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SOME RESULTS ON INTEGER-VALUED POLYNOMIALS OVER MODULES

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Abstract. Let M be a module over a commutative ring R. In this paper, we study $\mathrm{Int}(R, M)$, the module of integer-valued polynomials on M over R, and $\mathrm{Int}_M(R)$, the ring of integer-valued polynomials on R over M. We establish some properties of Krull dimensions of $\mathrm{Int}(R, M)$ and $\mathrm{Int}_M(R)$. We also determine when $\mathrm{Int}(R, M)$ and $\mathrm{Int}_M(R)$ are nontrivial. Among the other results, it is shown that $\mathrm{Int}(\mathbb{Z}, M)$ is not Noetherian module over $\mathrm{Int}_M(\mathbb{Z}) \cap \mathrm{Int}(\mathbb{Z})$, where M is a finitely generated $\mathbb{Z}$-module.

1. Introduction

Let D be a commutative integral domain with field of fractions K. The ring of integer-valued polynomials on D is defined by

$$\mathrm{Int}(D) = \{f \in K[X] \mid f(D) \subseteq D\}.$$

The first systematic studies of the algebraic properties of $\mathrm{Int}(D)$ were done by Pólya [21] and Ostrowski [18] in 1919. Both Pólya and Ostrowski were primarily concerned with the module structure of $\mathrm{Int}(D)$, and were interested in determining whether $\mathrm{Int}(D)$ had a regular basis. There is an extensive literature on $\mathrm{Int}(D)$, see for example [3, 4, 22]. The reader is referred to the textbooks [1] and [16] for a general introduction to integer-valued polynomials.

More recently, attention has turned to the consideration of integer-valued polynomials on algebras. See for example [7, 8, 15, 19, 20, 25]. The typical approach for this construction is to take a torsion-free D-algebra A that is finitely generated as a D-module and such that $A \cap K = D$. We also refer the reader to the survey papers [6] and [27].

Throughout the paper, R is a commutative ring with identity and M is a unitary R-module. The set of all zero-divisors of M denoted by $Z_R(M)$. The set of all non zero-divisors on M denoted by U (that is $U := R\backslash Z_R(M)$). Let N be a submodule of M. The *colon ideal* $(N :_R M)$ is the set of all elements r in R such that $rM \subseteq N$. The annihilator of M, denoted by $\mathrm{Ann}_R(M)$, is $(0 :_R M)$.

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If there is no ambiguity, we simply write $Z(M)$, $(N : M)$ and $\mathrm{Ann}(M)$ instead of $Z_R(M)$, $(N :_R M)$ and $\mathrm{Ann}_R(M)$, respectively. The module M is called *faithful* if $\mathrm{Ann}(M) = 0$. For each $x \in M$, the annihilator of Rx, is denoted by $\mathrm{Ann}(x)$,

The *total quotient* $T(M)$ of M is the localization $U^{-1}M$ (see for example [24]). Note that the canonical R-module mapping $M \longrightarrow T(M)$ is an inclusion and we consider M as an R-submodule of $T(M)$.

Suppose that X is an indeterminate that commutes with the elements of M and R. Let $M[X]$ denote the set of formal polynomials of the form $\sum_{i=1}^{n} m_i X^i$, where $m_i \in M$ (for polynomials with central variable over a noncommutaive ring, see for example [9]). Obviously, $M[X]$ is an Abelian group under usual addition. Moreover, $M[X]$ is naturally an $R[X]$-module under the $R[X]$-scalar multiplication defined by

$$(\sum_{i=1}^{m} a_i X^i)(\sum_{j=1}^{n} m_j X^j) = \sum_{k=0}^{m+n} (\sum_{i+j=k} a_i m_j) X^k.$$

For a polynomial $f(X) = \sum_{i=1}^{n} m_i X^i \in M[X]$ the value of f at the element $a \in R$ is defined by $f(a) = \sum_{i=1}^{n} m_i a^i \in M$. Recently, Elliott [5] studied the integer-valued polynomials on commutative rings and modules. Elliott defined the *ring of integer-valued polynomials on R* as follows:

$$\mathrm{Int}(R) = \{f \in T(R)[X] \mid f(R) \subseteq R\}.$$

He defined the *module of integer-valued polynomials on M over R* as follows:

$$\mathrm{Int}(R, M) = \{f \in T(M)[X] \mid f(R) \subseteq M\}.$$

Elliott also defined the *ring of integer-valued polynomials on R over M* as follows:

$$\mathrm{Int}_M(R) = \{f \in U^{-1}R[X] \mid f(R)M \subseteq M\}.$$

Note that

$$M[X] \subseteq \mathrm{Int}(R, M) \subseteq T(M)[X],$$

and

$$(R.1)[X] \subseteq \mathrm{Int}_M(R) \subseteq U^{-1}R[X],$$

where $R.1$ is the image of R in $U^{-1}R$. It is easy to see that $\mathrm{Int}_M(R)$ is a ring and $\mathrm{Int}(R, M)$ is an $\mathrm{Int}_M(R)$-module. We say that $\mathrm{Int}(R, M)$ (respectively, $\mathrm{Int}_M(R)$) is *nontrivial* if $\mathrm{Int}(R, M) \neq M[X]$ (respectively, $\mathrm{Int}_M(R) \neq (R.1)[X]$).

This paper consists of two sections. In Section 2, we prove some preliminary facts about $\mathrm{Int}(R, M)$. In particular, we determine when $\mathrm{Int}(R, M)$ is nontrivial (see Theorem 2.5). We also give some properties of Krull dimension of $\mathrm{Int}(R, M)$ (see Corollary 2.7). In Section 3, we prove some basic results of $\mathrm{Int}_M(R)$. In particular, we show that $\mathrm{Int}_M(R)$ is not Noetherian $\mathrm{Int}_M(\mathbb{Z}) \cap \mathrm{Int}(\mathbb{Z})$-module, where M is a finitely generated $\mathbb{Z}$-module (see Theorem 3.2(1)). We also give some properties of Krull dimension of $\mathrm{Int}_M(R)$

(see Theorem 3.4 and Corollary 3.5). Finally, we determine when $\mathrm{Int}_M(R)$ is nontrivial (see Theorem 3.6).

