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Global solution to the compressible non-isothermal nematic liquid crystal equations with constant heat conductivity and vacuum

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Abstract

This paper considers the initial-boundary value problem of the one-dimensional full compressible nematic liquid crystal flow problem. The initial density is allowed to touch vacuum, and the viscous and heat conductivity coefficients are kept to be positive constants. Global existence of strong solutions is established for any H^2 initial data in the Lagrangian flow map coordinate, which holds for both vacuum and non-vacuum case. The key difficulty is caused by the lack of the positive lower bound of the density. To overcome such difficulty, it is observed that the ratio of $\frac{\rho_{0(y)}}{\rho(t,y)}$ is proportional to the time integral of the upper bound of temperature and vector director field, along the trajectory. Density weighted Sobolev type inequalities are constructed for both temperature and director field in terms of $\frac{\rho_{0(y)}}{\rho(t,y)}$ and small dependence on their dissipation estimates. Besides this, to deal with cross terms arising due to liquid crystal flow, higher order priori estimates are established by using effective viscous flux.

Keywords: Liquid crystal equations; Global solutions; Constant heat conductivity; Vacuum

1 Introduction

Liquid crystal material can be known as an intermediate phase between solid and liquid. It flows like liquid and exhibits additional microscopic structural properties. For example, rigid polymers solution, DNA, and many other materials show phase change for different ranges of temperature. On the base of structural properties, liquid crystals are categorized as nematic, sematic, and cholesteric phases. Nematic liquid crystal are made up of rod-like molecules, and their structure induces preferred average directional order. The historic example of nematic phase is N-p-*methoxybenzlidene*-p-*butylaniline* (MBBA) which appears in between 20°C and 47°C temperatures. The kinematic behavior of nematic liquid crystals due to the strong coupling between the microscopic and macroscopic interaction field is very complicated and interesting. The hydrodynamic theory for nematic liquid crystal is derived by extending the static theory in generalizations of body and surface forces as in the initial derivation of macroscopic hydrodynamics. The system that describes the hydrodynamics of nematic liquid crystal was proposed during the period between 1958 and 1968 by Ericksen [1, 2] and Leslie [3] and reduces to the Oseen–Frank theory [4] in the static case. It is a macroscopic continuum description of materials evolution under the effects of both the flow velocity field $u(x,t)$ and the average molecular orientation $d(x,t)$ (the microscopic configuration) of rod-like liquid molecules and their mutual interactions, whereas $d(x,t)$ is a unit vector. Recently, there has been modeling study of a generalized non-isothermal Ericksen–Leslie system by Hieber and Prüss [5] (incompressible non-isothermal case), [6] (compressible non-isothermal case) with consistency on the laws of thermodynamic, and De Anna–Liu [7] derived the generalized compressible non-isothermal Ericksen–Leslie system with full Oseen–Frank energy density. We also refer to [8] where the author derived more sophisticated thermodynamically consistent models of nematic liquid crystal flows both in tensorial and vectorial forms by using the *conservation dissipation formalism*.

Suppose that $\rho(x,t)$, $u(x,t)$, $d(x,t)$, and $\theta(x,t)$ are mass density, fluid velocity vector field, director vector field (represents preferred average directional order), and temperature, respectively. Analogous to one constant approximation model [9], the couple system of thermally driven fluid flow and director field satisfies the following conservation laws:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho,\theta) = \operatorname{div}\mathbb{S} - \operatorname{div}(\nabla d \odot \nabla d - \frac{1}{2}|\nabla d|^2\mathbb{I}_3), \\ \partial_t d + u\cdot\nabla d = \Delta d + |\nabla d|^2 d, \qquad |d|^2 = 1, \\ \partial_t(\rho\theta) + \operatorname{div}(\rho u\theta) + p(\rho,\theta)\operatorname{div} u + \operatorname{div} q = \mathbb{S} : \nabla u + |\Delta d + |\nabla d|^2 d|^2, \end{cases} \tag{1.1}$$

whereas the Cauchy stress tensor $\mathbb{S}$ is defined as

$$\mathbb{S} = \mu'\left(\nabla u + \nabla u^{\mathsf{T}}\right) + \lambda'(\operatorname{div} u)\mathbb{I},$$

the viscosity coefficients μ' and λ' satisfy

$$\mu' > 0, \quad 2\mu' + 3\lambda' \geq 0,$$

and ∇u^{T} is the transpose of the matrix ∇u, $\mathbb{I}$ is a 3×3 identity matrix. The notation $\nabla d \odot \nabla d$ is a 3×3 matrix with (i,j)th entries, given by $\partial_i d\cdot\partial_j d$ $(1\leq i,j\leq 3)$. The equation of state for ideal polytropic gas is given by

$$p(\rho,\theta) = R\rho\theta,$$

$R > 0$ is the gas constant. The internal energy flux is defined by the Fourier law as follows:

$$q = \kappa(\theta)\nabla\theta\kappa > 0.$$

Total energy $\mathcal{E}_T$ is the sum of internal energy and kinetic energy, given as

$$\mathcal{E}_T = \left(\rho\left(e + \frac{1}{2}|u|^2\right) + \frac{1}{2}|\nabla d|^2\right), \tag{1.2}$$

whereas $e = c_v\theta$.

When the director field is a constant vector field, system (1.1) reduces to compressible non-isentropic Navier–Stokes equations. First, let us recall some literature works on the well-posedness theory for the compressible Navier–Stokes equations in a three-dimensional space. When the initial density is away from vacuum, Nash [10] and Itaya [11] established the local existence and uniqueness of classical solutions in 1962 and 1977, respectively. In 1980, Matsumura and Nishida [12] first obtained the global classical solutions for the initial data close to a non-vacuum equilibrium in a Sobolev space. When vacuum is allowed, in 2006, Cho and Kim [13] proved the existence of unique local strong solutions in bounded and unbounded domains. In 2018, Huang and Li [14] proved the global existence of classical solutions in the whole space with smooth initial data which are of small energy but possibly large oscillations. For more results on the existence of solutions, see [15–24] and the references therein.

For liquid crystal in higher dimension, in 2012, Huang et al. [9] constructed the local existence of strong solution with sufficient regular initial data. In the same year, Hu and Wu [25] proved the existence and uniqueness of the global strong solution in critical Besov spaces provided that the initial datum is close to constant equilibrium state $(1, 0, \bar{d})$. The spherically symmetric solutions to compressible hydrodynamic flow of liquid crystals can be found in reference [26]. In 2015, Yang [27] obtained global in time strong solution and justified the low Mach number limit in a bounded domain. For the results of compressible liquid crystal flows model with Ginzburg–Landau energy, existence, and large-time behavior of global weak or strong solution, we refer to [28–30] and the references therein. For more details about the development in the analysis of liquid crystal, an interested reader can see a very systematic review by Lin and Wang [31]. There are fewer results about non-isothermal nematic liquid crystal equations compared to those on compressible isentropic nematic liquid crystal. In 2017, Guo et al. [32] established the existence of global-in-time smooth solutions in a three-dimensional space provided that the initial datum is close to a steady state. In 2018, the local well-posedness of nematic liquid crystal equations (1.1) was studied by Fan et al. [33] in a bounded domain $\Omega \subset \mathbb{R}^3$. In 2019, Francesco and Liu [7] derived a general Ericksen–Leslie system under the action of thermal effects and proved the global-in-time well-posedness of the system for small initial data in the framework of Besov spaces. Recently, Zhong [34] studied the singularity formation of strong solutions to the two-dimensional nematic liquid crystal flows in a bounded domain and proved that the strong solution exists globally if the temporal integral of the maximum norm of the divergence of the velocity is bounded.

All the global well-posedness results obtained for compressible liquid crystal flow in higher dimensional space variable are restricted to smallness assumptions on initial data, energy, and so on, the global existence with smooth initial data in dimension $n = 3$ is an open problem analogous to Navier–Stokes equations. Although the well-posedness theory for one-dimensional Navier–Stokes is well known, the physical phenomenon of liquid crystal dynamics creates serious difficulties in rigorous analysis. When the director field is a constant vector, the system is reduced to one-dimensional Navier–Stokes equations. For such a particular system of equations, in 1977, Kazhikhov and Shelukhin [35] established the global existence and uniqueness of strong solutions with positive density and constant viscosity coefficient, for the corresponding Cauchy problem, see [36]. Later, asymptotic behavior of the solution was studied by Okada [37] in 1987. Zlotnik and Amosov [38, 39] constructed the global weak solutions, and later, Chen, Hoff, and Trivisa [40] the initial and boundary conditions. In 1998, Jiang [41] studied the equations with density-dependent viscosity and proved that if the viscosity does not decrease to zero too rapidly, then smooth solutions exist globally in time. For more related works with density-dependent viscosity, see [42–46]. In 2000, Luo and Xin [47] obtained the regularity and solutions behavior near gas and vacuum interfaces. Large time behavior of solutions with large initial data was recently proved in an unbounded domain by Li and Liang [48]. In 2015, Pan and Zhang [49] proved the existence of global strong solutions when heat conductivity depends on temperature by the following Chapman–Enskog power law:

$$q = -\kappa(\theta)\nabla\theta, \quad \kappa(\theta) = C\theta^{\beta}, \beta > 0. \tag{1.3}$$

The global well-posedness of solution with $\beta = 0$ and vacuum was obtained by Li [50] in 2019, this result holds both for vacuum and non-vacuum case and has improved the well-known result of Kazhikhov [35](non-vacuum case). For the Cauchy problem, with constant coefficients and far field vacuum, Li and Xin [51] obtained the global existence for entropy bounded solution to non-isentropic Navier–Stokes equations with zero heat conduction.

The study of hydrodynamics of liquid crystal has attracted the attention of many researchers for the past few years. For isentropic compressible liquid crystal flow one-dimensional space, Ding et al. [52], in 2011, gave the existence of weak solution (ρ, u, d) with $0 \le \rho_0 \in L^{\gamma}[0,1]$ for $\gamma > 1$, $u_0 \in L^2[0,1]$, and $d_0 \in H^1[0,1]$. Later, Ding, Wang, and Wen [53] obtained the existence and uniqueness of global classical solution for Hölder continuous initial data and initial density away from vacuum. In 2015, Huang and Ding [54] studied a free boundary problem for the compressible liquid crystal flow model which connects to vacuum continuously. For non-isothermal compressible liquid crystal flow, in 2019, Tang and Sun [55] proved the global existence of strong solutions allowing vacuum, provided that the initial data satisfy some compatibility condition and the heat conductivity satisfies

$$C^{-1}\left(1+\theta^{\beta}\right) \le \kappa(\theta) \le C\left(1+\theta^{\beta}\right), \quad \beta > 0, \tag{1.4}$$

which plays an essential role in obtaining the regularity of the temperature. The result [54] was extended by Mei [56] in 2020, where the author proved global classical solution to the free boundary value problem in the presence of temperature equation with heat conductivity of type (1.4). By using the stronger assumption on heat conductivity (1.3), recently, in 2021, Li, Mahmood, and Shang [57] obtained the global strong solution with $\rho_0 \ge C$.

To our best knowledge, global strong solution to one-dimensional non-isothermal compressible nematic liquid crystal equations for arbitrary large initial data is not known for constant coefficients and vacuum. The global solution obtained for system (1.5) in Mei [56], Li et al., [57] holds for $\beta > 0$ with $\rho_0 \ge C$. Similarly, the result obtained in [55] with vacuum in Euler coordinates is restricted to the condition on heat conductivity of type (1.4). Motivated by Kazhikhov [35] and Li [50], in this paper we aim to study the global well-posedness of strong solutions to the one-dimensional non-isothermal compressible nematic liquid crystal flow equations, i.e., system (1.5), with constant viscosity and heat conductivity in the presence of vacuum.

1.1 Coordinates transformation and main results

Let $\rho(x,t)$, $u(x,t)$, $d(x,t)$, and $\theta(x,t)$ denote the unknown functions of density, velocity, director vector field($d(x,t) = (d^1, d^2, d^2)(x,t)$), and temperature, then the liquid crystal hydrodynamics for all $x \in (0,l)$ and time $t > 0$ is governed by the following set of partial differential equations in Eulerian coordinates:

$$\begin{cases} \partial_t \rho + (\rho u)_x = 0, \\ \partial_t(\rho u) + ((\rho u^2) + p(\rho,\theta))_x - \mu u_{xx} = -\frac{1}{2}(|d_x|^2)_x, \\ \partial_t d + u d_x = d_{xx} + |d_x|^2 d, \qquad |d|^2 = 1, \\ \partial_t(\rho\theta) + (\rho u\theta)_x + p u_x - \kappa(\theta_x)_x = \mu(u_x)^2 + |d_{xx} + |\partial_x d|^2 d|^2, \end{cases} \tag{1.5}$$

where μ and κ are viscosity and heat conductivity coefficients and $\mu = 2\mu' + \lambda'$.

The results of this paper will be proven in the Lagrangian flow map coordinate being stated in what follows. Let us define the coordinate transformation between the Lagrangian coordinate y and Euler coordinates x as

$$x = \xi(y,t),$$

where $\xi(y,t)$ denotes the flow map governed by u, that is,

$$\begin{cases} \xi_t(y,t) = u(\xi(y,t),t), \\ \xi(y,0) = y. \end{cases}$$

Let the new variables $\bar{\rho}$, $\bar{u}$, $\bar{d}$ $\bar{p}$, $\bar{\theta}$ denote the density, velocity, director field, pressure, and temperature, respectively, in the Lagrangian coordinate, that is,

$$\begin{aligned} &\rho(y,t) := \bar{\rho}\big(\xi(y,t),t\big), \qquad u(y,t) := \bar{u}\big(\xi(y,t),t\big), \qquad p(y,t) := \bar{p}\big(\xi(y,t),t\big), \\ &d(y,t) := \bar{d}\big(\xi(y,t),t\big), \qquad \theta(y,t) := \bar{\theta}\big(\xi(y,t),t\big). \end{aligned}$$

By the definition of $\xi(y,t)$, it is not hard to see that

$$\begin{aligned} &(\partial_x u, \partial_x d, \partial_x\theta, \partial_x p) = \left(\frac{\partial_y \bar{u}}{\partial_y \xi}, \frac{\partial_y \bar{d}}{\partial_y \xi}, \frac{\partial_y \bar{\theta}}{\partial_y \xi}, \frac{\partial_y p}{\partial_y \xi}\right), \\ &\big(\partial_x^2 u, \partial_x^2 d, \partial_x^2 \theta\big) = \left(\frac{1}{\partial_y \xi}\partial_y\left(\frac{\partial_y \bar{u}}{\partial_y \xi}\right), \frac{1}{\partial_y \xi}\partial_y\left(\frac{\partial_y \bar{d}}{\partial_y \xi}\right), \frac{1}{\partial_y \xi}\partial_y\left(\frac{\partial_y \bar{\theta}}{\partial_y \xi}\right)\right) \\ &\partial_t\rho + u\partial_x\rho = \partial_t\bar{\rho}, \qquad \partial_t u + u\partial_x u = \partial_t \bar{u}, \qquad \partial_t d + u\partial_x d = \partial_t \bar{d}, \qquad \partial_t\theta + u\partial_x\theta = \partial_t\bar{\theta}, \end{aligned}$$

and introduce a function $g(y,t) = \xi_y(y,t)$. Then it follows

$$g_t = u_y. \tag{1.6}$$

For the sake of simplicity, new variables are denoted by the same physical variables as before, then system (1.5) can be rewritten in the Lagrangian coordinate as follows:

$$\begin{cases} \rho_t + \rho \frac{u_y}{g} = 0, \\ \rho u_t + \frac{1}{g}(p + \frac{1}{2g^2}|d_y|^2)_y = \frac{\mu}{g}(\frac{u_y}{g})_y, \\ d_t = \frac{1}{g}(\frac{d_y}{g})_y + \frac{|d_y|^2}{g^2}d, \qquad |d|^2 = 1, \\ c_v\rho\theta_t + p\frac{u_y}{g} = \frac{\kappa}{g}(\frac{\theta_y}{g})_y + \mu\frac{u_y^2}{g^2} + |\frac{1}{g}(\frac{d_y}{g})_y + \frac{1}{g^2}|d_y|^2 d|^2. \end{cases} \tag{1.7}$$

Due to (1.6) and $(1.7)_1$, it is straightforward that

$$(g\rho)_t = g_t\rho + g\rho_t = v_y\rho - g\frac{v_y}{g}\rho = 0,$$

from which, by setting $\rho|_{t=0} = \rho_0$ and noticing that $g|_{t=0} = 1$, we have $g\rho = \rho_0$. Therefore, one can replace ρ by $\rho = \frac{\rho_0}{g}$ in the above equations, and a new modified system is written as follows:

$$\begin{cases} g_t - u_y = 0, \\ \rho_0 u_t + (p + \frac{1}{2g^2}|d_y|^2)_y = (\frac{\mu u_y}{g})_y, \\ d_t = \frac{1}{g}(\frac{d_y}{g})_y + \frac{|d_y|^2}{g^2}d, \qquad |d|^2 = 1, \\ \rho_0\theta_t + pu_y = \kappa(\frac{\theta_y}{g})_y + \frac{\mu u_y^2}{g} + g|\frac{1}{g}(\frac{d_y}{g})_y + \frac{1}{g^2}|d_y|^2 d|^2, \end{cases} \tag{1.8}$$

with the initial conditions

$$(g,u,d,\theta)(y,t)|_{t=0} = (1,u_0,d_0,\theta_0)(y), \quad y \in \Omega, \tag{1.9}$$

and the boundary conditions

$$(u,d_y,\theta_y)|_{y=0,l} = (0,0,0). \tag{1.10}$$

For $1 \leq r \leq \infty$ and positive integer k, we use $L^r = L^r((0,l))$ and $W^{k,r} = W^{k,r}((0,l))$ to denote the standard Lebesgue and Sobolev spaces, respectively, and in the case that $r = 2$, we use H^k instead of $W^{k,2}$. H_0^1 consists of all functions $v \in H^1$ satisfying $u(0) = u(l) = 0$. We always use $\|u\|_r$ to denote the L^r norm of u. Throughout this paper, C denotes a general nonnegative constant which may be different from line to line.

