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Demystifying Proofs Through Structured Interaction: A Case Study of One Instructor's Teaching in an Undergraduate Analysis Course

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ABSTRACT

The concept of proof is central to every undergraduate mathematics curriculum, but it is also a difficult concept for university students to understand and for university instructors to teach. Prior studies have contributed useful research knowledge about the nature and sources of undergraduate students' difficulties with proof, but they have paid less attention to the development of effective instructional practices to enhance student learning of proof. As a step towards this direction, this paper makes a contribution to the limited body of case studies of promising or potentially effective teaching practices in undergraduate mathematics by reporting a case study of an instructor's teaching, locally characterized as "effective," in an undergraduate analysis course at a leading Indian university. The instructor did not deviate from the so called "Definition-Theorem-Proof" (DTP) format that is followed in most proof-oriented university mathematics courses, but her teaching presented a set of features that, collectively, form a rather innovative teaching practice at the undergraduate level. Specifically, our case study shows that, even in the context of a rather crowded university classroom, proof can be demystified for students through structured interaction between the instructor and the students, that is, an interactive, conversational style of proof instruction invoking the participation of students. This is based on a solid foundation in symbolic logic at the very outset and on a significant time investment in definitions being explained in depth using informal language, visual aids, and real-life analogies. In addition to contributing to existing images of potentially effective teaching practices in undergraduate mathematics, this paper draws attention to the Indian educational context that has had little representation in international forums of mathematics education research thus far.

Key Words: Definition, Proof, Theorem, Undergraduate mathematics

INTRODUCTION

The concept of proof is at the core of every undergraduate mathematics curriculum, but it continues to be a difficult concept for university students to understand and for university instructors to teach. Prior studies at various levels of education have contributed useful research knowledge about the nature and sources of students' difficulties with proof, but on the whole less attention has been paid to the development of effective teaching practices to enhance students' learning of proof (for a review, see Stylianides, Stylianides, & Weber, 2017). This is especially problematic at the undergraduate level where there are relatively fewer studies than at the school

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level that documented promising or potentially effective teaching practices. Indeed, a rather extended body of research has offered images of what it might mean and look like when school teachers effectively engage their students with proof-related activities (e.g., Ball & Bass, 2000, 2003; Lampert, 1992; Reid, 2002; Stylianides, 2007, 2016; Zack, 1997), but these images cannot be carried over into the university level where the style of instruction is considerably different from school mathematics and often follows the so called "Definition-Theorem-Proof" (DTP) format (cf. Davis & Hersh, 1981; Dreyfus, 1991).

Thus, there is a need to expand the limited body of case studies of promising or potentially effective teaching practices at the undergraduate mathematics level (Fukawa-Connelly, 2012; Weber, 2004), especially teaching practices that do not deviate from the DTP format (broadly conceived). Maintaining some connection to the DTP format is strategic in terms of the potential usability or usefulness of the derived research knowledge, for this format is followed (one way or another) in most proof-oriented university mathematics courses and thus many university instructors relate to it. According to Weber (2012), "mathematicians are unlikely to implement teaching suggestions if these suggestions are at variance with their pedagogical goals and beliefs or if the suggested pedagogy is perceived to be outside the norms of appropriate pedagogical practice" (p. 464).

In this paper, we report a case study of one instructor's teaching, locally characterized as "effective," in an undergraduate analysis course at a leading Indian university. The instructor did not deviate from the DTP format, but her teaching presented a set of features that, collectively, form a rather innovative teaching practice in undergraduate mathematics that we believe is worth documentation and broader consideration. In addition to helping expand the thin research basis of empirical studies on teaching practices at the undergraduate level (Speer, Smith, & Horvath, 2010) and contributing to existing images of potentially effective proof-related instruction, we use this paper to draw attention to the Indian educational context that has had little representation in international forums of mathematics education research thus far (see Mesa & Wagner, 2019).

RELATED LITERATURE

1. Undergraduate students' difficulties with proof

It is a well-documented fact that many undergraduates face considerable difficulty in handling proof. A main factor to which this difficulty can be attributed is a sudden shift in mathematical epistemologies in the transition from school mathematics to university mathematics (e.g., Moore, 1994; Tall, 2010).

According to Tall (2010, p. 21), the curriculum throughout school education is largely confined to two forms of mathematical thinking or mental "worlds" of mathematics as he called them: *conceptual embodiment*, which is motivated by visual or physical perception, and *operational symbolism*, which involves a sequential procedure of computational operations. Right from kindergarten to upper-level school mathematics, the entire system is geared towards inculcating and nurturing the ways of thinking required for these two worlds. At university, however, "all this is turned on its head and reformulated in terms of axiomatic systems and formal deduction" (p. 21), which relate to a third world of mathematics, namely, *axiomatic formalism*. Thus, in the transition from school to university, there is a paradigm shift in mathematical epistemologies. The ways of thinking to which students had been accustomed during their school years are inadequate to cope with this third world of mathematics. Indeed, Tall (2010) noted that students' new experience with the world of axiomatic formalism is "accompanied by mental confusion as links, previously connected in perception and action, now require reorganization as formal deductions, and subtle implicit links from experience may be at variance with the new formal setting" (p. 21).

Tall (2010) illustrated the above using the topic of vectors. Under conceptual embodiment, a vector is defined as an entity with both magnitude and direction, which can be geometrically represented as a directed line segment. The second notion of a vector as simply an n-tuple of real numbers, represented as a column-vector, falls entirely within the ambit of *operational symbolism*, and is used for systems of equations in matrix algebra. Yet in the *axiomatic formalism* of undergraduate mathematics, a vector is just an element of an axiomatic structure called a vector space; students now have to ignore most of what they know about vectors and deduce the properties using the axioms only.

Undergraduates who intend to major in mathematics have to study about six or seven such axiomatic structures in great depth, which constitute about half their degree program. These are: the real number system; groups; rings and fields; vector spaces; Boolean algebras; metric spaces; and probability spaces. Each of these has about fifty to hundred proofs of various levels of complexity (and their application to problem solving). However, students are not prepared for this transition to proof, which Moore (1994) and others described as "abrupt." According to Moore (1994, p. 249), "many students begin their upper-level mathematics courses […] having seen no general perspective of proof or method of proof," and "[t]his abrupt transition to proof is a source of difficulty for many students, even for those who have done superior work with ease in their lower-level mathematics courses."

Proof is a multifaceted concept that has been studied from various perspectives in the research literature (for a discussion of some of these, see Stylianides et al., 2017) and thus students' difficulties with proof are broad-ranging and present themselves at undergraduate mathematics in various forms during proof comprehension, proof construction, proof validation, proof writing, etc. A common source of many student difficulties with proof, especially its form required at the undergraduate level where the validity of a proof's *modes of argumentation* (cf. Stylianides, 2007) cannot be compromised, is students' inadequate understanding of the basics of symbolic logic, such as quantifiers like "there exists" and "for every"; conditional statements and their converse, inverse, and contrapositive; negation; and implication (e.g., Harel & Sowder, 2007; Stylianides et al., 2017). For example, several students including undergraduates are unsure about interpreting the "direction" of a conditional statement, phrases like "if and only if" and "necessary and sufficient conditions," the implication symbol, and the negation of a statement, and they end up proving the converse of what is required, being convinced that a conditional statement is equivalent to its inverse but not its contrapositive, having difficulty formulating the negation of a claim in a proof by contradiction, and so on (e.g., Antonini & Mariotti, 2008; Durand-Guerrier, 2003; Hoyles & Küchemann, 2002; Inglis & Alcock, 2012; Selden & Selden, 2003; Stylianides, Stylianides, & Philippou, 2004; Weber, 2010; Yu, Chin, & Lin, 2004).

Students' difficulties with proof coupled with the strong emphasis of undergraduate mathematics on proof both emphasize the need for the identification or development of effective teaching practices to support undergraduate students' understanding of and engagement with proof. Unfortunately, however, effective practices for proof instruction are still scarce in university mathematics classes.

2. The DTP format of instruction

Davis and Hersh (1981) asserted that, "[i]n college, a typical lecture in advanced mathematics […] consists entirely of definition, theorem, proof, definition, theorem, proof, in solemn and unrelieved concatenation" (p. 151). It is widely accepted by mathematics educators and mathematicians that most proof-oriented courses are taught in this Definition-Theorem-Proof (DTP) format whose *traditional* style was described by Weber (2004) as follows:

> The instruction largely consists of the professor lecturing and the students passively taking notes, the material is presented in a strictly logical sequence, the logical nature (e.g., formal definitions, rigorous proofs) of the covered material is given precedent over its intuitive nature, and the main goal of the course is for the students to [be] capable of producing rigorous proofs about the covered mathematical concepts. (Weber, 2004, p. 116)

Weber's (2004) description of the traditional DTP style suggests that this format of instruction is typically "lecture-based" (Fukawa-Connelly, 2012) and that the style of proof presentation is primarily a combination of the *logico-structural* and *procedural styles* (Weber, 2004), that is, the emphasis is on establishing the proofs' logical veracity based on algebraic and symbolic manipulations without much attention to the intuitive motivation of concepts or the use of diagrams. This combined style of proof presentation is contrasted with the *semantic style* (Weber, 2004), which is characterized by the intuitive motivation of concepts and relationships and is generally supported by visual aids and diagrams.