2. Properties of $\mathrm{Int}(R, M)$

We begin with the following theorem.

Theorem 2.1. *Let M_i, $1 \le i \le n$ be R-modules with the same zero-divisors. Then there is an R-module isomorphism*

$$\mathrm{Int}(R, \bigoplus_{i=1}^{n} M_i) \cong_R \bigoplus_{i=1}^{n} \mathrm{Int}(R, M_i).$$

Proof. Let $U =: R \setminus Z_R(\bigoplus_{i=1}^n M_i) = R \setminus Z_R(M_i)$. We define:

$$\begin{aligned} \phi : \mathrm{Int}(R, \bigoplus_{i=1}^{n} M_i) &\longrightarrow \bigoplus_{i=1}^{n} \mathrm{Int}(R, M_i) \\ \sum_{k=0}^{p} \frac{A_k}{s_k} X^k &\mapsto (\sum_{k=0}^{p} \frac{a_{k1}}{s_k} X^k, \dots, \sum_{k=0}^{p} \frac{a_{kn}}{s_k} X^k), \end{aligned}$$

where $A_k = (a_{k1}, \dots, a_{kn}) \in \bigoplus_{i=1}^n M_i$ and $s_k \in U$ for each $0 \le k \le p$. First we show ϕ is well-defined. Let $f = \sum_{k=0}^{p} \frac{A_k}{s_k} X^k \in \mathrm{Int}(R, \bigoplus_{i=1}^n M_i)$. It is easy to see that $\phi(f) = (\sum_{k=0}^{p} \frac{a_{k1}}{s_k} X^k, \dots, \sum_{k=0}^{p} \frac{a_{kn}}{s_k} X^k) \in \bigoplus_{i=1}^n \mathrm{Int}(R, M_i)$. Now let $g = \sum_{k=0}^{q} \frac{B_k}{t_k} X^k$ be another element of $\bigoplus_{i=1}^n \mathrm{Int}(R, M_i)$ such that $f = g$. So $p = q$ and $\frac{A_k}{s_k} = \frac{B_k}{t_k}$ for all $0 \le k \le p$. Therefore there is a $u \in U$ such that $u(t_k A_k - s_k B_k) = 0$ for all $0 \le k \le p$. It follows that $u(t_k a_{ki} - s_k b_{ki}) = 0$ for all $0 \le k \le p$, $1 \le i \le n$ and hence $\phi(f) = \phi(g)$. So ϕ is well-defined. It is easy to see that ϕ is a homomorphism of R-modules. Now we show ϕ is injective. Let f be an element of $\sum_{k=0}^{p} \frac{A_k}{s_k} X^k \in \mathrm{Int}(R, \bigoplus_{i=1}^n M_i)$ such that $\phi(f) = 0$. Then $(\sum_{k=0}^{p} \frac{a_{k1}}{s_k} X^k, \dots, \sum_{k=0}^{p} \frac{a_{kn}}{s_k} X^k) = 0$. So there is a $u_{ki} \in U$ such that $u_{ki} a_{ki} = 0$, where $0 \le k \le p$ and $1 \le i \le n$. If $u_k = \prod_i u_{ki}$, then $u_k A_k = 0$ for all $0 \le k \le p$ and hence $f = 0$. It follows that ϕ is injective. Let $f = (\sum_{k=0}^{p} \frac{a_{k1}}{s_k} X^k, \dots, \sum_{k=0}^{p} \frac{a_{kn}}{s_k} X^k) \in \bigoplus_{i=1}^n \mathrm{Int}(R, M_i)$. Let $0 \le \alpha \le p$ and set

$$\widehat{s_\alpha} := \prod_{i \neq \alpha} s_i, \text{ and } u := \prod_i s_i.$$

We have

$$\begin{aligned} (\sum_{k=0}^{p} \frac{a_{k1}}{s_k} X^k, \dots, \sum_{k=0}^{p} \frac{a_{kn}}{s_k} X^k) &= (\sum_{k=0}^{p} \frac{\widehat{s_k} a_{k1}}{\widehat{s_k} s_k} X^k, \dots, \sum_{k=0}^{p} \frac{\widehat{s_k} a_{kn}}{\widehat{s_k} s_k} X^k) \\ &= (\sum_{k=0}^{p} \frac{\widehat{s_k} a_{k1}}{u} X^k, \dots, \sum_{k=0}^{p} \frac{\widehat{s_k} a_{kn}}{u} X^k). \end{aligned}$$

So $\phi(\sum \frac{B_k}{u} X^k) = f$, where $B_k = (\widehat{s_k} a_{k1}, \dots, \widehat{s_k} a_{kn})$. It follows that ϕ is surjective. Hence ϕ is an isomorphism and the proof is complete. ☐

Lemma 2.2. *Let M and N be two isomorphic Abelian groups. If M is an R-module, then N is also an R-module and $M \cong N$ as R-modules.*

Proof. Let $\phi : M \longrightarrow N$ be a $\mathbb{Z}$-module isomorphism. We define a scalar multiplication as follows:

$$\begin{array}{rcl} \mu : R \times N & \longrightarrow & N \\ (r, n) & \mapsto & \phi(r\phi^{-1}(n)) \end{array}$$

Then N is an R-module with this scalar multiplication and it is easy to see that $\phi : M \longrightarrow N$ is an R-module isomorphism. □

Theorem 2.3. *Let M be an R-module and I be an ideal of R such that $I \subseteq$ $\mathrm{Ann}(M)$. Then*

(1) $\mathrm{Int}(R, M) \cong_R \mathrm{Int}(R/I, M)$,
(2) $\mathrm{Int}(R, M) \cong_{\mathrm{Int}_M(R)} \mathrm{Int}(R/I, M)$.