The main result of this paper is stated as follows.

Theorem 1.1 *Assume that the initial data* $(\rho_0,u_0,d_{y0},\theta_0)(y) \in H^2 \times H^2 \times H^2 \times H^2$ *satisfy* $(\rho_0,\theta_0)(y) \geq 0$ *for all* $y \in [0,l]$, *with compatibility conditions*

$$\mu u_0'' - R(\rho_0\theta_0)' - \frac{1}{2}\left(|d_0'|^2\right)' = \sqrt{\rho_0}h_1$$

$$\kappa\theta_0'' + \mu\left(u_0'\right)^2 - Ru_0'\rho_0\theta_0 + |d_{0t}|^2 = \sqrt{\rho_0}h_2,$$

for $(h_1, h_2) \in L^2[0, l]$*. Then there is a unique global solution* (g, u, d, θ) *to system* (1.8) *subject to initial-boundary conditions* (1.9–1.10) *satisfying*

$$\begin{aligned}
&g \in C\big([0,T];H^2\big), \quad g_t \in L^2\big(0,T;H^2\big),\\
&u \in C\big(0,T;H^2\big) \cap L^2\big(0,T;H^3\big), \quad u_t \in L^2\big(0,T;H^1\big),\\
&d_y \in C\big(0,T;H^2\big), \quad d_{yt} \in L^2\big(0,T;H^1\big),\\
&0 \le \theta \in C\big([0,T];H^2\big) \cap L^2\big(0,T;H^3\big), \quad \theta_t \in L^2\big(0,T;H^1\big),
\end{aligned}$$

for any $T \in (0,\infty)$*. In particular, it holds that*

$$\inf_{0\le t\le T} \inf_{y\in(0,l)} g(y,t) \ge C$$

and

$$\max_{t\in[0,T]} \left(\|g\|_{H^2}^2 + \left\| \left(u, \frac{d_y}{g}, \theta \right) \right\|_{H^2}^2 \right) + \int_0^T \mathcal{D}(t)\,dt \le C \tag{1.11}$$

for any $T > 0$*, where* $C > 0$ *is a positive constant depending only on* R*,* c_v*,* μ*,* κ*,* m_1*,* $\mathcal{N}_1$*,* $\mathcal{N}_2$*,* $\mathcal{N}_3$*, and* T*, and dissipation* $\mathcal{D}(t)$ *is defined as*

$$\mathcal{D}(t) = \|g_t\|_{H^2}^2 + \big\|(u,d,\theta)\big\|_{H^3}^2 + \big\|(u_t,\theta_t)\big\|_{H^1}^2 + \left\| \frac{d_{yt}}{\sqrt{g}}, \frac{1}{\sqrt{g}} \left(\frac{d_y}{g} \right)_{yt} \right\|_{L^2}^2.$$

Remark 1 The result of this paper can be seen as a generalization of Kazhikhov–Shelukhin [35] (constant coefficients, non-vacuum) and Li [50] (constant coefficients and vacuum) towards liquid crystal dynamics.

Remark 2 The arguments of this paper also work for the free boundary value problem in which the boundary condition for velocity u is replaced by

$$\left(\mu \frac{u_y}{g} - \theta - \frac{1}{2} \frac{|d_y|^2}{g^2} \right) \bigg|_{y=0,l} = 0.$$

Thus our result improves the results of [56, 57] for $\kappa(\theta) = \theta^\beta$ with $\beta = 0$, which does not include the constant case.

Remark 3 The same result as in Theorem 1.1 still holds if we replace the boundary condition $\partial_y\theta(0,t) = \partial_y\theta(l,t) = 0$ by one of the following three:

$$\begin{aligned}
&\theta(0,t) = \theta(l,t) = 0,\\
&\theta(0,t) = \partial_y\theta(l,t) = 0,\\
&\partial_y\theta(0,t) = \theta(l,t) = 0,
\end{aligned}$$

and the proof is exactly the same as the one established in this paper, the only difference is that the basic energy identity in Lemma 1 will be replaced by an inequality.

Remark 4 The method used in this paper can be applied to another model, e.g., magnetohydrodynamics, by slightly modifying the proof of this paper.

Remark 5 The method can easily be adopted for the non-isothermal compressible liquid crystal model in which Dirichlet free energy ($|\nabla d|^2$) is replaced by Ginzburg–Landau energy ($|\nabla d|^2 + \frac{1}{4\epsilon^2}(|d|^2-1)^2$) in the one-dimensional case.

The key issue of proving Theorem 1.1 is to establish the appropriate a priori energy estimates for the solution of (1.8) up to any finite time, subject to initial and boundary conditions (1.9)–(1.10). Compared to Navier–Stokes, the presence of highly nonlinear director field equation makes the analysis more difficult due to production of $g|\frac{1}{g}(\frac{d_y}{g})_y + \frac{1}{g^2}|d_y|^2 d|^2$ in the temperature equation, the term $\frac{|d_y|^2}{g^2}d$, constraint $|d|^2 = 1$ in the director field equation, and coupling of the torque balance equation with the momentum balance equation with the term $\frac{1}{g^2}|d_y|^2$. In addition to the difficulty caused by the lack of the positive lower bound of the density, it is observed that upper bound density strongly depends on the L^∞-norm of $\frac{d_y}{g}$ and the quantity $\rho_0\theta$. Strong coupling of cross terms and vacuum causes serious difficulties in the control of $L^\infty(L^2)$ of $\sqrt{\rho_0}\theta$. To overcome such difficulties, density-weighted embedding inequalities are used both for temperature and the director field.

The desired a priori energy estimates are carried out as, first, we obtain from (1.8) an identity as follows:

$$g(y,t) = X(t)^{-1}Y(y,t)^{-1}Z(y,t)^{-1}\left(1 + R\int_0^t X(s)Y(y,s)Z(y,s)\rho_0(y)\theta(y,s)\,ds\right)$$

for some functions $X(t)$, $Y(y,t)$, and $Z(y,t) = \exp(-\int_0^t \frac{1}{2g^2}|d_y|^2\,ds)$. The temperature equation is not used at all in deriving the above identity, and this identity is in the spirit of the one in [35], but in different Lagrangian coordinates. The basic energy estimate implies that both X and Y are uniformly away from zero and uniformly bounded up to any finite time. As a direct corollary of the above identity, one can obtain the uniform positive lower bound of g and the control of the upper bound of g in terms of $\int_0^t \theta\,ds$ and $\int_0^t \frac{1}{2g^2}|d_y|^2\,ds$. By using the positive lower bound of g, we obtain a density-weighted embedding inequality which implies that the upper bound of $\sqrt{\rho_0}\theta$ and $\frac{1}{\sqrt{\rho_0}}|\frac{d_y}{g}|^2$ can be controlled by that of g, up to a small dependence on $\|\frac{\partial_y\theta}{\sqrt{g}}\|_2$, and $\frac{1}{\sqrt{g}}\partial_y|\frac{d_y}{g}|^2$. This will be used for the $L^\infty(L^2)$ type estimates on $\sqrt{\rho_0}\theta$.

Second, we carry out the $L^\infty(L^2)$ energy estimate on $\sqrt{\rho_0}\theta$ and, at the same time, the $L^\infty(L^2)$ energy estimate will be evolved naturally, due to the coupling structure between u, d, and θ in the system. Compared to Navier–Stokes equations, we do not have control for $L^\infty(L^2)$ energy estimate on $\sqrt{\rho_0}\theta$ because of time integral term of the director field in product with time integral of temperature. It is noted that there is no dissipation estimates on the director field equation due to energy conservation. As a conclusion, by extracting dissipation estimates from momentum and torque balance equation in terms of $\|\theta\|_\infty$, we are able to obtain the a priori upper bound of g and the a priori $L^\infty(L^2)\cap L^2(H^1)$ type estimates on $(u, \frac{d_y}{g}, \theta)$.

Third, by using the effective viscous flux $F := \mu\frac{\partial_y u}{g} - p - \frac{1}{2}|\frac{d_y}{g}|^2$ and working on its $L^\infty(L^2)\cap L^2(H^1)$ type a priori estimate that strongly depends on the term $\|\frac{d_{yt}}{\sqrt{g}}\|_{L^2}^2$ of the director field, we are able to get the a priori $L^\infty(H^1)$ estimate on (g,u); however, due to the presence of the term $\frac{\mu}{g}(\partial_y u)^2$ and $g|d_t|^2$ and the degeneracy of the leading term $\rho_0\partial_t\theta$ in the θ equation, we are not able to obtain the corresponding $L^\infty(H^1)$ estimate on θ without appealing to higher order energy estimates than H^1.

Lastly, we obtained the a priori $L^\infty(H^2)$ type estimates on (u,θ), which are achieved through performing the $L^\infty(L^2)$ type energy estimate on $\sqrt{\rho_0}\partial_t\theta$, $L^\infty(H^1)$ type estimate on F, and by using a priori $L^\infty(H^3)$ estimates on the director field. It should be mentioned that the desired a priori $L^\infty(H^2)$ estimates on θ are obtained without knowing their a priori $L^\infty(H^1)$ bound in advance.

In fact, all the energy estimates obtained in this paper hold if we replace boundary condition (1.10) with the above one by copying or slightly modifying the proof.

2 Proof of the main results

2.1 A priori L^2 estimates

We start with the basic energy identity.

Lemma 1 *For time and space variables* $(t,x)\in[0,T]\times[0,l]$, *the conservation of mass and energy is given as*

$$\int_0^l \rho(y,t)\,dy = \int_0^l \rho(y,0)\,dy, \tag{2.1}$$

$$\int_0^l \left(c_v\rho_0\theta + \frac{1}{2}|\rho_0 u|^2 + \frac{1}{2g}|d_y|^2\right)(y,t)\,dy = E_0, \tag{2.2}$$

where

$$E_0 = \int_0^l \left(c_v\rho_0\theta + \frac{1}{2}\rho_0|u|^2 + \frac{1}{2g}|d_y|^2\right)(y,0)\,dy.$$

Proof Integrating $(1.8)_1$ with respect to space and time and using boundary condition (1.10), it is easy to see that equality (2.1) holds.

In order to prove (2.2), we multiply $(1.8)_2$ by u. The resulting equation is then integrated over [0,l]; after integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^l \rho_0 u^2\,dy + \int_0^l \frac{u_y^2}{g}\,dy = \int_0^l \left(p + \frac{1}{2g^2}|d_y|^2\right)u_y\,dy. \tag{2.3}$$

On the other hand, taking dot product of equation $(1.8)_3$ with $-(\frac{1}{g}(\frac{d_y}{g})_y + \frac{|d_y|^2 d}{g^2})$, as a result one can get

$$-d_t\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{|d_y|^2}{g^2}d\right) = -\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{1}{g^2}|d_y|^2 d\right|^2. \tag{2.4}$$

The left-hand side of (2.4) can be written as

$$-d_t\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{|d_y|^2}{g^2}d\right) = -d_t\cdot\frac{1}{g}\left(\frac{d_y}{g}\right)_y - d_t\cdot\frac{|d_y|^2}{g^2}d = -d_t\cdot\frac{1}{g}\left(\frac{d_y}{g}\right)_y, \tag{2.5}$$

where we have used constraint $|d|^2 = 1$. Plugging (2.5) into (2.4) and then multiplying the resulting equation by g, we get

$$-d_t \cdot \left(\frac{d_y}{g}\right)_y = -g\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{1}{g^2}|d_y|^2 d\right|^2. \tag{2.6}$$

Then, we integrate equality (2.6) over [0,l], and we have

$$\frac{1}{2}\frac{d}{dt}\int_0^l \frac{|d_y|^2}{g}\,dy + \int_0^l g\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{1}{g^2}|d_y|^2 d\right|^2 dy = -\int_0^l \frac{1}{2g^2}|d_y|^2 u_y\,dy. \tag{2.7}$$

Combining (2.3) and (2.7) with integral form of the temperature equation $(1.8)_4$ implies equality (2.2). □

Next, we carry out the estimate on the lower bound of g. To this end, we perform some calculations in the spirit of [35] as preparations. Now, we integrate momentum conservation equation with respect to time t, and using first equation of (1.8), we can get

$$\rho_0 u(y,t) - \rho_0 u_0(y) + \int_0^t \left(p + \frac{1}{2g^2}|d_y|^2\right)_y (y,s)\,ds = \frac{d}{dy}\big(\ln g(y,t) - \ln g_0(y)\big). \tag{2.8}$$

Integrating (2.8) with respect to y from the point $\xi(t)$ to an arbitrary point y for any fixed time t, we get

$$\begin{aligned}&\int_{\xi(t)}^y \big(\rho_0 u(y,t) - \rho_0 u_0(y)\big)\,dy + \int_0^t \left(p + \frac{1}{2g^2}|d_y|^2\right)(y,s)\,ds \\ &\quad - \int_0^t \left(p + \frac{1}{2g_0^2}|d_y|^2\right)\big(\xi(t),s\big)\,ds = \ln g(y,t) - \ln g_0\big(\xi(t),t\big).\end{aligned} \tag{2.9}$$

After rewriting, we have

$$\exp\left(\int_0^t p\,ds\right) = g(y,t)X(t)Y(y,t)Z(y,t), \tag{2.10}$$

where

$$\begin{aligned}X(t) &= \frac{1}{g(\xi(t),t)}\exp\left(\int_0^t p + \frac{1}{2g^2}|d_y|^2\big(\xi(t),s\big)\,ds\right),\\ Y(y,t) &= \exp\left(\int_{\xi(t)}^y \rho_0 u_0(x) - \rho_0 u(t,x)\,dx\right),\\ Z(y,t) &= \exp\left(-\int_0^t \frac{1}{2g^2}|d_y|^2\,ds\right).\end{aligned}$$

Multiplying equation (2.10) by $R\rho_0\theta$ and integrating the result over $[0,t]$, we have

$$\exp\left(\int_0^t p\,ds\right) - 1 = R\rho_0\int_0^t X(s)Y(y,s)Z(y,s)\theta\,ds. \tag{2.11}$$

From (2.10) and (2.11) we get

$$g(y,t) = X(t)^{-1} Y(y,t)^{-1} Z(y,t)^{-1}\left(1 + R\int_0^t X(s)Y(y,s)Z(y,s)\rho_0(y)\theta(y,s)\,ds\right). \tag{2.12}$$

A prior positive lower and upper bound of g is stated in the following proposition.

Proposition 1 *Given $T \in (0,\infty)$, it holds that*

$$g \geq C,$$
$$\|g\|_\infty(t) \leq C\|Z^{-1}\|_\infty\left(1 + \frac{R}{\mu}\int_0^t \|\rho_0\theta\|_\infty\,d\tau\right)$$

for any $t \in [0,\infty)$.