The DTP format, in its traditional style as we described it above, has been widely criticized with some of the criticisms focusing on particular aspects of it, such as its lecture-based component. Weber (2004, pp. 116-117) and Fukawa-Connelly (2012, pp. 325-326) summarized some of these criticisms, including that this format of instruction intimidates students (Thurston, 1994), it denies the role of intuition in the learning process (Dreyfus, 1991), it is incapable of inducing meaningful learning (Leron & Dubinsky, 1995), and it conceals much of the complexity that characterizes mathematical practice (Davis & Hersh, 1981).

Overall the traditional DTP style is not conducive to students forming an "epistemic fluency" that would allow students to recognize and productively engage with the undergraduate institutional practices related to proof, thus failing to facilitate also students' smooth transition from school to university mathematics (Solomon, 2006). For example, the undergraduate students in Solomon's (2006) study lamented that proofs were presented as a finished product to be learned, not constructed by them, and they expressed a disenfranchisement in the learning process and adverse comparisons with their school experience. Of course, the DTP format of instruction does not have to follow, and indeed does not always follow, the traditional style; published case studies of university instructors' teaching that we review next (Fukawa-Connelly, 2012; Weber, 2004), as well as the case study that we report in this paper, all show that the DTP format need not be a rigid one and that it can be adapted to support meaningful student learning in undergraduate mathematics.

3. Two earlier case studies

Even though the concept of proof has attracted considerable research attention over the past few decades (Harel & Sowder, 2007; Stylianides et al., 2017), there has been little research on proof-related teaching practices at the undergraduate level (Speer et al., 2010; Weber, 2012). Two notable exceptions to this trend are the case studies reported by Weber (2004) and Fukawa-Connelly (2012) of the teaching practices of two instructors in the context of a real analysis course and an abstract algebra course, respectively. Weber and Fukawa-Connelly do not present the instructors in their case studies as "effective," though at least the instructor in Weber's study did appear to be locally characterized as effective as he had been awarded a major university teaching award. In any case, the teaching practices of both instructors exhibited some of the characteristics of the traditional DTP format but also some characteristics that deviated from the traditional style and appeared to be promising or potentially effective, thus justifying their detailed documentation and careful consideration.

1) The case study in a real analysis course

Weber (2004) conducted a case study of one professor called Dr. T teaching an introductory real analysis course to a group of 16 students at an American university. Dr. T was known to be popular with his students and had won a major university teaching award. An observation and analysis of his teaching practice revealed that he had many characteristics of the traditional DTP format of instruction, such as writing the definitions, examples, theorems, and proofs on the board with the students copying his writings into their notebooks. Also, he rarely asked questions or initiated discussions. However, Weber observed that Dr. T was different from traditional instructors in several significant ways. For example, his lectures were not linearly presented as a finished product but interspersed with explanations. These were either verbal or written as "scratch work" on a clearly demarcated side of the board. This was done for a better understanding, and so that students could construct similar proofs themselves.

Dr. T adopted a variety of teaching styles, depending on what he perceived to be as most suitable for the topic being covered. For instance, he mainly used the logico-structural style for the foundation topic of set theory, the procedural style for sequences, and the semantic style for the topology of real numbers. His teaching practice thus covered a broad canvas, even though it could be considered to be within the ambit of the DTP format of instruction.

For the *logico-structural style*, Dr. T stressed the importance of carefully using the definitions to understand how to begin and conclude a proof. He enunciated the following *unpacking principle* (p. 121): "A guiding principle when writing these proofs is to write down what we have and where we are headed. Many of these proofs are really just a matter of following through the definitions until we reach the conclusion," and "doing this can take you a long way on many of the problems." In keeping with this principle, Dr. T would start by writing the relevant definitions on the side as scratch work. He would list the hypotheses at the top and the conclusion at the bottom of the board. He would then use the definitions to unpack both, proceeding down from the hypotheses and up from the conclusion till they met somewhere in between, and the proof was constructed. In the end he would go through the entire written work linearly, explaining its logical validity.

Dr. T demonstrated that this "meet in the middle" strategy supported by scratch work could also be suitably adapted for the *procedural style*. For instance, while teaching limits of sequences, the scratch work – which is later incorporated into the body of the proof – comprises mainly of manipulating inequalities. He advocated this strategy as a handy "crutch" while constructing proofs involving limits.

It may be observed that Dr. T's teaching of sets, functions, and sequences was devoid of any semantic thought, and seldom made use of diagrams. However, for the advanced real line topology, he made a complete switchover to the *semantic style*: When introducing each topological concept, Dr. T would first give an intuitive description of the idea that the concept was trying to capture. For this, he usually used two-dimensional diagrams (even though the real line is unidimensional), as he felt that they were richer and more illuminating than one-dimensional ones. He then gave the formal definitions, illustrated them with examples, and linked them both to the diagrams. Again, for presenting the proofs of theorems, he first drew pictures illustrating the plausibility of the situation before providing the proof.

Weber asserts that Dr. T's pedagogical actions were based on a coherent set of mathematical and pedagogical beliefs. For example, at the very outset Dr. T honed the logical skills of his students through a quiz involving basics of logic like implication and negation, and this was because he believed that these skills were a prerequisite for students being comfortable with proofs. Dr. T apprehended also that "[i]f students find analysis too difficult, they will become frustrated and give up on the course" (p. 128), so he geared his teaching to prevent that. Thus Dr. T had "knowledge of the skills and understanding that students require in order to produce competent performance at proof-writing" (p. 128). He also had genuine concern for his students' academic interests, and for education in general.

2) The case study in an abstract algebra course

The second case study to be discussed here, conducted by Fukawa-Connelly (2012), is of a highly qualified lecturer called Dr. Tripp who taught an abstract algebra course to a group of 15 undergraduates at an American university. The instructor "was interested in educational issues, having co-authored a teaching manual as a graduate student" and she participated in "a professional development program for new or recent Ph.D.s in the mathematical sciences with the goal of changing her instructional practices in introductory mathematics courses" (p. 332). She was selected for study "because in her teaching of abstract algebra she claimed to use traditional teaching practices [notably lecturing in front of the class], she was a subject matter expert, and because her pedagogical expertise suggested she would be a good teacher" (p. 333). Based on Fukawa-Connelly's analysis, "this instructor, who claimed to give traditional lectures, actually gave lectures which were more student-centered than some might expect" (p. 343). The main factor for this was that her lectures took the form of proof presentation with dialog.

In contrast to Dr. T's monologs, Dr. Tripp's classes were predominantly conversational in style. She unfolded the proofs through a sequence of questions that were directed at the students and "modeled appropriate mathematical thinking by a master of the discipline" (p. 324). In this sense, "[t]he question sequences can be understood as the internal dialog that a mathematician might engage in when approaching a proving task" (p. 342), such as "What does that mean?", "What comes next?", and "What do I still need to do?". She expertly "funneled" her questions towards eliciting correct responses, and constantly acknowledged these with a "great!" or a "yeah!". All in all, the class experience provided by Dr. Tripp was uncommonly interactive, at least at a verbal level.

However, all proofs were presented linearly from beginning to end without any intuitive motivation or epistemic digression. Unlike Dr. T, there were no separate columns for thinking and writing. The dialog, woven into the presentation, was funneled to require factual responses only. Questions involving anything beyond the routine were answered by Dr. Tripp herself, without wait time for the students to think about them. Thus, while her lectures were highly participative at a functional level, they were largely confined to the logico-structural style, which did not leave much scope for a deeper semantic involvement of the students with the subject matter.

4. Concluding remarks and research question

A comparison of the classroom practices of Dr. T and Dr. Tripp as described in the two case studies shows that, while both broadly fell within the DTP format of instruction, each of them included some pedagogical techniques that enriched the students' epistemic experience. Thus, the practices of the two instructors illustrate Weber's (2004) observation that "DTP instruction is not a single teaching paradigm, but rather a diverse collection of pedagogical techniques sharing some core features" (p. 131). In particular lecture styles that fall under the umbrella of the DTP format of instruction "may vary widely and lead to drastically different learning on the part of the students" (Weber, 2004, p. 131), a point that was made well in Fukawa-Connelly's (2012) study.

The case study that we report next contributes further to the field's understanding of the DTP format of instruction as being more nuanced than is commonly assumed. The research question that guided our research was the following: How might the DTP format of instruction be adapted to productively engage students in proof-related work in the context of a rather crowded undergraduate analysis course in an Indian university, and what are the main features of the practice of the teacher, locally characterized as "effective," who is implementing this instruction?

RESEARCH CONTEXT AND METHODS

1. The Indian context – a backdrop

There are some problems peculiar to the Indian educational system, which make the task of undergraduate instruction in India a daunting one (though the college selected for our case study, being one of the few "elite" ones, does not face these to the extent that the state-run institutions do). These are mainly:

(1) *Large class size*: All popular undergraduate STEM courses in state-run institutions have an average class size of 80–100 students, making dialogic approaches to teaching difficult.
(2) *Language dichotomy*: While the medium of instruction in state-run schools right up to advanced-level courses is the mother-tongue (Hindi for the entire of North and Central India), university education in STEM courses is almost fully in English, and most of the prescribed texts are by American authors. So, in order to be an effective teacher for STEM courses, one has to give bilingual instruction, that is, to write all concepts in English but explain them in Hindi (or whatever the regional language is). Some colleges tackle this problem by forming their tutorial groups according to the language preferred for explanations. The language factor gives a decisive advantage to those privileged few (about 10%) who have had their entire school education in the very expensive, private English medium schools. It is this "elite" class of students who predominantly excel in university and who can subsequently compete effectively with their western counterparts.
(3) *Caste-based quotas*: As per the constitution of India, at least 40% of seats in all state-run educational institutions (including universities), and 20% in private ones, are reserved for the "lower and backward castes" who constitute almost 70% of India's population. This has caused deep resentment amongst the hitherto privileged upper castes, who have a vested interest in the maintenance of the status-quo. The unfortunate consequence of this is the social alienation of these "quota" students, leading sometimes to disharmony and tension within the classroom.

