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ON THE EXACT DIMENSION OF MANDELBROT MEASURE

BY

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Abstract. We develop, in the context of the boundary of a supercritical Galton–Watson tree, a uniform version of the argument used by Kahane (1987) on homogeneous trees to estimate almost surely and simultaneously the Hausdorff and packing dimensions of the Mandelbrot measure over a suitable set $\mathcal{J}$. As an application, we compute, almost surely and simultaneously, the Hausdorff and packing dimensions of the level sets $E(\alpha)$ of infinite branches of the boundary of the tree along which the averages of the branching random walk have a given limit point.

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1. INTRODUCTION AND MAINS RESULTS

Let $(N, W_1, W_2, \ldots)$ be a random vector taking values in $\mathbb{N}_+ \times \mathbb{R}_+^{*\mathbb{N}_+}$. Then consider $\{(N_{u0}, W_{u1}, W_{u2}, \ldots)\}_{u \in \bigcup_{n \geqslant 0} \mathbb{N}_+^n}$, a family of independent copies of this random vector indexed by the finite sequences $u = u_1 \ldots u_n, n \geqslant 0, u_i \in \mathbb{N}^*$ ($n = 0$ corresponds to the empty sequence denoted by $\emptyset$). Let T be the Galton–Watson tree with defining element $\{N_u\}$: we have $\emptyset \in \mathsf{T}$, and if $u \in \mathsf{T}$ and $i \in \mathbb{N}_+$, then ui, the concatenation of u and i, belongs to T if and only if $1 \leqslant i \leqslant N_u$. Similarly, for each $u \in \bigcup_{n \geqslant 0} \mathbb{N}_+^n$, denote by $\mathsf{T}(u)$ the Galton–Watson tree rooted at u and defined by $\{N_{uv}\}$, $v \in \bigcup_{n \geqslant 0} \mathbb{N}_+^n$.

For each $u \in \bigcup_{n \geqslant 0} \mathbb{N}_+^n$ we denote by $|u|$ its length, i.e. the number of letters of u, and by $[u]$ the cylinder $u \cdot \mathbb{N}_+^{\mathbb{N}_+}$, i.e. the set of $t \in \mathbb{N}_+^{\mathbb{N}_+}$ such that $t_1 t_2 \ldots t_{|u|} = u$. If $t \in \mathbb{N}_+^{\mathbb{N}_+}$, we put $|t| = \infty$, and the set of prefixes of t consists of $\{\emptyset\} \cup \{t_1 t_2 \ldots t_n : n \geqslant 1\} \cup \{t\}$. Also we set $t_{|n} = t_1 \ldots t_n$ if $n \geqslant 1$ and $t_{|0} = \emptyset$.

The probability space over which the previous random variables are built is denoted by $(\Omega, \mathcal{A}, \mathbb{P})$, and the expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$.

We assume that $\mathbb{E}(N) > 1$ so that the Galton–Watson tree is supercritical. Without loss of generality, we also assume that the probability of extinction equals zero, so that $\mathbb{P}(N \geqslant 1) = 1$.

The boundary of T is the subset of $\mathbb{N}_+^{\mathbb{N}_+}$ defined as

$$\partial\mathsf{T} = \bigcap_{n\geqslant 1} \bigcup_{u\in\mathsf{T}_n} [u],$$

where $\mathsf{T}_n = \mathsf{T} \cap \mathbb{N}_+^n$. The set $\mathbb{N}_+^{\mathbb{N}_+}$ is endowed with the standard ultrametric distance

$$d_1 : (s,t) \mapsto \exp(-|s \wedge t|),$$

where $s \wedge t$ stands for the longest common prefix of s and t, and with the convention that $\exp(-\infty) = 0$. The set $\partial\mathsf{T}$ endowed with the induced distance is almost surely (a.s.) compact.

For the sake of simplicity we will assume throughout that the logarithmic moment generating function

$$\tau(q) = \log \mathbb{E}\big(\sum_{i=1}^{N} W_i^q\big)$$

is finite over $\mathbb{R}$. Then, we define, for $u \in \bigcup_{n\geqslant 0} \mathbb{N}_+^n$, the random variable

$$W_{q,u} = \frac{W_u^q}{\mathbb{E}\big(\sum_{i=1}^{N} W_i^q\big)} = W_u^q \, e^{-\tau(q)}.$$

Consider the set

$$J = \{q \in \mathbb{R} : \quad \tau(q) - q\tau'(q) > 0\} = \big\{q \in \mathbb{R} : \quad \tau^*\big(\tau'(q)\big) > 0\big\},$$

where τ^* is the Legendre transform of the function τ defined, for all $\alpha \in \mathbb{R}$, as

$$\tau^*(\alpha) = \inf_{q\in\mathbb{R}} \big(\tau(q) - q\alpha\big).$$

Let

$$\Omega_\gamma^1 = \operatorname{int}\big\{q : \mathbb{E}\big[\big|\sum_{i=1}^{N} W_i^q\big|^\gamma\big] < \infty\big\}, \quad \Omega^1 = \bigcup_{\gamma\in(1,2]} \Omega_\gamma^1 \quad \text{and} \quad \mathcal{J} = J \cap \Omega^1.$$

Then, for $n \geqslant 1$ and $u \in \mathbb{N}_+^n$, we define the sequence $\big(Y_p(q,u)\big)_{p\geqslant 1}$ as

$$Y_p(q,u) = \sum_{v\in\mathsf{T}_p(u)} \prod_{k=1}^{n} W_{q,uv_1\ldots v_k};$$

when $u = \emptyset$, this quantity will be denoted by $Y_n(q)$, and when $n = 0$, its value equals one.