Proof. (1) For convenience, let the element $r + I$ of R/I be denoted by $\overline{r}$. We define

$$\begin{array}{rcl} \phi : \mathrm{Int}(R, M) & \longrightarrow & \mathrm{Int}(R/I, M) \\ \displaystyle\sum_{k=0}^{p} \frac{a_k}{s_k} X^k & \mapsto & \displaystyle\sum_{k=0}^{p} \frac{a_k}{\overline{s_k}} X^k. \end{array}$$

It is easy to see that ϕ is an R-module isomorphism.

(2) Immediately follows from Part (1) and Lemma 2.2. □

Let M be an R-module and $f \in M[X]$ and S be a multiplicative subset of R. Then $S^{-1}\langle f(R)\rangle$ is the localization of the R-module generated by the value of f on R and $\langle f(S^{-1}R)\rangle$ is the $S^{-1}R$-module generated by the value of f on $S^{-1}R$.

In following theorem, we generalize [1, Propositions I.2.5, I.2.7(ii)].

Theorem 2.4. *Let M be an R-module and $f \in M[X]$ and S be a multiplicative subset of R. Then*

(1) *$S^{-1}\,\mathrm{Int}(R, M) \subseteq \mathrm{Int}(S^{-1}R, S^{-1}M)$,*
(2) *If R is Noetherian, then $S^{-1}\,\mathrm{Int}(R, M) = \mathrm{Int}(S^{-1}R, S^{-1}M)$.*

Proof. (1) In view of [1, Theorem I.2.10], we have $\langle f(S^{-1}R)\rangle = S^{-1}\langle f(R)\rangle \subseteq S^{-1}M$. It follows that $S^{-1}\,\mathrm{Int}(R, M) \subseteq \mathrm{Int}(S^{-1}R, S^{-1}M)$.

(2) In view of the part (1), it is enough to show that $\mathrm{Int}(S^{-1}R, S^{-1}M) \subseteq S^{-1}\,\mathrm{Int}(R, M)$. Let $f \in \mathrm{Int}(S^{-1}R, S^{-1}M)$. Then $\langle f(R)\rangle \subseteq S^{-1}M \cap C(f)$, where $C(f)$, the content of f, is the R-module generated by the coefficients of f. Since R is a Noetherian ring, $S^{-1}M \cap C(f)$ is a Noetherian R-module and hence $\langle f(R)\rangle$ is a finitely generated R-module. If $s \in S$ is a common denominator of the generators of $\langle f(R)\rangle$, then $s\langle f(R)\rangle \subseteq M$. Therefore $sf \in \mathrm{Int}(R, M)$ and hence $f \in S^{-1}\,\mathrm{Int}(R, M)$. □

Let M be an R-module. Recall that the set of *associated primes* of M, denoted by $\mathrm{Ass}_R(M)$, is the set of prime ideals $\mathfrak{p}$ such that $\mathfrak{p} = \mathrm{Ann}(x)$ for some $x \in M$. If $\mathfrak{p}$ is a minimal prime ideal over $\mathrm{Ann}(x)$ for some $x \in M$, then $\mathfrak{p}$ is called a *weakly associated prime* of M. Sometimes $\mathfrak{p}$ is called a *weak Bourbaki prime* of M. The set of weakly associated primes of M is denoted by $\widetilde{\mathrm{Ass}}_R(M)$. If there is no ambiguity, we simply write $\mathrm{Ass}(M)$ and $\widetilde{\mathrm{Ass}}(M)$ instead of $\mathrm{Ass}_R(M)$ and $\widetilde{\mathrm{Ass}}_R(M)$, respectively.

Theorem 2.5. *Let M be an R-module. If $\mathrm{Int}(R, M) \neq M[X]$, then there exists $\mathfrak{p} \in \widetilde{\mathrm{Ass}}(T(M)/M)$ such that $R/\mathfrak{p}$ is finite. The converse is true when $\mathfrak{p}$ is finitely generated.*

Proof. Let $\mathfrak{p} \in \widetilde{\mathrm{Ass}}(T(M)/M)$ be such that $R/\mathfrak{p}$ is infinite. We claim that $\mathrm{Int}(R, M) \subseteq M_{\mathfrak{p}}[X]$. Let f be a polynomial of degree n in $\mathrm{Int}(R, M)$ and let $d = \prod_{0 \leq i < j \leq n}(a_j - a_i)$, where $a_0, \ldots, a_n$ be $n+1$ elements in distinct classes modulo $\mathfrak{p}$; in particular, $d \notin \mathfrak{p}$. In view of [1, Proposition I.3.18], we have $df \in M[X]$ and hence $\mathrm{Int}(R, M) \subseteq M_{\mathfrak{p}}[X]$. Now suppose that every element of $\widetilde{\mathrm{Ass}}(T(M)/M)$ has infinite residue field and there is a polynomial $f \in \mathrm{Int}(R, M)$ with some coefficient $x \in T(M) \setminus M$. Then there is a $\mathfrak{p} \in \widetilde{\mathrm{Ass}}(T(M)/M)$ such that $\mathrm{Ann}(\overline{x}) \subseteq \mathfrak{p}$, where $\overline{x}$ is the residue of x in the quotient module $T(M)/M$. Since $\mathrm{Int}(R, M) \subseteq M_{\mathfrak{p}}[X]$, there is $s \in R \setminus \mathfrak{p}$ such that $sx \in M$. It follows that $s\overline{x} = \overline{0}$ in $T(M)/M$ and so $s \in \mathrm{Ann}(\overline{x}) \subseteq \mathfrak{p}$, which is a contradiction.

Conversely, suppose that $\mathfrak{p}$ is finitely generated. By [28, Lemma 1.8], $\mathfrak{p} \in \mathrm{Ass}(T(M)/M)$. Therefore there exists a nonzero element $x \in T(M)/M$ such that $\mathfrak{p} = \mathrm{Ann}(x)$. Let $\{a_0, a_1, \ldots, a_n\}$ be a set of representatives modulo $\mathfrak{p}$. Then the polynomial $f = x\prod_{0 \leq i \leq n}(X - a_i)$ is a polynomial in $\mathrm{Int}(R, M)$ and its leading coefficient is not in M. □

A proper submodule P of M is called a *prime submodule* of M if for any x of M and element r of R, $rx \in P$ implies $x \in P$ or $rM \subseteq P$. A module M is called a *prime module* if its zero submodule is a prime submodule. This notion of prime submodule was first introduced and systematically studied in Dauns [2]. The reader is referred to [10] and [12] for more information about prime submodules.