Proof By Lemma 1, it follows from the Hölder inequality that

$$\begin{aligned}\left|\int_0^l \rho_0(u-u_0)\,d\xi\right| &\leq \left(\int \rho_0\,d\xi\right)^{\frac{1}{2}}\left[\left(\int_0^l \rho_0 u^2\,d\xi\right)^{\frac{1}{2}} + \left(\int_0^l \rho_0 u_0^2\,d\xi\right)^{\frac{1}{2}}\right]\\ &\leq 2\sqrt{2\|\rho_0\|_1 E_0}.\end{aligned}$$

Therefore, it follows from the definition of Y in Lemma 1 that

$$\exp\left\{-C\sqrt{\|\rho_0\|_1 E_0}\right\} \leq Y(y,t) \leq \exp\left\{C\sqrt{\|\rho_0\|_1 E_0}\right\},$$
$$C^{-1} \leq Y(y,t) \leq C.$$

Similarly,

$$0 < Z = e^{\int_0^t -\frac{|d_y|^2}{g^2}} < 1. \tag{2.13}$$

Now we show that $X(t)$ is bounded above and below by the initial data. Assume that

$$\mathcal{F}(y,t) = \int_0^t \left(\frac{u_y}{g} - p - \frac{1}{2g^2}|d_y|^2\right)(y,s)\,ds + \int_0^y \rho_0 u_0(x)\,dx, \tag{2.14}$$

then

$$\mathcal{F}_y = \rho_0 u, \qquad \mathcal{F}_t = \frac{u_y}{g} - p - \frac{1}{2g^2}|d_y|^2.$$

With the aid of the continuity equation, we obtain

$$(g\mathcal{F})_t - (u\mathcal{F})_y = u_y - \rho_0\theta - \frac{1}{2g}|d_y|^2 - \rho_0 u^2.$$

Integrating it over $[0,l] \times [0,t]$, using non-slip boundary (1.10) condition, we get

$$\int_0^l g\mathcal{F}\,dy = \int_0^l g_0\mathcal{F}_0\,dy - \int_0^t\int_0^l \left(\rho_0\theta + \frac{1}{2g}|d_y|^2 + \rho_0 u^2\right)dy\,ds.$$

By using mass conservation (2.1), we have

$$\int_0^l g\,dy = \int_0^l g_0\,dy.$$

Using the continuity of g, for $t > 0$, there exists $\xi(t) \in [0,l]$ such that

$$\mathcal{F}\big(\xi(t),t\big) = \frac{1}{g_0}\int_0^l g\mathcal{F}(y,t)\,dy.$$

On the other hand, since $u_y = g_t$, then from the definition of $\mathcal{F}$ we have

$$\mathcal{F}\big(\xi(t),t\big) = \ln g\big(\xi(t),t\big) - \int_0^t \left(p + \frac{1}{2g^2}|d_y|^2\right)\big(\xi(t),s\big)\,ds + \int_0^{\xi(t)} \rho_0 u_0(x)\,dx. \tag{2.15}$$

By rewriting in the form

$$\begin{aligned}
&\frac{1}{g(\xi(t),t)}\exp\left(\int_0^t \left(\rho_0\theta + \frac{1}{2g^2}|d_y|^2\right)\big(\xi(t),s\big)\,ds\right)\\
&\quad = \exp\left(\frac{1}{g_0}\int_0^t\int_0^l \left(\rho_0\theta + \frac{1}{2g}|d_y|^2 + \rho_0 u^2\right)dy\,ds + \int_0^{a(t)} \rho_0 u\big(\xi(t),t\big)\,dx\right).
\end{aligned} \tag{2.16}$$

Thus it is not hard to see that

$$C^{-1} \leq X(t) \leq C. \tag{2.17}$$

Recalling $p \geq 0$, we obtain

$$\begin{aligned}
g &= X^{-1}Y^{-1}Z^{-1}e^{\frac{1}{\mu}\int_0^t p\,d\tau},\\
&\geq C,
\end{aligned}$$

the conclusion follows. □

Next we establish the upper bound of g, where we need to prove $\|\rho_0\theta\|_{L^\infty}$ and for the director field $\|(\frac{|d_y|}{g})^2\|_{L^\infty}$ and $\|\frac{1}{\sqrt{\rho_0}}(\frac{|d_y|}{g})^2\|_\infty^2$ type a priori estimate. The need of the weighted estimate for temperature and director field is necessary for the upper bound of g. As a preparation of deriving the a priori upper bound of g and the a priori $L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ type estimates on (u, d_y, θ), for simplicity, the density-weighted estimates of θ and d_y are given in the following proposition.

Proposition 2 *We have the following two items*: (*i*) *It holds that*

$$\begin{aligned}
&\left\|\frac{1}{\rho_0^2(y_0)}\left(\frac{d_y}{g}\right)^2\right\|_\infty^2 \leq C + C\left\|\frac{1}{\sqrt{g}}\partial_y\left(\frac{d_y}{g}\right)^2\right\|_2^{\frac{4}{3}}\|g\|_\infty^{\frac{2}{3}},\\
&\left\|\left(\frac{|d_y|}{g}\right)^2\right\|_\infty \leq \left\|\frac{1}{\sqrt{g}}\partial_y\left(\frac{d_y}{g}\right)^2\right\|_2 + C,\\
&\|\rho_0^2\theta\|_\infty^2 \leq C + C\left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2^{\frac{4}{3}}\|g\|_\infty^{\frac{2}{3}},
\end{aligned}$$

$$\|\theta\|_\infty \le \sqrt{l}\left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2 + C,$$

where C depends on $(\bar{\rho}_0, \rho_0', E_0, l, |\Omega_0|)$ *and*

$$\bar{\rho} = \|\rho_0\|_\infty, \qquad \Omega_0 := \left\{y \in (0,l)\middle|\rho_0(y) \ge \frac{\bar{\rho}}{2}\right\}.$$

(ii) *As a consequence of* (i), *we have*

$$\left\|\frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\right\|_\infty^2 \le \eta\left\|\frac{1}{\sqrt{g}}\partial_y\left|\frac{d_y}{g}\right|^2\right\|_2^2 + C_\eta\left(\|g\|_\infty^2 + 1\right),$$
$$\|\sqrt{\rho_0}\theta\|_\infty^2 \le \eta\left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2^2 + C_\eta\left(\|g\|_\infty^2 + 1\right)$$

for any $\eta \in (0,\infty)$, *where* C_η *is a positive constant depending only on* η *and* $\mathcal{N}_1(\bar{\rho}_0, E_0, \Omega_0, \rho_0')$.

Proof Now the proof of Proposition 2 is given as follows: (i) Let

$$\left|\frac{d_y}{g}(y,t)\right|^2 = \mathcal{R}(y,t)$$

and

$$\Omega_0 := \left\{y \in (0,l)\middle|\rho_0(y) \ge \frac{\bar{\rho}}{2}\right\}.$$

Noticing that

$$\mathcal{R}(y,t) = \frac{1}{\Omega_0}\int_{\Omega_0}\mathcal{R}\,dz + \frac{1}{\Omega_0}\int_{\Omega_0}\int_z^y \partial_y\mathcal{R}\,dy\,dz,$$

we deduce that

$$\begin{aligned}\|\mathcal{R}\|_\infty &\le \frac{1}{\Omega_0}\int_{\Omega_0}\mathcal{R}\,dz + \int_0^l |\partial_y\mathcal{R}|\,dz \\ &\le \frac{2}{\Omega_0}\|\mathcal{R}\|_1 + \left(\int_0^l\left|\frac{\partial_y\mathcal{R}}{\sqrt{g}}\right|^2 dz\right)^{\frac{1}{2}}\left(\int_0^l g\,dz\right)^{\frac{1}{2}},\end{aligned}$$

thus one can get

$$\|\mathcal{R}\|_\infty \le \sqrt{l}\left\|\frac{\partial_y\mathcal{R}}{\sqrt{g}}\right\|_2 + \frac{2E_0}{\Omega_0}.$$

Density-weighted estimate for the director field:

By the Hölder and Young inequalities, and similarly as above, we deduce

$$\left(\frac{\mathcal{R}}{\rho_0^2}\right)^2(y,t) \le \Omega_0^{-1}\left(\frac{2}{\bar{\rho}^2}\right)^2\|\mathcal{R}\|_1^2 + \frac{4}{\bar{\rho}^3}\left\|\frac{\mathcal{R}}{\rho_0^2}\right\|_\infty\|\mathcal{R}\|_1\|\rho_0'\|_\infty$$

$$
\begin{aligned}
&+\frac{2}{\bar\rho^2}\left\|\frac{\partial_y\mathcal{R}}{\sqrt{g}}\right\|_2\|g\|_\infty^{\frac12}\left\|\frac{\mathcal{R}}{\rho_0^2}\right\|_2\\
&\le \Omega_0^{-1}\left(\frac{2}{\bar\rho^2}\right)^2\|\mathcal{R}\|_1^2+\frac{4}{\bar\rho^3}\left\|\frac{\mathcal{R}}{\rho_0^2(y_0)}\right\|_\infty\|\mathcal{R}\|_1\|\rho_0'\|_\infty\\
&+\frac{2}{\bar\rho^2}\left\|\frac{\partial_y\mathcal{R}}{\sqrt{g}}\right\|_2\|g\|_\infty^{\frac12}\left\|\frac{\mathcal{R}}{\rho_0^2}\right\|_1^{\frac12}\left\|\frac{\mathcal{R}}{\rho_0^2}\right\|_\infty^{\frac12}\\
&\le \Omega_0^{-1}\left(\frac{2}{\bar\rho^2}\right)^2\|\mathcal{R}\|_1^2+\frac{4}{\bar\rho^3}\left\|\frac{\mathcal{R}}{\rho_0^2(y_0)}\right\|_\infty\|\mathcal{R}\|_1\|\rho_0'\|_\infty\\
&+\frac{2}{\bar\rho^3}\left\|\frac{\partial_y\mathcal{R}}{\sqrt{g}}\right\|_2\|g\|_\infty^{\frac12}\|\mathcal{R}\|_1^{\frac12}\left\|\frac{\mathcal{R}}{\rho_0^2}\right\|_\infty^{\frac12}\\
&\le \delta\left\|\frac{\mathcal{R}}{\rho_0^2(y_0)}\right\|_\infty^2+\Omega_0^{-1}\left(\frac{2}{\bar\rho^2}\right)^2\|\mathcal{R}\|_1^2+C(\delta)\|\mathcal{R}\|_1^2\|\rho_0'\|_\infty^2\\
&+\frac{2}{\bar\rho^4}\left\|\frac{\partial_y\mathcal{R}}{\sqrt{g}}\right\|_2^{\frac43}\|g\|_\infty^{\frac23}\|\mathcal{R}\|_1^{\frac23}
\end{aligned}
$$

for any $y\in(0,l)$. Thus, from Lemma 1 and suitably small $\delta>0$, we have

$$
\left\|\frac{\mathcal{R}}{\rho_0^2(y_0)}\right\|_\infty^2\le C+C\left\|\frac{\partial_y\mathcal{R}}{\sqrt{g}}\right\|_2^{\frac43}\|g\|_\infty^{\frac23}. \tag{2.18}
$$

(ii) Thanks to (i), we have

$$
\begin{aligned}
&\left\|\frac{\mathcal{R}}{\rho_0^2(y_0)}\right\|_\infty^2\le C+C\left\|\frac{\partial_y\mathcal{R}}{\sqrt{g}}\right\|_2^{\frac43}\|g\|_\infty^{\frac23},\\
&\|\mathcal{R}\|_\infty\le\sqrt{l}\left\|\frac{\partial_y\mathcal{R}}{\sqrt{g}}\right\|_2+\frac{2E_0}{\Omega_0}
\end{aligned}
$$

for a positive constant C depending only on $\mathcal{N}_1$. Therefore, we have

$$
\begin{aligned}
\left\|\frac{\mathcal{R}}{\sqrt{\rho_0}}\right\|_\infty^2&=\left\|\left(\frac{\mathcal{R}}{\rho_0^2}\right)^{\frac14}\mathcal{R}^{\frac34}\right\|_\infty^2\le\left\|\left(\frac{\mathcal{R}}{\rho_0^2}\right)\right\|_\infty^{\frac12}\|\mathcal{R}\|_\infty^{\frac32}\\
&\le C\left(1+\left\|\frac{\partial_y\mathcal{R}}{\sqrt{g}}\right\|_2^{\frac43}\|g\|_\infty^{\frac23}\right)^{\frac14}\left(\left\|\frac{\partial_y\mathcal{R}}{\sqrt{g}}\right\|_2+1\right)^{\frac32}.
\end{aligned}
$$

Density-weighted estimate for temperature:
Assume $\|\rho\|_\infty=\bar\rho$, let there exist $y_0\in(0,l)$ such that

$$
\rho_0^2(y_0)\theta(y_0,t)\le\frac{2}{l}\int_0^l\rho_0^2\theta\,d\xi\le\frac{2\bar\rho}{l}\|\rho_0\theta\|_1.
$$

By the Hölder and Young inequalities, we deduce

$$
\begin{aligned}
\left(\rho_0^2\theta\right)^2(y,t)&\le(\rho_0\theta)^2(y_0,t)+2\int_0^l\rho_0^2\theta\left|\partial_y\left(\rho_0^2\theta\right)\right|d\xi\\
&\le\left(\frac{2\bar\rho}{l}\right)^2\|\rho_0\theta\|_1^2+2\int_0^l\left(2\rho_0^2\theta\rho_0\theta\left|\rho_0'\right|+\rho_0^2\theta\rho_0^2|\partial_y\theta|\right)d\xi
\end{aligned}
$$

$$
\begin{aligned}
&\le \left(\frac{2\bar\rho}{l}\right)^2 \|\rho_0\theta\|_1^2 + 4\left\|\rho_0^2\theta\right\|_\infty \|\rho_0\theta\|_1 \left\|\rho_0'\right\|_\infty \\
&\quad + 2\bar\rho^2 \left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2 \|g\|_\infty^{\frac12} \left\|\rho_0^2\theta\right\|_2 \\
&\le \left(\frac{2\bar\rho}{l}\right)^2 \|\rho_0\theta\|_1^2 + 4\left\|\rho_0^2\theta\right\|_\infty \|\rho_0\theta\|_1 \left\|\rho_0'\right\|_\infty + +4\left\|\rho_0^2\theta\right\|_\infty \|\rho_0\theta\|_1 \left\|\rho_0'\right\|_\infty \\
&\quad + 2\bar\rho^2 \left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2 \|g\|_\infty^{\frac12} \left\|\rho_0^2\theta\right\|_1^{\frac12} \left\|\rho_0^2\theta\right\|_\infty^{\frac12} + 2\bar\rho^{\frac52} \left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2 \|g\|_\infty^{\frac12} \|\rho_0\theta\|_1^{\frac12} \left\|\rho_0^2\theta\right\|_\infty^{\frac12} \\
&\le \frac12 \left\|\rho_0^2\theta\right\|_\infty^2 + \left(\frac{2\bar\rho}{l}\right)^2 \|\rho_0\theta\|_1^2 + 16\|\rho_0\theta\|_1^2 \left\|\rho_0'\right\|_\infty^2 \\
&\quad + 3\bar\rho^{\frac{10}{3}} \|\rho_0\theta\|_1^{\frac23} \left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2^{\frac43} \|g\|_\infty^{\frac23}
\end{aligned}
$$

for any $y \in (0,l)$, it is not hard to see that

$$
\left\|\rho_0^2\theta\right\|_\infty^2 \le C + C\left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2^{\frac43} \|g\|_\infty^{\frac23}.
$$

Noticing that

$$
\theta(y,t) = \frac{1}{\Omega_0}\int_{\Omega_0} \theta\,dz + \frac{1}{\Omega_0}\int_{\Omega_0}\int_z^y \partial_y\theta\,d\xi\,dz,
$$

we deduce, by the Hölder inequality, that

$$
\begin{aligned}
\|\theta\|_\infty &\le \frac{1}{\Omega_0}\int_{\Omega_0} \frac{\rho_0\theta}{\rho_0}\,dz + \int_0^l |\partial_y\theta|\,dz \\
&\le \frac{2}{\Omega_0\bar\rho}\|\rho_0\theta\|_1 + \left(\int_0^l \left|\frac{\partial_y\theta}{\sqrt{g}}\right|^2 dz\right)^{\frac12} \left(\int_0^l g\,dz\right)^{\frac12}. \qquad \square
\end{aligned}
$$

The other estimates for θ in Proposition 2 can be obtained similar to the director field.

By density-weighted estimate of θ and d_y in hand, the desired a priori $L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ estimates on $(u, \frac{d_y}{g}, \theta)$ are given as in the following lemma.

Lemma 2 *Given $T \in (0,\infty)$. It holds that*

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\sqrt{\rho_0}\mathcal{E}\|_2^2 + \frac{C_1}{\mu}\|\sqrt{\rho_0}u^2\|_2^2 + \left\|\left|\frac{d_y}{g^{\frac34}}\right|^2\right\|_2^2\right) \\
&\quad + \int_0^t \left(\|\sqrt{\rho_0}\theta\|_\infty^2 + \frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\|_\infty^2\right) d\tau \\
&\quad + \int_0^l \left[\left(\left|\frac{d_y}{g}\right|^2\right)_y\right]^2 dy + 5C_1\left\|\frac{u\partial_y u}{\sqrt{g}}\right\|_2^2 + \kappa c_v\left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2^2 \\
&\le \frac{C}{\eta}\left[(e^{\int_0^t \|\frac{|d_y|}{g}\|_{L^\infty}^2 d\tau}\left(1 + \frac{R}{\mu}\int_0^t \|\rho_0\theta\|_\infty\,d\tau\right)\right]^2 + \frac{C}{2}\frac{d}{dt}\int_0^l \rho_0(\mathcal{E}')^2\,dy \qquad (2.19)
\end{aligned}
$$

for any $t \in (0,T)$.