2. The course

The case study of this paper is in the context of a one-semester introductory real analysis course that was taught to a group of 48 students at an "elite" college of a leading Indian university in Spring 2018. Most of the students had done their entire schooling up to advanced-level courses in English medium, but 7 (of the underprivileged reserved quota) had studied in Hindi medium. This university offers STEM courses in English medium only, which poses a major challenge to the "quota" students, especially as such "elite" institutions pay little heed to their special language needs.

The topics covered in the course were: (1) real line topology; (2) real sequences (convergence); and (3) infinite series. The prescribed textbooks were Bartle and Sherbert (2015), Chapters 2, 3, and 11.1, and Bilodeau, Thie, and Keough (2011), Chapter 6. The course was offered over a 14-week semester, with classes (of 50 min each) held thrice a week. The lectures were supplemented by weekly tutorials in four groups of 12 students each (with one of these comprising of students who preferred to have the explanations in Hindi).

3. The instructor

The lecturer, whom we call Ms. X and whose teaching is being examined in this case study, holds a Master's degree in Mathematics from a leading Indian university, and is a gold medalist for securing the first position. She has been teaching for over three decades, and is an Associate Professor in a top-ranked college (her alma mater) affiliated to the university. During this period, she has taught courses in both pure and applied mathematics, ranging from real analysis and abstract algebra to mechanics and statistics. She does not have a doctoral degree, nor any published papers to her credit. However, she has consistently received outstanding evaluations from her students in whatever she has taught, has a formidable local reputation as a teacher par excellence, and is a crusader for educational reform. Comparative data between Ms. X's student evaluations and outcomes and those of other instructors of the same courses at her institution are not available, and thus we speak about reputation of teaching effectiveness rather than demonstrable teaching effectiveness. Ms. X's recognition locally as an effective teacher was the main reason for us selecting her for this case study. Also Ms. X is highly "committed to her work, able to articulate her point of view, and interested in doing so" – which are desirable qualities for a pedagogical case study (Elbaz, 1981, p. 51).

Ms. X's teaching practice is within the ambit of the DTP format of instruction, but there is no doubt her practice is not typical. The non-typical nature of her practice is not a problem for our purposes in this paper, as we are not focusing on the instructional treatment of proof in ordinary or representative undergraduate classes in Indian universities. Indeed, such a focus would have been of little interest to the field or to local practitioners, for it is common knowledge that the traditional style of the DTP format of instruction (as we described it earlier) dominates Indian undergraduate mathematics classes as much as it dominates similar classes elsewhere in the world. Following recognition that this state of affairs is problematic, a key issue that arises, and which this paper takes a step to address, is a need for greater understanding of how university instructors might productively engage their students in proof-related work. From a methodological standpoint this issue calls for examination of non-typical teaching practices at the undergraduate level, similar to the study of deviant, information-rich teaching practices at the elementary school level of teachers like Ball, Lampert, and Zack (see, e.g., Ball & Bass, 2000, 2003; Lampert, 1992; Reid, 2002; Stylianides, 2007, 2016; Zack, 1997). Examination of this sort of practices can afford researchers an opportunity to better understand what is involved in trying to improve the teaching of proof at the undergraduate level, which, in our view, is the first, foundational, step in an ambitious, long-term research program that would aim to help other university instructors to engage their students more productively in proof-related work.

4. Data sources

The main sources of data for this case study are video-recordings of Ms. X's lectures. Due to time constraints, only the first 24 (out of 42) lectures were video recorded, which almost fully covered the first two (of three) topics of the syllabus, namely real line topology and real sequences (convergence). Of these, ten were carefully selected for detailed analysis, so as to include most of the key definitions and theorems, and also illustrating Ms. X's use of all three presentation styles identified by Weber (2004). We presume that Ms. X's treatment of the third topic (infinite series) was similar to the other two.

A secondary source of data was the video-recording of a 1-hour interview with Ms. X towards the middle of the semester, in which she spoke on her broad objectives as a teacher and the pedagogical techniques she used to meet these objectives. The main purpose of this interview was to document Ms. X's own perspective on her teaching practice, which was useful for triangulation with our own analysis of her practice based on the video data.

5. Data analysis

Given that Ms. X's teaching practice followed the general DTP format of instruction, our first categorization of the observed lectures (or parts thereof) was according to the three main DTP components: (1) teaching of *Definitions*; (2) teaching of *Theorem statements*; and (3) teaching of *Proofs*. These categories were further sub-divided according to Weber's (2004) three styles of presentation (although Weber used those in relation to proof presentation, we found them useful also in characterizing Ms. X's teaching of definitions): (a) the *Semantic* style, which is characterized by intuitive motivation and visual aids including diagrams; (b) the *Logico-structural* style, which involves formal mathematical statements and symbolic logic; and (c) the *Procedural* style, which is built around computational and algebraic manipulation.

These categorizations were done in advance, before our more detailed analysis of the lectures to discern specific features of Ms. X's teaching practice. The broad structuring, in accordance with the three DTP components, was based on knowledge of the topics/goals of the lectures and on the first author's prior familiarity with Ms. X's teaching practice as well as his own long-standing experience as an undergraduate instructor. The sub-categorization according to Weber's (2004) three styles of presentation was also conceived of in advance, motivated largely by Weber's case study of Dr. T. From his own pedagogical experience and prior familiarity with Ms. X's teaching practice, corroborated by the analysis of Dr. T's case study, the first author could – fairly accurately – predict the style that was likely to be adopted by Ms. X in the various lectures according, again, to the topics/goals of those lectures. These predictions were subsequently confirmed during the analysis, thus offering confidence in our understanding of Ms. X's broad use of the various styles of presentation. The topics, as well as the specific definitions, theorems, and proofs, to be discussed later in the paper were carefully selected so as to bring out the diversity in the presentation styles adopted by Ms. X and the situations for which she adopted each style.

Other aspects of Ms. X's teaching practice that came up during the interview or observations, notably, the provision of a 2-week foundation course in symbolic logic at the very outset, were also documented. The result of this process was a nuanced characterization of Ms. X's teaching practice that is grounded to fit the available data.

MS. X'S TEACHING PRACTICE

1. General observations

Ms. X's teaching practice has the following main features:

(1) A solid foundation in *symbolic logic* at the very outset;
(2) A major focus on definitions explained in depth in a *semantic* style using informal language, visual aids, and real-life analogies;
(3) An interactive, conversational style of proof-instruction, invoking the *participation* of students.

With regard to the first main feature, Ms. X explained during the interview that the first step towards supporting students to understand and actually enjoy university-level mathematics is a solid foundation in symbolic logic:

> Ms. X: "The first step is proper *foundations*. Students who intend to major in mathematics have to be familiarized with the 'language of mathematics.' They need to be given a proper training in *symbolic logic* before they encounter any proof-based courses. Sadly, this is not done by most teachers of abstract algebra and analysis. In particular, clarity between converse and contrapositive, and fluency in negating statements are absolutely essential in the construction of proofs."

We will illustrate in detail the second and third main features of Ms. X's teaching practice in the following sections. Evidence from the interview suggests that these features are central to Ms. X's pedagogical practice from her own point of view too. In response to a question about what else, besides training in logic, is at the core of her teaching, Ms. X explained the importance she places on teaching definitions, describing essentially a semantic style of teaching (cf. feature 2):

> Ms. X: "I attach a lot of importance to the teaching of definitions, because without a proper grasp of definitions, the proofs won't make any sense [to the students]. So I spend a lot of time giving them *informal explanations and real-life analogies* with situations to which they can relate. This arouses their interest and involvement, and then the concept usually sticks in their mind."

During the interview Ms. X also asserted that her highly interactive style of teaching (cf. feature 3), in the form of a conversation with students, "comes so naturally [to her] that [she] do[es] not even regard it as a 'style' of teaching"; she "firmly believe[s] that all teaching must be like that." She invokes the participation of students in various ways. One way is by creating space for them to make mistakes and correct the mistakes themselves; as she put it, "Once you actually make a mistake and correct it yourself, you are not likely to repeat that mistake." Another way in which she invokes students' participation is by fostering their critical ability and observation power by making some deliberate mistakes for students to detect:

> Ms. X: "I also want to build their critical ability and observation power. In particular, to not accept every word simply because your teacher says so or [because] it is written in some textbook, but to always be alert to possible mistakes made by others. So, I sometimes *deliberately make a logical error while presenting a proof in class, and expect the students to detect the flaw*."

In response to a question by the interviewer (first author) about whether students are able to detect the logical errors, Ms. X responded in the affirmative and she attributed the students' ability to do that to "the 'grilling in logic' which [she] subjected them to in the beginning [of the course]."

Another way in which she invokes student participation during proof presentations, in addition to the "catch the error" pedagogical strategy, is what she calls the "whispering rounds," which she explained at the interview as follows:

> Ms. X (laughs): "The rationale for [the whispering rounds] is pretty obvious. When I pose a question to the whole class, I want *each student to think of an appropriate answer independently*, so I cannot let anyone hear somebody else's reply. So those who wish to answer have to simply raise their hands, and then I take a round of the classroom in which each of them has to whisper their answer into my ears. After mentally collecting all their answers, I come back to the board to discuss them."