Notation 1. Let M be an R-module and $a \in R$ and N be a submodule of M. We set:

$$\mathcal{S}_{N,a} = \{f \in \mathrm{Int}(R, M) \mid f(a) \in N\}.$$

Theorem 2.6. *Let M be an R-module and $a \in R$. Then*

1. *If N is a submodule of M, then $\mathcal{S}_{N,a}$ is a submodule of $\mathrm{Int}(R, M)$,*
2. *If P is a prime submodule of M, then $\mathcal{S}_{P,a}$ is a prime submodule of $\mathrm{Int}(R, M)$.*

Proof. (1) Obvious.

(2) Let P be a prime submodule of M. It is easy to see that $\mathcal{S}_{P,a}$ is a proper submodule of $\mathrm{Int}(R, M)$. Now let $r \in R$ and $f \in \mathrm{Int}(R, M)$ such that

$rf \in \mathcal{S}_{P,a}$. Suppose that $f \notin \mathcal{S}_{P,a}$. Since $rf(a) \in P$, we have $rM \subseteq P$. Hence $rg(a) \in P$ for every $g \in \text{Int}(R, M)$. It follows that $r\,\text{Int}(R, M) \subseteq \mathcal{S}_{P,a}$. This completes the proof. □

The *Krull dimension* of a ring R, denoted by $\dim R$, is the maximal length n of a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ of prime ideals of R. Similarly, the *Krull dimension* of an R-module M, denoted by $\dim M$, is the maximal length n of a chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of prime submodules of M (see for example [10]).

Corollary 2.7. *Let M be an R-module. Then*

(1) $\dim \text{Int}(R, M) \geq \dim M$,
(2) *If M is a prime R-module, then* $\dim \text{Int}(R, M) \geq \dim M + 1$.

Proof. (1) Let $P_0 \subset P_1 \subset \cdots \subset P_n$ be a chain of prime submodules of M and let $a \in R$. If $m_i \in P_i \setminus P_{i-1}$, then the constant polynomial $f(X) = m_i$ is in $\mathcal{S}_{P_i,a} \setminus \mathcal{S}_{P_{i-1},a}$. Therefore $\mathcal{S}_{P_0,a} \subset \mathcal{S}_{P_1,a} \subset \cdots \subset \mathcal{S}_{P_n,a}$ is a chain of prime submodules of $\text{Int}(R, M)$. So $\dim \text{Int}(R, M) \geq \dim M$.

(2) Let M be a prime R-module and let $(0) = P_0 \subset P_1 \subset \cdots \subset P_n$ be a chain of prime submodules of M. Let $a \in R$. Then $mX - ma$ is a nonzero element of $\mathcal{S}_{P_0,a}$ for every nonzero element $m \in M$. Now we show that $\text{Int}(R, M)$ is a prime R-module. Let $r \in R$ and $f(X) = \sum_{i=0}^{k} \frac{m_i}{s_i} X^i \in \text{Int}(R, M)$ such that $rf(X) = 0$. Suppose that $f(X) \neq 0$ so we may assume $\frac{m_k}{s_k} \neq 0$. Since $rf = 0$, there exists $u \in R \setminus Z(M)$ such that $rum_k = 0$. Since $um_k \neq 0$ and M is a prime R-module, we have $rM = 0$. It follows that $r\,\text{Int}(R, M) = 0$ and hence $\text{Int}(R, M)$ is a prime R-module. So the assertion follows from Part (1). □

An R-module M is said to be *polynomially torsion-free*, in short a PF R-module, if $f(R) = (0)$ implies $f = 0$ for all $f \in M[X]$ (see [1, Definition I.4.1]). We recall that an R-module M is called *torsion-free* if every zero-divisor on M is a zero-divisor on R (see for example [9, Page 44]).

Theorem 2.8. *Let M be a module over a Noetherian ring R. Then*

(1) *If M is torsion-free and R is PF, then M is PF,*
(2) *If M is finitely generated and PF, then R is PF.*

Proof. (1) It follows from the definition that $Z(M) \subseteq Z(R)$. Suppose on the contrary that M is not PF. By [1, Exercise 25], there exist a positive integer n and a nonzero $x \in M$ such that $g(R^{n+1}) \subseteq \text{Ann}(x)$, where $g = \prod_{0 \leq i \leq j \leq n} (X_j - X_i)$. So $\langle g(R^{n+1}) \rangle \subseteq \text{Ann}(x) \subseteq Z(R)$. By [26, Corollary 9.36] and [11, Theorem 6.5(1)], $Z(R)$ is the finite union of the associated primes in $\text{Ass}(R)$. Hence by the Prime Avoidance Theorem [26, Theorem 3.61], we have $\langle g(R^{n+1}) \rangle \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \text{Ass}(R)$. Therefore there exists a nonzero element $r \in R$ such that $g(R^{n+1}) \subseteq \text{Ann}(r)$. It follows from [1, Exercise 25] that R is not PF, which is a contradiction.

