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# On one BVP for a thermo-microstretch elastic space with spherical cavity

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**Abstract:** The present paper considers the equilibrium theory of thermo-microstretch elastic solids with microtemperatures. The method to solve the Neumann-type boundary value problem (BVP) for the whole space with spherical cavity is presented. The solution of this BVP in the form of absolutely and uniformly convergent series is obtained.

**Key words:** Thermo-microstretch elastic solids, microtemperatures, explicit solution, sphere

## 1. Introduction

The theory of thermoelasticity for elastic materials with microtemperatures, whose particles contain a displacement vector and temperature field, was established by Grot [11].

Eringen developed the theory of micromorphic bodies and the theory of thermo-microstretch elastic solids. An extensive review and basic results in the microcontinuum field theories for solids (micromorphic, microstretch, and micropolar) including electromagnetic and thermal interactions are given in his works [9,10].

In [16], Ieşan and Quintanilla formulated the boundary value problems of the theory of thermoelasticity with microtemperatures and presented a unique result and a solution of Boussinesq–Somigliana–Galerkin type. The theory of micromorphic elastic solids with microtemperatures was presented by Ieşan in [12,15]. In [18], Ieşan and Quintanilla discussed various problems using thermoelasticity with microtemperatures.

Many investigators studied different types of problems for thermo-microstretch medium in detail (some of those works can be seen in [1-8,13,14,17,20-24,26,27] and the references therein).

The present paper considers the equilibrium theory of thermo-microstretch elastic solids with microtemperatures. The method to solve the Neumann-type boundary value problem (BVP) for the whole space with spherical cavity is presented. The solution of this BVP in the form of absolutely and uniformly convergent series is obtained.

## 2. Basic equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be a point of Euclidean three-dimensional space $E^3$. Let us assume that $D^+$ is a ball with radius $R$, centered at point $O(0,0,0)$ in space $E^3$, and $S$ is a spherical surface with radius $R$. Denote the whole space with a spherical cavity and with boundary $S$ by $D$. ( $D = E_3 \backslash \overline{D^+}$).

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The system of equations of the linear equilibrium theory of thermoelasticity for isotropic and homogeneous thermo-microstretch elastic solids may be written as [12,26]

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\mathrm{graddiv}\mathbf{u} - \beta\mathrm{grad}\theta + \beta_1\mathrm{grad}\varphi = 0, \tag{1}$$

$$k_6\Delta\mathbf{w} + (k_4+k_5)\mathrm{graddiv}\mathbf{w} - k_3\mathrm{grad}\theta - k_2\mathbf{w} = 0, \tag{2}$$

$$k\Delta\theta + k_1\mathrm{div}\mathbf{w} = 0, \tag{3}$$

$$\gamma\Delta\varphi - \beta_1\mathrm{div}\mathbf{u} - d\mathrm{div}\mathbf{w} + m\theta - \varsigma\varphi = 0, \tag{4}$$

where $\mathbf{u} = (u_1, u_2, u_3)^\top$ is the displacement vector; $\mathbf{w} = (w_1, w_2, w_3)^\top$ is the microtemperature vector; $\theta(\mathbf{x})$ is the temperature measured from the constant absolute temperature $T_0$ $(T_0 > 0)$ by the natural state (i.e. by the state of the absence of loads); $\varphi$ is the microstretch function (microdilatation function); $\lambda,\ \mu,\ \beta,\ \beta_1,\ k,\ k_j,\ d,\ m,\ \gamma,\ \varsigma$ j=1,...,6, are constitutive coefficients; and $\Delta$ is the 3D Laplace operator. Throughout this paper, superscript $\top$ denotes transposition.

**Definition 2.1** *A vector-function* $\boldsymbol{U} = (\boldsymbol{u}, \boldsymbol{w}, \theta, \varphi)$ *defined in domain* $D$ *is called regular if*

$$\boldsymbol{U} \in C^2(D) \cap C^1(\overline{D})$$

*and at infinity it satisfies the following conditions:*

$$\mathbf{U}(\mathbf{x}) = O(|x|^{-1}) \qquad \frac{\partial\mathbf{U}}{\partial x_j} = O(|\mathbf{x}|^{-2}) \qquad |x|^2 = x_1^2 + x_2^2 + x_3^2 >> 1, \qquad j = 1,2,3.$$

The Neumann-type BVP for Eqs.(1)–(4) is formulated as follows:

*Find a regular solution* $\boldsymbol{U}(\boldsymbol{u}, \boldsymbol{w}, \theta, \varphi)$ *to Eqs. (1)–(4), for* $\boldsymbol{x} \in D$, *satisfying the following boundary conditions:*

$$\lim_{D\ni\mathbf{x}\to\mathbf{z}\in S} \mathbf{u}(\mathbf{x}), = \mathbf{G}(\mathbf{z}), \qquad \lim_{D\ni\mathbf{x}\to\mathbf{z}\in S} \mathbf{P}^{(2)}\left(\partial_{\mathbf{x}}, \mathbf{n}\right)\mathbf{w}(\mathbf{x}), = \mathbf{f}(\mathbf{y}),$$

$$\lim_{D\ni\mathbf{x}\to\mathbf{z}\in S}\left(k\frac{\partial\theta}{\partial\mathbf{n}} + k_1\mathbf{n}\mathbf{w}\right) = f_4(\mathbf{y}), \quad \lim_{D\ni\mathbf{x}\to\mathbf{z}\in S}\left(\gamma\frac{\partial\varphi}{\partial\mathbf{n}} - d\mathbf{n}\mathbf{w}\right) = f_5(\mathbf{y}), \quad \mathbf{y} \in S,$$

where $\mathbf{n}(\mathbf{z})$ is the external unit normal vector on $\mathbf{z} \in S$, the vector-functions $\mathbf{G}(\mathbf{z}) = (G_1, G_2, G_3)$, $\mathbf{f}(\mathbf{z}) = (f_1, f_2, f_3)$, and the functions $f_4(\mathbf{z})$, $f_5(\mathbf{z})$, ) are prescribed on $S$, at $\mathbf{z}$, and the vector $\mathbf{P}^{(2)}(\partial\mathbf{x}, \mathbf{n})\mathbf{w}$ has the following form [23]:

$$\mathbf{P}^{(2)}(\partial\mathbf{x}, \mathbf{n})\mathbf{w} = (k_5 + k_6)\frac{\partial\mathbf{w}}{\partial\mathbf{n}} + k_4\mathbf{n}\mathrm{div}\mathbf{w} + k_5[\mathbf{n}\cdot\mathrm{rot}\mathbf{w}]. \tag{5}$$

$[\mathbf{x}\cdot\mathbf{g}]$ denotes the vector product of the two vectors $\mathbf{x}$ and $\mathbf{g}$.

The following assertion holds.

**Theorem 2.1**([23]) *The Neumann-type BVP has at most one regular solution in domain* $D$.

To solve the above-mentioned problem, we proceed as follows: first is to study the BVP for equations (2) and (3) separately. By supposing that $\mathbf{w}$ and $\theta$ are known, we can study the BVP for equations (1) and (4), with respect to $\mathbf{u}$ and $\varphi$. By combining the obtained results, we obtain the solution of the BVP for equations (1)–(4).

## 3. Preliminaries

In this section, some basic results from previous papers [1,3] are given:

The spherical coordinates are:

$$x_1 = \rho \sin \xi \cos \eta, \quad x_2 = \rho \sin \xi \sin \eta, \quad x_3 = \rho \cos \xi, \quad x \in D$$

$$y_1 = R \sin \xi_0 \cos \eta_0, \quad y_2 = R \sin \xi_0 \sin \eta_0, \quad y_3 = R \cos \xi_0, \quad y \in S$$

$$|\mathbf{x}| = \rho = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad 0 \leq \xi \leq \pi, \quad 0 \leq \eta \leq 2\pi.$$

$(\mathbf{x} \cdot \mathbf{w}) = \sum\limits_{k=1}^{3} x_k w_k$ denotes the usual scalar product of two vectors $\mathbf{x}$ and $\mathbf{w}$.

Operator $\dfrac{\partial}{\partial S_k(x)}$ is defined as follows:

$$[\mathbf{x} \cdot \nabla]_k = \frac{\partial}{\partial S_k(x)}, \qquad k = 1, 2, 3, \qquad \nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right).$$

If $g_m$ is the spherical harmonic, then [19]:

$$\sum_{k=0}^{3} \frac{\partial^2 g_m(\mathbf{x})}{\partial S_k^2(\mathbf{x})} = -m(m+1) g_m(\mathbf{x}).$$

The following theorems hold true:

**Theorem 3.1** ([1]) *The regular solution* $\boldsymbol{W} = (\boldsymbol{w}, \theta)$*, where* $\boldsymbol{w} = (w_1, w_2, w_3)$*, of the homogeneous equations (2) and (3), in* $D$*, can be represented in the following form* (for details see [1]):

$$\begin{aligned} &\mathbf{w}(\mathbf{x}) = a \operatorname{grad}\vartheta(\mathbf{x}) + b \operatorname{grad}\vartheta_1(\mathbf{x}) + c \operatorname{rot}\boldsymbol{\varphi}^{\mathbf{3}}(\mathbf{x}), \\ &\theta(\mathbf{x}) = \vartheta(\mathbf{x}) + \vartheta_1(\mathbf{x}), \end{aligned} \tag{6}$$