Since, for all $q \in \mathcal{J}$, we have

$$\begin{cases} \mathbb{E}\big(\sum_{i=1}^{N} W_{q,i}\big) = 1, \\ \mathbb{E}\big(\sum_{i=1}^{N} W_{q,i} E \log W_{q,i}\big) = q\tau'(q) - \tau(q) < 0, \\ \mathbb{E}\Big(\big(\sum_{i=1}^{N} W_{q,i}\big) \log^{+}\big(\sum_{i=1}^{N} W_{q,i}\big)\Big) < \infty, \end{cases}$$

it follows that $\big(Y_p(q,u)\big)$ converges to a positive limit $Y(q,u)$ with probability one, while the limit exists and vanishes if the condition is violated. This fact was proven by Kahane in [14] when N is constant and by Biggins in [5] in general. Then, we can associate the Mandelbrot measure defined on the σ-field $\mathcal{C}$ generated by the cylinders of $\mathbb{N}_+^{\mathbb{N}_+}$ as

$$\mu_q([u]) = \begin{cases} W_{q,u_1} W_{q,u_2} \ldots W_{q,u_1\ldots u_n} Y(q,u) & \text{if } u \in \mathsf{T}_n, \\ 0 & \text{otherwise}, \end{cases} \tag{1.1}$$

and supported on $\partial\mathsf{T}$. Moreover, under the property $E\big(Y(q)\log^{+} Y(q)\big) < \infty$, hence in particular when $E\big(Y(q)^h\big) < \infty$ for some $h > 1$, where $Y(q) = Y(q,\emptyset)$, we have, following [14], [16], [4], for all $q \in \mathcal{J}$, a.s., for μ_q-almost every $t \in \partial\mathsf{T}$,

$$\liminf_{n\to\infty} \frac{\log \mu_q([t_{|n}])}{-n} \geqslant \tau(q) - q\tau'(q).$$

Hence, for all $q \in \mathcal{J}$, a.s., the lower Hausdorff dimension of μ_q is

$$\underline{\dim}\, \mu_q \geqslant \tau(q) - q\tau'(q),$$

see Section 6 for the definition.

The Mandelbrot measure μ_q is naturally considered when studying the multifractal analysis of some random sets (see [10], [19], [1]–[3], [7]). By exploiting the simultaneous construction of the Mandelbrot measure μ_q, $q \in \mathcal{J}$, and using a uniform version of the argument applied by Kahane in [13] on homogeneous trees, we get the following result.

THEOREM 1.1. *With probability one, for all* $q \in \mathcal{J}$, $\underline{\dim}\, \mu_q \geqslant \tau(q) - q\tau'(q)$.

As an application we study, for $q \in \mathcal{J}$, the set $E\big(\tau'(q)\big)$ associated with the branching random walk with $\big(X_i = \log(W_i)\big)_{1 \leqslant i \leqslant N}$ (see Section 4). Since, with probability one, for all $q \in \mathcal{J}$, the set $E\big(\tau'(q)\big)$ is supported by μ_q and its packing dimension is smaller than $\tau^*\big(\tau'(q)\big)$ (see Proposition 2.7 in [2]), we get

$$\text{a.s.}, \forall q \in \mathcal{J},\ \overline{\operatorname{Dim}}\, \mu_q \leqslant \tau(q) - q\tau'(q),$$

where $\overline{\operatorname{Dim}}\, \mu_q$ is the upper packing dimension of μ_q (see Section 6 for the definition). As a consequence, we infer that the measures are exact dimensional.

COROLLARY 1.1. *With probability one, for all* $q \in \mathcal{J}$,

$$\dim \mu_q = \operatorname{Dim} \mu_q = \tau(q) - q\tau'(q),$$

where $\dim \mu_q$ *and* $\operatorname{Dim} \mu_q$ *denote the Hausdorff and packing dimensions of* μ_q, *respectively.*

REMARK 1.1. *These results are known* (see [1], [3]). *Using a uniform version of a percolation argument, we will give a new proof of the sharp lower bounds for the lower Hausdorff dimension of these measures.*

2. PRELIMINARIES

Given an increasing sequence $\{\mathcal{A}_n\}_{n\geqslant 1}$ of sub-σ-fields of $\mathcal{A}$ and a sequence of random functions $\{P_n(t,\omega)\}_{n\geqslant 1}$ $(t \in \partial\mathsf{T})$ such that

1. $P_n(t) = P_n(t,\omega)$ are non-negative and independent processes; $P_n(\cdot,\omega)$ is Borelian for almost all ω; $P_n(t,\cdot)$ is $\mathcal{A}_n$-mesurable for each t;
2. $\mathbb{E}\big(P_n(t)\big) = 1$ for all $t \in \partial\mathsf{T}$.

Such a sequence $\{P_n\}$ is called a *sequence of weights* adopted to $\{\mathcal{A}_n\}$. Let

$$Q_n(t) = Q_n(t,\omega) = \prod_{k=1}^{n} P_k(t,\omega).$$

For any $n \geqslant 1$ and any positive Radon measure σ on $\partial\mathsf{T}$ (we write $\sigma \in \mathcal{M}^+(\partial\mathsf{T})$), we consider the random measures $Q_n\sigma$ defined as

$$Q_n\sigma(A) = \int_A Q_n(t)d\sigma(t) \quad (A \in \mathcal{B}(\partial\mathsf{T})),$$

where $\mathcal{B}(\partial\mathsf{T})$ is the Borel field on $\partial\mathsf{T}$. For all $A \in \mathcal{B}(\partial\mathsf{T})$, $Q_n\sigma(A)$ is a positive martingale so it converges almost surely. Also, for all $\sigma \in \mathcal{M}^+(\partial\mathsf{T})$, the random measure $Q_n\sigma$ converges weakly, almost surely, to the random measure $Q\sigma$.

There are two possible extreme cases. The first one is that $Q_n\sigma(\partial\mathsf{T})$ converges almost surely to zero, i.e. $Q\sigma = 0$ a.s. In this case, we say that Q *degenerates* on σ or σ is said to be Q-*singular*. The second one is that $Q_n\sigma(\partial\mathsf{T})$ converges in L^1 so that $\mathbb{E}\big(Q_n(\sigma)(\partial\mathsf{T})\big) = \sigma(\partial\mathsf{T})$. In this case we say that Q *fully acts* on σ or σ is said to be Q-*regular*.

THEOREM 2.1. *Let* α *be a positive number such that* $\mathcal{H}^\alpha(\partial\mathsf{T}) < \infty$, *where* $\mathcal{H}^\alpha$ *denotes the* α-*dimensional Hausdorff measure. Let* $0 < h < 1$ *and* $C > 0$. *Suppose*

$$\sup_{t\in\bar{B}} \big(Q_n(t)^h\big) \leqslant C|B|^{(1-h)\alpha} \tag{2.1}$$

for all balls B *and some* $n = n(B)$ *depending on* B. *Then* Q *is completely degenerate, that is,* $Q\sigma = 0$ *a.s. for all* $\sigma \in \mathcal{M}^+(\partial\mathsf{T})$.

This provides a good tool to verify the Q-singularity of σ. Indeed, if a measure is not killed, it means that it has a lower Hausdorff dimension at least α.

