to avoid excluding important cases.
From now on, *Lie groupoid* will stand for almost differentiable groupoid, *manifold* will stand for *manifold with corners* and *smooth manifold* will stand for manifold wihout corners.
A Lie groupoid is called *d-simply connected* if its d-fibers $G_x = d^{-1}(x)$ are simply connected ([5]).

Definition 1.3. A *Lie algebroid* A over a manifold M is a vector bundle A over M, together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of A and a bundle map $\rho : A \to TM$, extended to a map $\rho_\Gamma : \Gamma(A) \to \Gamma(TM)$ between sections of these bundles, such that

1. $\rho_\Gamma([X,Y]) = [\rho_\Gamma(X), \rho_\Gamma(Y)]$

2. $[X, fY] = f[X,Y] + (\rho_\Gamma(X)f)Y$

for any smooth sections X and Y of A and any smooth function f on M.

There is a Lie algebroid $A(G)$ associated to a Lie groupoid G, constructed as follows: let $T_{\text{vert}}G = \ker d_* = \cup_{x\in G^1} TG_x \subset TG^1$ be the vertical bundle over G^1. Then $A(G) = T_{\text{vert}}G|_{G^0}$ is the structural bundle of the Lie algebroid over G^0. The anchor map is given by

$$r_*|_A : A \to TG^0$$

([2]). The Lie bracket of $\Gamma(A)$ is the Lie bracket of $\Gamma(T_{\text{vert}}G)$ restricted to right invariant sections.

Definition 1.4. A Lie algebroid A over a manifold M is said to be *integrable* if there exists a Lie groupoid G such that $G^0 = M$ and A is isomorphic to the Lie algebroid associated to G. G is said to *integrate* A.

Remark 1.5. There might be more than one Lie groupoid integrating a Lie algebroid. However, by [5, Lie I], if a Lie algebroid over a smooth manifold is integrable, there is a unique d-simply connected Lie groupoid integrating it.

Example 1.6. 1. Any Lie group is a Lie groupoid with the set of units being a singleton.

2. ([8, Example 4, Section 4]) Let M be a smooth manifold. Let $\widetilde{M}$ be the universal covering of M. Let $H = (\widetilde{M} \times \widetilde{M})/\pi_1(M)$. Then H is naturally a d-simply connected Lie groupoid with the set of units being M, and the associated Lie algebroid being $\mathrm{id} : TM \to TM$. It is called the homotopy groupoid.

3. The space of continuous paths on a topological space modulo homotopy equivalence forms a groupoid which is called the fundamental groupoid.

We recall the definitions and basic properties of manifolds with Lie structures at infinity. For details and proofs, we refer to [1].

Definition 1.7. A *structural Lie algebra* of vector fields on a manifold M (possibly with corners) is a subspace $\mathcal{V} \subset \Gamma(TM)$ of the real vector space of vector fields on M with the following properties:

1. $\mathcal{V}$ is closed under Lie brackets;
2. $\mathcal{V}$ is a finitely generated projective $\Gamma(M)$-module;
3. The vector fields in $\mathcal{V}$ are tangent to all faces in M.

Denote by $\mathcal{V}_b(M) \subset \Gamma(TM)$ the subspace of vector fields tangent to all faces in M. This is a structural Lie algebra of vector fields, and any structural Lie algebra is a subspace of $\mathcal{V}_b(M)$ ([1, Example 2.5]). By the Serre-Swan theorem, given a structural Lie algebra of vector fields $\mathcal{V}$ on M, there exists a vector bundle $A = A_{\mathcal{V}} \to M$ such that $\mathcal{V} \simeq \Gamma(A_{\mathcal{V}})$, and there exists a natural vector bundle map $\rho : A_{\mathcal{V}} \to TM$ such that the induced map $\rho_\Gamma : \Gamma(A_{\mathcal{V}}) \to \Gamma(TM)$ is identified with the inclusion map $\mathcal{V} \subset \Gamma(TM)$. The vector bundle $A_{\mathcal{V}}$ is then a Lie algebroid with anchor map ρ.

Definition 1.8. A *Lie structure at infinity* on a smooth manifold M_0 is a pair $(M, \mathcal{V})$, where

1. M is a compact manifold, possibly with corners, and M_0 is the interior of M;
2. $\mathcal{V}$ is a structural Lie algebra of vector fields on M;
3. $\rho : A_{\mathcal{V}} \to TM$ induces an isomorphism on M_0, that is, $\rho|_{M_0} : A|_{M_0} \to TM_0$ is an isomorphism of vector bundles.