*where*

$$\begin{aligned} &\Delta\vartheta = 0, \qquad (\Delta - s_1^2)\vartheta_1 = 0, \quad (\Delta - s_2^2)\boldsymbol{\varphi}^{\mathbf{3}} = 0, \qquad \operatorname{div}\boldsymbol{\varphi}^{\mathbf{3}} = 0, \\ &s_1^2 = \frac{kk_2 - k_1k_3}{kk_7} > 0, \quad s_2^2 = \frac{k_2}{k_6} > 0, \quad a = -\frac{k_3}{k_2}, \quad b = -\frac{k}{k_1}, \quad c = -\frac{k_6}{k_2}, \\ &\boldsymbol{\varphi}^{\mathbf{3}}(\mathbf{x}) = [\mathbf{x} \cdot \nabla]\varphi_3(\mathbf{x}) + \operatorname{rot}[\mathbf{x}.\nabla]\varphi_4(\mathbf{x}), \quad (\Delta - s_2^2)\varphi_j = 0, \qquad j = 3, 4. \end{aligned} \tag{7}$$

*In addition, if*

$$\int\limits_{S(0,a_1)} \varphi_j ds = 0, \quad j = 3, 4,$$

*where* $S(0, a_1) \subset D$ *is an arbitrary spherical surface with radius* $a_1$*, then between the vector* $\boldsymbol{W}(\boldsymbol{x}) = (\boldsymbol{w}, \theta)$ *and the functions* $\vartheta$*,* $\vartheta_1$*,* $\varphi_j$*,* $j = 3, 4$*, there exists one-to-one correspondence.*

**Remark.** *The solutions of Eqs. (2) and (3) can be rewritten in the following form* [1]:

$$\begin{aligned} &\mathbf{w}(\mathbf{x}) = a\,\mathrm{grad}\vartheta(\mathbf{x}) + b\,\mathrm{grad}\vartheta_1(\mathbf{x}) + [\mathbf{x}\cdot\nabla]\varphi_4(\mathbf{x}) + c\,\mathrm{rot}[\mathbf{x}\cdot\nabla]\varphi_3(\mathbf{x}),\\ &\theta(\mathbf{x}) = \vartheta(\mathbf{x}) + \vartheta_1(\mathbf{x}). \end{aligned} \tag{8}$$

**Theorem 3.2**([3]) *The regular solution* $\boldsymbol{U} = (\boldsymbol{u}, \varphi)$, *where* $\boldsymbol{u} = (u_1, u_2, u_3)$, *of the following equations:*

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\mathrm{graddiv}\mathbf{u} + \beta_1\mathrm{grad}\varphi = \beta\mathrm{grad}(\vartheta+\vartheta_1), \tag{9}$$

$$\gamma\Delta\varphi - \beta_1\mathrm{div}\mathbf{u} - \varsigma\varphi = (dbs_1^2 - m)\vartheta_1 - m\vartheta, \tag{10}$$

*in* $D$, *can be represented in the following form (for details see* [3]:)

$$\mathbf{u} = \mathbf{\Phi} + \mathbf{u}_0, \tag{11}$$

$$\varphi = \psi + \psi_3 + \psi_0, \tag{12}$$

*where* $\mathbf{u}_0$ *is a particular solution of equation* (9):

$$\begin{aligned} &\mathbf{u}_0 = \mathrm{grad}\left\{\frac{\beta\beta_1 - m(\lambda+\mu)}{\mu\beta_1}\vartheta_0 + \frac{(\lambda+\mu)\varsigma - \beta_1^2}{\mu\beta_1}\Psi_0 - \frac{\beta_1}{\mu_0 s_3^2}\psi_3 + \frac{a_{22}}{s_1^2}\vartheta_1\right\},\\ &a_{22} = \frac{\beta(\gamma s_1^2 - \varsigma) + \beta_1(m - dbs_1^2)}{\gamma\mu_0(s_1^2 - s_3^2)}. \end{aligned} \tag{13}$$

$\psi_0$ *is a particular solution of equation* (10):

$$\psi_0 = \frac{a_{11}\vartheta_1}{s_1^2 - s_3^2}, \quad s_3^2 = \frac{\mu_0\varsigma - \beta_1^2}{\mu_0\gamma}, \quad a_{11} = \frac{\beta\beta_1 - \mu_0(m - dbs_1^2)}{\gamma\mu_0}.$$

*The vector-function* $\mathbf{\Phi}$ *is a harmonic function and is chosen such that*

$$\Delta\mathbf{\Phi} = 0, \quad \mathrm{div}\mathbf{\Phi} = \frac{m\mu_0 - \beta\beta_1}{\mu\beta_1}\vartheta - \frac{\mu_0\gamma s_3^2}{\beta_1\mu}\psi, \quad \Delta\mathrm{div}\Phi = 0.$$

$\Psi_0$ *and* $\vartheta_0$ *are chosen such that*

$$\Delta\Psi_0 = \psi, \quad \Delta\vartheta_0 = \vartheta, \quad \Delta\Delta\Psi_0 = 0, \quad \Delta\Delta\vartheta_0 = 0.$$

*The functions* $\psi$ *and* $\psi_3$ *are solutions of the following equations:*

$$\Delta\psi = 0, \quad (\Delta - s_3^2)\psi_3 = 0,$$

*and respectively,* $\mathrm{div}\boldsymbol{u}$ *satisfies the following condition:*

$$\mathrm{div}\mathbf{u} = \frac{m\vartheta - \varsigma\psi}{\beta_1} - \frac{\beta_1}{\mu_0}\psi_3 + a_{22}\vartheta_1. \tag{14}$$

Thus, the general solutions of Eqs. (1)–(4) are represented by means of harmonic, biharmonic, and metaharmonic functions and are given by formulas (8),(11), and (12).