Yet another way in which Ms. X invokes student participation is by calling on students to answer questions, without them having volunteered to do so. The practice of students answering questions by teacher nomination is common in Asian classrooms (see, e.g., Tan, 2007) and culturally acceptable in the Indian context.

Beyond the three main features of Ms. X's pedagogy already summarized, Ms. X holds "the unflinching belief that *every single student*, given the right kind of teaching and guidance, can be made to understand and actually enjoy higher mathematics." This belief appears to underpin Ms. X's pedagogy, notably the whispering rounds that help engage as many students as are willing to participate, but also the practice of nominating non-volunteer students to answer teacher questions, and contributes to her effort to *demystify proof for all of the students in her class*.

In what follows, we delve deeper into Ms. X's teaching practice using our analysis of the lecture videotapes and considering, in turn, her teaching of Definitions, Theorem statements, and Proofs, as per the core aspects of the DTP style of instruction. We will consider several examples under each – from a number of topics covered in the course related to real line topology and real sequences (convergence) – so as to make more transparent Ms. X's pedagogical approach.

2. The teaching of definitions

The definitions are the edifice on which the entire theory rests: without a firm grasp of the relevant definitions, the theorem statements would not make much sense, and the proof-construction would be a challenging task. With this belief, Ms. X spends a considerable amount of time and effort in motivating and explaining the definitions of every topic. To this end, she makes use of *informal terminology and simple diagrams*, and often enriches these with *real-life analogies* (such as in Topic 1, Limit Points, discussed below). The *formal definition* is provided only after the intuitive motivation "has sunk in." For some topics (such as in Topic 2, Convergence of real sequences, discussed below), Ms. X introduces the main concept after *discussing several examples and generalizing* from them. Here again, an informal, intuitive definition invariably precedes the formal definition required for the proofs. The formal definition is immediately followed by its *negation*, i.e., when the definition is *false*. This not only helps in strengthening understanding of the original definition, but also it is required later for proofs by contradiction.

Next we present two illustrations of Ms. X's teaching of definitions. Whenever Ms. X or a student speaks a word or a phrase loudly, or with emphasis, that word or phrase has been put in italics. All student names are pseudonyms.

1) Topic 1: Limit points (in real-line topology)

<u>Informal introduction with diagram</u>

Ms. X writes the topic on the board and proceeds with the informal introduction of the concept using a diagram.

Ms. X: "Let **S** be a subset of **R**, and let p be a real number. Informally, p is a limit point of **S** means that p is 'very close' to **S** in the following sense: every '*vicinity*' of p has some point of **S** (other than itself, of course)."

Ms. X draws on the board a two-dimensional sketch (even though **R** is one-dimensional), as in *Figure 1*, for she believes that this is not only more illustrative, but also

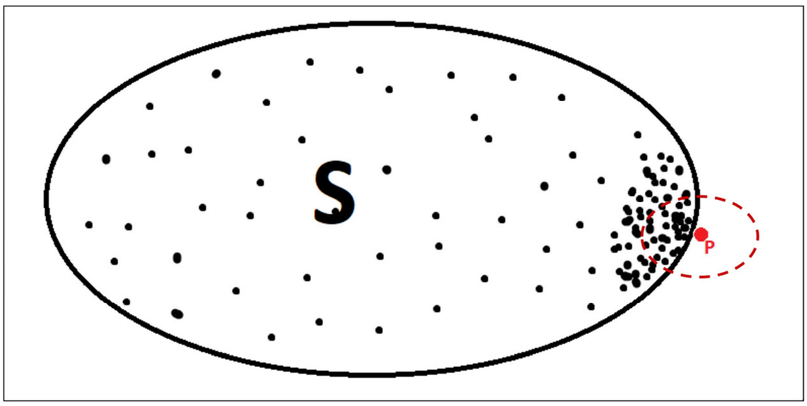

Figure 1. A two-dimension sketch of S and a limit point p.

generalizable for the multi-dimensional and axiomatic abstraction required in advanced analysis and topology.

Real-life analogy

Ms. X proceeds to give a real-life analogy of a student being a limit point of a college:

Ms. X: "Suppose that **S** is this college (i.e., the set of all its students), and p is some student (not necessarily a member of **S**). Then student p is called a limit point of the college **S** if and only if every 'social circle' (friends' group) of p has some member of **S** (besides herself/himself). Observe that for p to be a limit point of **S**, p need not be a member of **S**; there can be students who are not members of college **S** but are 'very close' to **S** in the sense defined above. […] [She elaborates with a hypothetical example.] Also, however, an element (member) of **S** may or may not be a limit point of **S**; there can be students who are members of **S**, but are yet not 'very close' to **S** in the above sense. […] [She elaborates with a real example.]"

Formal definition

Ms. X introduces next the formal definition:

Ms. X: "Okay, we now formalize the definition in mathematical language. Let **S**⊆ **R** & $p \in$ **R**. A 'vicinity' of p can be described in terms of an open interval with center p, i.e. of the form $(p-\varepsilon, p+\varepsilon)$ which can be denoted by $I_\varepsilon(p)$ for some $\varepsilon > 0$."

Ms. X elaborates further on the definition:

Ms. X: "Thus, p is a limit point of **S** means that every open interval with center p has some point of **S** (other than itself). The formal definition can be written in terms of epsilon."

She writes on the board what appears in *Figure 2*, and she explains further by drawing a small open disc with center p, which has some other point of **S**.

Definition: p is said to be a limit point of **S** iff for every $\varepsilon > 0$ the interval I_ε(p) has some point of **S** $\sim \{p\}$
i.e. for every $\varepsilon > 0$, $I_\varepsilon(p) \cap S \sim \{p\} \neq \phi$

Figure 2. A formal definition of a limit point of S

Negation of definition

Ms. X: "As I keep emphasizing, to properly understand any concept, we must do its negation also. So tell me, Ira, what is the meaning of 'p is not a limit point of S'?"

Ira: "Ma'am, there exists $\varepsilon > 0$ such that $I_\varepsilon(p) \cap S \sim \{p\} = \phi$."

Ms. X: "Yes, good!"

2) Topic 2: Convergence of real sequences

Informal introduction with diagram

The concept of a sequence, its diagrammatic representations, and a "tail" of a sequence have all been introduced informally by Ms. X in the preceding lecture, illustrated with numerous examples.

Generalization from several examples (using diagrams again)

Ms. X: "In the previous class, we had done several examples of sequences. Let us now try to discern a common behavioral pattern in some of them. First consider the sequence…."

She writes a sequence on the board and draws a one-dimensional diagram as in *Figure 3*.

Ex 1) <1/n> = 1, 1/2, 1/3,1/4,……………

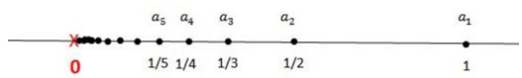

Figure 3. A sequence and a respective diagram

Ms. X: "As you notice, the sequence is moving 'backwards' and '*approaching*' (i.e., moving '*very close'* to) what?"

Several students: "Ma'am, to *zero*."

Ms. X: "Yes! What does this 'very close' means *rigorously* – that we will formulate shortly. No hurry! Meanwhile, let us see some more examples."

Ms. X moves on to offer a few more examples such as

$$<(-1)^n/n> \;=\; -1, 1/2, -1/3, 1/4, \ldots$$

which "approaches" zero from *both* sides, and

$$< n/n{+}1 > \;=\; 1/2, 2/3, 3/4, 4/5, \ldots$$

which "approaches" 1.

Ms. X: "We are now going to examine and describe this common behavior of all these sequences. Informally (as, say, economics students would put it), we can say that a_n comes '*arbitrarily close*' to p for '*sufficiently large*' values of n. But, as mathematics majors, you should also have a more rigorous formulation of these phrases using precise symbols, set theory etc. I am now going to draw an arbitrary sequence which looks like it's coming 'very close' to a particular number, and then all of us will try to make a rigorous description of that behavior."

Ms. X then draws a diagram as presented in *Figure 4*, which she develops step-by-step as she keeps explaining. First she marks a point p on the real line. Then she draws a sequence whose initial few terms are randomly placed, but soon develops a sort of pattern and 'approaches' p in a zig-zag manner.

Terminology: We write $a_n \to p$ as $n \to \infty$ or limit of $a_n = p$ (as $n \to \infty$)

Figure 4. An "arbitrary sequence" coming "very close" to "a particular number."

Ms. X: "We now make use of this diagram to formulate a rigorous definition of the sequence coming 'very close' to p. Now, 'the degree of closeness' to p can be described through intervals surrounding p (centered at p, for convenience)."

She draws an open interval J with center p and continues:

Ms. X: "Now *whatever* interval I take surrounding p (howsoever small), I can make a_n lie in it, provided that I take n to be *large enough*, i.e., greater than some particular natural number, say n_0. This gives us one formal definition."

Ms. X writes on the board the text in *Figure 5*, and she speaks out as she writes.

Then she draws a few intervals in decreasing order of length (cf. *Figure 4*), and she asks the students what n_0 "works" for each. Also, she points out that once an n_0 works for a given interval J, then so does any subsequent natural number.

A sequence $< a_n >$ is said to converge to a real number p iff for every open interval J with center p, there exists $n_0 \in \boldsymbol{N}$ such that $a_n \in J$ for every $n \geq n_0$
i.e., the tail $\{a_n : n \geq n_0\}$ is fully contained in J.