3. PROOF OF THEOREM 1.1

For each $\beta \in (0,1]$, let W_β be a random variable taking the value $1/\beta$ with probability β and the value 0 with probability $1-\beta$. Then, let $\{W_{\beta,u}\}_{u\in\bigcup_{n\geqslant 0}\mathbb{N}_+^n}$ be a family of independent copies of W_β. Denote by $(\Omega_\beta,\mathcal{A}_\beta,\mathbb{P}_\beta)$ the probability space on which this family is defined.

We naturally extend to $\big(\Omega_\beta\times\Omega,\mathcal{A}_\beta\otimes\mathcal{A},\mathbb{P}_\beta\otimes\mathbb{P}\big)$ the random variables $W_{\beta,u}$ and the random vectors $(N_{u0},W_{u1},\ldots)$ as

$$W_{\beta,u}(\omega_\beta,\omega)=W_{\beta,u}(\omega_\beta)$$

and

$$\big(N_{u0}(\omega_\beta,\omega),W_{u1}(\omega_\beta,\omega),\ldots\big)=\big(N_{u0}(\omega),W_{u1}(\omega),\ldots\big),$$

so that the families $\{W_{\beta,u}\}_{u\in\bigcup_{n\geqslant 0}\mathbb{N}_+^n}$ and $\{(N_{u0},W_{u1},\ldots)\}_{u\in\bigcup_{n\geqslant 0}\mathbb{N}_+^n}$ are independent.

The expectation with respect to $\mathbb{P}_\beta\otimes\mathbb{P}$ will also be denoted by $\mathbb{E}$. For $n\geqslant 1$ and $\beta\in(0,1]$, we set $\mathcal{F}_n=\sigma\big((N_u,W_{u1},W_{u2},\ldots):u\in\bigcup_{k=0}^n\mathbb{N}_+^{k-1}\big)$ and $\mathcal{F}_{\beta,n}=\sigma\big((W_{\beta,u1},W_{\beta,u2},\ldots):u\in\bigcup_{k=0}^n\mathbb{N}_+^{k-1}\big)$. We denote by $\mathcal{F}_0$ and $\mathcal{F}_{\beta,0}$ the trivial σ-field.

If $\beta\mathbb{E}(N)>1$, the random variables

$$N_{\beta,u}(\omega_\beta,\omega)=\sum_{i=1}^{N_u(\omega)}\mathbf{1}_{\{\beta^{-1}\}}\big(W_{\beta,ui}(\omega_\beta)\big)$$

define a new supercritical Galton–Watson process with which the trees $\mathsf{T}_{\beta,n}\subset\mathsf{T}_n$ and $\mathsf{T}_{\beta,n}(u)\subset\mathsf{T}_n(u)$, $u\in\bigcup_{n\geqslant 0}\mathbb{N}_+^n$, $n\geqslant 1$, are associated, as well as the infinite tree $\mathsf{T}_\beta\subset\mathsf{T}$ and the boundary $\partial\mathsf{T}_\beta\subset\partial\mathsf{T}$ conditional on non-extinction.

For $u\in\bigcup_{n\geqslant 0}\mathbb{N}_+^n$, $1\leqslant i\leqslant N(u)$, and $q\in\mathcal{J}$ we define

$$W_{\beta,q,ui}=W_{\beta,ui}W_{q,ui}.$$

For $q\in\mathcal{J}$, $\beta\mathbb{E}(N)>1$, $n\geqslant 0$ and $u\in\bigcup_{n\geqslant 0}\mathbb{N}_+^n$, we define

$$Y_n(\beta,q,u)=\sum_{v_1\ldots v_n\in\mathsf{T}_n(u)}\prod_{k=1}^n W_{\beta,q,u\cdot v_1\ldots v_k}.$$

When $u=\emptyset$, this quantity will be denoted by $Y_n(\beta,q)$, and when $n=0$, its value equals one.

3.1. A family of measures indexed by $\mathcal{J}$. For $\beta \in \big(\mathbb{E}(N)^{-1}, 1\big]$ and $\epsilon > 0$ we set

$$\mathcal{J}_{\beta,\epsilon} = \big\{q \in \mathcal{J} : \tau^*\big(\tau'(q)\big) > -\log\beta + \epsilon\big\}.$$

Notice that $\tau^*\big(\tau'(q)\big)$ takes values between zero and $\tau(0) = \log\big(E(N)\big)$ over $\mathcal{J}$. Then

$$\mathcal{J} = \bigcup_{\beta\in(\mathbb{E}(N)^{-1},1],\epsilon>0} \mathcal{J}_{\beta,\epsilon}. \tag{3.1}$$

The following propositions will be established in Section 5.

PROPOSITION 3.1. (1) *For all $u \in \bigcup_{n\geqslant 0}\mathbb{N}_+^n$, the sequence of continuous functions $Y_n(\cdot, u)$ converges uniformly, almost surely and in L^1 norm, to a positive limit $Y(\cdot, u)$ on $\mathcal{J}$.*

(2) *With probability one, for all $q \in \mathcal{J}$, the mapping*

$$\mu_q([u]) = \Big(\prod_{k=1}^{n} W_{q,u_1\ldots u_k}\Big)Y(q,u) \tag{3.2}$$

defines a positive measure on $\partial\mathsf{T}$.

PROPOSITION 3.2. *Let $\beta\in(0,1]$ such that $\beta\mathbb{E}(N)>1$. Then, for all $\epsilon \in \mathbb{Q}_+^*$:*

(1) *the sequence of continuous functions $Y_n(\beta,\cdot)$ converges uniformly, almost surely and in L^1 norm, to a positive limit $Y(\beta,\cdot)$ on $\mathcal{J}_{\beta,\epsilon}$;*

(2) *the sequence of continuous functions*

$$q \mapsto \widetilde{Y}_n(\beta,q) = \sum_{u\in\mathsf{T}_n}\Big(\prod_{k=1}^{n} W_{\beta,u_1\ldots u_k}\Big)\mu_q([u])$$

converges uniformly, almost surely and in L^1 norm, toward $Y(\beta,\cdot)$ on $\mathcal{J}_{\beta,\epsilon}$.