Denote $\mathcal{E} := \frac{1}{2}(u^2 + \frac{1}{\rho_0}\frac{|d_y|^2}{g}) + c_v\theta$, then one can derive from $(1.8)_2$, $(1.8)_3$, and $(1.8)_4$ that

$$\begin{aligned}
&\rho_0\partial_t\mathcal{E} + \partial_y\left(u\left(p + \frac{1}{2g^2}|d_y|^2\right)\right) - \kappa\,\partial_y\left(\frac{\partial_y\theta}{g}\right) \\
&\quad = \mu\partial_y\left(\frac{1}{g}\partial_y\left(\frac{u^2}{2}\right)\right) + \left(d_t\cdot\frac{d_y}{g}\right)_y.
\end{aligned}\tag{2.20}$$

Multiplying (2.20) by $\mathcal{E}' = \frac{1}{2}(u^2 + \frac{|d_y|^2}{g^2}) + c_v\theta$ and integrating the resultant over $(0,l)$, one gets from integration by parts that

$$\begin{aligned}
&\frac{d}{dt}\int_0^l \left(\rho_0\mathcal{E}\mathcal{E}'\right)dy + \int_0^l \frac{1}{g}\left(\kappa\,\partial_y\theta + \mu u\partial_y u + \left(d_t\cdot\frac{d_y}{g}\right)\right)\partial_y\mathcal{E}'\,dy \\
&\quad = \int_0^l \rho_0\mathcal{E}\left(\mathcal{E}'\right)_t dy + \int_0^l u\left(p + \frac{1}{2g^2}|d_y|^2\right)\partial_y\mathcal{E}'\,dy, \\
&\frac{d}{dt}\int_0^l \left(\rho_0\mathcal{E}\mathcal{E}'\right)dy + \int_0^l \frac{1}{g}\left(\kappa\,\partial_y\theta + \mu u\partial_y u + \left(\left|\frac{d_y}{g}\right|^2\right)_y\right)\partial_y\mathcal{E}'\,dy \\
&\quad \le \frac{C}{2}\frac{d}{dt}\int_0^l \rho_0\left(\mathcal{E}'\right)^2 dy + \int_0^l u\left(p + \frac{1}{2g^2}|d_y|^2\right)\partial_y\mathcal{E}'\,dy,
\end{aligned}\tag{2.21}$$

where we have used that $\mathcal{E} \le \mathcal{E}'$ and the fact $d_t \cdot \frac{d_y}{g} = \frac{1}{g}(|\frac{d_y}{g}|^2)_y$. By the Young inequality, we have

$$\begin{aligned}
&\int_0^l \frac{1}{g}\left(\kappa\,\partial_y\theta + \mu u\partial_y u + \left(\left|\frac{d_y}{g}\right|^2\right)_y\right)\partial_y\mathcal{E}'\,dy \\
&\quad = \int_0^l \frac{1}{g}\left(\kappa\,\partial_y\theta + \mu u\partial_y u + \left(\left|\frac{d_y}{g}\right|^2\right)_y\right)\left[u\partial_y u + \left(\frac{|d_y|^2}{g^2}\right)_y + c_v\partial_y\theta\right]dy \\
&\quad \ge \frac{3\kappa c_v}{4}\int_0^l \left|\frac{\partial_y\theta}{\sqrt{g}}\right|^2 dy - C\int_0^l \left[\left|\frac{u\partial_y u}{\sqrt{g}}\right|^2 + \left|\left(\left|\frac{d_y}{g}\right|^2\right)_y\right|^2\right]dy
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^l u\left(p + \frac{1}{2g^2}|d_y|^2\right)\partial_y\mathcal{E}\,dy \\
&\quad = R\int_0^l u\frac{\rho_0}{g}\theta\left(u\partial_y u + \left(\left|\frac{d_y}{g}\right|^2\right)_y + c_v\partial_y\theta\right)dy \\
&\qquad + \int_0^l u\frac{1}{2g^2}|d_y|^2\left(u\partial_y u + \left(\left|\frac{d_y}{g}\right|^2\right)_y + c_v\partial_y\theta\right)dy \\
&\quad \le \frac{\kappa c_v}{4}\int_0^l \left|\frac{\partial_y\theta}{\sqrt{g}}\right|^2 dy + C\int_0^l \frac{1}{g}\left(\rho_0^2u^2\theta^2 + (u\partial_y u)^2 + \left[\left(\left|\frac{d_y}{g}\right|^2\right)_y\right]^2\right)dy, \\
&\qquad + \frac{\kappa c_v}{4}\int_0^l \left|\frac{\partial_y\theta}{\sqrt{g}}\right|^2 dy + C\int_0^l \frac{1}{g}\left(u^2\left|\frac{d_y}{g}\right|^4 + (u\partial_y u)^2 + \left[\left(\left|\frac{d_y}{g}\right|^2\right)_y\right]^2\right)dy,
\end{aligned}$$

for a positive constant C depending only on R, c_v, μ, and κ. Substituting the above two inequalities into (2.21) and applying Lemma 1 and Proposition 1, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho_0}\mathcal{E}\|_2^2+\kappa c_v\left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2^2\\
&\le\delta\left(\left\|\frac{u\partial_y u}{\sqrt{g}}\right\|_2^2+\left\|\frac{1}{\sqrt{g}}\left(\left|\frac{d_y}{g}\right|^2\right)_y\right\|_2^2\right)\\
&\quad+C\left(\int_0^l\frac{\rho_0u^2}{g}\rho_0\theta^2+\frac{1}{g}\left(u^2\left|\frac{d_y}{g}\right|^4\right)dy\right)+\frac{C}{2}\frac{d}{dt}\int_0^l\rho_0(\mathcal{E}')^2\,dy\\
&\le\delta\left(\left\|\frac{u\partial_y u}{\sqrt{g}}\right\|_2^2+\left\|\frac{1}{\sqrt{g}}\left(\left|\frac{d_y}{g}\right|^2\right)_y\right\|_2^2\right)\\
&\quad+C\|\sqrt{\rho_0}u\|_2^2\left(\|\sqrt{\rho_0}\theta\|_\infty^2+\left\|\frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\right\|_\infty^2\right)+\frac{C}{2}\frac{d}{dt}\int_0^l\rho_0(\mathcal{E}')^2\,dy\\
&\le\delta\left(\left\|\frac{u\partial_y u}{\sqrt{g}}\right\|_2^2+\left\|\frac{1}{\sqrt{g}}\left(\left|\frac{d_y}{g}\right|^2\right)_y\right\|_2^2\right)\\
&\quad+C\left(\|\sqrt{\rho_0}\theta\|_\infty^2+\left\|\frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\right\|_\infty^2\right)+\frac{C}{2}\frac{d}{dt}\int_0^l\rho_0(\mathcal{E}')^2\,dy
\end{aligned}$$

for a positive constant C depending only on R, c_v, μ, and κ, and thus

$$\frac{d}{dt}\|\sqrt{\rho_0}\mathcal{E}\|_2^2+\kappa c_v\left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2^2 \tag{2.22}$$

$$\le C_1\left(\left\|\frac{u\partial_y u}{\sqrt{g}}\right\|_2^2+\left\|\frac{1}{\sqrt{g}}\left(\left|\frac{d_y}{g}\right|^2\right)_y\right\|_2^2\right)+C\left(\|\sqrt{\rho_0}\theta\|_\infty^2+\left\|\frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\right\|_\infty^2\right) \tag{2.23}$$

$$+\frac{C}{2}\frac{d}{dt}\int_0^l\rho_0(\mathcal{E}')^2\,dy \tag{2.24}$$

for a positive constant C_1 depending only on R, c_v, μ, and κ.

Multiplying the momentum equation by $4u^3$ and integrating the resultant over $(0,l)$, after integration by parts and the Young inequality, one gets that

$$\begin{aligned}
&\frac{d}{dt}\int_0^l\rho_0u^4\,dy+12\mu\int_0^l\left|\frac{u\partial_y u}{\sqrt{g}}\right|^2dy\\
&=12\int_0^l\left(p+\frac{1}{2g^2}|d_y|^2\right)u^2\partial_yu\,dy\\
&=12\int_0^l\left(\frac{R\rho_0}{g}\theta+\frac{1}{2g^2}|d_y|^2\right)u^2\partial_yu\,dy\\
&\le 6\mu\int_0^l\left|\frac{u\partial_y u}{\sqrt{g}}\right|^2dy+\frac{6R^2}{\mu}\int_0^l\frac{\rho_0u^2}{g}\rho_0\theta^2\,dy+\frac{6}{\mu}\int_0^l\frac{1}{g}\left(u^2\left|\frac{d_y}{g}\right|^4\right)dy,\\
&\le C_1\left(\left\|\frac{u\partial_y u}{\sqrt{g}}\right\|_2^2\right)+C\|\sqrt{\rho_0}u\|_2^2\left(\|\sqrt{\rho_0}\theta\|_\infty^2+\left\|\frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\right\|_\infty^2\right)\\
&\le C_1\left(\left\|\frac{u\partial_y u}{\sqrt{g}}\right\|_2^2\right)+C\left(\|\sqrt{\rho_0}\theta\|_\infty^2+\left\|\frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\right\|_\infty^2\right),
\end{aligned}$$

from which, by Proposition 1, the Hölder inequality, and Proposition 1, one obtains

$$\frac{d}{dt}\int_0^l \rho_0 u^4\,dy + 6\mu\int_0^l \left|\frac{u\partial_y u}{\sqrt{g}}\right|^2 dy \le C\left(\|\sqrt{\rho_0}\theta\|_\infty^2 + \left\|\frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\right\|_\infty^2\right),$$

that is,

$$\frac{d}{dt}\|\sqrt{\rho_0}u^2\|_2^2 + 6\mu\left\|\frac{u\partial_y u}{\sqrt{g}}\right\|_2^2 \le C\left(\|\sqrt{\rho_0}\theta\|_\infty^2 + \left\|\frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\right\|_\infty^2\right). \tag{2.25}$$

Now we first operate the ∂_y on both sides of the director field equation

$$(d_y)_t = \left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y + \left(\frac{|d_y|^2}{g^2}d\right)_y.$$

Taking the dot product of the equation by $(\frac{|d_y|^2}{g^3}d_y)$, we get

$$(d_y)_t\cdot\left|\frac{d_y}{g}\right|^2\frac{d_y}{g} = \left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\cdot\left|\frac{d_y}{g}\right|^2\frac{d_y}{g} + \left(\frac{|d_y|^2 d}{g^2}\right)_y\cdot\left|\frac{d_y}{g}\right|^2\frac{d_y}{g}. \tag{2.26}$$

Now we compute each term on the left- and right-hand side of equation (2.26) one by one. The term on the left-hand side after integration by parts can be written as

$$\begin{aligned}
\int_0^l (d_y)_t\cdot\left|\frac{d_y}{g}\right|^2\frac{d_y}{g}\,dy &= \frac{1}{4}\int_0^l\left(d_y\cdot\left|\frac{d_y}{g}\right|^2\frac{d_y}{g}\right)_t dy + \frac{3}{4}\int_0^l\left|\frac{d_y}{g}\right|^4 g_t\,dy\\
&= \frac{1}{4}\int_0^l\left(\frac{|d_y|^4}{g^3}\right)_t dy + \frac{3}{4}\int_0^l\left|\frac{d_y}{g}\right|^4 u_y\,dy\\
&= \frac{1}{4}\int_0^l\left(\frac{|d_y|^4}{g^3}\right)_t dy - \frac{3}{4}\int_0^l\left(\left|\frac{d_y}{g}\right|^4\right)_y u\,dy\\
&= \frac{1}{4}\int_0^l\left(\frac{|d_y|^4}{g^3}\right)_t dy - \frac{3}{2}\int_0^l\left|\frac{d_y}{g}\right|^2\left(\left|\frac{d_y}{g}\right|^2\right)_y u\,dy\\
&= \frac{1}{4}\int_0^l\left(\frac{|d_y|^4}{g^3}\right)_t dy - \frac{3}{2}\int_0^l\frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\left(\left|\frac{d_y}{g}\right|^2\right)_y\sqrt{\rho_0}u\,dy.
\end{aligned}$$

In the above calculation we have used the boundary conditions $d_y = 0$, $u = 0$. For the terms on the right-hand side, we calculate them one by one. The first term after integration by parts and using boundary condition is as follows:

$$\begin{aligned}
&\int_0^l\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\cdot\left|\frac{d_y}{g}\right|^2\frac{d_y}{g}\,dy\\
&= -\int_0^l\frac{1}{g}\left(\frac{d_y}{g}\right)_y\cdot\left(\left|\frac{d_y}{g}\right|^2\frac{d_y}{g}\right)_y dy\\
&= -\int_0^l\frac{1}{g}\left(\frac{d_y}{g}\right)_y\cdot\left|\frac{d_y}{g}\right|^2\left(\frac{d_y}{g}\right)_y dy - \int_0^l\frac{1}{g}\left(\frac{d_y}{g}\right)_y\cdot\left(\left|\frac{d_y}{g}\right|^2\right)_y\frac{d_y}{g}\,dy\\
&= -\int_0^l\frac{1}{g}\left|\left(\frac{d_y}{g}\right)_y\right|^2\left|\frac{d_y}{g}\right|^2 dy - \frac{1}{2}\int_0^l\left[\left(\left|\frac{d_y}{g}\right|^2\right)_y\right]^2 dy
\end{aligned}$$

$$\leq -\int_0^l \frac{1}{g}\left[\left(\left|\frac{d_y}{g}\right|^2\right)_y\right]^2 dy.$$

Now the last term of (2.26) can be written as

$$\left(\frac{|d_y|^2 d}{g^2}\right)_y \cdot \left|\frac{d_y}{g}\right|^2 \frac{d_y}{g} = \frac{|d_y|^2}{g^2} \cdot \left|\frac{d_y}{g}\right|^2 \frac{|d_y|^2}{g},$$

where we have used the constraint $|d|^2 = 1$ for the director field. Substituting the computed term in (2.26), integrating the resultant with respect to space variable, and using Holder's inequality result in

$$\begin{aligned} &\frac{d}{dt}\left\|\left(\frac{d_y}{g^{\frac{3}{4}}}\right)^2\right\|_2^2 + \left\|\frac{1}{\sqrt{g}}\left(\left|\frac{d_y}{g}\right|^2\right)_y\right\|_2^2 \\ &\quad \leq C\left(\left\|\frac{d_y}{\sqrt{g}}\right\|_2^2 + \|\sqrt{\rho_0}u\|_2^2\right)\left\|\frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\right\|_\infty^2. \end{aligned} \tag{2.27}$$

Adding (2.27) and (2.25) with (2.24), one can get

$$\begin{aligned} &\frac{d}{dt}\left(\|\sqrt{\rho_0}\mathcal{E}\|_2^2 + \frac{C_1}{\mu}\|\sqrt{\rho_0}u^2\|_2^2 + \left\|\left(\frac{d_y}{g^{\frac{3}{4}}}\right)^2\right\|_2^2\right) \\ &\qquad + \left\|\frac{1}{\sqrt{g}}\left(\left|\frac{d_y}{g}\right|^2\right)_y\right\|_2^2 + 5C_1\left\|\frac{u\partial_y u}{\sqrt{g}}\right\|_2^2 + \kappa c_v\left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2^2 \\ &\quad \leq C\left(\|\sqrt{\rho_0}\theta\|_\infty^2 + \left\|\frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\right\|_\infty^2\right) + \frac{C}{2}\frac{d}{dt}\int_0^l \rho_0\left(\mathcal{E}'\right)^2 dy \end{aligned} \tag{2.28}$$

for any $t \in (0, T)$.

By Proposition 1 and (ii) of Proposition 2, we have

$$\begin{aligned} &\|\sqrt{\rho_0}\theta\|_\infty^2(t) + \left\|\frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\right\|_\infty^2(t) \\ &\quad \leq \eta\left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2^2(t) + \eta\left\|\frac{1}{\sqrt{g}}\left(\left|\frac{d_y}{g}\right|^2\right)_y\right\|_2^2 \\ &\qquad + \frac{C}{\eta}\left(e^{\int_0^t \|(\frac{|d_y|}{g})^2\|_{L^\infty} d\tau}\left(1 + \frac{R}{\mu}\int_0^t \|\rho_0\theta\|_\infty d\tau\right)\right)^2. \end{aligned} \tag{2.29}$$

Collecting the estimates provides the proof of the lemma.

In order to close the estimates of the lemma, we need to control the unbounded operator $e^{\int_0^t \|(\frac{|d_y|}{g})^2\|_{L^\infty} d\tau}$ and $\frac{R}{\mu}\int_0^t \|\rho_0\theta\|_\infty d\tau$. We see that the director field term is in product with the time integral of $L^\infty - norm$ of temperature, so it cannot be controlled by right-hand side terms, thus we need new estimates to control these terms. In order to control the right-hand side of the above inequality, dissipation estimates are obtained on the director field in terms of $L^\infty - norm$ of temperature.

Lemma 3 *Given $T \in (0,\infty)$. It holds that*

$$\sup_{0\le t\le T}\left\|\left(\sqrt{\rho_0}u, \frac{|d_y|}{\sqrt{g}}, \sqrt{\rho_0}\theta\right)\right\|_2^2 + \int_0^T\left(\|\theta\|_\infty^2 + \left\|\left(\partial_y\theta, \partial_y u, \frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right)\right\|_2^2\right)dt$$
$$\le C\big((1+T) + \big\|\big(\sqrt{\rho_0}u_0, \sqrt{\rho_0}u_0^2, \sqrt{\rho_0}\theta_0\big)\big\|_2^2\big) + \int_0^T \|\theta\|_\infty \, d\tau$$

for positive constants C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, and T.