Figure 5. An informal definition of convergence of a sequence

Ms. X: "Now, any open interval J centered at p is of the form $(p - \varepsilon, p + \varepsilon)$ for some $\varepsilon > 0$. Also, the statement $a_n \in (p - \varepsilon, p + \varepsilon)$ is equivalent to $|a_n - p| < \varepsilon$ (recall properties of absolute value). So we get the following equivalent formulation, which is universally accepted as the standard definition of convergence."

Formal definition

Ms. X introduces the formal definition in *Figure 6*.

Definition: A sequence $< a_n >$ is said to **converge** to a real number p iff
for every $\varepsilon > 0$, there exists $n_0 \in \boldsymbol{N}$ such that $|a_n - p| < \varepsilon$ for every $n \geq n_0$

In words:

The distance (absolute difference) between a_n and p can be made arbitrarily small (for sufficiently large values of *n*).

Figure 6. A formal definition of convergence of a sequence

Negation of definition

Ms. X: "As always, we must formulate the negation of this definition also. Mansi, can you please try this?"

Mansi: "For every $n_0 \in \boldsymbol{N}$, there exists an $\varepsilon > 0$ such that…" [Ms. X interrupts her.]

Ms. X: "Please stop, Mansi. A cardinal principle of negation is to not change the order in which the quantifiers occur; so please try again."

Mansi: "There exists $\varepsilon > 0$ such that for every $n_0 \in$ N, $|a_n - p| \geq \varepsilon$ for some $n < n_0$."

Ms. X: "There is still an error in the end. The statement 'P for every Q' is equivalent to 'Q implies P', whose negation is what?"

Several students: "Q but not P."

Ms. X: "Yes, there exists (some) situation in which Q is true but P is false. Mansi, now speak out the entire negation."

Mansi: "Okay ma'am. There exists $\varepsilon > 0$ such that for every $n_0 \in \boldsymbol{N}$, there exists $n \geq n_0$ such that $|a_n - p| \geq \varepsilon$."

Ms. X: "That's excellent!"

3) Concluding remarks

While instructors who follow the traditional DTP format tend to present formal definitions to the students without any background or explanation, Ms. X spends a considerable amount of time and effort towards a gradual "build up" to the definition. This involves the use of informal language, illustrative diagrams, real-life analogies, and a variety of examples. A definition that can be intelligible to a non-specialist always precedes the formal definition, which is like the "grand finale." Ms. X always concludes the discussion of a definition with its negation so as to strengthen clarity as well as prepare proofs by contradiction. This entire process is done in an interactive, conversational manner; the negation is invariably provided by the students.

Overall, Ms. X's teaching of definitions is predominantly in the *semantic style* (Weber, 2004), with the *logico-structural* style (ibid) coming only towards the end for writing the formal definition and its negation. Ms. X's investment in strengthening students' understanding of symbolic logic is also evident in the discussion of the negation of definitions. The in-depth, clear introduction of definitions, in conjunction with the foundation course in symbolic logic at the outset, pave the way for demystifying proofs, as we will discuss later in the paper.

3. The teaching of theorem statements

The teaching of theorem statements tends to be the most routine part of the entire process of DTP instruction; most teachers take just a few minutes over it by simply writing the statements on the board. However, here too Ms. X displays some innovation: first she does a few carefully selected *examples*, and then she asks the students to observe and try to *generalize* from those examples the possible statement of a theorem.

Next we illustrate Ms. X's teaching of theorem statements by presenting her introduction to the statement of the Monotone Convergence Theorem and its counterpart, the Monotone Divergence Theorem.

1) The Monotone Convergence/Divergence Theorems

Ms. X: "I am going to write 2 sequences on the board. Tell me a point of similarity between them:

< n/n+1> = 1/2, 2/3, 3/4, 4/5, ….

< 2n > = 2, 4, 6, 8, …."

Several students: "Both are increasing."

Ms. X: "Yes. And now tell me a point of difference. [Students discuss amongst themselves for a bit.] Okay, as regards boundedness?"

Some students: "The first one is bounded, but the second one is not."

Ms. X: "Yes, good. What about convergence and divergence? Deepak, you answer."

Deepak (in Hindi): "Ma'am, the first one is convergent, and the second one is divergent."

Ms. X: "Very good, Deepak! So, can you all make a general observation? I would like you to make two separate (but similar) statements. Anyone wants to try?"

Mirza: "Ma'am, I'll try. First one: Every increasing and bounded sequence will converge. Second one: Every increasing and unbounded sequence will diverge."

Ms. X: "Perfect, Mirza. So we get the following two theorems."

Ms. X begins to write The Monotone Convergence Theorem on the board but does not complete the statement (*Figure 7*).

The Monotone Convergence Theorem (MCT): A sequence which is monotonically increasing and bounded above must converge…

Figure 7. An incomplete statement of the Monotone Convergence Theorem

Ms. X: "[Must converge] to what, Pia?"

Pia: "To its supremum, ma'am."

Ms. X: "Excellent! And the second statement observed by Mirza is…."

Ms. X writes "The Monotone Divergence Theorem" on the board and invites a student to offer the full statement:

Ms. X: "Mansi, please state it properly and fully."

Mansi: "A sequence which is monotonically increasing and unbounded above must diverge to infinity."

Ms. X: "That's absolutely perfect, very good Mansi."

2) Concluding remarks

Rather than simply *telling* to students the statement of a theorem to be proved, as many instructors who follow the DTP format would do, Ms. X tries to extract it from the students. By presenting some relevant examples and asking the students to discern a pattern, she is usually able to achieve this objective. This sort of presentation is entirely in the *semantic* style (Weber, 2004). Also, we observe again Ms. X's interactive, conversational manner; Ms. X is asking questions that invoke students' participation (not always voluntary but firmly within established cultural norms) in deriving the theorem statement.

4. The teaching of proof

We now analyze Ms. X's handling of the main task, the teaching of proof. Having (1) laid the foundations of *symbolic logic* at the very outset and (2) spent considerable time and effort motivating the *definitions* and in some cases the theorem statements too, the task of demystifying the proofs has been prepared. Let us recall that in the teaching of definitions, Ms. X had primarily used the semantic style, with the logico-structural style taking over at the final stage of formalization only. However, for proof presentation she used *all three styles* that were identified by Weber (2004), depending on the suitability of each for the purpose. We illustrate all styles in the following sections.

There are two notable pedagogical strategies employed by Ms. X during some of her proof presentations, with the objective of strengthening her students' critical ability and independent thinking as well as broadening student participation during the lesson. At the interview Ms. X talked about these pedagogical strategies and her rationale for them; we summarize the strategies again here for easy reference:

(1) "*Catch the error*" *pedagogical strategy*: Ms. X often deliberately made a logical error and expected the students to detect it. By and large, most students were able to point out the fallacy in the argument, largely due to their rigorous training in symbolic logic at the beginning of the semester.

(2) "*Whispering round*" *pedagogical strategy*: When Ms. X posed a question to the entire class, the students would raise their hands, and then she would take a round of the classroom in which the students would whisper their responses into her ear. The purpose behind this was that *all* students would have an opportunity to think for themselves without being influenced by other students' responses.

1) Illustration of semantic style

For the teaching of proofs, Ms. X uses the semantic style rather selectively, mostly only where she believes that a *visual representation* of the proof is feasible and useful. One reason for this could be that the semantic style takes up more time than the other two; and she has already invested a substantial amount of time in the teaching of definitions. After painstakingly constructing the "semantic proof," Ms. X would also write down the "formal proof" rapidly in the logico-structural style. This serves the triple purpose of revision, crystallization of ideas, and providing concise study-material to the students.

First illustration: Theorem for characterization of limit point

Occasionally (as in the theorem below), Ms. X does not reveal the statement of the theorem prior to its proof, she but allows for it to "emerge" or "be discovered" from the proof.

Ms. X: "Recall the definition: p is said to be a limit point of S if every open interval centered at p has some point of S $\sim \{p\}$. We are now going to obtain another equivalent characterization of limit point. I will not tell you its statement in advance; rather, we will discover the statement from its proof."

Ms. X writes on the board, "Theorem (Characterization of Limit Point) – Statement later," and proceeds by speaking and demonstrating on a one-dimensional diagram (*Figure 8*), which she builds as the proof goes along.

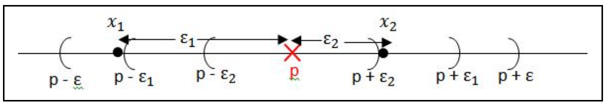

Figure 8. Diagram used in the process of proving theorem for characterization of limit point

Ms. X: "Let p be a limit point of S. So for any $\varepsilon > 0$, the interval $I_\varepsilon(p)$ has some point of $S \sim \{p\}$, say x_1. My question now is: How many points of $S \sim \{p\}$ will the interval $I_\varepsilon(p)$ have? We have already got one, i.e. x_1, using the definition. Now how many more such points can I get?"

There is no response from the class, and Ms. X proceeds with some more fundamental questions:

Ms. X: "Okay, can I first get one more? Please note that the definition holds for every $\varepsilon > 0$; so I can now apply it on some '*new*' ε. What should I take this new ε to be to get another point in the original interval? [There is now some discussion among the students.] Chogsi, please try."

Chogsi: "Ma'am, I should take the new epsilon to be greater than the original epsilon."

Ms. X: "But in that case, the new point x_2 need not lie in the original interval [demonstrates in the same diagram]. Clearly, the new epsilon should be *smaller* than the original; the question is, how much precisely should it be to get a point of $S \sim \{p\}$ in the original interval different from x_1?"