## 4. Solution of the Neumann boundary value problem

In this section, we present the method to solve the Neumann BVP for a whole space with a spherical cavity.

Taking into account (8) and the following identities:

$$\frac{\partial}{\partial \mathbf{n}}\mathrm{grad} h(\mathbf{x}) = \frac{1}{\rho}\mathrm{grad}\left[\left(\rho\frac{\partial}{\partial \rho}-1\right)h(\mathbf{x})\right],$$
$$\frac{\partial}{\partial \mathbf{n}}\mathrm{rot}\mathbf{h}(\mathbf{x}) = \frac{1}{\rho}\mathrm{rot}\left[\left(\rho\frac{\partial}{\partial \rho}-1\right)\mathbf{h}(\mathbf{x})\right], \tag{15}$$

the vector $(\mathbf{P}^{(2)}(\partial\mathbf{x},\mathbf{n})\mathbf{w}$ takes the following form [2]:

$$\mathbf{P}^{(2)}(\partial\mathbf{x},\mathbf{n})\mathbf{w} =$$
$$\frac{k_5+k_6}{\rho}\left\{\mathrm{grad}\left[\left(\rho\frac{\partial}{\partial\rho}-1\right)(a\vartheta(\mathbf{x})+b\vartheta_1(\mathbf{x}))\right]+c\,rot\left(\rho\frac{\partial}{\partial\rho}-1\right)\boldsymbol{\varphi}^{\mathbf{3}}(\mathbf{x})\right\} \tag{16}$$
$$+k_4bs_1^2\frac{\mathbf{x}\vartheta_1}{\rho}+\frac{k_5}{\rho}[\mathbf{x}\cdot\boldsymbol{\varphi}^{\mathbf{3}}(\mathbf{x})].$$

On account of formulas (16) and (8), the following identities hold true:

$$\left(\mathbf{x}\cdot\mathbf{P}^{(2)}(\partial\mathbf{x},\mathbf{n})\mathbf{w}\right) = (k_5+k_6)a\rho\frac{\partial^2\vartheta(\mathbf{x})}{\partial\rho^2}+\left[(k_5+k_6)b\rho\frac{\partial^2}{\partial\rho^2}+k_4bs_1^2\rho\right]\vartheta_1(\mathbf{x})$$
$$+c(k_5+k_6)\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\sum_{k=1}^{3}\frac{\partial^2\varphi_3(\mathbf{x})}{\partial S_k^2(x)},$$
$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}\left[\mathbf{x}\cdot\mathbf{P}^{(2)}(\partial\mathbf{x},\mathbf{n})\mathbf{w}\right]_k = (k_5+k_6)\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\sum_{k=1}^{3}\frac{\partial^2(a\vartheta(\mathbf{x})+b\vartheta_1(\mathbf{x}))}{\partial S_k^2(x)}$$
$$-\left\{c(k_5+k_6)\left[\rho\frac{\partial^2}{\partial\rho^2}+\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right]+k_5\rho\right\}\sum_{k=1}^{3}\frac{\partial^2\varphi_3(\mathbf{x})}{\partial S_k^2(x)}, \tag{17}$$
$$\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}\left[\mathbf{P}^{(2)}(\partial\mathbf{x},\mathbf{n})\mathbf{w})\right]_k = k_5\left(\frac{\partial}{\partial\rho}-\frac{1}{\rho}\right)\sum_{k=1}^{3}\frac{\partial^2\varphi_4(\mathbf{x})}{\partial S_k^2(x)},$$
$$\frac{\partial\theta}{\partial n}=\frac{\partial\vartheta}{\partial\rho}+\frac{\partial\vartheta_1}{\partial\rho},\quad (\mathbf{x}\cdot\mathbf{w}) = a\rho\frac{\partial\vartheta}{\partial\rho}+b\rho\frac{\partial\vartheta_1}{\partial\rho}+c\sum_{k=1}^{3}\frac{\partial^2\varphi_3}{\partial S_k^2(\mathbf{x})}.$$