Proof Multiplying $(1.8)_2$ by u and taking the dot product of director field equation $(1.8)_3$ with $-(\frac{1}{g}(\frac{d_y}{g})_y + \frac{|d_y|^2 d}{g^2})$ respectively, the resultant is integrated over $(0,l)$. From the momentum equation it is not hard to see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho_0}u\|_2^2 + \mu\left\|\frac{\partial_y u}{\sqrt{g}}\right\|_2^2 \\
&= \int_0^l\left(p + \left|\frac{d_y}{g}\right|^2\right)\partial_y u\, dy \\
&= R\int_0^l\left(\frac{\rho_0}{g}\theta + \left|\frac{d_y}{2g}\right|^2\right)\partial_y u\, dy \\
&\le \frac{\mu}{2}\left\|\frac{\partial_y u}{\sqrt{g}}\right\|_2^2 + \frac{R}{2\mu}\int_0^l \frac{\rho_0^2}{g}\theta^2\, dy \\
&\quad + \frac{1}{\mu}\int_0^l\left|\frac{d_y}{g}\right|^4 dy. \qquad (2.30)
\end{aligned}$$

Similarly, from the director field equation, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1\frac{|d_y|^2}{g}\,dy + \int_0^l g\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{1}{g^2}|d_y|^2 d\right|^2 dy = -\int_0^l \frac{1}{2g^2}|d_y|^2 u_y\, dy. \qquad (2.31)$$

By using the constraint $|d|^2 = 1$, it is not not hard to see that

$$g\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{1}{g^2}|d_y|^2 d\right|^2 = \frac{1}{g}\left(\frac{d_y}{g}\right)_y^2 - \left|\frac{d_y}{g}\right|^4.$$

Now equation (2.31), can be written as

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^l \frac{|d_y|^2}{g}\,dy + \int_0^l \frac{1}{g}\left(\frac{d_y}{g}\right)_y^2 dy &= -\int_0^l\left|\frac{d_y}{g}\right|^2 u_y\, dy + \int\left|\frac{d_y}{g}\right|^4 \\
&\le \frac{\mu}{2}\left\|\frac{\partial_y u}{\sqrt{g}}\right\|_2^2 + C\int\left|\frac{d_y}{g}\right|^4. \qquad (2.32)
\end{aligned}$$

Combining (2.30) and (2.32) and using Sobolev embedding result in

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho_0}u\|_2^2 + \left\|\frac{|d_y|}{\sqrt{g}}\right\|_2^2\right) + \mu\left\|\frac{\partial_y u}{\sqrt{g}}\right\|_2^2 + \left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right\|_2^2, \\
&\quad \le \frac{\mu}{2}\left\|\frac{\partial_y u}{\sqrt{g}}\right\|_2^2 + \frac{R}{2\mu}\int_0^l\frac{\rho_0^2}{g}\theta^2\,dy + \frac{1}{2}\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right\|_2^2 + C. \qquad (2.33)
\end{aligned}$$

The constraint $|d|^2 = 1$ is used in the above inequality, thus, by Proposition 1,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho_0}u\|_2^2 + \left\|\frac{|d_y|}{\sqrt{g}}\right\|_2^2\right) + \mu\left\|\frac{\partial_y u}{\sqrt{g}}\right\|_2^2 + \left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right\|_2^2 \\ &\quad\le C\big(\|\rho_0\|_\infty\big)\big(\|\theta\|_\infty + 1\big). \end{aligned} \tag{2.34}$$

□

Thus we see that, in order to close the above estimates, we need a control on temperature, which will be given in the following lemma.

Control of term $\|\theta\|_\infty$: Now we prove the following key lemma.

Lemma 4 *For any $t \ge 0$, it holds that*

$$\int_0^t \|\theta\|_{L^\infty}\,d\tau \le C(1+t)^2,$$

whereas C depends on $\|\rho_0\|_\infty$ and the initial data.

Proof Multiplying the temperature equation by $(\theta+\delta)^{-1}$ for some $\delta \in (0,1)$, the resultant equation can be written as

$$\begin{aligned} \rho_0\big(\ln(\theta+\delta)\big)_t + \frac{\rho_0\theta}{(\theta+\delta)g}u_y &= \kappa\left(\frac{\theta_y}{(\theta+\delta)g}\right)_y + \kappa\left(\frac{\theta_y^2}{(\theta+\delta)^2 g}\right) \\ &\quad + \left(\frac{u_y^2}{g} + g\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{1}{g^2}|d_y|^2 d\right|^2\right)\frac{1}{\theta+\delta}. \end{aligned} \tag{2.35}$$

Integrating the above inequality over time and space variables for $(\theta+\delta) \ge 1$, then using boundary condition for temperature, we get

$$\begin{aligned} &\int_0^l \rho_0\big(\ln(\theta+\delta)\big) + \int_0^l\int_0^t \kappa\left(\frac{\theta_y^2}{(\theta+\delta)^2 g}\right)dy\,d\tau \\ &\quad + \int_0^l\int_0^t \left(\frac{u_y^2}{g} + g\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{1}{g^2}|d_y|^2 d\right|^2\right)\frac{1}{\theta+\delta}\,dy\,d\tau \\ &= 2\int_0^l \rho_0\big(\ln(\theta+\delta)\big)\,dy - \int_0^l \rho_0\big(\ln(\theta_0+\delta)\big)\,dy + \int_0^l\int_0^t \frac{\rho_0\theta}{(\theta+\delta)g}u_y\,dyd\tau \\ &\le 2\int_0^l \rho_0\big((\theta+\delta)\big)\,dy - \int_0^l \rho_0\big(\ln(\theta_0+\delta)\big)\,dy + \int_0^l\int_0^t \frac{\rho_0\theta}{(\theta+\delta)g}u_y\,dyd\tau \\ &\le 2\int_0^l \rho_0\big((\theta+\delta)\big)\,dy + \int_0^l\int_0^t \frac{\rho_0^2\theta^2}{(\theta+\delta)g}\,dy\,d\tau + \frac{1}{2}\int_0^l\int_0^t \frac{u_y^2}{\theta+\delta)g}\,dy\,d\tau + C \\ &\le \frac{1}{2}\int_0^l\int_0^t \frac{u_y^2}{(\theta+\delta)g}\,dy\,d\tau + C\big(\|\rho\|_\infty, m_0, E_0\big)(1+t). \end{aligned}$$

Thus we get

$$\int_0^l \rho_0\big(\ln(\theta+\delta)\big) + \int_0^l\int_0^t \kappa\left(\frac{\theta_y^2}{(\theta+\delta)^2 g}\right)dy\,d\tau \le C(1+t). \tag{2.36}$$

From Proposition 2, inequality $(i)_4$, it holds that

$$\|\theta\|_{L^\infty} \le C(1+t)^2. \tag{2.37}$$

This completes Lemma 4. □

Combining all these lemmas, we have the following proposition.

Proposition 3 *Given $T \in (0,\infty)$. It holds that*

$$\begin{aligned}&\sup_{0\le t\le T}\left\|\left(\sqrt{\rho_0}u^2, \sqrt{\rho_0}\theta, \left|\frac{d_y}{g^{\frac{3}{4}}}\right|^2\right)\right\|_2^2 + \int_0^T \left(\|\sqrt{\rho_0}\theta\|_\infty^2 + \frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\|_\infty^2\right) dt \\ &\quad + \int_0^T \left\|\left(\partial_y\theta, u\partial_y u, \frac{1}{\sqrt{g}}\left(\left|\frac{d_y}{g}\right|^2\right)_y, \mu\frac{\partial_y u}{\sqrt{g}}, \frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right)\right\|_2^2 dt \\ &\le C\end{aligned} \tag{2.38}$$

and

$$\sup_{0\le t\le T}\|g\|_\infty^2 + \int_0^T \|\theta\|_\infty^2\, dt \le C \tag{2.39}$$

for a positive constant C depending only on the initial data R, c_v, μ, κ, m_1, $\mathcal{N}_1$, and T, where m_1 and $\mathcal{N}_1$ are the numbers in Proposition 1 and Proposition 2, respectively.

Proof Summing with (2.28), one obtains

$$\begin{aligned}&\frac{d}{dt}\left(\|\sqrt{\rho_0}\mathcal{E}\|_2^2 + \frac{C_1}{\mu}\|\sqrt{\rho_0}u^2\|_2^2 + \left\|\left(\frac{d_y}{g^{\frac{3}{4}}}\right)^2\right\|_2^2\right) \\ &\quad + \int_0^t \|\sqrt{\rho_0}\theta\|_\infty^2\, d\tau + \int_0^t \left\|\frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\right\|_\infty^2 (t)\, d\tau \\ &\quad + 5C_1\left\|\frac{u\partial_y u}{\sqrt{g}}\right\|_2^2 + \frac{\kappa c_v}{2}\left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2^2 + \left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\cdot\left(\frac{d_y}{g}\right)_y\right)\right\|_2^2 \\ &\le C\left(1 + \frac{R}{\mu}\int_0^t \|\rho_0\theta\|_\infty\, d\tau\right)\end{aligned}$$

for any $t \in (0,T)$, where C is a positive constant depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, and T. Applying the Gronwall inequality to the above inequality, one gets

$$\begin{aligned}&\sup_{0\le t\le T}\left\|\left(\sqrt{\rho_0}u^2, \sqrt{\rho_0}\theta, \left(\frac{d_y}{g^{\frac{3}{4}}}\right)^2\right)\right\|_2^2 + \int_0^T \left(\|\sqrt{\rho_0}\theta\|_\infty^2 + \frac{1}{\sqrt{\rho_0}}\left|\frac{d_y}{g}\right|^2\|_\infty^2\right) dt \\ &\quad + \int_0^T \left\|\left(\partial_y\theta, u\partial_y u, \frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\left(\frac{d_y}{g}\right)_y\right)\right)\right\|_2^2 dt \\ &\le C\end{aligned} \tag{2.40}$$

for a positive constant C depending only on the initial data $R, c_v, \mu, \kappa, m_1, \mathcal{N}_1$, and T. Thus the desired estimates

$$\sup_{0\le t\le T}\|g\|_\infty^2+\int_0^T\|\theta\|_\infty^2\,dt\le C$$

follow from (2.40) by applying Proposition 1 and (i) of Proposition 2. □

2.2 A priori H^1 estimates

This section is devoted to the a priori H^1 type estimates on (g,u,d_y,θ). Precisely, we will carry out the a priori $L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$ estimate on u, a priori $L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$ estimate on d_y, and the $L^\infty(0,T;H^1)$ estimate on g; however, due to the presence of the term $\frac{\mu}{g}(\partial_y u)^2$ and $g|\frac{1}{g}(\frac{d_y}{g})_y+\frac{1}{g^2}|d_y|^2d|^2$ on the right-hand side of the equation for θ $(1.8)_4$, one cannot get the desired a priori H^1 estimate of θ independent of the lower bound of the density without appealing to the higher than H^1 energy estimates. Before going to prove the H^1 bound for velocity, we first give the following estimates of director field, because the velocity field estimates strongly depend on the director field.

Lemma 5 *There exists a positive constant $C>0$ such that, for any $t\in[0,T]$,*

$$\int_0^l\left(\frac{1}{g}\left|\left(\frac{d_y}{g}\right)_y\right|^2+g|d_t|^2\right)dy+\int_0^T\int_0^l\frac{|d_{yt}|^2}{g}\,dy\,dt\le C, \tag{2.41}$$

$$\max_{y\in[0,1]}\left|\frac{d_y}{g}\right|\le C. \tag{2.42}$$

Proof Differentiating $(1.8)_3$ with respect to time, taking the dot product by $(\frac{d_y}{g})_y$, and integrating over interval $y\in[0,l]$, we obtain

$$\int_0^l d_{tt}\cdot\left(\frac{d_y}{g}\right)_y dy=\int_0^l\frac{1}{g}\left(\frac{d_y}{g}\right)_{yt}\cdot\left(\frac{d_y}{g}\right)_y dy+\int_0^l\left(\frac{|d_y|^2d}{g^2}\right)_t\cdot\left(\frac{d_y}{g}\right)_y dy. \tag{2.43}$$

The term on the left-hand side of (2.43) can be written with the aid of $(1.8)_1$ and constraints $|d|^2=1$ as follows:

$$\begin{aligned}\int_0^l d_{tt}\cdot\left(\frac{d_y}{g}\right)_y dy&=\frac{d}{dt}\int_0^l d_t\cdot\left(\frac{d_y}{g}\right)_y dy-\int_0^l d_t\cdot\left(\frac{d_y}{g}\right)_{yt}dy\\&=\frac{d}{dt}\int_0^l\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y+\frac{|d_y|^2d}{g^2}\right)\cdot\left(\frac{d_y}{g}\right)_y dy\\&\quad+\int_0^l d_{yt}\cdot\left(\frac{d_y}{g}\right)_t dy\\&=\frac{d}{dt}\int_0^l\frac{1}{g}\left|\left(\frac{d_y}{g}\right)_y\right|^2dy-\frac{d}{dt}\int_0^l\frac{1}{g^3}|d_y|^4\,dy+\int_0^l\frac{1}{g}|d_{yt}|^2\,dy\\&\quad-\int_0^l\frac{1}{g^2}(u_yd_{yt}\cdot d_y)\,dy.\end{aligned} \tag{2.44}$$

The first term on the right-hand side of (2.43) can be written as

$$\int_0^l \frac{1}{g}\left(\frac{d_y}{g}\right)_{yt}\cdot\left(\frac{d_y}{g}\right)_y dy = \frac{1}{2}\frac{d}{dt}\int_0^l \frac{1}{g}\left|\left(\frac{d_y}{g}\right)_y\right|^2 dy + \int_0^l \frac{u_y}{2g^2}\left|\left(\frac{d_y}{g}\right)_y\right|^2 dy, \tag{2.45}$$

where we have used equation $(1.8)_1$. Inserting (2.44) and (2.45) into (2.43), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^l \frac{1}{g}\left|\left(\frac{d_y}{g}\right)_y\right|^2 dy + \int_0^l \frac{|d_{yt}|^2}{g} dy \\
&\quad = \frac{d}{dt}\int_0^l \frac{1}{g^3}|d_y|^4 dy + \int_0^l \frac{1}{g^2}u_y(d_{yt}\cdot d_y)\,dy + \int_0^l \frac{u_y}{2g^2}\left|\left(\frac{d_y}{g}\right)_y\right|^2 dy \\
&\qquad + \int_0^l \left(\frac{|d_y|^2 d}{g^2}\right)_t\cdot\left(\frac{d_y}{g}\right)_y dy \\
&\quad =: \frac{d}{dt}\int_0^l \frac{1}{g^3}|d_y|^4 dy + \sum_{i=1}^3 Q_i.
\end{aligned} \tag{2.46}$$

Next the estimate of each term is given as follows: The term Q_1 is estimated as

$$\begin{aligned}
Q_1 &\le \delta\int_0^l \frac{1}{g}|d_{yt}|^2 dy + C(\delta)\int_0^l \frac{1}{g^3}u_y^2|d_y|^2 dy \\
&\le \delta\int_0^l \frac{1}{g}|d_{yt}|^2 dy + C(\delta)\max_{y\in[0,1]}\frac{|d_y|^2}{g^2}\int_0^l \frac{1}{g}u_y^2 dy.
\end{aligned} \tag{2.47}$$

Now the estimate on the last term Q_3 is given as

$$\begin{aligned}
Q_3 &= \int_0^l \left(\frac{|d_y|^2 d}{g^2}\right)_t\cdot\left(\frac{d_y}{g}\right)_y dy \\
&\le \int_0^l \left(-\frac{2|d_y|^2 dg_t}{g^3} + \frac{2(d_y\cdot d_{yt})d}{g^2} + \frac{|d_y|^2 d_t}{g^2}\right)\cdot\left(\frac{1}{g}d_y\right)_y dy \\
&\le C\left(\int_0^l \frac{|d_y|^4}{g^4}|u_y|\,dy + \int_0^l \frac{1}{g^3}|d_y|^3|d_{yt}|\,dy\right) \\
&\quad + C\int_0^l \frac{|d_y|^2}{g^2}|\sqrt{g}d_t|\left|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right| dy \\
&\le C\left(\max_{y\in[0,1]}\frac{|d_y|^2}{g^2}\right)^{\frac{3}{2}}\left\|\frac{u_y}{\sqrt{g}}\right\|_{L^2}\left\|\frac{d_y}{\sqrt{g}}\right\|_{L^2} + C\max_{y\in[0,1]}\left(\frac{|d_y|^2}{g^2}\right)\left\|\frac{d_y}{\sqrt{g}}\right\|_{L^2}\left\|\frac{d_{yt}}{\sqrt{g}}\right\|_{L^2} \\
&\quad + C\max_{y\in[0,1]}\left(\frac{|d_y|^2}{g^2}\right)\|\sqrt{g}d_t\|_{L^2}\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right\|_{L^2} \\
&\le \delta\left\|\frac{d_{yt}}{\sqrt{g}}\right\|_{L^2}^2 + C\left(\left\|\frac{u_y}{\sqrt{g}}\right\|_{L^2}^2 + \|\sqrt{g}d_t\|_{L^2}^2 + 1\right)\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right\|_{L^2}^2.
\end{aligned} \tag{2.48}$$