Another student: "Ma'am, the difference between x_1 and p?"

Ms. X: "Yes, excellent! [repeats loudly] The difference between x_1 and p [explains using the diagram]. By taking $\varepsilon_1 = |x_1 - p| > 0$ and applying the definition to the smaller interval $I_{\varepsilon 1}(p)$, we get a point, say x_2, of $S \sim \{p\}$ different to x_1.
Why is x_2 different to x_1? [Pauses for a bit.] It is because x_2 is inside, i.e., in the interior of this new interval, whereas x_1 is an end-point of it [demonstrates in the diagram]."

Ms. X: "Next, how do I get a point x_3 of $S \sim \{p\}$ in the original interval different from both x_1 and x_2? The lone Humanities representative [in the class], Smita, will answer this."

Smita: "Ma'am I take $\varepsilon_2 = |x_2 - p|$ and apply the definition."

Ms. X: "Yes, correct. This will give a point x_3 of $S \sim \{p\}$ different from both x_1 and x_2. Now I can keep repeating this process indefinitely to get a whole sequence $x_1, x_2, x_3, \ldots$ of distinct points of $S \sim \{p\}$ all lying in the original interval $I_\varepsilon(p)$. So how many points of $S \sim \{p\}$ does the original interval have?"

Students (chorus): "Infinite."

Ms. X: "Yes, [repeats loudly] infinite. So we have got our required statement [writes the statement of the theorem as in *Figure 9*]."

Theorem (Characterization of limit point): p is a limit point of S iff every open interval centered at p contains infinitely many points of S i.e., for each $\varepsilon > 0$, $I_\varepsilon(p) \cap S$ is infinite.

Figure 9. Statement of theorem for characterization of limit point

The episode finishes with Ms. X writing the proof linearly on the board in a logico-structural style, without using any diagram and repeating only briefly the verbal explanations.

<u>Second illustration: Monotone Convergence Theorem</u>

In this episode Ms. X announced in advance the statement to be proved (*Figure 10*).

Monotone Convergence Theorem: A sequence which is monotonically increasing and bounded above must converge to its supremum.

Figure 10. Statement of Monotone Convergence Theorem

Ms. X mainly faces the students as she speaks; she is using the board only for the diagram (*Figure 11*).

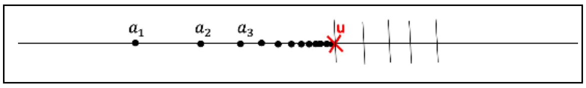

Figure 11. Diagram used in the process of proving Monotone Convergence Theorem

Ms. X: "Let us imagine that there is this sequence <a_n> which is increasing [meaning monotonically] and bounded above [gestures both these attributes through appropriate hand movements]. As it is bounded above, so there are a whole lot of upper bounds, i.e., 'walls' which cannot be crossed. So, here is my sequence which is 'moving forward' only (as it is increasing), ambling along, till suddenly it sees this wall coming up in front [draws diagram, see *Figure 11*]. So what does it do? As it cannot go backwards, so it starts taking smaller and smaller steps till it… what?"

A student: "Bangs against the wall?"

Ms. X: "Is that what we want?"

Several students: "No ma'am, it converges to the wall."

Ms. X: "Which wall, Anirvan? There are infinitely many walls [points to diagram]."

Anirvan: "The first wall, ma'am, i.e. the least upper bound."

Ms. X: "Excellent, Anirvan! But how do we know that this exists? Rajesh, you answer. [She receives no response, and she repeats question in Hindi.]"

Rajesh: "Ma'am, Complete Property."

Ms. X: "Yes Rajesh, Complete-ness Property. As the set of a_ns is obviously non-empty and given to be bounded above, hence it has a least upper bound, i.e., supremum, say u."

Ms. X: "Now that we believe that $<a_n>$ converges to its supremum u, we must prove it. So how do we begin? It is the same standard way which you all should know on your fingertips."

Chorus: "Let $\varepsilon > 0$ be given. [To show $\exists n_0 \epsilon \boldsymbol{N}$ such that $|a_n - u| < \varepsilon \ \forall \ n \geq n_0$.]"

Ms. X: "Here the interval-form would be more convenient; so to show the existence of an n_0 such that $u - \varepsilon < a_n < u + \varepsilon$ for every $n \geq n_0$. Now which of these two inequalities obviously holds for every *n*? Nikhil, please answer."

Nikhil: "The second one, ma'am, because all terms are $\leq$ u, as it is an upper bound; and $u < u + \varepsilon$."

Ms. X: "Excellent, Nikhil! So we need to only prove the one involving $u - \varepsilon$. As u is the least upper bound of the sequence, what can we say about $u - \varepsilon$ [points to diagram]?"

Chorus: "Not upper bound."

Ms. X: "And what does that mean? Tarini will answer."

Tarini: "There exists an element which is greater than $u - \varepsilon$."

Ms. X: "Very good! That is, $\exists \ n \in \boldsymbol{N}$ such that $a_n > u - \varepsilon$ [marks with a dot on diagram]. So, are we done?"

There is a mixed response from the class with most students saying "no" but a few saying "yes."

Ms. X: "Shivam, what do you say?"

Shivam: "No ma'am, we are not done, as we have got only one term greater than $u - \varepsilon$."

Ms. X: "Yes, we have only one term which is greater than $u - \varepsilon$; but we need this inequality to hold for all terms after some stage. So, how do we justify that; please raise your hands only."

At this point Ms. X conducts a "whispering round." After taking a round of the classroom, she says:

Ms. X: "Most of you have got it right. We hadn't yet used the main info given, that the sequence is increasing. So, if that one term which is already greater than $u - \varepsilon$ is, say, the n_0th term, then all subsequent terms will, obviously, also be greater than $u - \varepsilon$ [demonstrates in diagram], i.e., $a_n > u - \varepsilon \ \forall \ n \geq n_0$. So we are done."

The episode finishes again with Ms. X writing the proof linearly on the board in a logico-structural style, without any explanation, which gets done promptly.

2) Illustration of logico-structural style

While Ms. X primarily uses the semantic style for introducing and explaining definitions, she uses it only occasionally while teaching proofs. Her predominant style of proof presentation is logico-structural, which relies heavily on the expert use of Set Theory and Symbolic Logic without the support of any intuition or diagrams.

We now illustrate Ms. X's use of the logico-structural style to prove the "Union Theorem for Derived Sets." The episode exemplifies also Ms. X's "catch the error" pedagogical strategy: she deliberately writes an invalid argument on the board and asks the students to detect the flaw; she expects the students to be able to do so, as she had already laid the foundations of logic. In the episode the "catch the error" pedagogical strategy is combined with a "whispering round."

We first describe briefly the part of the lecture leading up to the theorem.

Ms. X: "The symbol d(S) or D(S) denotes the set of all limit points of S, called the 'Derived Set' of S. Thus p $\in$ d(S) means p is a limit point of S. Now we can restate our characterization of limit point as… [writes on board as in *Figure 12*]."

Characterization of limit point (Restatement): $p \in d(S)$ iff for each $\varepsilon > 0$, $I_\varepsilon(p) \cap S$ is infinite
Corollary: If there exists a p in d(S), then S must be infinite
Contrapositive: A finite set S cannot have any limit points, i.e., $d(S) = \phi$

Figure 12. Notes on the board in preparation for proof of Union Property of Derived Set

Ms. X does several examples of derived sets, including $d(\mathbf{Z}) = \phi$, $d(\mathbf{Q}) = \mathbf{R}$, $d(0,1) = [0,1]$, & $d\{1/n: n\in\boldsymbol{N}\} = \{0\}$, and announces that the class shall now prove some results about derived sets, beginning with the proof of the observation

$$S \subseteq T \Rightarrow d(S) \subseteq d(T),$$

which she explains briefly and writes on the board in a logico-structural style.

Ms. X: "We shall now prove a property of derived sets [writes on board the statement of the theorem as in *Figure 13* and asks students to verify the property for some specific sets]."

Theorem (Union Property of Derived Set):
d(S U T) = d(S) U d(T)

Figure 13. Statement of Union Property of Derived Set

Ms. X: "Let us now discuss how to prove this. That the righthand side is contained in the left-hand side is obvious using the above observation [gives brief reason]. For the other way, I am going to give you all a proof, and you please *examine* the proof."

Ms. X writes on the board in a formal, linear manner the text presented in *Figure 14*, and she explains briefly as she writes.

$p \in d(S \cup T) \Rightarrow$ for each $\varepsilon > 0$, $I_\varepsilon(p) \cap (S \cup T)$ is infinite
$\Rightarrow$ for each $\varepsilon > 0$, $[I_\varepsilon(p) \cap S] \cup [I_\varepsilon(p) \cap T]$ is infinite [By distributivity of $\cap$ over U]
$\Rightarrow$ for each $\varepsilon > 0$, $I_\varepsilon(p) \cap S$ is infinite or $I_\varepsilon(p) \cap T$ is infinite [discusses reason verbally]
$\Rightarrow p \in d(S)$ or $p \in d(T)$
$\Rightarrow p \in d(S) \cup d(T)$
It follows that $d(S \cup T) \subseteq d(S) \cup d(T)$

Figure 14. Notes on the board for proving Union Property of Derived Set

Ms. X: "I can see that Mirza has his hand up. Does anyone else have any objection to make?"

Several hands go up. Ms. X walks up to each one of these students, who whisper their "objections" in her ear. After finishing the "whispering round," she comes back to the board.