Let us assume that functions $\vartheta, \vartheta_1$ and $\varphi_j, j=3,4$ are sought in the following form [25]:

$$\vartheta(\mathbf{x}) = \sum_{n=0}^{\infty}\frac{R^{n+2}}{(2n+1)\rho^{n+1}}Y_{1n}(\vartheta,\varphi),$$
$$\vartheta_1(\mathbf{x}) = \sum_{n=0}^{\infty}\Psi_n(is_1\rho)Y_{2n}(\vartheta,\varphi), \tag{18}$$
$$\varphi_j(\mathbf{x}) = \sum_{n=0}^{\infty}\Psi_n(is_2\rho)Y_{jn}(\vartheta,\varphi),\quad \rho>R,\quad j=3,4,$$

respectively, where $Y_{1m}$, $Y_{2m}$ m and $Y_{jm}$ are the unknown spherical harmonic of order $n$,

$$\Psi_n(is_k\rho) = \frac{\sqrt{R_1} H^{(1)}_{n+\frac{1}{2}}(is_k\rho)}{\sqrt{\rho} H^{(1)}_{n+\frac{1}{2}}(is_k R)}, \quad k = 1, 2.$$

By substituting (18) into (17) we obtain:

$$\begin{aligned}
&\left(\mathbf{x} \cdot \mathbf{P}^{(2)}(\partial\mathbf{x}, \mathbf{n})\mathbf{w}\right) \\
&= \sum_{n=0}^{\infty} \left\{ (k_5 + k_6) a \frac{(n+1)(n+2)R^{n+2}}{(2n+1)\rho^{n+2}} Y_{1n} + \left[ (k_5 + k_6) b\rho \frac{\partial^2}{\partial\rho^2} + k_4 b s_1^2 \rho \right] \Psi_n(is_1\rho) Y_{2n}(\xi, \eta) \right. \\
&\left. - c(k_5 + k_6) \left( \frac{\partial}{\partial\rho} - \frac{1}{\rho} \right) n(n+1) \Psi_n(is_2\rho) Y_{3n} \right\},
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=1}^{3} \frac{\partial}{\partial S_k(x)} \left[ \mathbf{x} \cdot \mathbf{P}^{(2)}(\partial\mathbf{x}, \mathbf{n})\mathbf{w} \right]_k = \\
&\sum_{n=0}^{\infty} n(n+1) \left\{ (k_5 + k_6) \left[ a \frac{(n+2)R^{n+2}}{(2n+1)\rho^{n+2}} Y_{1n} - \left( \frac{\partial}{\partial\rho} - \frac{1}{\rho} \right) b \Psi_n(is_1\rho) Y_{2n} \right] + \right. \\
&\left. \left[ c(k_5 + k_6) \left( \rho \frac{\partial^2}{\partial\rho^2} + \frac{\partial}{\partial\rho} - \frac{1}{\rho} \right) + k_5 \rho \right] \Psi_n(is_2\rho) Y_{3n} \right\}, \qquad (19) \\
&\sum_{k=1}^{3} \frac{\partial}{\partial S_k(x)} \left[ \mathbf{P}^{(2)}(\partial\mathbf{x}, \mathbf{n})\mathbf{w} \right]_k = -k_5 \sum_{n=0}^{\infty} n(n+1) \left( \frac{\partial}{\partial\rho} - \frac{1}{\rho} \right) \Psi_n(is_2\rho) Y_{4n}, \\
&(\mathbf{x} \cdot \mathbf{w}) = \sum_{n=0}^{\infty} \left\{ -\frac{a(n+1)R^{n+2}}{(2n+1)\rho^{n+1}} Y_{1n} + b\rho \frac{\partial}{\partial\rho} \Psi_n(is_1\rho) Y_{2n} - cn(n+1) \Psi_n(is_2\rho) Y_{3n} \right\}, \\
&k \frac{\partial\theta}{\partial\mathbf{n}} + k_1(\mathbf{n}\mathbf{w}) = \sum_{n=0}^{\infty} \left[ -\frac{(k + ak_1)(n+1)R^{n+2}}{(2n+1)\rho^{n+2}} Y_{1n} - \frac{ck_1 n(n+1)}{R} \Psi_m(is_2\rho) Y_{3n} \right].
\end{aligned}$$

Let us introduce the following functions:

$$(\mathbf{z} \cdot \mathbf{G})^- = h_1(\mathbf{z}), \qquad \sum_{k=1}^{3} \frac{\partial}{\partial S_k(\mathbf{z})} [\mathbf{z} \cdot \mathbf{G}]_k^- = h_2(\mathbf{z}),$$

$$\sum_{k=1}^{3} \left( \frac{\partial G_k}{\partial S_k(\mathbf{z})} \right)^- = h_3(\mathbf{z}), \qquad (\mathbf{z} \cdot \mathbf{f})^- = h_4(\mathbf{z}),$$

$$\sum_{k=1}^{3} \frac{\partial}{\partial S_k(\mathbf{z})} [\mathbf{z} \cdot \mathbf{f}]_k^- = h_5(\mathbf{z}), \qquad \sum_{k=1}^{3} \left( \frac{\partial f_k}{\partial S_k(\mathbf{z})} \right)^- = h_6(\mathbf{z}),$$

$$\left( k \frac{\partial\theta}{\partial\mathbf{n}} + k_1(\mathbf{n}\mathbf{w}) \right)^- = h_7(\mathbf{y}), \qquad \left( \gamma \frac{\partial\varphi}{\partial\mathbf{n}} - d(\mathbf{n}\mathbf{w}) \right)^- = h_8(\mathbf{y}), \quad \mathbf{y} \in S.$$