Definition 1.9. A *Riemannian manifold with a Lie structure at infinity* is a manifold with a Lie structure at infinity $(M, \mathcal{V})$ endowed with a bundle metric g on $A = A_{\mathcal{V}}$. In particular, g defines a Riemannian metric on M_0 via the anchor map.

A Riemannian manifold with a Lie structure at infinity has infinite volume ([1, Proposition 4.1]), bounded curvature ([1, Corollary 4.3]) and is complete ([1, Corollary 4.9]). Sufficient conditions for the positivity of the injectivity radius are given in [1, Theorem 4.14] and [1, Theorem 4.17].

2 Injectivity radius of a manifold with Lie structure at infinity

The following theorem is due to Debord ([6, Theorem 2], see also [5, Corollary 5.9]).

Theorem 2.1 (Debord)**.** *Every almost injective Lie algebroid over a smooth manifold is integrable.*

This has the following implication for Lie structures at infinity.

Theorem 2.2. *Any Lie algebroid over a manifold with corners associated with a Lie structure at infinity is integrable.*

Proof. This extension of Theorem 2.1 to manifolds with corners is well-known to experts. However, since no explicit proof seems to be available in the literature, we will provide one for the convenience of the readers.
Let $(M, \mathcal{V})$ be a Lie structure at infinity of M_0 and $A = A_{\mathcal{V}}$ be the corresponding structural vector bundle. Taking two copies of M and gluing them along a maximal subset of disjoint boundary hypersurfaces, we obtain a compact manifold with corners M_1 with at least one hypersurface less. Repeating this operation finitely many times, we obtain a closed manifold $\widetilde{M}$ with a finite group Γ acting on $\widetilde{M}$ such that $\widetilde{M}/\Gamma \simeq M$ topologically. Now, by [7, Exercise 1.6.2], M is naturally an orbifold. Changing the smooth structure on $\widetilde{M}$, one can in fact ensure that the quotient map $q : \widetilde{M} \to M$ is such that $q^*(\mathcal{C}^\infty(M)) = \mathcal{C}^\infty(\widetilde{M})_\Gamma = \{f \in \mathcal{C}^\infty(\widetilde{M}) : \forall g \in \Gamma, f \circ g = f\}$. In a suitable local chart, q can be written as $(x_1, \dots, x_k, x_{k+1}, \dots, x_n) \mapsto (x_1^2, \dots, x_k^2, x_{k+1}, \dots, x_n)$ where k is the depth of the point $(0, 0, \dots, 0)$. Let $\widetilde{\mathcal{V}} = \mathcal{C}^\infty(\widetilde{M}) \otimes_{\mathcal{C}^\infty(\widetilde{M})_\Gamma} q^*\mathcal{V} \subset \mathfrak{X}(T\widetilde{M})$ be the pull-back of the structural vector fields. For instance, if $\mathcal{V}_b(M)$ is the space of vector fields tangent to the faces of M, then $\widetilde{\mathcal{V}}_b(M)$ is the space of vector fields on $\widetilde{M}$ which are tangent to $q^{-1}(\partial M)$ (the union of some closed submanifolds of $\widetilde{M}$).
Now, $\widetilde{\mathcal{V}}$ is a finitely generated projective $\mathcal{C}^\infty(\widetilde{M})$-module. To see this, it suffices to show that $\widetilde{\mathcal{V}}$ is locally free of rank k for some k. Given $p \in \widetilde{M}$, then since $\mathcal{V}$ is locally free of rank k for some k, there exist $v_1, \dots, v_k \in \mathcal{V}$ which locally and freely span $\mathcal{V}$ near $q(p)$. This means $\widetilde{\mathcal{V}}$ is locally and freely spanned by $q^*v_1, \dots, q^*v_n \in \widetilde{\mathcal{V}}$ near p, showing that $\widetilde{\mathcal{V}}$ is locally free of rank k as claimed.
By the Serre-Swan theorem, we have a vector bundle $A_{\widetilde{\mathcal{V}}}$ over $\widetilde{M}$ with the space of smooth sections $\mathcal{C}^\infty(\widetilde{M}, A_{\widetilde{\mathcal{V}}}) = \widetilde{\mathcal{V}}$. Clearly the inclusions $\widetilde{\mathcal{V}} \subset \widetilde{\mathcal{V}_b} \subset \mathcal{C}^\infty(\widetilde{M}, T\widetilde{M})$ induce an anchor map, so that $A_{\widetilde{\mathcal{V}}}$ is naturally an almost injective Lie algebroid. By the Theorem 2.1 and the Remark 1.5, there exists therefore a d-simply connected groupoid $\widetilde{G}$ integrating $A_{\widetilde{\mathcal{V}}}$. Each element $g \in \Gamma$ induces an automorphism $\rho(g) : A_{\widetilde{\mathcal{V}}} \to A_{\widetilde{\mathcal{V}}}$, and by Lie II, an automorphism on $\widetilde{G}$. Hence we have an action of the group Γ over $\widetilde{G}$. The quotient $\widetilde{G}/\Gamma$ is then the desired d-simply connected Lie groupoid integrating $(M, \mathcal{V})$. ∎