Ms. X: "Many of you have got it [the error in the presented proof]. We will now discuss it. Okay, to detect the flaw, let us start from the conclusion and move 'backwards.' We are trying to show that:

$$p \in d(S) \cup d(T)$$

i.e., we are trying to show that

$$p \in d(S) \quad \text{or} \quad p \in d(T)$$

i.e., we are trying to show that

[for each $\varepsilon > 0, I_\varepsilon(p) \cap S$ is infinite] or [for each $\varepsilon > 0, I_\epsilon(p) \cap T$ is infinite].

Let us insert this extra step in the proof [writes it at the appropriate place, i.e., between the third and fourth lines, in the 'proof' in *Figure 14*]. Now does this [newly added step] follow from the previous step [third line in the 'proof' in *Figure 14*]?"

The students discuss between themselves and Ms. X raises the question:

Ms. X: "In general, does the following hold?"

(P or Q) for every element

implies

(P for every element) or (Q for every element)

Students (loud chorus): "No Ma'am."
Ms. X: "Why?"
Several students: "Ma'am, Pink and Blue."
Ms. X: "Yes, recall that this is what we called the 'Pink and Blue' problem which was discussed during the logic sessions. What it said was that the statement 'Everyone (coming to some party) must wear either Pink or Blue' does not imply that either 'Everyone must wear Pink' or

'Everyone must wear Blue'. Okay, we were so far trying the 'direct' approach: I am here and I want to reach there, so I start from *here* and set off logically, hoping to reach there. As we did not succeed, we have to try some other approach. An alternative way is 'proof by contradiction': Suppose *that* (what I want) is *not* true; then what happens? And then work towards getting some contradiction. Let us try that."

Ms. X (speaks and writes): "To prove d(S U T) ⊆ d(S) U d(T)
Let p ∈ d(S U T) To show p ∈ d(S) U d(T), i.e. to show p ∈ d(S) or p ∈ d(T) *Suppose not*, then what?"

Several students: "Ma'am, p ∉ d(S) *and* p ∉ d(T)."

Ms. X: "Yes, correct! Because recall that, when you negate, then 'or' changes to 'and' (and vice versa). Now, what is the meaning of p ∉ d(S) , that is the negation of p ∈ d(S)? Please use its 'restatement'. You please answer [points towards a student]. I have forgotten your name."

Student: "Ma'am, Rohan. I don't know, as I missed the last class."

Ms. X: "Rohan, you've missed too many classes; please cover up. Okay, Maya, please do this negation."

Maya: "Ma'am, there exists $\varepsilon > 0$ such that $I_\varepsilon(p) \cap S$ is finite."

Ms. X: "Correct! Let us call it ε_1. So, $\exists\ \varepsilon_1 > 0$ such that $I_{\varepsilon 1}(p) \cap S$ is finite. Similarly, $\exists\ \varepsilon_2 > 0$ such that $I_{\varepsilon 2}(p) \cap T$ is finite [writes on board]. Now if I want a single epsilon for which both these statements are true, then what should I take it to be?"

Several students: "Their minimum."

Ms. X: "That's right [gives brief reason]. Thus $\exists\ \varepsilon > 0$ such that $I_\varepsilon(p) \cap S$ and $I_\varepsilon(p) \cap T$ are both finite."

Ms. X: "Now if two sets are *both* finite, then what will also be finite, Rohit?"

Rohit: "Ma'am, their intersection."

Ms. X: "That's trivial and would hold even if just one of them was finite."

Several students: "Ma'am, their union."

Ms. X: "Yes! So we get [writes]: $[I_\varepsilon(p) \cap S]$ U $[I_\varepsilon(p) \cap T]$ is finite [asks students to use distributivity], i.e. $I_\varepsilon(p) \cap$ (S U T) is finite. Now, what does this mean?"

A student: "Ma'am, this means that p is not a limit point of S U T."

Ms. X: "Excellent! Which means p ∉ d(S U T). So….?"

Chorus: "Contradiction!"

Ms. X: "Yes, this contradicts our hypothesis; so we are done."

We see that students' prior training in logic has allowed them to find the logical error in the presented proof and how their prior training in negating statements (see teaching of definitions) has allowed the class to embark upon a new approach to proving the theorem using proof by contradiction.

3) Illustration of procedural style

Where she deems appropriate, Ms. X uses the procedural style of proof presentation, that is, she uses mainly *algebraic manipulation* with no role of intuition or diagrams, and minimal use of symbolic logic. A vast topic which she has covered is the Algebra of Convergence, which relies primarily on the manipulation of inequalities involving absolute value, mainly the Triangle Inequality $|x + y| \leq |x| + |y|$.

Even while using the relatively humdrum procedural style, Ms. X is being rather innovative – for instance by conceiving of a general format that covers several results in a single stroke. This greatly increases clarity by allowing students to see a broader picture, and also saves time – both for the teaching and for its revision.

Ms. X: "The topic for today is the Algebra of Convergence. For this, the only prerequisite – besides of course the definition of convergence – is the statement of the Theorem: 'Every convergent sequence is bounded.' We shall also be using the Triangle Inequality for absolute value, which is what? [Several students reply: $|x + y| \leq |x| + |y|$ for all real numbers x and y]."

Ms. X then writes "Algebra of Convergence" and the statement of the theorems as in *Figure 15.*

If $\langle a_n \rangle \to a$ & $\langle b_n \rangle \to b$, then:
1) $\langle a_n + b_n \rangle \to a + b$ (Sum Theorem)
2) $\langle a_n - b_n \rangle \to a - b$ (Difference Theorem)
3) $\langle a_n b_n \rangle \to ab$ (Product Theorem)
4) $\langle a_n / b_n \rangle \to a/b$, provided $b_n \neq 0\ \forall n$ and $b \neq 0$ (Quotient Theorem)

Figure 15. Statements of theorems related to Algebra of Convergence.

Then Ms. X proceeds with the proofs, writing on the board and speaking as she writes:

Ms. X: "Let us think of a general format which covers all these results. All of them come under the format: Given that $< a_n > \to a$ and $< b_n > \to b$
To prove $< c_n > \to c$, where c_n is some combo (through operation) of a_n and b_n
i.e., given that $|a_n - a|$ and $|b_n - b|$ can both be made arbitrarily small, we have to show that $|c_n - c|$ an also be made arbitrarily small. So tell me what should we do?
[no response] Okay, I'll tell you.
We will try and show $|c_n - c| \leq \lambda |a_n - a| + \mu |b_n - b|$ for some positive constants λ and μ [i.e., a linear combination of the 'crucial expressions'].
Once this is done, our task becomes routine. Here is the format: [Speaks and writes main expressions] Let $\varepsilon > 0$ be given
[We have to show $\exists$ $n_0 \in N$ such that $|c_n - c| < \varepsilon \ \forall \ n \geq n_0$] We have already obtained an inequality of the form:
$|c_n - c| \leq \lambda |a_n - a| + \mu |b_n - b|$, where $\lambda > 0, \ \mu > 0$
Now to prove the LHS $< \varepsilon$, it suffices to get the RHS $< \varepsilon$
For this it suffices to show that each expression in the RHS is $< \frac{\varepsilon}{2}$
This is obviously equivalent to $|a_n - a| < \frac{\varepsilon}{2\lambda}$ and $|b_n - b| < \frac{\varepsilon}{2\mu}$
Now as $< a_n > \to a$,
so corr. to $\frac{\varepsilon}{2\lambda} > 0$, $\exists$ $n_1 \in N$ such that $|a_n - a| < \frac{\varepsilon}{2\lambda}$ $\forall$ $n \geq n_1$
Also as $< b_n > \to b$, so corr. to $\frac{\varepsilon}{2\mu} > 0$, $\exists$ $n_2 \in N$ such that $|b_n - b| < \frac{\varepsilon}{2\mu}$ $\forall$ $n \geq n_2$
Now what would be a common n_0 such that both the above inequalities hold $\forall$ $n \geq n_0$?"

Chorus: "Ma'am, the maximum of n_1 and n_2."

Ms. X: "Yes, good. So both the above hold for all $n \geq n_0$.
Finally, consider $|c_n - c| \leq \lambda |a_n - a| + \mu |b_n - b|$ which is $\leq \lambda . \frac{\varepsilon}{2\lambda} + \mu . \frac{\varepsilon}{2\mu} = \varepsilon$ $\forall$ $n \geq n_0$
As $\varepsilon > 0$ was arbitrary, therefore we are done."

Ms. X: "Let us now apply this general format to the particular results. The Sum and Difference Theorems are very easy. We will write down the formal proof of the Product Theorem now.
[Speaks and writes:]
Given: $< a_n > \to a$ and $< b_n > \to b$
To show $< a_n b_n > \to ab$. Let $\varepsilon > 0$ be given Consider $|a_n b_n - ab|$.
[Speaks:] What should we do to get $a_n - a$ and $b_n - b$? [no response]
Okay, if there was ab_n instead of ab, then I could have taken b_n common to get $a_n - a$. So, I subtract and add ab_n.
[Writes:] We have $|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| = |b_n (a_n - a) + a (b_n - b)| \leq |b_n||a_n - a| + |a||b_n - b|$ ------ (1) [by Triangle Inequality]
[Speaks:] Now can we take $|b_n|$ to be the λ and $|a|$ to be the μ of the general format proof?"

Some student: "$|b_n|$ is not a constant."

Ms. X: "Excellent observation! It is not a constant as it involves the variable n. Now can it be made less than some constant? What is that property called?"

Several students: "Bounded/ boundedness."

Ms. X: "Yes; and why is that true for sequence b_n?"