(2) Without loss of generality, we may assume $\text{Ann}(M) = 0$ (i.e., M is faithful). First we show that $Z(R) \subseteq Z(M)$. Let a be a nonzero element of

$Z(R)$. So there exists a nonzero element $b \in R$ such that $ab = 0$. Since b is nonzero, $bM \neq 0$ and hence there exits a nonzero $x \in M$ such that $bx \neq 0$. We have $a \in \mathrm{Ann}(bx) \subseteq Z(M)$. Hence $Z(R) \subseteq Z(M)$. Now suppose on the contrary that R is not PF. By [1, Exercise 25], there exist a positive integer n and a nonzero $r \in M$ such that $g(R^{n+1}) \subseteq \mathrm{Ann}(r)$. Again, by [26, Corollary 9.36] and [11, Theorem 6.5(1)], $Z(M)$ is the finite union of associated primes in $\mathrm{Ass}(R)$. Hence by the Prime Avoidance Theorem [26, Theorem 3.61], we have $g(R^{n+1}) \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}(M)$. Therefore there exists a nonzero $x \in M$ such that $g(R^{n+1}) \subseteq \mathrm{Ann}(x)$. It follows from [1, Exercise 25] that M is not PF, which is a contradiction. □

From the above theorem, we have immediate important corollary:

Corollary 2.9. *Let M be a finitely generated torsion-free module over a Noetherian ring R. Then M is PF if and only if R if PF.*

We close this section by the following theorem.

Theorem 2.10. *Let M be a module over a Noetherian ring R. Then the following statements are equivalent*:

(1) $\mathrm{Int}(R, M) = M[X]$,
(2) *$T(M)/M$ is a PF R-module,*
(3) *$R/\mathfrak{p}$ is infinite for every associated prime $\mathfrak{p}$ of $T(M)/M$.*

Proof. (1)⇒(2): Let $\overline{f} = \sum_{i=1}^{n}(\frac{m_i}{s_i} + M)X^i$ be an element of $T(M)/M[X]$ such that $\overline{f}(R) = 0$. If $f := \sum_{i=1}^{n} \frac{m_i}{s_i} X^i$, then $f(R) \subseteq M$ and hence $f \in M[X]$. It follows that $\overline{f} = 0$.

(2)⇒(1): Let $f \in \mathrm{Int}(R, M)$. If $\overline{f}$ is the image of f in $T(M)/M$, then $\overline{f}(R) = 0$ in $T(M)/M$, since $f(R) \subseteq M$. So $\overline{f} = 0$ and hence $f \in M[X]$.

(2)⇒ (3): The assertion follows from [1, Proposition I.4.10]. □

3. Properties of $\mathrm{Int}_M(R)$

We begin this section by the following theorem.

Theorem 3.1. *Let $\{M_i : i \in \Lambda\}$ be an indexed family of R-modules with the same zero-divisors. Then*

$$\mathrm{Int}_{\sum_i M_i}(R) \cong_R \bigcap_i \mathrm{Int}_{M_i}(R).$$

Proof. Obvious. □

Theorem 3.2. *Let M be a finitely generated $\mathbb{Z}$-module. Then*

(1) $\mathrm{Int}_M(\mathbb{Z})$ *is not a Noetherian* $\mathrm{Int}_M(\mathbb{Z}) \cap \mathrm{Int}(\mathbb{Z})$*-module,*
(2) *If M is a faithful $\mathbb{Z}$-module, then* $\mathrm{Int}(\mathbb{Z}, M)$ *is not a Noetherian* $\mathrm{Int}_M(\mathbb{Z}) \cap \mathrm{Int}(\mathbb{Z})$*-module.*

Proof. (1) Since M is a finitely generated $\mathbb{Z}$-module, [11, Theorem 6.5(1)] implies that $\mathrm{Ass}(M)$ is finite. Let $\mathrm{Ass}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. We claim that $Z(M)$ contains at most n prime numbers. Suppose on the contrary that $\{q_1, q_2, \ldots, q_{n+1}\} \subseteq Z(M)$, where the q_i $(1 \leq i \leq n+1)$ are distinct prime numbers. Therefore, there are distinct numbers i, j and $\mathfrak{p}_k \in \mathrm{Ass}(M)$ such that $q_i, q_j \in \mathfrak{p}_k$. Hence $1 \in \mathfrak{p}_k$, which is a contradiction. Now let $\{p_1, p_2, \ldots, p_n, \ldots\} \subseteq \mathbb{Z} \setminus Z(M)$ be an infinite set of prime numbers. For each $i \geq 1$, let $f_i(X) = \frac{X^{p_i} - X}{p_i}$. For each $n \geq 1$, let I_n be the ideal of $\mathrm{Int}_M(\mathbb{Z}) \cap \mathrm{Int}(\mathbb{Z})$ generated by $\{f_1, \ldots, f_n\}$ (note that $f_i \in \mathrm{Int}_M(\mathbb{Z})$ because $p_i \in \mathbb{Z} \setminus Z(M)$ and $f_i \in \mathrm{Int}(\mathbb{Z})$). We claim that $f_{n+1} \notin I_n$ for each $n \geq 1$. Suppose on the contrary that $f_{n+1} \in I_n$ for some n. Then, there exist $g_1, \ldots, g_n \in \mathrm{Int}_M(\mathbb{Z}) \cap \mathrm{Int}(\mathbb{Z})$ such that

$$f_{n+1} = g_1 f_1 + \cdots + g_n f_n.$$

By equating the coefficients of $-X$, we have

$$\frac{1}{p_{n+1}} = \frac{g_1(0)}{p_1} + \cdots + \frac{g_n(0)}{p_n},$$

which is a contradiction since $g_i(0) \in \mathbb{Z}$ and $p_1, \ldots, p_{n+1}$ are all distinct prime numbers.