3.2. Proof of Theorem 1.1. Let $\epsilon \in \mathbb{Q}_+^*$ and $\beta \in (0,1]$ such that $\beta\mathbb{E}(N) > 1$. For every $t \in \partial\mathsf{T}$ and $\omega_\beta \in \Omega_\beta$ set

$$Q_{\beta,n}(t,\omega_\beta) = \prod_{k=1}^{n} W_{\beta,t_{|k}},$$

so that for $q \in \mathcal{J}_{\beta,\epsilon}$, $\widetilde{Y}_n(\beta,q)$ is the total mass of the measure $Q_{\beta,n}(t,\omega_\beta)\cdot \mathrm{d}\mu_q^\omega(t)$.

Now, Proposition 3.2 claims that there exists a measurable subset A of $\Omega\times\Omega_\beta$ of full probability in the set of those (ω,ω_β) such that $(\mathsf{T}_{\beta,n})_{n\geqslant 1}$ survives and for all $(\omega,\omega_\beta)\in A$, for all $q\in\mathcal{J}_{\beta,\epsilon}$, $\widetilde{Y}_n(\beta,q)$ does not converge to zero. Moreover, since the branching number of the tree T is $\mathbb{P}$-almost surely equal to the constant $\mathbb{E}(N)$ and $\beta\mathbb{E}(N) > 1$, conditional on T, the $\mathbb{P}_\beta$-probability of non-extinction of $(\mathsf{T}_{\beta,n})_{n\geqslant 1}$ is positive ([17], Theorem 6.2). Thus, the projection of A to Ω has

$\mathbb{P}$-probability one and there exists a measurable subset $\Omega(\beta,\epsilon)$ of Ω such that $\mathbb{P}\big(\Omega(\beta,\epsilon)\big)=1$ and for all $\omega\in\Omega(\beta,\epsilon)$, there exists $\Omega_\beta^\omega\subset\Omega_\beta$ of positive probability such that for all $\omega\in\Omega(\beta,\epsilon)$, for all $q\in\mathcal{J}_{\beta,\epsilon}$, for all $\omega_\beta\in\Omega_\beta^\omega$, $\widetilde{Y}_n(\beta,q)$ does not converge to zero. In terms of the multiplicative chaos theory developed in [12], this means that for all $\omega\in\Omega(\beta,\epsilon)$ and $q\in\mathcal{J}_{\beta,\epsilon}$, the set of those ω_β such that the multiplicative chaos $\big(Q_{\beta,n}(\cdot,\omega)\big)_{n\geqslant 1}$ has not killed μ_q on the compact set $\partial\mathsf{T}$ has a positive $\mathbb{P}_\beta$-probability. Now, the good property of $\big(Q_{\beta,n}(\cdot,\omega)\big)_{n\geqslant 1}$ is

$$\mathbb{E}_\beta\Big(\sup_{t\in B}\big(Q_{\beta,n}(t)\big)^h\Big)=e^{n(1-h)\log(\beta)}=(|B|)^{-(1-h)\log(\beta)}$$

for any $h\in(0,1)$ and any ball B of generation n in $\partial\mathsf{T}$, where $|B|$ stands for the diameter of B and $\mathbb{E}_\beta$ stands for the expectation with respect to $\mathbb{P}_\beta$. Thus, we can apply Theorem 3 of [12] and claim that for all $\omega\in\Omega(\beta,\epsilon)$ and all $q\in\mathcal{J}_{\beta,\epsilon}$, no piece of μ_q is carried by a Borel set of Hausdorff dimension less than $-\log(\beta)$.

Let $\Omega'=\bigcap_{\beta\in(\mathbb{E}(N)^{-1},1]\cap\mathbb{Q}_+^*,\epsilon\in\mathbb{Q}_+^*}\Omega(\beta,\epsilon)$. This set is of $\mathbb{P}$-probability one. Let $q\in\mathcal{J}$. By (3.1), there exists a sequence of points $(\beta_n,\epsilon_n)\in\big(\mathbb{E}(N)^{-1},1\big]\times\mathbb{Q}_+^*$ such that $\tau(q)-q\tau'(q)>-\log(\beta_n)+\epsilon_n/2$ for all $n\geqslant 1$, $\lim_{n\to\infty}-\log(\beta_n)=\tau(q)-q\tau'(q)$, $\lim_{n\to\infty}\epsilon_n=0$ and $q\in\bigcap_{n\geqslant 1}\mathcal{J}_{\beta_n,\epsilon_n}$. Consequently, the previous paragraph implies that for all $\omega\in\Omega'$,

$$\underline{\dim}(\mu_q^\omega)\geqslant\limsup_{n\to\infty}-\log(\beta_n)=\tau(q)-q\tau'(q).$$

4. APPLICATION

Let $(N,X_1,X_2,\ldots)$ be a random vector taking values in $\mathbb{N}_+\times(\mathbb{R})^{\mathbb{N}_+}$. Then consider $\{(N_u,X_{u1},X_{u2},\ldots)\}_{u\in\bigcup_{n\geqslant 0}\mathbb{N}_+^n}$ a family of independent copies of the vector $(N,X_1,X_2,\ldots)$ indexed by the set of finite words over the alphabet $\mathbb{N}_+$. We assume that $\mathbb{E}(N)>1$ and $\mathbb{P}(N\geqslant 1)=1$. Suppose that, for all $u\in\mathsf{T}$, X_u is integrable and the sequences $(X_u)_{u\in\bigcup_{n\geqslant 0}\mathbb{N}_+^n}$ are i.i.d. Given $t\in\partial\mathsf{T}$, by the strong law of large numbers, we have $\lim_{n\to\infty}n^{-1}S_n(t)=\mathbb{E}(X_1)$ almost surely, where $S_n(t)=\sum_{k=1}^n X_{t_1\ldots t_k}$. Since $\partial\mathsf{T}$ is not countable, the following question naturally arises: are there some $t\in\partial\mathsf{T}$ so that $\lim_{n\to\infty}n^{-1}S_n(t)=\alpha\neq\mathbb{E}(X_1)$? Multifractal analysis is a framework adapted to answer this question. Consider the set $\mathcal{I}$ of those $\alpha\in\mathbb{R}$ such that

$$E(\alpha)=\Big\{t\in\partial\mathsf{T}:\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n X_{u_1\ldots u_k}=\alpha\Big\}\neq\emptyset.$$

These level sets can be described geometrically through their Hausdorff dimensions. They have been studied by many authors, see [10], [19], [1]–[3], [7]; all these papers also deal with the multifractal analysis of associated Mandelbrot measure (see also [14], [21], [16] for the study of Mandelbrot measures dimension).