Let us assume that functions $h_k, \quad k=1,..,8,$ are representable in the form of the following series:

$$h_k(\mathbf{y}) = \sum_{n=0}^{\infty} h_{kn}(\xi_0, \eta_0),$$

where $h_{kn} \quad k=1,..,8$ are the spherical harmonics of order $n$ :

$$h_{kn} = \frac{2n+1}{4\pi R^2} \int\limits_S P_n(\cos\gamma) h_k(\mathbf{y}) dS_y,$$

$P_n$ is the Legender polynomial of the $n$th order, and $\gamma$ is an angle formed by the radius-vectors $Ox$ and $Oy$,

$$\cos\gamma = \frac{1}{|\mathbf{x}||\mathbf{y}|} \sum_{k=1}^{3} x_k y_k.$$

**Remark.** ([3]) *The condition* $\int\limits_{S(0,a_1)} \varphi_j ds = 0$ *implies that* $Y_{30} = Y_{40} = 0$.

Keeping in mind the boundary conditions, from (19), when $\rho \to R$, we get the following system of algebraic equations:

$$\begin{aligned}
&(k_5+k_6)a\frac{(n+1)(n+2)}{(2n+1)}Y_{1n} + \left[(k_5+k_6)bR\left(\frac{\partial^2\Psi_n(is_1\rho)}{\partial\rho^2}\right)_{\rho=R} + k_4 b s_1^2 R\right] Y_{2n}(\xi,\eta) \\
&-c(k_5+k_6)n(n+1)\left[\left(\frac{\partial}{\partial\rho} - \frac{1}{\rho}\right)\Psi_n(is_2\rho)\right]_{\rho=R} Y_{3n} = h_{4n}, \\
&n(n+1)\left\{(k_5+k_6)\left[a\frac{n+2}{2n+1}Y_{1n} - b\left(\frac{\partial\Psi_n(is_1\rho)}{\partial\rho}\right)_{\rho=R} Y_{2n} + \frac{b}{R}Y_{2n}\right] + \right. \\
&\qquad \left. \left[c(k_5+k_6)\left(R\frac{\partial^2\Psi_n(is_2\rho)}{\partial\rho^2} + \frac{\partial\Psi_n(is_2\rho)}{\partial\rho} - \frac{1}{R}\right)_{\rho=R} + k_5 R\right] Y_{3n}\right\} = h_{5n}, \\
&\qquad -k_5 n(n+1)\left(\frac{\partial\Psi_n(is_2\rho)}{\partial\rho} - \frac{1}{\rho}\right)_{\rho=R} Y_{4n} = h_{6n}, \quad h_{50}=0, \quad h_{60}=0, \\
&\qquad \left[\frac{(k+ak_1)(n+1)}{(2n+1)}Y_{1n} + \frac{ck_1 n(n+1)}{R}Y_{3n}\right] = -h_{7n}.
\end{aligned} \tag{20}$$

According to Theorem 2.1, we conclude that system (20) for $n \geq 0$ is uniquely solvable and we shall find functions $Y_{jn}$. This means that functions $\mathbf{w}$ and $\theta$ are known.

In the same way, from formulas (11) and (12), as above, we can easily derive [3]:

$$\begin{aligned}
&(\mathbf{x}\cdot\mathbf{u})=(\mathbf{x}\cdot\mathbf{\Phi})+\rho\frac{\partial}{\partial\rho}\left\{\frac{\beta\beta_1-m(\lambda+\mu)}{\mu\beta_1}\vartheta_0+\frac{(\lambda+\mu)\varsigma-\beta_1^2}{\mu\beta_1}\Psi_0-\frac{\beta_1}{\mu_0 s_3^2}\psi_3+\frac{a_{22}}{s_1^2}\vartheta_1\right\},\\
&\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{u}]_k=\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{\Phi}]_k\\
&+\sum_{k=1}^{3}\frac{\partial^2}{\partial S_k^2(x)}\left\{\frac{\beta\beta_1-m(\lambda+\mu)}{\mu\beta_1}\vartheta_0+\frac{(\lambda+\mu)\varsigma-\beta_1^2}{\mu\beta_1}\Psi_0-\frac{\beta_1}{\mu_0 s_3^2}\psi_3+\frac{a_{22}}{s_1^2}\vartheta_1\right\},\\
&\sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(x)}=\sum_{k=1}^{3}\frac{\partial \Phi_k}{\partial S_k(x)},\qquad \gamma\frac{\partial}{\partial\mathbf{n}}\varphi-d(\mathbf{nw})=\gamma\frac{\partial}{\partial\rho}\left[\psi+\psi_3+\frac{a_{11}\vartheta_1}{s_1^2-s_3^2}\right]-d(\mathbf{nw}),
\end{aligned}\tag{21}$$