(2) As in Part (1), let $\{p_1, p_2, \ldots, p_n, \ldots\} \subseteq \mathbb{Z} \setminus Z(M)$ be an infinite set of prime numbers. Let $M = \mathbb{Z}m_1 + \cdots + \mathbb{Z}m_k$ and let $f_{ij}(X) = \frac{X^{p_i} - X}{p_i} m_j \in \mathrm{Int}(\mathbb{Z}, M)$, where $1 \leq j \leq k$. For each $n \geq 1$, let I_{nj} be the $\mathrm{Int}_M(\mathbb{Z}) \cap \mathrm{Int}(\mathbb{Z})$-submodule of $\mathrm{Int}(\mathbb{Z}, M)$ generated by $\{f_{1j}, \ldots, f_{nj}\}$. We claim that $f_{(n+1)j} \notin I_{nj}$ for each $n \geq 1$. Suppose on the contrary that $f_{(n+1)j} \in I_{nj}$ for some n. Then there exist $g_{1j}, \ldots, g_{nj} \in \mathrm{Int}_M(\mathbb{Z}) \cap \mathrm{Int}(\mathbb{Z})$ such that $f_{(n+1)j} = g_{1j} f_{1j} + \cdots + g_{nj} f_{nj}$. By equating the coefficients of $-X$, we have $\frac{1}{p_{n+1}} m_j = \frac{g_{1j}(0)}{p_1} m_j + \cdots + \frac{g_{nj}(0)}{p_n} m_j$. So $\left(\frac{1}{p_{n+1}} - \left(\frac{g_{1j}(0)}{p_1} + \cdots + \frac{g_{nj}(0)}{p_n}\right)\right) m_j = 0$. Since M is a faithful R-module, we have

$$\prod_{j=1}^{k} \left(\frac{1}{p_{n+1}} - \left(\frac{g_{1j}(0)}{p_1} + \cdots + \frac{g_{nj}(0)}{p_n}\right)\right) = 0.$$

So there is $1 \leq t \leq k$ such that $\left(\frac{1}{p_{n+1}} - \left(\frac{g_{1t}(0)}{p_1} + \cdots + \frac{g_{nt}(0)}{p_n}\right)\right) = 0$, which is a contradiction by the fact that $g_{it}(0) \in \mathbb{Z}$. □

Notation 2. Let M be an R-module and $a \in R$ and N be a submodule of M. We set:

$$\mathcal{I}_{N,a} = \{f \in \mathrm{Int}_M(R) \mid f(a)M \subseteq N\}.$$

Theorem 3.3. *Let M be an R-module and $a \in R$. Then*

(1) *If N is a submodule of M, then $\mathcal{I}_{N,a}$ is an ideal of $\mathrm{Int}_M(R)$,*
(2) *If P is a prime submodule of M such that $(P : M) \subseteq Z(M)$, then $\mathcal{I}_{P,a}$ is a prime ideal of $\mathrm{Int}_M(R)$.*

Proof. (1) Obvious.

(2) Let P be a prime submodule of M. It is easy to see that $\mathcal{I}_{P,a}$ is a proper ideal of $\text{Int}_M(R)$. Now let $f, g \in \text{Int}_M(R)$ such that $fg \in \mathcal{I}_{P,a}$. Suppose that $g(a)M \nsubseteq P$. So there exists $m_0 \in g(a)M \setminus P$ (note that $m_0 \in M$). Let $U = R \setminus Z(M)$ and let $f(a) = \frac{b}{s}$ for some $b \in R$ and $s \in U$. Since $\frac{b}{s}m_0 \in P$, we have $bm_0 \in P$ and hence $bM \subseteq P$. Now let m be an arbitrary element of M. Since $\frac{b}{s}M \subseteq M$, there exists $m' \in M$ such that $\frac{b}{s}m = \frac{m'}{1}$. It follows that $sm' = bm \in P$. Since $s \notin (P : M)$, we must have $m' \in P$ and hence $\frac{b}{s}M \subseteq P$. It follows that $f \in \mathcal{I}_{P,a}$. This completes the proof. □

Let M be an R-module. A proper submodule P of M is called a *strongly prime submodule* if $(P + Rx : M)y \subseteq P$ for $x, y \in M$, implies that either $x \in P$ or $y \in P$. The collection of all strongly prime submodules of M is called the strongly spectrum of M and is denoted by $\text{S.Spec}(M)$. The *strong dimension* of M ($\text{s.dim}_R(M)$) in terms of ascending chains of strongly prime submodules as follow:

$$s.\dim_R(M) = \sup\{n \mid \exists P_0, P_1, \ldots, P_n \in \text{S.Spec}(M) \text{ such that } P_0 \subset P_1 \subset \cdots \subset P_n\}.$$

For more information about strongly prime submodules, we refer the reader to [13] and [23].

Theorem 3.4. *Let M be a nonzero R-module. Then*

(1) $\dim \text{Int}_M(R) \geq \text{s.dim}_{U^{-1}R} U^{-1}M$,

(2) *If M is a prime R-module, then* $\dim \text{Int}_M(R) \geq \text{s.dim}_{U^{-1}R} U^{-1}M + 1$.

Proof. (1) Let

$$Q_0 \subset Q_1 \subset \cdots \subset Q_n$$

be a chain of strongly prime submodules of $U^{-1}M$. Then by [14, Proposition 2.5], there exist strongly prime submodules $P_0, \ldots, P_n$ of M such that $U^{-1}P_i = Q_i$ and $(P_i : M) \subseteq Z(M)$ for all $i = 0, 1, \ldots, n$. By Theorem 3.3(2), $\mathcal{I}_{P_i,a}$ is a prime ideal for all $i = 0, \ldots, n$. Let $1 \leq i \leq n$. It is easy to see that $P_{i-1} \subset P_i$ and so [14, Lemma 3.1] implies that, $(P_{i-1} : M) \subset (P_i : M)$. Let $a_i \in (P_i : M) \setminus (P_{i-1} : M)$, then the constant polynomial $f(X) = a_i$ is in $\mathcal{I}_{P_i,a} \setminus \mathcal{I}_{P_{i-1},a}$. Therefore $\mathcal{I}_{P_0,a} \subset \mathcal{I}_{P_1,a} \subset \cdots \subset \mathcal{I}_{P_n,a}$ is a chain of prime ideals of $\text{Int}_M(R)$. So $\dim \text{Int}_M(R) \geq \text{s.dim}_{U^{-1}R} U^{-1}M$.