Take, for $u \in \bigcup_{n\geqslant 0} \mathbb{N}_+^n$, the random variable $W_u = e^{X_u}$ and set

$$I = \{\tau'(q);\ q \in \mathcal{J}\}.$$

THEOREM 4.1. *With probability one, for all $\alpha \in I$, the multifractal formalism holds at α, i.e.*,

$$\dim E(\alpha) = \operatorname{Dim} E(\alpha) = \tau^*(\alpha);$$

in particular, $E(\alpha) \neq \emptyset$.

P r o o f. A simple covering argument yields, with probability one, for all $\alpha \in I$, $\operatorname{Dim} E(\alpha) \leqslant \tau^*(\alpha)$ (see, for example, Proposition 2.7 in [2]). In addition, consider the Mandelbrot measure μ_q, $q \in \mathcal{J}$, defined by (1.1). It is known (see, for example, Corollary 2.5 in [1]) that with probability one, $\mu_q\big(E\big(\tau'(q)\big)\big) = 1$. In addition, according to Theorem 1.1, we have, with probability one, for all $q \in \mathcal{J}$, $\underline{\dim}\, \mu_q \geqslant \tau(q) - q\tau'(q)$. We deduce the result from the mass distribution principle (Theorem 6.2 below). ∎

REMARK 4.1. *This result has been proved in* [3] *when N is not random, and in the weaker form, for each fixed $\alpha \in I$, almost surely* $\dim E(\alpha) = \tau^*(\alpha)$ *in* [10], [19], [7], *when N is random.*

REMARK 4.2. *Using the Cauchy formula, we can prove Theorem* 1.1 (see [1]). *Then our result gives a new approach to estimate, almost surely and simultaneously, the lower Hausdorff dimension of the Mandelbrot measure over $\mathcal{J}$.*

5. PROOF OF PROPOSITIONS 3.1 AND 3.2

Define, for $(q, p, \beta) \in \mathcal{J} \times [1, \infty) \times (0, 1]$, the function

$$\varphi_\beta(p, q) = \exp\big(\tau(pq) - p\tau(q) + (1-p)\log\beta\big).$$

LEMMA 5.1. *For all nontrivial compact $K \subset \mathcal{J}_{\beta,\epsilon}$ there exists a real number $1 < p_K < 2$ such that for all $1 < p \leqslant p_K$ we have*

$$\sup_{q\in K} \varphi_\beta(p_K, q) < 1.$$

P r o o f. Let $q \in \mathcal{J}_{\beta,\epsilon}$; we have $\frac{\partial \varphi_\beta}{\partial p}(1^+, q) < 0$ and there exists $p_q > 1$ such that $\varphi_\beta(p_q, q) < 1$. Therefore, in a neighborhood V_q of q, we have $\varphi_\beta(p_q, q') < 1$ for all $q' \in V_q$. If K is a nontrivial compact of $\mathcal{J}_{\beta,\epsilon}$, it is covered by a finite number of such V_{q_i}. Let $p_K = \inf_i p_{q_i}$. If $1 < p \leqslant p_K$ and $\sup_{q\in K} \varphi_\beta(p, q) \geqslant 1$, there exists $q \in K$ such that $\varphi_\beta(p, q) \geqslant 1$, and $q \in V_{q_i}$ for some i. By log-convexity of the mapping $p \mapsto \varphi_\beta(p, q)$ and the fact that $\varphi_\beta(1, q) = 1$, since $1 < p \leqslant p_{q_i}$, we have $\varphi_\beta(p, q) < 1$, which is a contradiction. ∎

LEMMA 5.2. *For all compact $K \subset \mathcal{J}$, there exists $\tilde{p}_K > 1$ such that*

$$\sup_{q \in K} \mathbb{E}\Big(\big(\sum_{i=1}^{N} W_i^q \big)^{\tilde{p}_K} \Big) < \infty.$$

Proof. Since K is compact and the family of open sets $J \cap \Omega_\gamma^1$ increases to $\mathcal{J}$ as γ decreases to one, there exists $\gamma \in (1,2]$ such that $K \subset \Omega_\gamma^1$. Take $\tilde{p}_K = \gamma$. The conclusion comes from the fact that the function $q \mapsto \mathbb{E}\big(\big(\sum_{i=1}^{N} W_i^q \big)^{\tilde{p}_K} \big)$ is continuous over $\Omega_{\tilde{p}_K}^1$. ∎

LEMMA 5.3 (Biggins [6]). *If $\{X_i\}$ is a family of integrable and independent complex random variables with $\mathbb{E}(X_i) = 0$, then $\mathbb{E}\big| \sum X_i \big|^p \leqslant 2^p \sum \mathbb{E}|X_i|^p$ for $1 \leqslant p \leqslant 2$.*

The same lines as in Lemma 2.11 in [1], we get the following lemma.

LEMMA 5.4. *Let $(N, V_1, V_2, \ldots)$ be a random vector taking values in $\mathbb{N}_+ \times \mathbb{C}^{\mathbb{N}_+}$ and such that $\sum_{i=1}^{N} V_i$ is integrable and $\mathbb{E}\big(\sum_{i=1}^{N} V_i \big) = 1$. Consider a sequence $\{(N_u, V_{u1}, V_{u2}, \ldots)\}_{u \in \bigcup_{n \geqslant 0} \mathbb{N}_+^n}$ of independent copies of $(N, V_1, \ldots, V_N)$. We define the sequence $(Z_n)_{n \geqslant 0}$ by $Z_0 = 1$ and for $n \geqslant 1$*

$$Z_n = \sum_{u \in \mathsf{T}_n} \big(\prod_{k=1}^{n} V_{u_{|k}} \big).$$

Let $p \in (1,2]$. There exists a constant C_p depending on p only such that for all $n \geqslant 1$,

$$\mathbb{E}(|Z_n - Z_{n-1}|^p) \leqslant C_p \Big(\mathbb{E}\big(\sum_{i=1}^{N} |V_i|^p \big) \Big)^{n-1} \Big(\mathbb{E}\big(\big| \sum_{i=1}^{N} V_i \big|^p \big) + 1 \Big).$$

Proof of Proposition 3.2. (1) Recall that the uniform convergence result uses an argument developed in [6]. Fix a compact $K \subset \mathcal{J}_{\beta,\epsilon}$. By Lemma 5.2 we can fix a compact neighborhood K' of K and $\widetilde{p}_{K'} > 1$ such that

$$\sup_{q \in K'} \mathbb{E}\Big(\big(\sum_{i=1}^{N} W_i^q \big)^{\tilde{p}_{K'}} \Big) < \infty.$$