By using the Hölder inequality and embedding $W^{1,1}(0,l)\hookrightarrow L^\infty(0,l)$, Q_2 is estimated as

$$Q_2 \le \left(\int_0^l \left|\frac{u_y}{\sqrt{g}}\right|^2 dy\right)^{\frac{1}{2}}\left(\int_0^l \left|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right|^2 dy\right)^{\frac{1}{2}}\left(\max_{y\in[0,1]}\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right|^2\right)^{\frac{1}{2}}$$

$$\leq \left\| \frac{1}{\sqrt{g}} u_y \right\|_{L^2} \left\| \frac{1}{\sqrt{g}} \left(\frac{d_y}{g} \right)_y \right\|_{L^2} \left(\max_{y\in[0,1]} \left| \frac{1}{g} \left(\frac{d_y}{g} \right)_y \right|^2 \right)^{\frac{1}{2}}. \tag{2.49}$$

By using the Sobolev embedding and equations $(1.8)_3$, it follows that

$$\begin{aligned}
&\left(\max_{y\in[0,1]} \left| \frac{1}{g} \left(\frac{d_y}{g} \right)_y \right|^2 \right)^{\frac{1}{2}} \qquad (2.50)\\
&\leq \left\| \frac{1}{g} \left(\frac{d_y}{g} \right)_y \right\|_{L^2} + \left(\int_0^l \left| \frac{1}{g} \left(\frac{d_y}{g} \right)_y \cdot \left(\frac{1}{g} \left(\frac{d_y}{g} \right)_y \right)_y \right| dy \right)^{\frac{1}{2}} \\
&\leq \left\| \frac{1}{g} \left(\frac{d_y}{g} \right)_y \right\|_{L^2} + \left(\int_0^l \left| \frac{1}{g} \left(\frac{d_y}{g} \right)_y \cdot \left(d_{yt} - \left(\frac{1}{g^2} |d_y|^2 d \right)_y \right) \right| dy \right)^{\frac{1}{2}} \\
&\leq \left\| \frac{1}{g} \left(\frac{d_y}{g} \right)_y \right\|_{L^2} + \left\| \frac{1}{\sqrt{g}} \left(\frac{d_y}{g} \right)_y \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{1}{\sqrt{g}} d_{yt} \right\|_{L^2}^{\frac{1}{2}} \\
&\quad + \left(\int_0^l \left| \frac{1}{g} \left(\frac{d_y}{g} \right)_y \cdot \left(2 \frac{d_y}{g} \cdot \left(\frac{d_y}{g} \right)_y d + \frac{1}{g^2} |d_y|^2 d_y \right) \right| dy \right)^{\frac{1}{2}} \\
&\leq \left\| \frac{1}{g} \left(\frac{d_y}{g} \right)_y \right\|_{L^2} + \left\| \frac{1}{\sqrt{g}} \left(\frac{d_y}{g} \right)_y \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{1}{\sqrt{g}} d_{yt} \right\|_{L^2}^{\frac{1}{2}} \\
&\quad + C \left(\int_0^l \left| \frac{d_y}{g} \frac{|d_y|^2}{g^2} \cdot \left(\frac{d_y}{g} \right)_y \right| dy \right)^{\frac{1}{2}} \\
&\leq \left\| \frac{1}{g} \left(\frac{d_y}{g} \right)_y \right\|_{L^2} + \left\| \frac{1}{\sqrt{g}} \left(\frac{d_y}{g} \right)_y \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{1}{\sqrt{g}} d_{yt} \right\|_{L^2}^{\frac{1}{2}} \\
&\quad + C \max_{y\in[0,1]} \left(\frac{|d_y|^2}{g^2} \right) \left\| \frac{d_y}{\sqrt{g}} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{1}{\sqrt{g}} \left(\frac{d_y}{g} \right)_y \right\|_{L^2}^{\frac{1}{2}}, \qquad (2.51)
\end{aligned}$$

whereas in the fourth inequality the following fact is used: $\frac{1}{g}(\frac{d_y}{g})_y \cdot d + \frac{|d_y|^2}{g^2} = 0$. Plugging (2.50) into (2.49), using Young's inequality, we get

$$\begin{aligned}
Q_2 &\leq \left\| \frac{u_y}{\sqrt{g}} \right\|_{L^2} \left\| \frac{1}{\sqrt{g}} \left(\frac{d_y}{g} \right)_y \right\|_{L^2}^2 + \left\| \frac{u_y}{\sqrt{g}} \right\|_{L^2} \left\| \frac{1}{\sqrt{g}} \left(\frac{d_y}{g} \right)_y \right\|_{L^2}^{\frac{3}{2}} \left\| \frac{d_{yt}}{\sqrt{g}} \right\|_{L^2}^{\frac{1}{2}} \\
&\leq \delta \left\| \frac{d_{yt}}{\sqrt{g}} \right\|_{L^2}^2 + C \left(\left\| \frac{u_y}{\sqrt{g}} \right\|_{L^2}^2 + 1 \right) \left\| \frac{1}{\sqrt{g}} \left(\frac{d_y}{g} \right)_y \right\|_{L^2}^2. \qquad (2.52)
\end{aligned}$$

Plugging the estimates Q_1, Q_2, Q_3 into (2.46) and choosing δ small enough, it holds

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_0^l \frac{1}{g} \left| \left(\frac{d_y}{g} \right)_y \right|^2 dy + \int_0^l \frac{|d_{yt}|^2}{g} dy \\
&= \frac{d}{dt} \int_0^l \frac{|d_y|^4}{g^3} dy + C \left(\left\| \frac{u_y}{\sqrt{g}} \right\|_{L^2}^2 + \|\sqrt{g} d_t\|_{L^2}^2 + 1 \right) \left\| \frac{1}{\sqrt{g}} \left(\frac{d_y}{g} \right)_y \right\|_{L^2}^2. \qquad (2.53)
\end{aligned}$$

Integrating (2.53) with respect to time over the interval $[0,t]$, the resultant reads as

$$
\begin{aligned}
&\frac{1}{2}\int_0^l \frac{1}{g}\left|\left(\frac{d_y}{g}\right)_y\right|^2 dy + \int_0^t\int_0^l \frac{|d_{yt}|^2}{g}\,dy\,ds \\
&\quad\le C\left(1+\int_0^l \frac{|d_y|^4}{g^3}\,dy + \int_0^t\left(\left\|\frac{u_y}{\sqrt{g}}\right\|_{L^2}^2 + \|\sqrt{g}d_t\|_{L^2}^2 + 1\right)\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right\|_{L^2}^2 ds\right) \\
&\quad\le C\left(1+\max_{y\in[0,1]}\left(\frac{|d_y|^2}{g^2}\right)\left\|\frac{d_y}{\sqrt{g}}\right\|_{L^2}^2\right) \\
&\qquad + C\int_0^t\left(\left\|\frac{u_y}{\sqrt{g}}\right\|_{L^2}^2 + \left\|\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right\|_{L^2}^2 + 1\right)\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right\|_{L^2}^2 ds,
\end{aligned}
\tag{2.54}
$$

where we have used the fact that

$$
\int_0^l g|d_t|^2\,dy \le C\left(1+\int_0^l \frac{1}{g}\left|\left(\frac{d_y}{g}\right)_y\right|^2 dy\right).
\tag{2.55}
$$

Using the estimates in hand, energy conservation (2.2), and the Gronwall inequality, it follows

$$
\frac{1}{2}\int_0^l \frac{1}{g}\left|\left(\frac{d_y}{g}\right)_y\right|^2 dy + \int_0^t\int_0^l \frac{|d_{yt}|^2}{g}\,dy\,ds \le C,
\tag{2.56}
$$

which deduces that $\int_0^l g|d_t|^2\,dy \le C$. Similarly, by the Sobolev embedding $W^{1,1}([0,l]) \hookrightarrow L^\infty([0,l])$, inequality (2.42) is estimated. The proof of Lemma 5 is completed. □

<u>H^1 estimates for velocity:</u>

Define the effective viscous flux F as

$$
F := \mu\frac{\partial_y u}{g} - p - \frac{1}{2}\left|\frac{d_y}{g}\right|^2 = \mu\frac{\partial_y u}{g} - R\frac{\rho_0}{g}\theta - \frac{1}{2}\left|\frac{d_y}{g}\right|^2.
$$

Then, one can derive from (1.8) that

$$
\begin{aligned}
&\partial_t F - \frac{\mu}{g}\partial_y\left(\frac{\partial_y F}{\rho_0}\right) \\
&\quad = -\frac{\kappa R}{c_v g}\partial_y\left(\frac{\partial_y\theta}{g}\right) - \left(\frac{R}{c_v}+1\right)\frac{\partial_y u}{g}F - \frac{d_y}{g}\cdot\left(\frac{d_y}{g}\right)_t - \frac{R}{c_v}|d_t|^2 - \frac{R}{2c_v}\left|\frac{d_y}{g}\right|^2\frac{u_y}{g} \\
&\qquad - \frac{1}{2}\left|\frac{d_y}{g}\right|^2\frac{u_y}{g} \\
&\quad = -\frac{\kappa R}{c_v g}\partial_y\left(\frac{\partial_y\theta}{g}\right) - \left(\frac{R}{c_v}+1\right)\frac{\partial_y u}{g}F - \frac{d_y}{g}\cdot\frac{d_{yt}}{g} - \frac{R}{c_v}|d_t|^2 \\
&\qquad - \left(\frac{R}{2c_v}+\frac{1}{2}\right)\left|\frac{d_y}{g}\right|^2\frac{u_y}{g}.
\end{aligned}
\tag{2.57}
$$

Moreover, by equation (1.8)$_2$, one has $\partial_y F = \rho_0 \partial_t u$, from which, recalling the boundary conditions, we have

$$\partial_y F(0,t) = \partial_y F(l,t) = 0, \quad t \in (0,\infty).$$

We have the a priori L^2 estimates on F stated in the following.

Proposition 4 *Given $T \in (0,\infty)$. It holds that*

$$\sup_{0\le t\le T} \|F\|_2^2 + \int_0^T \left\| \frac{\partial_y F}{\sqrt{\rho_0}} \right\|_2^2 dt \le C$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, N, and T, where

$$N := \left\|\sqrt{\rho_0} u_0^2\right\|_2 + \|\sqrt{\rho_0}\theta_0\|_2 + \left\|u_0'\right\|_2 + \left\|d_0'\right\|_2,$$

and m_1 and $\mathcal{N}_1$ are the numbers in Proposition 1 and Proposition 2, respectively.

Proof Multiplying equation (2.57) by gF, integrating the resultant over $(0,l)$, and recalling $\partial_y F|_{y=0,l} = 0$, one gets from integration by parts that

$$\begin{aligned}
&\int_0^l \partial_t F g F\,dy + \mu \int_0^l \left| \frac{\partial_y F}{\sqrt{\rho_0}} \right|^2 dy \\
&= \kappa \frac{R}{c_v} \int_0^l \frac{\partial_y \theta \partial_y F}{g}\,dy - \left(\frac{R}{c_v} - 1\right) \int_0^l \partial_y u F^2\,dy \\
&\quad - \int_0^l \frac{d_y}{g} \cdot d_{yt} F\,dy - \frac{R}{c_v} \int_0^l |d_t|^2 g F\,dy - \left(\frac{R}{2c_v} - 1\right) \int_0^l \left|\frac{d_y}{g}\right|^2 u_y F\,dy.
\end{aligned}$$

Using (1.8)$_1$, one has

$$\begin{aligned}
\int_0^l \partial_t F g F\,dy &= \frac{1}{2}\frac{d}{dt} \int_0^l g F^2\,dy - \frac{1}{2} \int_0^l \partial_t g F^2\,dy \\
&= \frac{1}{2}\frac{d}{dt} \int_0^l g F^2\,dy - \frac{1}{2} \int_0^l \partial_y u F^2\,dy.
\end{aligned}$$

Therefore, it follows from the Hölder, Young, and Gagliardo–Nirenberg inequalities and Corollary 3 that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_0^l g F^2\,dy + \mu \int_0^l \left| \frac{\partial_y F}{\sqrt{\rho_0}} \right|^2 dy \\
&= \kappa \left(\frac{R}{c_v} - 1\right) \int_0^l \frac{\partial_y \theta \partial_y F}{g}\,dy + \left(\frac{3}{2} - \frac{R}{c_v}\right) \int_0^l \partial_y u F^2\,dy \\
&\quad - \int_0^l \frac{d_y}{g} \cdot d_{yt} F\,dy - \frac{R}{c_v} \int_0^l |d_t|^2 g F\,dy + \left(1 - \frac{R}{2c_v}\right) \int_0^l \left|\frac{d_y}{g}\right|^2 u_y F\,dy \\
&=: \sum_{i=1}^5 K_i.
\end{aligned}$$

Now we estimate each term as follows:

$$
\begin{aligned}
\sum_{i=1}^{5} K_i \le{}& \kappa \left|\frac{R}{c_v}-1\right|\sqrt{\bar\rho}\left\|\frac{\partial_y F}{\sqrt{\rho_0}}\right\|_2\left\|\frac{\partial_y\theta}{\sqrt{g}}\right\|_2+\left(\frac12-\frac{R}{c_v}\right)\|\partial_y u\|_2\|F\|_2\|F\|_\infty\\
&+\delta\left(\left\|\frac{d_{yt}}{\sqrt g}\right\|^2+\frac{1}{\sqrt g}\left\|\left(\frac{d_y}{g}\right)_y\right\|^2+1\right)+C(\delta)\left[\left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty\right.\\
&\left.+\left(1-\frac{R}{2c_v}\right)\|u_y\|_2^2+\frac{R}{c_v}\|d_t\|_2^2\right]\|F\|_2^2\\
\le{}& C\left(\left\|\frac{\partial_y F}{\sqrt{\rho_0}}\right\|_2\|\partial_y\theta\|_2+\|\partial_y u\|_2\|F\|_2^{\frac32}\left(\|F\|_2+\|\partial_y F\|_2\right)^{\frac12}\right)\\
&+\delta\left(\left\|\frac{d_{yt}}{\sqrt g}\right\|^2+\frac{1}{\sqrt g}\left\|\left(\frac{d_y}{g}\right)_y\right\|^2+1\right)\\
&+C(\delta)\left(\left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty+\left(1-\frac{R}{2c_v}\right)\|u_y\|_2^2+\frac{R}{c_v}\|d_t\|_2^2\right)\|F\|_2^2\\
\le{}& \frac{\mu}{2}\left\|\frac{\partial_y F}{\sqrt{\rho_0}}\right\|_2^2+C\left[\left(1+\left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty+\|d_t\|_2^2+\|\partial_y u\|_2^2\right)\|F\|_2^2+\|\partial_y\theta\|_2^2\right.\\
&\left.+\left(\left\|\frac{d_{yt}}{\sqrt g}\right\|^2+\frac{1}{\sqrt g}\left\|\left(\frac{d_y}{g}\right)_y\right\|^2+1\right)\right],
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\frac{d}{dt}\|\sqrt g F\|_2^2+\mu\left\|\frac{\partial_y F}{\sqrt{\rho_0}}\right\|_2^2\\
&\quad\le C\left[\left(1+\left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty+\|d_t\|_2^2+\|\partial_y u\|_2^2\right)\|F\|_2^2\right.\\
&\qquad\left.+\|\partial_y\theta\|_2^2+\left(\left\|\frac{d_{yt}}{\sqrt g}\right\|^2+\frac{1}{\sqrt g}\left\|\left(\frac{d_y}{g}\right)_y\right\|_2^2+1\right)\right]
\end{aligned}
\tag{2.58}
$$

for any $t\in(0,T)$, where C is a positive constant depending only on R, c_v, μ, κ, m_1. Applying the Gronwall inequality to (2.58) and using Corollary 3, the conclusion follows. □

Based on Proposition 4 and Corollary 3, we can obtain the desired H^1 type estimates on g, u, $\frac1g(\frac{d_y}{g})_y$, stated as follows.

Proposition 5 *Given $T\in(0,\infty)$. It holds that*

$$
\begin{aligned}
&\sup_{0\le t\le T}\left(\|\partial_y g\|_2^2+\|\partial_y u\|_2^2+\left\|\frac1g\left(\frac{d_y}{g}\right)_y\right\|_2^2+\|d_t\|_2^2\right)\\
&\qquad+\int_0^T\left(\|\sqrt{\rho_0}\partial_t u\|_2^2+\left\|\partial_y^2 u\right\|_2^2\right)dt+\int_0^T\left\|\frac{1}{\sqrt g}\left(\frac1g\left(\frac{d_y}{g}\right)_y+d\left|\frac{d_y}{g}\right|^2\right)_y\right\|^2 dt\\
&\quad\le C
\end{aligned}
\tag{2.59}
$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T, where m_1, $\mathcal{N}_1$, and $\mathcal{N}_2$ are the numbers in Propositions 1, 2, and 4, respectively.

Proof

$$\underbrace{\sup_{0\le t\le T}\left(\|\partial_y u\|_2^2+\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right\|_2^2+\|d_t\|_2^2\right)}_{I_1}+\underbrace{\int_0^T\|\sqrt{\rho_0}\partial_t u\|_2^2+\left\|\frac{d_{yt}}{\sqrt{g}}\right\|_2^2 dt}_{I_2}\le C.$$

The estimates of velocity term in I_1 and I_2 are straightforward from a priori estimates in hand and by the definition of F, noticing that $\rho_0\partial_t u=\partial_y F$. Note that, by the Sobolev embedding inequality, it follows from Proposition 4 that

$$\int_0^T\|F\|_\infty^2\,dt\le C,\qquad \int_0^l\|F\|_{H^1}^2\,dt\le C \tag{2.60}$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T.