Chorus: "Because every convergent sequence is bounded."

Ms. X: "Yes; I told you that we will be using that somewhere.
[Speaks and writes:]
As $< b_n >$ is bounded, so there exists $K > 0$ such that $|b_n| \leq K$ $\forall$ $n \in N$
So (1) gives: $|a_n b_n - ab| \leq K |a_n - a| + \{|a| + 1\} |b_n - b|$
I have added 1 as $|a|$ can be zero.
We now proceed exactly as in the general format with $\lambda = K$ and $\mu = |a| + 1$
[Speaks:] Now please try out the Quotient Theorem as an assignment. It is quite difficult, and all of you will get stuck at some point. But still you must try it, and we'll then discuss how to tackle it from that point onwards."

Chorus: "Okay, ma'am."

We see that, even when using the procedural style of proof presentation, Ms. X invokes student participation through appropriate questioning. Also, she strives to help students see a collection of related theorems as an interconnected network of results rather than as isolated facts.

DISCUSSION

Ms. X's teaching practice does not deviate from the DTP format that is followed in most proof-oriented university mathematics courses (Davis & Hersh, 1981; Dreyfus, 1991). However, the way in which the DTP format plays out in Ms. X's classroom is qualitatively different from the *traditional* DTP style (cf. Weber, 2004). Indeed, the main features of Ms. X's teaching practice present, collectively, an innovative teaching practice that is conducive to undergraduate students forming an "epistemic fluency" to productively engage with proof-related institutional practices from which many undergraduate students are disenfranchised (e.g., Solomon, 2006).

Figure 16 outlines a few key features of Ms. X's teaching practice. On the basis of our analysis of the observational and interview data, the teaching practice of Ms. X emerges as one of *structured interaction* between Ms. X and her students, that is, an interactive, conversational style of proof instruction, invoking the participation of students. This is based on a solid foundation in symbolic logic at the very outset (with particular emphasis on implication, converse, contrapositive, and negation), and a major focus on definitions that are explained in depth in a *semantic* style. At the beginning of each topic, definitions are motivated and explained using informal language, visual aids (e.g., diagrams), and real-life analogies, before formal definitions are finally introduced in a logico-structural style. Theorem statements also often get motivated in a *semantic* style instead of simply being announced to the students: Ms. X often invokes student participation in discerning theorem statements by generalizing from examples and patterns. Proof presentation in Ms. X's class follows the same highly interactive, conversational style, in which students are active participants in the construction of the proofs in the collective domain, benefitting from the preparatory activities that allow them to make sense of definitions and theorem statements and being equipped with sound logical reasoning skills. Ms. X uses all three styles of proof presentation identified by Weber (2004), depending on their suitability. *The logico-structural* style predominates proof presentations, but the *semantic* style is also used where visual representation is

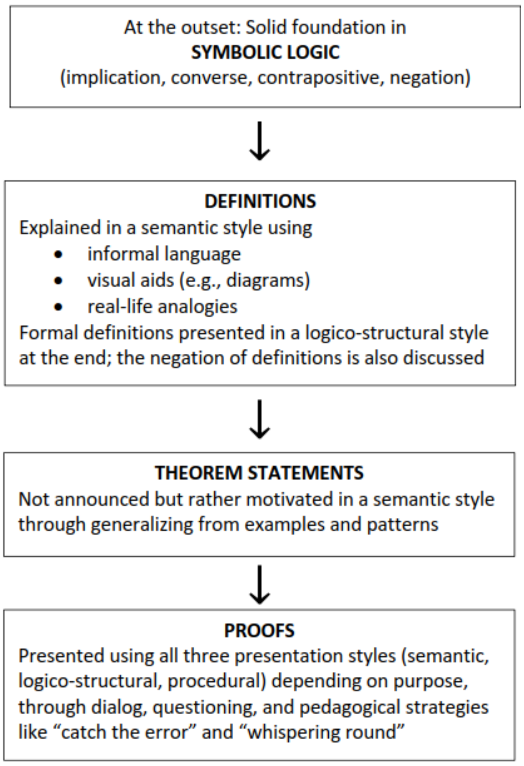

Figure 16. Outline of a few key features of Ms. X's teaching practice

feasible; the "routine" *procedural* style is used sometimes, but rather creatively so as to "cover" several results through a common format, emphasizing their connections. Throughout *all* stages of the DTP format of instruction, Ms. X invokes student participation by using appropriate questioning. Primarily during proof presentations, Ms. X uses also the pedagogical strategies of "catch the error" and "whisper round" that help foster students' critical ability and allow space for all students to think independently despite the rather large class size (there are 48 students in the room). Ms. X further expands the pool of student contributors to whole-class discussions by nominating students who have not volunteered to answer her questions, a practice that is common in Asian classrooms (e.g., Tan, 2007) and culturally acceptable in the Indian context.

Lai and Weber (2014) investigated the factors that mathematicians profess to consider when constructing or revising proofs for pedagogical purposes, which the authors called "pedagogical proofs." The authors contrasted pedagogical proofs with the proofs that mathematicians write for disciplinary purposes, and they defined the former kind of proofs as follows, drawing on Shulman (1987): "proofs that transform mathematical knowledge into ways of 'representing ideas so that the unknowing can come to know, those without understanding can comprehend and discern, and the unskilled can become adept' (Shulman, 1987, p. 7)" (Lai & Weber, 2014, p. 93). Following their interview-based study with ten mathematicians, Lai and Weber found that, "in pedagogical proofs, participants largely professed to value the teleological features of diagrams and highlighting main ideas; however, they did not always include these aspects in their proof constructions and revisions" (p. 106). Lai and Weber compared this finding with findings of other empirical studies, including Weber (2004), that noted that "professors sometimes presented highly formal arguments emphasizing logic and deduction – in other words, emphasizing epistemic and communicative factors – even though these professors recognized the importance of students seeing the informal side of mathematical proof" (Lai & Weber, 2014, p. 106). Thus, there was a mismatch between professors' beliefs about what a pedagogical proof should look like and the pedagogical proofs that they constructed or revised during the interview. Lai and Weber did not observe these professors' actual teaching, but it might be hypothesized that a similar mismatch would exist between their declared values in pedagogical proofs and the proofs that they actually presented to their students. Ms. X not only professed at the interview the importance of a unity between the formal and informal side of mathematical proof, including the value of teleological features of diagrams and main ideas in pedagogical proofs, but also, and most importantly, she put all of this into practice within the DTP format of instruction.

The latter reinforces our claim about the innovative nature of Ms. X's teaching practice and the point that the DTP format need not be a rigid one but rather can be adapted to support meaningful student learning. This point was also illustrated by the case studies of university instructors' teaching that were reported by Weber (2004) and Fukawa-Connelly (2012) and that we reviewed earlier in the paper: Dr. T and Dr. Tripp, respectively. Similar to ours, each of these case studies help expand the thin research basis of empirical studies on teaching practices at the undergraduate level (Speer et al., 2010) and contribute unique but complementary images of potentially effective proof-related instruction.

To highlight the distinctive features of Ms. X's teaching practice, let us compare Ms. X's practice with those of Dr. T and Dr. Tripp. With regard to laying a foundation of symbolic logic, done in depth by Ms. X, Dr. T did that in a cursory manner while there is no evidence of Dr. Tripp doing that. Regarding semantic explanation of definitions, always done by Ms. X, this was done by Dr. T for topological topics only while Dr. Tripp rarely did that. As far as pedagogical proofs are concerned, Ms. X and Dr. T always tried to help students see the informal side of mathematical proof before the formal one, while Dr. Tripp rarely did that. Ms. X was more in line with Dr. Tripp with regard to fostering interaction with students, a practice that was rarely followed by Dr. T. Rasmussen and Marrongelle (2006) described a "continuum of instructional perspectives from pure discovery to pure telling" and they suggested that a well-designed course would be "situated toward the middle of such a continuum" (p. 391). The teaching practices of Dr. T and Dr. Tripp, though closer to pure telling, were highly effective from the students' perspective. From this, we can infer that how to tell is a significant pedagogical dimension outside the framework of this continuum. We believe that the pedagogy of Ms. X is situated toward the middle of that continuum in that Ms. X seemed to have achieved a defensible balance between, on the one hand, engaging students in building up definitions, discovering theorem statements, and contributing ideas to the construction of proofs as part of the collective work of the class, and, on the other hand, maintaining control of the flow of classwork including when and how to tell while presenting definitions, theorem statements, and proofs.

Overall, we see that Ms. X's teaching practice combined features of those of Dr. T and Dr. Tripp, but in a rather unique manner and in a different setting: the cultural context of India (vs. the American) and in a class of a fairly large size (48 students vs. 16 in Dr. T's class and 15 in Dr. Tripp's). Of course, this is not to say that the practices of each of Dr. T and Dr. Tripp presented no further distinctive features; rather, it is to suggest that Ms. X's practice offers an image of a potentially effective instruction that complements images already reported in the literature. Such images are sorely needed as a first step towards a longer-term research and development program that would aim to scale-up effective instruction in undergraduate mathematics through the design of interventions or other means (cf. Stylianides et al., 2007; Stylianides & Stylianides, 2017). Given that mathematicians are unlikely to implement teaching practices unless these practices are attuned to normative pedagogical practice (cf. Weber, 2012), case studies of instructors such as Ms. X, Dr. T, and Dr. Tripp, whose practices fit generally under the popular DTP format, contribute research knowledge that can be useful to and usable in a possible reform of undergraduate mathematics instruction.

Author Note

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