By Lemma 5.1, we can fix $1 < p_K \leqslant \min(2, \tilde{p}_{K'})$ such that $\sup_{q \in K} \varphi_\beta(p_K, q) < 1$. Then for each $q \in K$, there exists a neighborhood $V_q \subset \mathbb{C}$ of q whose projection to $\mathbb{R}$ is contained in K' and such that for all $u \in \mathsf{T}$ and $z \in V_q$, the random variable

$$W_{\beta,z,u} = W_{\beta,u} \frac{e^{z \log W_u}}{\mathbb{E}\big(\sum_{i=1}^{N} e^{z \log W_i} \big)}$$

is well defined, and we have

$$\sup_{z\in V_q} \varphi_\beta(p_K, z) < 1,$$

where for all $z \in \mathbb{C}$

$$\varphi_\beta(p_K, z) = \beta^{1-p_K}\mathbb{E}\Big(\sum_{i=1}^{N} |e^{z\log W_i}|^{p_K}\Big)\Big|\mathbb{E}\Big(\sum_{i=1}^{N} e^{z\log W_i}\Big)\Big|^{-p_K}.$$

By extracting a finite covering of K from $\bigcup_{q\in K} V_q$, we find a neighborhood $V \subset \mathbb{C}$ of K such that $\sup_{z\in V} \varphi_\beta(p_K, z) < 1$. Since the projection of V to $\mathbb{R}$ is included in K' and the mapping $z \mapsto \mathbb{E}\big(\sum_{i=1}^{N} e^{z\log W_i}\big)$ is continuous and does not vanish on V, by considering a smaller neighborhood of K included in V if necessary, we can assume that

$$A_V = \sup_{z\in V}\mathbb{E}\Big(\Big|\sum_{i=1}^{N} e^{z\log W_i}\Big|^{p_K}\Big)\Big|\mathbb{E}\Big(\sum_{i=1}^{N} e^{z\log W_i}\Big)\Big|^{-p_K} + 1 < \infty.$$

Now, for $u \in \mathsf{T}$, we define the analytic extension of $Y_n(\beta, q, u)$ to V given by

$$Y_n(\beta, z, u) = \sum_{v\in\mathsf{T}_n(u)} \prod_{k=1}^{n} W_{\beta,z,uv_1\ldots v_k}.$$

We denote also $Y_n(\beta, z, \emptyset)$ by $Y_n(\beta, z)$. Now, applying Lemma 5.4 with $V_i = W_{\beta,z,i}$, we obtain

$$\mathbb{E}\big(|Y_n(\beta, z) - Y_{n-1}(\beta, z)|^{p_K}\big) \leqslant C_{p_K}\Big(\mathbb{E}\Big(\sum_{i=1}^{N}|V_i|^{p_K}\Big)\Big)^{n-1}\Big(\mathbb{E}\Big(\Big|\sum_{i=1}^{N} V_i\Big|^{p_K}\Big) + 1\Big).$$

Notice that $\mathbb{E}\big(\sum_{i=1}^{N}|V_i|^{p_K}\big) = \varphi_\beta(p_K, z)$. Then,

$$\mathbb{E}\big(|Y_n(\beta, z) - Y_{n-1}(\beta, z)|^{p_K}\big) \leqslant C_{p_K} A_V \sup_{z\in V}\varphi(p_K, z)^{n-1}.$$

With probability one, the functions $z \in V \mapsto Y_n(\beta, z), n \geqslant 0$, are analytic. Fix a closed disc $D(z_0, 2\rho) \subset V$. Theorem 6.1 below implies

$$\sup_{z\in D(z_0,\rho)} |Y_n(\beta, z) - Y_{n-1}(\beta, z)| \leqslant 2\int_{[0,1]} \big|Y_n\big(\beta, \zeta(\theta)\big) - Y_{n-1}\big(\beta, \zeta(\theta)\big)\big| d\theta,$$

where, for $\theta \in [0, 1]$, $\zeta(\theta) = z_0 + 2\rho e^{i2\pi\theta}$. Furthermore, Jensen's inequality and

Fubini's theorem give

$$\begin{aligned}\mathbb{E}\Big(\sup_{z\in D(z_0,\rho)}|Y_n(\beta,z)-Y_{\beta,n-1}(z)|^{p_K}\Big)&\\ &\leqslant \mathbb{E}\Big(\big(2\int_{[0,1]}\big|Y_n\big(\beta,\zeta(\theta)\big)-Y_{n-1}\big(\beta,\zeta(\theta)\big)\big|d\theta\big)^{p_K}\Big)\\ &\leqslant 2^{p_K}\mathbb{E}\Big(\int_{[0,1]}\big|Y_n\big(\beta,\zeta(\theta)\big)-Y_{n-1}\big(\beta,\zeta(\theta)\big)\big|^{p_K}d\theta\Big)\\ &\leqslant 2^{p_K}\int_{[0,1]}\mathbb{E}\big|Y_n\big(\beta,\zeta(\theta)\big)-Y_{n-1}\big(\beta,\zeta(\theta)\big)\big|^{p_K}d\theta\\ &\leqslant 2^{p_K}C_{p_K}A_V\sup_{z\in V}\varphi_\beta(p_K,z)^{n-1}.\end{aligned}$$

Since $\sup_{z\in V}\varphi_\beta(p_K,z)<1$, it follows that

$$\sum_{n\geqslant 1}\Big\|\sup_{z\in D(z_0,\rho)}|Y_n(\beta,z)-Y_{n-1}(\beta,z)|\Big\|_{p_K}<\infty.$$

This implies that $z\mapsto Y_n(\beta,z)$ converge uniformly, almost surely and in L^{p_K} norm over the compact $D(z_0,\rho)$, to a limit $z\mapsto Y(\beta,z)$. This also implies that

$$\|\sup_{z\in D(z_0,\rho)}Y(\beta,z)\|_{p_K}<\infty.$$

Since K can be covered by finitely many such discs $D(z_0,\rho)$, we get the uniform convergence, almost surely and in L^{p_K} norm, of the sequence $\big(q\in K\mapsto Y_n(\beta,q)\big)_{n\geqslant 1}$ to $q\in K\mapsto Y(\beta,q)$. Moreover, since $\mathcal{J}_{\beta,\epsilon}$ can be covered by a countable union of such compact K, we get the simultaneous convergence for all $q\in\mathcal{J}_\beta$. The same holds simultaneously for all the functions $q\in\mathcal{J}_\beta\mapsto Y_n(\beta,q,u)$, $u\in\bigcup_{n\geqslant 0}\mathbb{N}_+^n$, because $\bigcup_{n\geqslant 0}\mathbb{N}_+^n$ is countable.