(2) Without loss of generality, we may assume that M is a faithful R-module. Let M be a prime R-module and let $(0) = P_0 \subset P_1 \subset \cdots \subset P_n$ be a chain of prime submodules of M. Let $a \in R$. Then $X - a$ is a nonzero element of $\mathcal{I}_{P_0,a}$. Now we show that $\text{Int}_M(R)$ is an integral domain. Suppose on the contrary that $\text{Int}_M(R)$ is not an integral domain. So there are nonzero polynomials $f(X) = \sum_{i=0}^{p} \frac{a_i}{s_i} X^i$ and $g(X) = \sum_{i=0}^{q} \frac{b_i}{t_i} X^i$ in $\text{Int}_M(R)$ such that $fg = 0$. We may assume $\frac{a_p}{s_p} \neq 0$ and $\frac{b_q}{t_q} \neq 0$. Since $\frac{a_p}{s_p}\frac{b_q}{t_q} = 0$, there exists $u \in R \setminus Z(M)$ such that $ua_pb_q = 0$. Since M is a nonzero faithful module, we have $a_p = 0$

or $b_q = 0$, which is a contradiction. So $\mathrm{Int}_M(R)$ is an integral domain and the assertion follows from Part (1). □

The *classical Krull dimension* of an R-module M is defined as the Krull dimension of the ring $R/\mathrm{Ann}(M)$ and denoted by $\mathrm{cl.K.dim}_R(M)$ (see [17]). We close this paper by the following corollary.

Corollary 3.5. *Let M be a nonzero finitely generated R-module. Then*

(1) $\dim \mathrm{Int}_M(R) \geq \dim \frac{U^{-1}R}{\mathrm{Ann}_{U^{-1}R}(U^{-1}M)}$,

(2) *If M is a prime R-module, then* $\dim \mathrm{Int}_M(R) \geq \dim \frac{U^{-1}R}{\mathrm{Ann}_{U^{-1}R}(U^{-1}M)} + 1$.

Proof. (1) Since M is a finitely generated R-module, $U^{-1}M$ is finitely generated over $U^{-1}R$. So by [14, Theorem 3.3(2)] and Theorem 3.5(1), we have

$$\begin{aligned}\dim \mathrm{Int}_M(R) \geq \mathrm{s.dim}_{U^{-1}R}\, U^{-1}M &= \mathrm{cl.K.dim}_{U^{-1}R}(U^{-1}M)\\ &= \dim \frac{U^{-1}R}{\mathrm{Ann}_{U^{-1}R}(U^{-1}M)}.\end{aligned}$$

(2) Since $U^{-1}M$ is finitely generated over $U^{-1}R$, by [14, Theorem 3.3(2)] and Theorem 3.5(2), we have

$$\begin{aligned}\dim \mathrm{Int}_M(R) &\geq \mathrm{s.dim}_{U^{-1}R}\, U^{-1}M + 1\\ &= \mathrm{cl.K.dim}_{U^{-1}R}(U^{-1}M) + 1\\ &= \dim \frac{U^{-1}R}{\mathrm{Ann}_{U^{-1}R}(U^{-1}M)} + 1.\end{aligned}$$ □

We close this paper by the following theorem which is similar to Theorem 2.5. We may write a for the image of $a \in R$ and identify R with its image in $U^{-1}R$ under the canonical map $R \longrightarrow U^{-1}R$ (see for example [1, Page 9]).

Theorem 3.6. *Let R be integrally closed in $U^{-1}R$ and M be a finitely generated R-module. Let $U^{-1}M$ be a faithful $U^{-1}R$-module. If $\mathrm{Int}_M(R) \neq R[X]$, then there exists $\mathfrak{p} \in \widetilde{\mathrm{Ass}}_R((U^{-1}R)/R)$ such that $R/\mathfrak{p}$ is finite. The converse is true when $\mathfrak{p}$ is finitely generated.*

Proof. Suppose that every element of $\widetilde{\mathrm{Ass}}_R((U^{-1}R)/R)$ has an infinite residue field. We claim that $\mathrm{Int}_M(R) \subseteq R_{\mathfrak{p}}[X]$ for all $\mathfrak{p} \in \widetilde{\mathrm{Ass}}_R((U^{-1}R)/R)$. Let $f \in \mathrm{Int}_M(R)$, $r \in R$ and $t := f(r)$. Since $tM \subseteq M$, an argument similar to that of [26, Proposition 13.15] (determinant trick) shows that

$$t^n + a_1 t^{n-1} + \cdots + a_{n-1}t + a_n \in (0 :_{U^{-1}R} M),$$

where $a_i \in R$ for $i = 1, 2, \ldots, n$. It is easy to see that $(0 :_{U^{-1}R} M) = (0 :_{U^{-1}R} U^{-1}M)$. Since $U^{-1}M$ be a faithful $U^{-1}R$-module and R is integrally closed in $U^{-1}R$, we must have $t \in R$. Now assume that $\mathfrak{p} \in \widetilde{\mathrm{Ass}}_R((U^{-1}R)/R)$ and let $d = \prod_{0 \leq i < j \leq n}(a_j - a_i)$, where $a_0, \ldots, a_n$ be $n+1$ elements in distinct classes modulo $\mathfrak{p}$; in particular, $d \notin \mathfrak{p}$. In view of [1, Proposition I.3.18], we have

$df \in R[X]$ and hence $\mathrm{Int}_M(R) \subseteq R_{\mathfrak{p}}[X]$. Now let $\mathrm{Int}_M(R) \neq R[X]$. Then there is a polynomial $f \in \mathrm{Int}_M(R)$ with some coefficient $x \in (U^{-1}R) \setminus R$. So there is a $\mathfrak{p} \in \widetilde{\mathrm{Ass}}_R((U^{-1}R)/R)$ such that $\mathrm{Ann}(\overline{x}) \subseteq \mathfrak{p}$, where $\overline{x}$ is the residue of x in the quotient module $(U^{-1}R)/R$. Since $\mathrm{Int}_M(R) \subseteq R_{\mathfrak{p}}[X]$, there is $s \in R \setminus \mathfrak{p}$ such that $sx \in R$. It follows that $s\overline{x} = \overline{0}$ in $(U^{-1}R)/R$ and so $s \in \mathrm{Ann}(\overline{x}) \subseteq \mathfrak{p}$, which is a contradiction.