where function $\vartheta_1$ is defined from (20) and (18), and the function $\vartheta_0$ is defined as:

$$\vartheta_0(\mathbf{x})=\frac{1}{2}\sum_{n=0}^{\infty}\frac{R^{n+2}}{(1-4n^2)\rho^{n-1}}Y_{1n}(\vartheta,\eta).\tag{22}$$

Functions $(\mathbf{x}\cdot\mathbf{\Phi})$ and $\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{\Phi}]_k$ can be replaced with functions $\vartheta$ , $\vartheta_0$, $\Psi_0$, and $\psi$ in the following form [3]:

$$\begin{aligned}
&(\mathbf{x}\cdot\mathbf{\Phi})=\Omega+2\left[\frac{m\mu_0-\beta\beta_1}{\mu\beta_1}\vartheta_0-\frac{\mu_0\gamma s_3^2}{\beta_1\mu}\Psi_0\right],\\
&\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{\Phi}]_k=\rho^2\left[\frac{m\mu_0-\beta\beta_1}{\mu\beta_1}\vartheta-\frac{\mu_0\gamma s_3^2}{\beta_1\mu}\psi\right]-\left(\rho\frac{\partial}{\partial\rho}+1\right)(\mathbf{x}\cdot\mathbf{\Phi}),
\end{aligned}\tag{23}$$

where $\Omega$ is an arbitrary harmonic function $\Delta\Omega=0$.

Then (21) takes the following form:

$$\begin{aligned}
&(\mathbf{x}\cdot\mathbf{u})=\Omega-2\frac{\mu_0\gamma s_3^2}{\beta_1\mu}\Psi_0+\rho\frac{\partial}{\partial\rho}\left\{\frac{(\lambda+\mu)\varsigma-\beta_1^2}{\mu\beta_1}\Psi_0-\frac{\beta_1}{\mu_0 s_3^2}\psi_3\right\}\\
&+2\frac{m\mu_0-\beta\beta_1}{\mu\beta_1}\vartheta_0+\rho\frac{\partial}{\partial\rho}\left\{\frac{\beta\beta_1-m(\lambda+\mu)}{\mu\beta_1}\vartheta_0+\frac{a_{22}}{s_1^2}\vartheta_1\right\},\\
&\sum_{k=1}^{3}\frac{\partial}{\partial S_k(x)}[\mathbf{x}\cdot\mathbf{u}]_k=-\rho^2\frac{\mu_0\gamma s_3^2}{\beta_1\mu}\psi-\left(1+\rho\frac{\partial}{\partial\rho}\right)\left[\Omega-2\frac{\mu_0\gamma s_3^2}{\beta_1\mu}\Psi_0\right]\\
&+\sum_{k=1}^{3}\frac{\partial^2}{\partial S_k^2}\left[\frac{(\lambda+\mu)\varsigma-\beta_1^2}{\mu\beta_1}\Psi_0-\frac{\beta_1}{\mu_0 s_3^2}\psi_3\right]+\sum_{k=1}^{3}\frac{\partial^2}{\partial S_k^2}\left[\frac{\beta\beta_1-m(\lambda+\mu)}{\mu\beta_1}\vartheta_0++\frac{a_{22}}{s_1^2}\vartheta_1\right]\\
&+R^2\frac{m\mu_0-\beta\beta_1}{\mu\beta_1}\vartheta-2\frac{m\mu_0-\beta\beta_1}{\mu\beta_1}\left(1+\rho\frac{\partial}{\partial\rho}\right)\vartheta_0,\\
&\sum_{k=1}^{3}\frac{\partial u_k}{\partial S_k(x)}=\sum_{k=1}^{3}\frac{\partial \Phi_k}{\partial S_k(x)},\qquad \gamma\frac{\partial}{\partial\mathbf{n}}\varphi-d(\mathbf{nw})=\gamma\frac{\partial}{\partial\rho}\left[\psi+\psi_3+\frac{a_{11}\vartheta_1}{s_1^2-s_3^2}\right]-d(\mathbf{nw}).
\end{aligned}\tag{24}$$