To complete the proof of (1), we must show that a.s., $q\in K\mapsto Y(\beta,q)$ does not vanish. Without loss of generality we suppose that $K=[0,1]$. If I is a dyadic closed subinterval of $[0,1]$, we denote by E_I the event $\{\exists\, q\in I: Y(\beta,q)=0\}$. Let I_0,I_1 stand for two dyadic subintervals of I in the next generation. The event E_I being a tail event of probability zero or one, if we suppose that $P(E_I)=1$, there exists $j\in\{0,1\}$ such that $P(E_{I_j})=1$. Suppose now that $P(E_K)=1$. The previous remark allows us to construct a decreasing sequence $\big(I(n)\big)_{n\geqslant 0}$ of dyadic subintervals of K such that $P(E_{I(n)})=1$. Let q_0 be the unique element of $\bigcap_{n\geqslant 0}I(n)$. Since $q\mapsto Y(\beta,q)$ is continuous, we have $P\big(Y(\beta,q_0)=0\big)=1$, which contradicts the fact that $\big(Y_n(\beta,q_0)\big)_{n\geqslant 1}$ converges to $Y(\beta,q_0)$ in L^1.

(2) Here we develop, in the context of the boundary of a supercritical Galton–Watson tree, a uniform version of the argument used by Kahane in [13] on homogeneous trees, and written in complete rigor in [24]. Fix $\epsilon>0$ and a compact set

K in $\mathcal{J}_{\beta,\epsilon}$. Denote by E the separable Banach space of the real-valued continuous functions over K endowed with the supremum norm.

For $n \geqslant m \geqslant 1$ and $q \in K$ let

$$Z_{m,n}(\beta, q) = \sum_{u \in \mathsf{T}_m} Y_{n-m}(q, u) \prod_{k=1}^{m} W_{\beta,q,u_1\ldots u_k}.$$

Notice $Z_{n,n}(\beta, q) = Y_n(\beta, q)$. Moreover, since $Y_n(\beta,\cdot)$ converges almost surely and in L^1 norm to $Y(\beta,\cdot)$ as $n \to \infty$, $Y_n(\beta,\cdot)$ belongs to $L^1_E = L^1_E(\Omega_\beta \times \Omega, \mathcal{A}_\beta \times \mathcal{A}, \mathbb{P}_\beta \times \mathbb{P})$ (where we use the notation of Section V-2 in [20]), so that the continuous random function $\mathbb{E}\big(Z_{n,n}(\beta, q)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n\big)$ is well defined by Proposition V-2-5 in [20]; also, for any fixed $q \in K$, we can deduce from the definitions and the independence assumptions that

$$Z_{m,n}(\beta, q) = \mathbb{E}\big(Z_{n,n}(\beta, q)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n\big)$$

almost surely. By Proposition V-2-5 in [20] again, since $g \in E \mapsto g(q)$ is a continuous linear form over E, we thus have

$$Z_{m,n}(\beta, q) = \mathbb{E}\big(Z_{n,n}(\beta, \cdot)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n\big)(q)$$

almost surely. By considering a dense countable set of q in K, we can conclude that the random continuous functions $Z_{m,n}(\beta,\cdot)$ and $\mathbb{E}\big(Z_{n,n}(\beta,\cdot)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n\big)$ are equal almost surely.

Similarly, since for each $q \in K$ the martingale $\big(Y_n(\beta, q), \mathcal{F}_{\beta,n} \otimes \mathcal{F}_n\big)$ converges to $Y(\beta, q)$ almost surely and in L^1, and $Y(\beta,\cdot) \in L^1_E$, by using Proposition V-2-5 in [20] again we can get almost surely

(5.1)

$$Z_{n,n}(\beta,\cdot) = \mathbb{E}\big(Y(\beta,\cdot)|\mathcal{F}_{\beta,n} \otimes \mathcal{F}_n\big), \text{ hence } Z_{m,n}(\beta,\cdot) = \mathbb{E}\big(Y(\beta,\cdot)|\mathcal{F}_{\beta,m} \otimes \mathcal{F}_n\big).$$

Moreover, it follows from Proposition 3.2(1) and the definition of $\mu_q([u])$ that $Z_{m,n}(\beta,\cdot)$ converges almost surely uniformly and in L^1 norm, as $n \to \infty$, to $\widetilde{Y}_m(\beta,\cdot)$. This and (5.1) yield, by Proposition V-2-6 in [20],

$$\widetilde{Y}_m(\beta,\cdot) = \lim_{n\to\infty} Z_{m,n}(\beta,\cdot) = \mathbb{E}\Big(Y(\beta,\cdot)|\mathcal{F}_{\beta,m} \otimes \sigma\big(\bigcup_{n\geqslant 1} \mathcal{F}_n\big)\Big)$$

and finally

$$\lim_{m\to\infty} \widetilde{Y}_m(\beta,\cdot) = \mathbb{E}\Big(Y(\beta,\cdot)|\sigma\big(\bigcup_{m\geqslant 1} \mathcal{F}_{\beta,m}\big) \otimes \sigma\big(\bigcup_{n\geqslant 1} \mathcal{F}_n\big)\Big) = Y(\beta,\cdot)$$

almost surely (since, by construction, $Y(\beta,\cdot)$ is $\sigma\big(\bigcup_{m\geqslant 1} \mathcal{F}_{\beta,m}\big) \otimes \sigma\big(\bigcup_{n\geqslant 1} \mathcal{F}_n\big)$-measurable), where the convergences hold in the uniform norm. Moreover, since $\mathcal{J}_{\beta,\epsilon}$ can be covered by a countable union of such compact K, we get the simultaneous convergence for all $q \in \mathcal{J}_{\beta,\epsilon}$. ■

Proof of Proposition 3.1. The proof of the first point is similar to the proof of Proposition 3.2(1) ($\beta = 1$). The second point is a consequence of the branching property:

$$Y_{n+1}(q,u) = \sum_{i=1}^{N} W_{q,ui}\, Y_n(q,ui). \quad \blacksquare$$

6. APPENDICES

APPENDIX 1 — CAUCHY FORMULA

DEFINITION 6.1. Let $D(\zeta, r)$ be a disc in $\mathbb{C}$ with center ζ and radius r. The set ∂D is the boundary of D. Let $g \in \mathcal{C}(\partial D)$ be a continuous function on ∂D. We define the *integral* of g on ∂D as

$$\int_{\partial D} g(\zeta)d\zeta = 2i\pi r \int_{[0,1]} g\big(\zeta(t)\big)e^{i2\pi t}dt,$$

where $\zeta(t) = \zeta + re^{i2\pi t}$.