Conversely, suppose that $\mathfrak{p}$ is finitely generated. By [28, Lemma 1.8], $\mathfrak{p} \in \mathrm{Ass}_R((U^{-1}R)/R)$. Therefore there exists a nonzero element $x \in (U^{-1}R)/R$ such that $\mathfrak{p} = \mathrm{Ann}(x)$. Let $\{a_0, a_1, \ldots, a_n\}$ be a set of representatives modulo $\mathfrak{p}$. Then the polynomial $f = x\prod_{0\leq i\leq n}(X - a_i)$ is a polynomial in $\mathrm{Int}_M(R)$ and its leading coefficient is not in R. □

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References

[1] P.-J. Cahen and J.-L. Chabert, *Integer-valued polynomials*, Mathematical Surveys and Monographs, **48**, American Mathematical Society, Providence, RI, 1997.

[2] J. Dauns, *Prime modules*, J. Reine Angew. Math. **298** (1978), 156–181. https://doi.org/10.1515/crll.1978.298.156

[3] J. Elliott, *Integer-valued polynomial rings, t-closure, and associated primes*, Comm. Algebra **39** (2011), no. 11, 4128–4147. https://doi.org/10.1080/00927872.2010.519366

[4] ———, *Presentations and module bases of integer-valued polynomial rings*, J. Algebra Appl. **12** (2013), no. 1, 1250137, 25 pp. https://doi.org/10.1142/S021949881250137X

[5] ———, *Integer-valued polynomials on commutative rings and modules*, Comm. Algebra **46** (2018), no. 3, 1121–1137. https://doi.org/10.1080/00927872.2017.1388811

[6] S. Frisch, *Integer-valued polynomials on algebras: a survey*, Actes du CIRM **2** (2010), no. 2, 27–32.

[7] ———, *Integer-valued polynomials on algebras*, J. Algebra **373** (2013), 414–425. https://doi.org/10.1016/j.jalgebra.2012.10.003

[8] ———, *Polynomial functions on upper triangular matrix algebras*, Monatsh. Math. **184** (2017), no. 2, 201–215. https://doi.org/10.1007/s00605-016-1013-y

[9] T. Y. Lam, *Lectures on Modules and Rings*, Springer-Verlag, Berlin-Heidelberg-New York, 1998.

[10] S. H. Man and P. F. Smith, *On chains of prime submodules*, Israel J. Math. **127** (2002), 131–155. https://doi.org/10.1007/BF02784529

[11] H. Matsumura, *Commutative Ring Theory*, translated from the Japanese by M. Reid, Cambridge Studies in Advanced Mathematics, **8**, Cambridge University Press, Cambridge, 1986.

[12] R. L. McCasland and P. F. Smith, *Prime submodules of Noetherian modules*, Rocky Mountain J. Math. **23** (1993), no. 3, 1041–1062. https://doi.org/10.1216/rmjm/1181072540

[13] A. R. Naghipour, *Strongly prime submodules*, Comm. Algebra **37** (2009), no. 7, 2193–2199. https://doi.org/10.1080/00927870802467239

[14] ———, *Some results on strongly prime submodules*, J. Alg. Sys. **1** (2013), no. 2, 79–89.

[15] A. R. Naghipour, M. R. Rismanchian, and J. Sedighi Hafshejani, *Some results on the integer-valued polynomials over matrix rings*, Comm. Algebra **45** (2017), no. 4, 1675–1686. https://doi.org/10.1080/00927872.2016.1222407

[16] W. Narkiewicz, *Polynomial Mappings*, Lecture Notes in Mathematics, **1600**, Springer-Verlag, Berlin, 1995. https://doi.org/10.1007/BFb0076894

[17] D. G. Northcott, *Lessons on Rings, Modules and Multiplicities*, Cambridge University Press, London, 1968.

[18] A. Ostrowski, *Über ganzwertige Polynome in algebraischen Zahlkörpern*, J. Reine Angew. Math. **149** (1919), 117–124.

[19] G. Peruginelli and N. J. Werner, *Integral closure of rings of integer-valued polynomials on algebras*, in Commutative algebra, 293–305, Springer, New York, 2014.

[20] ———, *Non-triviality conditions for integer-valued polynomial rings on algebras*, Monatsh. Math. **183** (2017), no. 1, 177–189. https://doi.org/10.1007/s00605-016-0951-8

[21] G. Pólya, *Über ganzwertige Polynome in algebraischen Zahlkörpern*, J. Reine Angew. Math. **149** (1919), 97–116.

[22] D. E. Rush, *The conditions* $\mathrm{Int}(R) \subseteq R_S[X]$ *and* $\mathrm{Int}(R_S) = \mathrm{Int}(R)_S$ *for integer-valued polynomials*, J. Pure Appl. Algebra **125** (1998), no. 1-3, 287–303. https://doi.org/10.1016/S0022-4049(96)00107-7

[23] ———, *Strongly prime submodules, G-submodules and Jacobson modules*, Comm. Algebra **40** (2012), no. 4, 1363–1368. https://doi.org/10.1080/00927872.2010.551530

[24] K. Samei, *Reduced multiplication modules*, Proc. Indian Acad. Sci. Math. Sci. **121** (2011), no. 2, 121–132. https://doi.org/10.1007/s12044-011-0014-y

[25] J. Sedighi Hafshejani, A. R. Naghipour, and M. R. Rismanchian, *Integer-valued polynomials over block matrix algebras*, J. Algebra Appl. **19** (2020), no. 3, 2050053, 17 pp. https://doi.org/10.1142/S021949882050053X

[26] R. Y. Sharp, *Steps in Commutative Algebra*, second edition, London Mathematical Society Student Texts, **51**, Cambridge University Press, Cambridge, 2000.

[27] N. J. Werner, *Integer-valued polynomials on algebras*: *a survey of recent results and open questions*, in Rings, polynomials, and modules, 353–375, Springer, Cham, 2017.

[28] S. Yassemi, *Weakly associated primes under change of rings*, Comm. Algebra **26** (1998), no. 6, 2007–2018. https://doi.org/10.1080/00927879808826256

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