However, the a priori estimates on terms $\|\partial_y g\|_2^2$ and $\int_0^T\|\partial_y^2 u\|_2^2\,dt$ need to be computed. Rewrite $(1.8)_1$ in terms of F as

$$\partial_t g=\frac{1}{\mu}\left(gF+R\rho_0\theta+\frac{g}{2}\left|\frac{d_y}{g}\right|^2\right).$$

Differentiating the above equations in y, multiplying the resultant by $\partial_y g$, and integrating over $(0,l)$, it follows from the Hölder and Young inequalities that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_y g\|_2^2&=\frac{1}{\mu}\int_0^l\left(F|\partial_y g|^2+\partial_y Fg\partial_y g+R\left(\rho_0'\theta+\rho_0\partial_y\theta\right)\partial_y g\right)dy\\&\quad+\frac{1}{\mu}\int_0^l\left[d_y\cdot\left(\frac{d_y}{g}\right)_y\partial_y g+\frac{1}{2}\left|\frac{d_y}{g}\right|^2|\partial_y g|^2\right]dy=:\mathcal{T}.\end{aligned}$$

Now we compute $\mathcal{T}$ as

$$\begin{aligned}\mathcal{T}&\le\frac{1}{\mu}\left(\|F\|_\infty\|\partial_y g\|_2^2+\|g\|_\infty\|\partial_y F\|_2\|\partial_y g\|_2+R\left(\|\theta\|_\infty\left\|\rho_0'\right\|_2+\bar\rho\|\partial_y\theta\|_2\right)\|\partial_y g\|_2\right)\\&\quad+C\max_{y\in[0,1]}\left|\frac{d_y}{g}\right|\left\|\left(\frac{d_y}{g}\right)_y\right\|_2\|\partial_y g\|_2+\frac{1}{2}\left\|\frac{d_y}{g}\right\|_\infty^2\|\partial_y g\|_2^2\\&\le C\left(\|F\|_\infty^2+\left\|\frac{d_y}{g}\right\|_\infty^2+1\right)\|\partial_y g\|_2^2\\&\quad+C\left(\|g\|_\infty^2\|\partial_y F\|_2^2+\left\|\rho_0'\right\|_2^2\|\theta\|_\infty^2+\bar\rho^2\|\partial_y\theta\|_2^2\right)+C\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right\|_2^2\end{aligned}$$

for a positive constant C depending only on R and μ. Applying the Gronwall inequality, it follows from (2.60), Corollary 3, and Proposition 4 that

$$\sup_{0\le t\le T}\|\partial_y g\|_2^2\le C \tag{2.61}$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T.

A priori estimates on $\int_0^T \|\partial_y^2 u\|_2^2\,dt$:

Noticing that $\partial_y u = \frac{1}{\mu}(gF + R\rho_0\theta + \frac{g}{2}|\frac{d_y}{g}|^2)$, one has

$$\partial_y^2 u = \frac{1}{\mu}\left(\partial_y gF + g\partial_y F + R\rho_0'\theta + R\rho_0\partial_y\theta + d_y\cdot\left(\frac{d_y}{g}\right)_y + \frac{1}{2}\left|\frac{d_y}{g}\right|^2 g_y\right),$$

thus by the Hölder inequality, (2.60), (2.61), it follows from Corollary 3 and Proposition 4 that

$$\begin{aligned}
&\int_0^T \|\partial_y^2 u\|_2^2\,dt \\
&\quad\le C\int_0^T \left(\|\partial_y g\|_2^2\|F\|_\infty^2 + \|g\|_\infty^2\|\partial_y F\|_2^2 + \|\rho_0'\|_2^2\|\theta\|_\infty^2 + \|\partial_y\theta\|_2^2\right) \\
&\qquad + C\left[\left(\max_{t\in[0,T]}\left(\max_{y\in[0,1]}\left|\frac{d_y}{g}\right|\right)^2 + 1\right)\int_0^T\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right\|_2^2 dt + \int_0^T \|u_y\|_2^2\,dt\right] \\
&\quad\le C\int_0^T\left(\|F\|_\infty^2 + \|g\|_\infty^2\|\partial_y F\|_2^2 + \|\theta\|_\infty^2 + \|\partial_y\theta\|_2^2 + \left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right\|_2^2 + \|u_y\|_2^2\right)dt \\
&\quad\le C
\end{aligned}$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T, proving the conclusion. □

We summarize the estimates obtained in this subsection as follows.

Corollary 1 *Given $T\in(0,\infty)$. It holds that*

$$\begin{aligned}
&\sup_{0\le t\le T}\left\|\left(F,\partial_y g,\partial_y u,\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y,\sqrt{g}d_t\right)\right\|^2 \\
&\quad + \int_0^T\left\|\left(\frac{\partial_y F}{\sqrt{\rho_0}},\partial_y^2 u,\sqrt{\rho_0}\partial_t u,\frac{d_{yt}}{\sqrt{g}}\right)\right\|_2^2 dt \le C
\end{aligned}$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T, where m_1, $\mathcal{N}_1$, and $\mathcal{N}_2$ are the numbers in Propositions 1, 2, and 4, respectively.

2.3 A priori H^2 estimates

This subsection is devoted to the a prior H^2 estimates on (g,u,d_y,θ). As will be shown in this subsection, one can get the desired a priori $L^\infty(0,T;H^2)$ estimate of θ without using the a priori $L^\infty(0,T;H^1)$ bound of it.

As a preparation, we first give some estimates on $\|\partial_y\theta\|_2$ and $\|\partial_t\theta\|_\infty$ in terms of $\|\sqrt{\rho_0}\partial_t\theta\|_2$ and $\|\partial_y\partial_t\theta\|_2$, and $\|\partial_t u\|_\infty$, in terms of $\|\sqrt{\rho_0}\partial_t u\|_2$ and $\|\partial_y\partial_t u\|_2$, which will be used later in higher order a priori estimates.

Proposition 6 *Given $T\in(0,\infty)$; (i) It holds that*

$$\|\partial_y\theta\|_2^2 \le C\big(1 + \|\sqrt{\rho_0}\partial_t\theta\|_2\big)$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T, where m_1, $\mathcal{N}_1$, and $\mathcal{N}_2$ are the numbers in Propositions 1, 2, and 4, respectively.

(*ii*) *It holds that*

$$\|\partial_t\theta\|_\infty \le \sqrt{\frac{2}{\Omega_0\bar{\rho}}}\|\sqrt{\rho_0}\partial_t\theta\|_2 + \sqrt{l}\left\|\frac{\partial_y\partial_t\theta}{\sqrt{g}}\right\|_2,$$

$$\|\partial_t u\|_\infty \le \sqrt{\frac{2}{\Omega_0\bar{\rho}}}\|\sqrt{\rho_0}\partial_t u\|_2 + \sqrt{l}\left\|\frac{\partial_y\partial_t u}{\sqrt{g}}\right\|_2,$$

where Ω_0 is the number in Proposition 2.

Proof (i) Multiplying (2.86) by θ, integrating the resultant over $(0,l)$, and integrating by parts, it follows from the Hölder inequality that

$$\begin{aligned}
&\kappa\int_0^l\left|\frac{\partial_y\theta}{\sqrt{g}}\right|^2 dy\\
&\quad= \int_0^l\left(\partial_y uF + \frac{1}{2}\left|\frac{d_y}{g}\right|^2\partial_y uF + g\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{1}{g^2}|d_y|^2 d\right|^2 - c_v\rho_0\partial_t\theta\right)\theta\,dy\\
&\quad\le \left(\|\partial_y u\|_2\|F\|_2 + \|\partial_y u\|_2\|F\|_2\left\|\frac{d_y}{g}\right\|_\infty^2 + \|\sqrt{g}d_t\|_2^2\right)\|\theta\|_\infty + c_v\|\sqrt{\rho_0}\partial_t\theta\|_2\|\sqrt{\rho_0}\theta\|_2,
\end{aligned}$$

from which, by Corollaries 3–1 and (i) of Proposition 2, we have that

$$\begin{aligned}
\|\partial_y\theta\|_2^2 &\le C\int_0^l\left|\frac{\partial_y\theta}{\sqrt{g}}\right|^2 dy \le C\big(\|\theta\|_\infty + \|\sqrt{\rho_0}\partial_t\theta\|_2\big)\\
&\le C\big(\|\partial_y\theta\|_2 + 1 + \|\sqrt{\rho_0}\partial_t\theta\|_2\big) \le \frac{1}{2}\|\partial_y\theta\|_2^2 + C\big(1 + \|\sqrt{\rho_0}\partial_t\theta\|_2\big),
\end{aligned}$$

and thus,

$$\|\partial_y\theta\|_2^2 \le C\big(1 + \|\sqrt{\rho_0}\partial_t\theta\|_2\big)$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T.

(ii) Recall that $\Omega_0 := \{y \in (0,l)|\rho_0(y) \ge \frac{\bar{\rho}}{2}\}$ and $|\Omega_0| > 0$. Noticing

$$\partial_t\theta(y,t) = \frac{1}{|\Omega_0|}\int_{\Omega_0}\partial_t\theta(z,t)\,dz + \frac{1}{|\Omega_0|}\int_{\Omega_0}\int_z^y\partial_y\partial_t\theta(\xi,t)\,d\xi\,dz,$$

it follows from the Hölder inequality that

$$\begin{aligned}
\big|\partial_t\theta(y,t)\big| &\le \frac{1}{|\Omega_0|}\left|\int_{\Omega_0}\frac{\sqrt{\rho_0}\partial_t\theta}{\sqrt{\rho_0}}\,dz\right| + \int_0^l\big|\partial_y\partial_t\theta(\xi,t)\big|\,d\xi\\
&\le \sqrt{\frac{2}{|\Omega_0|\bar{\rho}}}\|\sqrt{\rho_0}\partial_t\theta\|_2 + \left(\int_0^l\left|\frac{\partial_y\partial_t\theta}{\sqrt{g}}\right|^2 d\xi\right)^{\frac{1}{2}}\left(\int_0^l g\,d\xi\right)^{\frac{1}{2}}\\
&= \sqrt{\frac{2}{|\Omega_0|\bar{\rho}}}\|\sqrt{\rho_0}\partial_t\theta\|_2 + \sqrt{l}\left\|\frac{\partial_y\partial_t\theta}{\sqrt{g}}\right\|_2,
\end{aligned}$$

which implies

$$\|\partial_t\theta\|_\infty \le \sqrt{\frac{2}{|\Omega_0|\bar{\rho}}}\|\sqrt{\rho_0}\partial_t\theta\|_2 + \sqrt{l}\left\|\frac{\partial_y\partial_t\theta}{\sqrt{g}}\right\|_2.$$

In the same way as above, the same conclusion holds for $\partial_t u$. □

Proposition 7 *Given $T \in (0,\infty)$. It holds that*

$$\sup_{0\le t\le T}\left\|\left(\sqrt{\rho_0}\partial_t\theta, \frac{\partial_y F}{\sqrt{\rho_0}}\right)\right\|_2^2 + \int_0^T \left\|(\partial_t F, \partial_y\partial_t\theta)\right\|_2^2 dt \le C\big(\|h_1\|_2 + \|h_2\|_2\big)$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T, where

$$h_1 := \frac{\mu u_0'' - R(\rho_0\theta_0)' + (|d_0'|^2)'}{\sqrt{\rho_0}}, \qquad h_2 := \frac{1}{\sqrt{\rho_0}}\left[\mu\left(u_0'\right)^2 + \kappa\theta_0'' + \left(|d_{0t}|^2\right)' - Ru_0'\rho_0\theta_0\right],$$

and m_1, $\mathcal{N}_1$, and $\mathcal{N}_2$ are the numbers in Propositions 1, 2, and 4, respectively.

Proof Rewrite $(1.8)_4$ as

$$c_v\rho_0\partial_t\theta - \kappa\partial_y\left(\frac{\partial_y\theta}{g}\right) = \partial_y uF + \frac{1}{2}\left|\frac{d_y}{g}\right|^2\partial_y u + g\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{1}{g^2}|d_y|^2 d\right|^2, \tag{2.62}$$

or equivalently,

$$\begin{aligned} c_v\rho_0\partial_t\theta - \kappa\partial_y\left(\frac{\partial_y\theta}{g}\right) &= \frac{1}{\mu}\left(gF + R\rho_0\theta + \frac{1}{2g}|d_y|^2\right)F \\ &\quad + \left(g\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{1}{g^2}|d_y|^2 d\right|^2 + \frac{1}{g^2}|d_y|^2 u_y\right), \end{aligned} \tag{2.63}$$

from which, differentiating in t and using $(1.8)_1$, one has

$$\begin{aligned} &c_v\rho_0\partial_t^2\theta - \kappa\partial_y\left(\frac{\partial_y\partial_t\theta}{g} - \frac{\partial_y u\partial_y\theta}{g^2}\right) \\ &= \frac{1}{\mu}\left(\partial_y uF^2 + 2gF\partial_t F\right) + \frac{R\rho_0}{\mu}(\partial_t\theta F + \theta\partial_t F) + \frac{1}{2\mu}\left(\partial_t\left(\frac{|d_y|^2}{g}\right)F + \frac{|d_y|^2}{g}\partial_t F\right) \\ &\quad + \left(g\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{1}{g^2}|d_y|^2 d\right|^2 + \frac{1}{g^2}|d_y|^2 u_y\right)_t \\ &= \frac{\partial_y u}{\mu}F^2 + \frac{1}{\mu}\left(2gF + R\rho_0\theta + \frac{1}{2}\frac{|d_y|^2}{g}\right)\partial_t F + \left(\frac{R}{\mu}\rho_0\partial_t\theta + \frac{1}{2\mu}\partial_t\left(\frac{|d_y|^2}{g}\right)\right)F \\ &\quad + \left(g\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{1}{g^2}|d_y|^2 d\right|^2 + \frac{1}{g^2}|d_y|^2 u_y\right)_t. \end{aligned}$$

Multiplying the above equation by $\partial_t\theta$, integrating the resultant over $(0,l)$, one gets from integration by parts that

$$\frac{c_v}{2}\frac{d}{dt}\int_0^l \rho_0|\partial_t\theta|^2\,dy + \kappa\int_0^l\left|\frac{\partial_y\partial_t\theta}{\sqrt{g}}\right|^2 dy$$

$$
\begin{aligned}
&= \kappa \int_0^l \frac{\partial_y u \partial_y \theta}{g^2} \partial_y \partial_t \theta \, dy + \int_0^l \left(\frac{1}{\mu} R\rho_0 F(\partial_t \theta) + \frac{1}{2\mu} \partial_t \left(\frac{|d_y|^2}{g} \right) F \right) \partial_t \theta \, dy \\
&\quad + \frac{1}{\mu} \int_0^l \left[\left(2gF + R\rho_0 \theta + \frac{1}{2} \frac{|d_y|^2}{g} \right) \partial_t F + \partial_y u F^2 \right] \partial_t \theta \, dy \\
&\quad + \int_0^l \left(g \left| \frac{1}{g} \left(\frac{d_y}{g} \right)_y + \frac{1}{g^2} |d_y|^2 d \right|^2 + \frac{1}{g^2} |d_y|^2 u_y \right)_t \partial_t \theta \, dy. \qquad (2.64)
\end{aligned}
$$