THEOREM 6.1. *Let $D = D(a,r)$ be a disc in $\mathbb{C}$ with radius $r > 0$, and f be a holomorphic function in a neighborhood of D. Then, for all $z \in D$*

$$f(z) = \frac{1}{2i\pi} \int_{\partial D} \frac{f(\zeta)d\zeta}{\zeta - z}.$$

It follows that

$$\sup_{z\in D(a,r/2)} |f(z)| \leqslant 2 \int_{[0,1]} \big|f\big(\zeta(t)\big)\big|dt. \tag{6.1}$$

APPENDIX 2 — MASS DISTRIBUTION PRINCIPLE

THEOREM 6.2 (Falconer [9]). *Let ν be a positive and finite Borel probability measure on a compact metric space (X,d). Assume that $M \subseteq X$ is a Borel set such that $\nu(M) > 0$ and*

$$M \subseteq \left\{ t \in X, \liminf_{r\to 0^+} \frac{\log \nu\big(B(t,r)\big)}{\log r} \geqslant \delta \right\}.$$

Then the Hausdorff dimension of M is bounded from below by δ.

APPENDIX 3 — HAUSDORFF AND PACKING MEASURES AND DIMENSIONS

Given a subset K of $\mathbb{N}_+^{\mathbb{N}_+}$ endowed with a metric d making it σ-compact, $g : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous non-decreasing function near zero and such that $g(0) = 0$, and E a subset of K, the *Hausdorff measure* of E with respect to the gauge function g is defined as

$$\mathcal{H}^g(E) = \lim_{\delta\to 0^+} \inf \Big\{ \sum_{i\in\mathbb{N}} g\big(\mathrm{diam}(U_i)\big) \Big\},$$

the infimum being taken over all the countable coverings $(U_i)_{i\in\mathbb{N}}$ of E by subsets of K of diameters less than or equal to δ.

If $s \in \mathbb{R}_+^*$ and $g(u) = u^s$, then $\mathcal{H}^g(E)$ is also denoted by $\mathcal{H}^s(E)$ and called the *s-dimensional Hausdorff measure* of E. Then, the *Hausdorff dimension* of E is defined as

$$\dim E = \sup\{s > 0 : \mathcal{H}^s(E) = \infty\} = \inf\{s > 0 : \mathcal{H}^s(E) = 0\},$$

with the convention $\sup\emptyset = 0$ and $\inf\emptyset = \infty$.

Packing measures and dimensions are defined as follows. Given g and $E \subset K$ as above, one first defines

$$\overline{\mathcal{P}}^g(E) = \lim_{\delta\to 0^+} \sup \Big\{ \sum_{i\in\mathbb{N}} g\big(\mathrm{diam}(B_i)\big) \Big\},$$

the supremum being taken over all the packings $\{B_i\}_{i\in\mathbb{N}}$ of E by balls centered on E and with diameter smaller than or equal to δ. Then, the *packing measure* of E with respect to the gauge g is defined as

$$\mathcal{P}^g(E) = \lim_{\delta\to 0^+} \inf \Big\{ \sum_{i\in\mathbb{N}} \overline{\mathcal{P}}^g(E_i) \Big\},$$

the infimum being taken over all the countable coverings $(E_i)_{i\in\mathbb{N}}$ of E by subsets of K of diameters less than or equal to δ. If $s \in \mathbb{R}_+^*$ and $g(u) = u^s$, then $\mathcal{P}^g(E)$ is also denoted by $\mathcal{P}^s(E)$ and called the *s-dimensional measure* of E. Then, the *packing dimension* of E is defined as

$$\mathrm{Dim} E = \sup\{s > 0 : \mathcal{P}^s(E) = \infty\} = \inf\{s > 0 : \mathcal{P}^s(E) = 0\},$$

with the convention $\sup\emptyset = 0$ and $\inf\emptyset = \infty$. For more details the reader is referred to [9].

If μ is a positive and finite Borel measure supported on K, then its *lower Hausdorff and packing dimensions* are defined as

$$\underline{\dim}(\mu) = \inf\{\dim F : \ F \text{ Borel}, \ \mu(F) > 0\},$$
$$\underline{\mathrm{Dim}}(\mu) = \inf\{\mathrm{Dim}\, F : \ F \text{ Borel}, \ \mu(F) > 0\},$$

and its *upper Hausdorff and packing dimensions* are defined as

$$\overline{\dim}(\mu) = \inf\{\dim F : \ F \text{ Borel}, \ \mu(F) = \|\mu\|\},$$
$$\overline{\mathrm{Dim}}(\mu) = \inf\{\mathrm{Dim}\, F : \ F \text{ Borel}, \ \mu(F) = \|\mu\|\}.$$

We have (see [8], [11])

$$\underline{\dim}(\mu) = \text{ess inf}_\mu \liminf_{r\to 0^+} \frac{\log \mu\big(B(t,r)\big)}{\log(r)},$$
$$\underline{\mathrm{Dim}}(\mu) = \text{ess inf}_\mu \limsup_{r\to 0^+} \frac{\log \mu\big(B(t,r)\big)}{\log(r)}$$

and

$$\overline{\dim}(\mu) = \text{ess sup}_\mu \liminf_{r\to 0^+} \frac{\log \mu\big(B(t,r)\big)}{\log(r)},$$
$$\overline{\mathrm{Dim}}(\mu) = \text{ess sup}_\mu \limsup_{r\to 0^+} \frac{\log \mu\big(B(t,r)\big)}{\log(r)},$$

where $B(t,r)$ stands for the closed ball of radius r centered at t. If $\underline{\dim}(\mu) = \overline{\dim}(\mu)$ (resp. $\underline{\mathrm{Dim}}(\mu) = \overline{\mathrm{Dim}}(\mu)$), this common value is denoted by $\dim \mu$ (resp. $\mathrm{Dim}(\mu)$), and if $\dim \mu = \mathrm{Dim}\mu$, one says that μ is *exact dimensional*.

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