Let M_0 be a connected smooth manifold with a Lie structure at infinity $(M, \mathcal{V})$. By Theorem 2.2, there exists a d-simply connected groupoid $G = (M, G^1, d, r, \mu, u, \iota)$ with units M such that $A(G) \simeq A$ as Lie algebroids over M. Therefore $A(G)$ is equipped with an inner product also noted g. The anchor map is given by $r_* : A(G) \to TM$.
We have an isomorphism $r^*A(G) \simeq T_{\text{vert}}G$ where $r^*A(G)$ is the pull-back of $A(G)$ via the range map $r : G \to M$ ([2, (19)]). Explicitly, for $p \in G$, $(r^*A(G))_p = A(G)_{r(p)} = T_{r(p)}G_{r(p)}$. The vector bundle $r^*A(G)$ is equipped with a metric induced by the metric g on $A(G)$, hence so is $T_{\text{vert}}G$. Therefore each G_x becomes a Riemannian manifold for all $x \in M$.
Let $G_x^x = \{g \in G_x : r(g) = x\}$. For $x \in M_0$, G_x^x is a discrete group since $T_xG_x^x$ is of dimension 0 (being the kernel of the map $r_* : A(G)_x \to T_xM_0$).

Lemma 2.3. *([2, page 733]) If $A \to TM$ is the Lie algebroid associated with a Lie structure at infinity and G is the corresponding d-simply connected Lie groupoid, then for all $x \in M_0, r : G_x \to M_0$ is a covering map with group G_x^x.*

Proof. By [5, Proposition 1.1], for all $x \in M_0$, $r(G_x) \subset M_0$ (which is the leaf of the singular foliation of A passing by x). On the other hand, $G|_{M_0}$ is the unique d-simply connected Lie groupoid which integrates TM_0, and therefore it isomorphic to the homotopy groupoid $(\widetilde{M_0} \times \widetilde{M_0})/\pi_1(M_0)$. Consequently, $M_0 = r(G_x)$ for all $x \in M_0$.
Now, by definition of a Lie structure at infinity, $r_* : T_yG_x \to T_{r(y)}M_0$ is an isomorphism. This means that $r : G_x \to M_0$ is a local diffeomorphism. Moreover, $g_1, g_2 \in G_x$ with $r(g_1) = r(g_2)$ if and only if there exists $h = g_1^{-1}g_2 \in G_x^x$ such that $g_2 = g_1h$. That is, $r : G_x \to M_0$ is a covering map with group G_x^x. ∎

Theorem 2.4. *Let M_0 be a connected smooth manifold with a Lie structure at infinity $(M, \mathcal{V})$. Then for any Riemannian metric g on A, the injectivity radius of (M_0, g) is positive.*

Proof. We prove the theorem by contradiction. Suppose that the injectivity radius of (M_0, g) is zero, then, as the curvature is bounded, there is a sequence of geodesic loops $c_i : [0, a_i] \to M_0$, parametrized by arc-length, with $a_i \to 0$. By compactness of M, we can suppose that $c_i(0)$ converges to a point $p \in M$. We have $p \in \partial M$ since the injectivity radius is positive in any compact region of M_0.
Let U be a local chart of M containing p such that U is contractible.

Lemma 2.5. *There exists a number $N > 0$ such that $\forall n > N$, the loop c_n is contained in U.*