Let functions $\psi$, $\Omega$, $\psi_3$, and $\sum_{k=1}^{3}\frac{\partial\Phi_k}{\partial S_k(x)}$ be sought in the following form [25]:

$$\begin{aligned}
&\psi(\mathbf{x})=\sum_{n=0}^{\infty}\frac{R^{n+2}}{(2n+1)\rho^{n+1}}Z_n(\xi,\eta),\qquad \rho>R,\\
&\Omega(\mathbf{x})=\sum_{n=0}^{\infty}\frac{R^{n+2}}{(2n+1)\rho^{n+1}}Z_{1n}(\xi,\eta),\qquad \rho>R,\\
&\psi_3(\mathbf{x})=\sum_{n=0}^{\infty}\Psi_n(is_3\rho)Z_{3n}(\xi,\eta),\qquad \rho>R,\\
&\sum_{k=1}^{3}\frac{\partial\Phi_k}{\partial S_k(x)}=\sum_{n=0}^{\infty}\frac{R^{n+2}}{(2n+1)\rho^{n+1}}Z_{4n}(\xi,\eta),\qquad \rho>R,
\end{aligned}\tag{25}$$

where $Z_n$, $Z_{1n}$ , and $Z_{jn}$, $j=3,4$, are the unknown spherical harmonics of order $n$. Using (25), the solution of equation $\Delta\Psi_0=\psi$ can be written in the following form:

$$\Psi_0(\mathbf{x})=\frac{1}{2}\sum_{n=0}^{\infty}\frac{R^{n+2}}{(1-4n^2)\rho^{n-1}}Z_n(\xi,\eta).\tag{26}$$

Substituting (26) and (25) into (24), passing to the limit as $\rho\to R$, for the determination of unknown functions we arrive at the following system of algebraic equations:

$$\begin{aligned}
&\frac{RZ_{1n}}{2n+1}+\frac{-\mu_0\gamma s_3^2(n+1)+\varsigma\mu(n-1)}{2\mu\beta_1(1-4n^2)}R^3Z_n-\frac{\beta_1R}{\mu_0s_3^2}\left[\frac{\partial}{\partial\rho}\Psi_n(is_3\rho)\right]_{\rho=R}Z_{3n}=\omega_{5n},\\
&\frac{nRZ_{1n}}{2n+1}+\frac{(n+1)R^3}{2\mu\beta_1(1-4n^2)}[(2-n)\mu_0\gamma s_3^2+n\mu\varsigma]Z_n+\frac{\beta_1n(n+1)}{\mu_0s_3^2}Z_{3n}=-\omega_{6n},\\
&\frac{RZ_{4n}}{2n+1}=h_{3n},\qquad -\gamma\frac{n+1}{2n+1}Z_n+\gamma\left[\frac{\partial}{\partial\rho}\Psi_n(is_3\rho)\right]_{\rho=R}Z_{3n}=\omega_{7n},
\end{aligned}\tag{27}$$

where

$$\omega_5=h_1-2\frac{m\mu_0-\beta\beta_1}{\mu\beta_1}\left[\vartheta_0\right]_{\rho=R}-\left\{\rho\frac{\partial}{\partial\rho}\left[\frac{\beta\beta_1-m(\lambda+\mu)}{\mu\beta_1}\vartheta_0+\frac{a_{22}}{s_1^2}\vartheta_1\right]\right\}_{\rho=R}=\sum_{m=0}^{\infty}\omega_{5m},$$

$$\omega_6=h_2-\left\{\sum_{k=1}^{3}\frac{\partial^2}{\partial S_k^2}\left[\frac{\beta\beta_1-m(\lambda+\mu)}{\mu\beta_1}\vartheta_0+\frac{a_{22}}{s_1^2}\vartheta_1\right]\right\}_{\rho=R}$$

$$-R^2\frac{m\mu_0-\beta\beta_1}{\mu\beta_1}\left[\vartheta\right]_{\rho=R}+2\frac{m\mu_0-\beta\beta_1}{\mu\beta_1}\left[\left(\rho\frac{\partial}{\partial\rho}+1\right)\vartheta_0\right]_{\rho=R}=\sum_{m=0}^{\infty}\omega_{6m},$$

$$\omega_7=h_8-\frac{\gamma a_{11}}{s_1^2-s_3^2}\left[\frac{\partial}{\partial\rho}\vartheta_1\right]_{\rho=R}+\left[d(\mathbf{nw})\right]_{\rho=R}=\sum_{m=0}^{\infty}\omega_{7m},$$

$$\omega_{kn}=\frac{2n+1}{4\pi R^2}\int_S P_n(\cos\gamma)\omega_k(\mathbf{y})dS_y,\quad k=5,6,7.$$

By virtue of Theorem 2.1, we get the following result: the system (27) for $n\geq 0$ is uniquely solvable.

## 5. Conclusions

By using the above-mentioned method, it is possible to construct explicitly the solutions of basic BVPs for systems (1)–(4) for simple cases of 2D domains (circle, plane with a circular hole) in the form of absolutely and uniformly convergent series.

This method can be extended to the systems of equations in modern linear theories of poroelasticity and thermoelasticity for materials with microstructures.

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