Proof. Let $(x_1, \ldots, x_k, y_1, \ldots, y_l)$ be a set of local coordinates centered at the point p with $x_i \geq 0$ for all i and $p = (0, \ldots, 0)$. Let $g_b = \sum_{i=1}^k \frac{dx_i^2}{x_i^2} + \sum_{i=1}^l dy_i^2$ be a local b-metric and $g_0 = \sum_{i=1}^k dx_i^2 + \sum_{i=1}^l dy_i^2$ be a local metric with boundary. Since the structural vector fields are tangential vector fields ($\mathcal{V} \subset \mathcal{V}_b$), taking U smaller if needed, there exist constants $C, K > 0$ such that $g \geq C g_b \geq CK g_0$ in $U \cap M_0$. Let $l^t(c_i), l_b^t(c_i), l_0^t(c_i)$ denote the lengths of the segment $[c_i(0), c_i(t)]$ (of the geodesic loop c_i) with respect to the metric g, the local b-metric g_b and the local metric with boundary g_0 respectively (suppose that the segment is contained in U). Let $\varepsilon > 0$ be such that $B_0(p, \varepsilon) = \{x \in \mathbb{R}_+^k \times \mathbb{R}^l : d_0(x, p) < \varepsilon\} \subset U$ (where d_0 is the distance with respect to the metric g_0, well-defined on $B_0(p, \varepsilon)$). Since $a_i \to 0$, there exists N_1 such that $a_i < \min(\frac{\varepsilon}{4}, CK\frac{\varepsilon}{4})$ for all $i > N_1$. Since $c_i(0) \to p$, there exists N_2 such that $d_0(p, c_i(0)) < \frac{\varepsilon}{4}$ for all $i > N_2$. Let $N = \max(N_1, N_2)$.
Now let n be any number greater than N. Suppose that the loop c_n is not contained in U. Then it is not contained in $B_0(p, \frac{\varepsilon}{2})$. Thus there exists $t \in [0, a_n]$ minimal such that $d_0(c_n(t), p) = \frac{\varepsilon}{2}$. Then we have $d_0(c_n(0), c_n(t)) \geq |d_0(c_n(t), p) - d_0(c_n(0), p)| \geq \frac{\varepsilon}{4}$, which implies $a_i = l(c_i) \geq l^t(c_i) \geq CK l_0^t(c_i) \geq CK d_0(c_n(0), c_n(t)) \geq CK\frac{\varepsilon}{4}$, which is a contradiction. Therefore the loop c_n is contained in U.
The lemma is proven. ∎

Hence, without loss of generality, we can suppose that the loops are contained in U.
Denote by $G = (M, G^1, d, r, \mu, u, \iota)$ the d-simply connected groupoid integrating $A_{\mathcal{V}} \to TM$. Since U is contractible, the fundamental class of each loop c_i is trivial, therefore by Lemma 2.3 we can lift c_i to a geodesic loop $\widetilde{c}_i$ in $G_{c_i(0)}$ (i.e. $\widetilde{c}_i : [0, a_i] \to r^{-1}(U) \cap G_{c_i(0)}$) such that the base points are $\widetilde{c}_i(0) = \widetilde{c}_i(a_i) = c_i(0) = c_i(a_i)$.
Let $S(T_{\mathrm{vert}}G) = \{x \in T_{\mathrm{vert}}G : \|x\| = 1\}$. We have a natural projection $\pi : S(T_{\mathrm{vert}}G) \to G^1$. On $S(T_{\mathrm{vert}}G)$ we have a flow Ψ which, over each d-fiber G_x of $d : G^1 \to G^0$, corresponds to the geodesic flow of G_x. The geodesic loops on G_x correspond to segments $[P_i, Q_i]$ of the flow Ψ on $S(TG_x)$ (with $Q_i = \Psi_{a_i}(P_i)$). We have two sequences $P_i = (\widetilde{c}_i(0), \dot{\widetilde{c}}_i(0))$ and $Q_i = (\widetilde{c}_i(a_i), \dot{\widetilde{c}}_i(a_i))$ in $S(A) \subset S(T_{\mathrm{vert}}G)$. By compactness of $S(A)$ and M, there exists a subsequence such that $P_i \to P \in S(TG_p)$ and $Q_i \to Q \in S(TG_p)$.
Since $a_i \to 0$, we have $P = Q$. In a local chart, we can write $(\frac{Q_i - P_i}{a_i}, c_i(0)) \to (w, p)$. Since $a_i \to 0$, $w = \dot{\Psi}(P)$. Since $P_i, Q_i \in (S(A))_{c_i(0)}$ for all i, w is tangent to the fiber $S(A)_p = S(TG_p)$, which is a contradiction (for Ψ is the geodesic flow over G_p).

∎

Remark 2.6. In [1], a flow Φ is defined on $S(A)$ extending the geodesic flow on $S(TM_0)$. However, Φ itself is not quite a geodesic flow since typically it has fixed points at the boundary. Our approach does not seem to work with this flow. Indeed, to each geodesic loop $c_i : [0; a_i] \to M_0$, we have a corresponding segment $\Phi_i : [0; a_i] \to S(A)$. By considering a convergent subsequence, the limit of $(c_i(0), \dot{c}_i(0))$ is a point v contained in $\partial S(A) = S(A)|_{\partial M}$. The limit of $c_i(0)$ is a point $p = \pi(v)$ in ∂M. In the notations of [1], we have $(\pi^{\#} r_*)(H_v(v)) = 0$ and $r_*(v) = 0$. In particular, the flow Φ at v is stationary: $\forall t, \Phi_t(v) = v$. This, however, is not sufficient to obtain a contradiction, since at the boundary, Φ may have some fixed points as mentioned above.

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