The terms on the right-hand side of (2.64) are estimated as follows. By Corollary 3, it follows from the Young inequality and (i) of Proposition 6 that

$$
\begin{aligned}
\kappa \int_0^l \frac{\partial_y u \partial_y \theta}{g^2} \partial_y \partial_t \theta \, dy &\le \frac{\kappa}{4} \left\| \frac{\partial_y \partial_t \theta}{\sqrt{g}} \right\|_2^2 + C \|\partial_y u\|_\infty^2 \|\partial_y \theta\|_2^2 \\
&\le \frac{\kappa}{4} \left\| \frac{\partial_y \partial_t \theta}{\sqrt{g}} \right\|_2^2 + C \left(\|F\|_\infty^2 + \|\theta\|_\infty^2 + \left\| \left| \frac{d_y}{g} \right|^2 \right\|_\infty^2 \right) \|\partial_y \theta\|_2^2 \\
&\le \frac{\kappa}{4} \left\| \frac{\partial_y \partial_t \theta}{\sqrt{g}} \right\|_2^2 \\
&\quad + C \left(\|F\|_\infty^2 + \|\theta\|_\infty^2 + \left\| \left| \frac{d_y}{g} \right|^2 \right\|_\infty^2 \right) \left(1 + \|\sqrt{\rho_0} \partial_t \theta\|_2 \right)
\end{aligned}
$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, and T. By Corollary 3, Corollary 1, and (ii) of Proposition 6, it follows from the Hölder and Young inequalities that

$$
\begin{aligned}
&\frac{1}{\mu} \int_0^l \left[\left(2gF + R\rho_0 \theta + \frac{1}{2} \frac{|d_y|^2}{g} \right) \partial_t F + \partial_y u F^2 \right] \partial_t \theta \, dy \\
&\le C \left[\left(\|g\|_\infty \|F\|_2 + \|\sqrt{\rho_0} \theta\|_2 + \left\| \frac{d_y}{\sqrt{g}} \right\|_2^2 \right) \|\partial_t F\|_2 + \|\partial_y u\|_2 \|F\|_2 \|F\|_\infty \right] \|\partial_t \theta\|_\infty \\
&\le C \left(\|\partial_t F\|_2 + \|F\|_\infty \right) \left(\|\sqrt{\rho_0} \partial_t \theta\|_2 + \left\| \frac{\partial_y \partial_t \theta}{\sqrt{g}} \right\|_2 \right) \\
&\le \frac{\kappa}{4} \left\| \frac{\partial_y \partial_t \theta}{\sqrt{g}} \right\|_2^2 + C \left(\|\sqrt{g} \partial_t F\|_2^2 + \|\sqrt{\rho_0} \partial_t \theta\|_2^2 + \|F\|_\infty^2 \right)
\end{aligned}
$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T. Before going to estimate the next term, we first compute

$$
\begin{aligned}
&\int_0^l \left(2g \left| \frac{1}{g} \left(\frac{d_y}{g} \right)_y + \frac{1}{g^2} |d_y|^2 d \right|^2 + \frac{1}{g^2} |d_y|^2 u_y \right)_t \theta_t \, dy \\
&= \int_0^l \left(2g |d_t|^2 + \frac{1}{g^2} |d_y|^2 u_y \right)_t \theta_t \, dy \\
&= \int_0^l \left[4g d_t d_{tt} + 2|d_t|^2 u_y + \frac{1}{g^2} |d_y|^2 u_{yt} + \frac{u_y}{g} \left(\frac{|d_y|^2}{g} \right)_t - \frac{(u_y)^2}{g^2} \frac{|d_y|^2}{g} \right] \theta_t \, dy \\
&=: \mathcal{U}. \qquad (2.65)
\end{aligned}
$$

The right-hand side is estimated as follows:

$$
\begin{aligned}
\mathcal{U} &\le \delta\big(\|\sqrt{g}d_{tt}\|^2 + \|u_{yt}\|_2^2\big) + C\|\sqrt{g}d_t\|^2\|\theta_t\|_\infty^2 + C\left\|\frac{d_y}{\sqrt{g}}\right\|_\infty^2\|\theta_t\|_\infty^2 \\
&\quad + C\|u_y\|_2^2\|\left|\frac{d_y}{g}\right|^2\|_\infty^2 + \frac{1}{2}\|u_y\|_2^2\|\theta_t\|_\infty^2 \\
&\le \delta\left(\|\sqrt{g}d_{tt}\|^2 + \left[\partial_t\left(\frac{|d_y|^2}{g}\right)\right]^2 + \|\sqrt{g}\partial_t G\|_2^2\right) + C\|u_y\|_2^2\|\left|\frac{d_y}{g}\right|^2\|_\infty^2 \\
&\quad + C\|\sqrt{g}d_t\|^2\|\theta_t\|_\infty^2 + \frac{1}{2}\|u_y\|_2^2\|\theta_t\|_\infty^2 + C\|\partial_t\theta\|_\infty^2 + C\left\|\frac{d_y}{\sqrt{g}}\right\|_\infty^2\|\theta_t\|_\infty^2 \\
&\le \delta\left(\|\sqrt{g}d_{tt}\|^2 + \left\|\partial_t\left(\frac{|d_y|^2}{g}\right)\right\|_2^2 + \|\sqrt{g}\partial_t G\|_2^2\right) + C\|\theta_t\|_\infty^2 \\
&\le \delta\left(\|\sqrt{g}d_{tt}\|^2 + C\left(\|u_y\|_2^2 + \left\|\frac{d_{yt}}{\sqrt{g}}\right\|_2^2\right) + \|\sqrt{g}\partial_t G\|_2^2\right) + C\|\theta_t\|_\infty^2 \\
&\le \delta\big(\|\sqrt{g}d_{tt}\|^2 + \|\sqrt{g}\partial_t G\|_2^2\big) + C\|\theta_t\|_\infty^2.
\end{aligned}
\tag{2.66}
$$

The second integral on the right-hand side of (2.64) is estimated as

$$
\begin{aligned}
&\int_0^l \left(\frac{1}{\mu}R\rho_0 F(\partial_t\theta) + \frac{1}{2\mu}\partial_t\left(\frac{|d_y|^2}{g}\right)F\right)\partial_t\theta\, dy \\
&\quad \le \delta\left[\partial_t\left(\frac{|d_y|^2}{g}\right)\right]^2 + C\|\partial_t\theta\|_\infty^2 \\
&\quad \le \delta\left(\|u_y\|_2^2 + \left\|\frac{d_{yt}}{\sqrt{g}}\right\|_2^2\right) + C\|\partial_t\theta\|_\infty^2,
\end{aligned}
\tag{2.67}
$$

where we have used the Hölder, Young inequalities and mass conservation equation. Therefore, one obtains from (2.64) that

$$
\begin{aligned}
&c_v\frac{d}{dt}\|\sqrt{\rho_0}\partial_t\theta\|_2^2 + \kappa\left\|\frac{\partial_y\partial_t\theta}{\sqrt{g}}\right\|_2^2 \le C_3\big(\|\sqrt{g}d_{tt}\|^2 + \|\sqrt{g}\partial_t F\|_2^2\big) \\
&\quad + \left[\left(1 + \|F\|_\infty^2 + \|\theta\|_\infty^2 + \left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty^2\right)\big(\|\sqrt{\rho_0}\partial_t\theta\|_2^2 + 1\big)\right]
\end{aligned}
\tag{2.68}
$$

for a positive constant C_3 depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T. In order to control the above estimates, there is a need to get dissipation estimates on $(\|\sqrt{g}d_{tt}\|^2, \|\sqrt{g}\partial_t F\|_2^2)$.

Next, we give the higher order derivative estimates of the director vector field d. The identity $\frac{1}{g}(\frac{d_y}{g})_y \cdot d + \frac{|d_y|^2}{g^2} = 0$, which is obtained from the director field equation with the help of constraint $|d|^2 = 1$, is frequently used in the proof of the lemma. This fact replaces the higher derivative with nonlinearity.

Lemma 6 *For any* $0 < t \le T$, *it holds that*

$$
\int_0^l \left(\frac{1}{g}\left|\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right|^2 + \frac{1}{g}|d_{yt}|^2\right) dy + \int_0^T\int_0^l \frac{1}{g}\left|\left(\frac{d_y}{g}\right)_{yt}\right|^2 dy\, dt \le C. \tag{2.69}
$$

Proof In order to prove higher order derivative estimates on the director field, the derivative with respect to y variable is taken on both sides of director field equation (1.8)$_3$ and the resultant is multiplied by $\frac{1}{g}$, then it follows

$$\frac{1}{g}d_{yt} = \frac{1}{g}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y + \frac{1}{g}\left(\frac{d|d_y|^2}{g^2}\right)_y. \tag{2.70}$$

Taking the dot product of (2.70) on both sides with $(\frac{1}{g}(\frac{d_y}{g})_y)_{yt}$ and integrating over space variable, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^l \left(\frac{1}{g}\left|\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right|^2\right) dy \\
&= -\int_0^l \frac{1}{g^2}g_t\frac{1}{g}\left|\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right|^2 dy + 2\int_0^l \frac{1}{g}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y \cdot \left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_{yt} dy \\
&= -\int_0^l \frac{1}{g^2}u_y\frac{1}{g}\left|\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right|^2 dy + 2\int_0^l \frac{d_{yt}}{g}\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_{yt} dy \\
&\quad - 2\int_0^l \frac{1}{g}\left(\frac{|d_y|^2}{g^2}d\right)_y \cdot \left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_{yt} dy =: \sum_{i=1}^3 R_i,
\end{aligned} \tag{2.71}$$

where in the second equality we have used $\frac{1}{g}(\frac{d_y}{g})_y \cdot d + \frac{|d_y|^2}{g^2} = 0$. This fact replaces the higher derivative with nonlinearity. The key point is to control the higher derivative with lower one by using the constraint $|d|^2 = 1$. Now the terms R_1, R_2, R_3 are estimated one by one.

The term R_1 is estimated as

$$\begin{aligned}
R_1 &\le \max_{y\in[0,1]}\left(\frac{u_y}{g}\right)\int_0^l \frac{1}{g}\left|\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right|^2 dy \\
&\le C\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}^2.
\end{aligned} \tag{2.72}$$

Before estimating the term R_2, we first make use of (1.8)$_1$ and compute the integral. The simplified form is written as

$$\begin{aligned}
R_2 &= 2\int_0^l \frac{d_{yt}}{g}\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_{yt} dy \\
&= 2\int_0^1 \left(\left(\frac{d_y}{g}\right)_t \cdot \left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_{yt} - \frac{g_t d_y}{g^2}\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right) dy \\
&= -2\int_0^1 \left(\frac{d_y}{g}\right)_{yt}\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_t dy + \int_0^1 \frac{2u_y d_y}{g^2}\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy \\
&= -2\int_0^l \left(\frac{1}{g}\left(\frac{d_y}{g}\right)_{yt}\cdot\left(\frac{d_y}{g}\right)_{yt} - \frac{u_y}{g^2}\left(\frac{d_y}{g}\right)_y\cdot\left(\frac{d_y}{g}\right)_{yt}\right) dy \\
&\quad + \int_0^1 \frac{2u_y d_y}{g^2}\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy.
\end{aligned} \tag{2.73}$$

Now we are in a position to estimate the term R_2, where we will use the definition of effective viscous flux:

$$
\begin{aligned}
R_2 \le & -2\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_{yt}\right\|_{L^2}^2 + 2\max_{y\in[0,1]}\left(\frac{u_y}{g}\right)\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_{yt}\right\|_{L^2}\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_{y}\right\|_{L^2} \\
& + 2\max_{y\in[0,1]}\left(\frac{u_y}{g}\right)\left\|\frac{d_y}{\sqrt{g}}\right\|_{L^2}\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2} \\
\le & -\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_{yt}\right\|_{L^2}^2 \\
& + C\left(\|F\|_\infty^2 + \|\theta\|_\infty^2 + \left\|\frac{d_y}{g}\right\|_\infty^4\right)\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}^2 + C.
\end{aligned} \tag{2.74}
$$

It is hard to estimate the term R_3, so there is the need to simplify the integral so that the required bounds can be obtained, thus we simplify it as

$$
\begin{aligned}
R_3 = & -2\frac{d}{dt}\int_0^l \frac{1}{g^2}\left(\frac{|d_y|^2}{g^2}d\right)_y \cdot \left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy \\
& -2\int_0^l \frac{u_y}{g^2}\left(\frac{|d_y|^2}{g^2}d\right)_y \cdot \left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy \\
& +2\int_0^l \frac{1}{g}\left(\frac{|d_y|^2}{g^2}d\right)_{yt} \cdot \left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy \\
=: & -2\frac{d}{dt}\int_0^l \frac{1}{g^2}\left(\frac{|d_y|^2}{g^2}d\right)_y \cdot \left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy + \sum_{i=1}^2 R_{3i}.
\end{aligned} \tag{2.75}
$$

In order to compute the term R_3, we estimate terms R_{31} and R_{32} one by one. The term R_{31} is controlled as follows:

$$
\begin{aligned}
R_{31} = & -2\int_0^l \frac{u_y}{g^2}\left(\frac{2d_y}{g}\cdot\left(\frac{d_y}{g}\right)_y d + \frac{|d_y|^2}{g^2}d_y\right)\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy \\
\le & \, 2\max_{y\in[0,1]}\left|\frac{u_y}{g}\right|\max_{y\in[0,1]}\left|\frac{d_y}{g}\right|\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right\|_{L^2}\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2} \\
& + 2\max_{y\in[0,1]}\left|\frac{u_y}{g}\right|\max_{y\in[0,1]}\left|\frac{d_y}{g}\right|^2\left\|\frac{d_y}{\sqrt{g}}\right\|_{L^2}\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2} \\
\le & \, C\left(\|F\|_\infty^2 + \|\theta\|_\infty^2 + \left\|\frac{d_y}{g}\right\|_\infty^4\right)\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}^2 + C,
\end{aligned} \tag{2.76}
$$

where Lemma 1, Lemma 5, and the director field equation are used.

The term R_{32} cannot be estimated directly, so we first need to simply the integral, $(1.8)_1$ and $(1.8)_3$ are used frequently:

$$
\begin{aligned}
R_{32} = & \, 2\int_0^l \frac{1}{g}\left(\frac{2d_y}{g}\cdot\left(\frac{d_y}{g}\right)_y d + \frac{|d_y|^2}{g^2}d_y\right)_t \cdot \left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy \\
= & \, 2\int_0^l \frac{1}{g}\left(2\left(\frac{d_y}{g}\right)_t\cdot\left(\frac{d_y}{g}\right)_y d + \frac{2d_y}{g}\cdot\left(\frac{d_y}{g}\right)_{yt} d\right)\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy
\end{aligned}
$$

$$+2\int_0^l \frac{1}{g}\left(\frac{2d_y}{g}\cdot\left(\frac{d_y}{g}\right)_y d_t\right)\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy$$
$$+2\int_0^l \frac{1}{g}\left(\frac{2d_y}{g}\cdot\left(\frac{d_y}{g}\right)_t d_y+\frac{|d_y|^2}{g^2}d_{yt}\right)\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy.$$

Thus, after simplifying and collecting the terms, we have

$$\begin{aligned}R_{32} &= 4\int_0^l \frac{1}{g}\left(\left(\frac{d_y}{g}\right)_t\cdot\left(\frac{d_y}{g}\right)_y d\right)\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy\\
&\quad+4\int_0^l \frac{1}{g}\left(\frac{d_y}{g}\cdot\left(\frac{d_y}{g}\right)_{yt} d\right)\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy\\
&\quad+4\int_0^l \frac{1}{g}\left(\frac{d_y}{g}\cdot\left(\frac{d_y}{g}\right)_y d_t\right)\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy\\
&\quad+2\int_0^l \frac{1}{g}\left(\frac{2d_y}{g}\cdot\left(-\frac{u_y d_y}{g^2}+\frac{d_{yt}}{g}\right)d_y\right)\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy\\
&\quad+2\int_0^l \frac{1}{g}\left(\frac{|d_y|^2}{g^2}d_{yt}\right)\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy =: \sum_{i=1}^5 R_{32i}.\end{aligned} \tag{2.77}$$

Next, we are in a position to estimate terms $\sum_{i=1}^5 R_{32i}$ one by one. The term R_{321} is estimated as follows:

$$\begin{aligned}R_{321} &= 4\int_0^l \frac{1}{g}\left(\left(-\frac{u_y d_y}{g^2}+\frac{d_{yt}}{g}\right)\cdot\left(\frac{d_y}{g}\right)_y d\right)\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy\\
&\le C\left(\|F\|_\infty^2+\|\theta\|_\infty^2+\left\|\frac{d_y}{g}\right\|_\infty^4\right)\max_{y\in[0,1]}\left|\frac{d_y}{g}\right|\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_y\right\|_{L^2}\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}\\
&\quad+C\max_{y\in[0,1]}\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right|\left\|\frac{d_{yt}}{\sqrt{g}}\right\|_{L^2}\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}\\
&\le C\left(\|F\|_\infty^2+\|\theta\|_\infty^2+\left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty^2+\left\|\frac{d_{yt}}{\sqrt{g}}\right\|_{L^2}^2\right)\left(1+\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}^2\right).\end{aligned} \tag{2.78}$$

The term R_{322} of (2.3) is controlled as follows:

$$\begin{aligned}R_{322} &\le C\max_{y\in[0,1]}\left(\frac{d_y}{g}\right)\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_{yt}\right\|_{L^2}\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}\\
&\le \delta\left\|\frac{1}{\sqrt{g}}\left(\frac{d_y}{g}\right)_{yt}\right\|_{L^2}^2+C\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}^2,\end{aligned} \tag{2.79}$$

where the Holder and Young inequalities are used along with bounds of the director field in hand.

The term R_{324} is estimated as follows:

$$\begin{aligned}R_{324} &= 2\int_0^l \frac{1}{g}\left(\frac{2d_y}{g}\cdot\left(-\frac{u_y d_y}{g^2}+\frac{d_{yt}}{g}\right)d_y\right)\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy\\
&\le C\max_{y\in[0,1]}\left|\frac{u_y}{g}\right|\max_{y\in[0,1]}\left|\frac{d_y}{g}\right|^2\left\|\frac{d_y}{\sqrt{g}}\right\|_{L^2}\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}\end{aligned}$$

$$+ \max_{y\in[0,1]}\left|\frac{d_y}{g}\right|^2\left\|\frac{d_{yt}}{\sqrt{g}}\right\|_{L^2}\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}$$
$$\le C\left(\|F\|_\infty^2+\|\theta\|_\infty^2+\left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty^2+\left\|\frac{d_{yt}}{\sqrt{g}}\right\|_{L^2}^2\right)$$
$$\times\left(1+\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}^2\right), \tag{2.80}$$

where the definition of effective viscous flux and the estimates of director field and velocity are used. The terms R_{323} and R_{325} are estimated together as follows:

$$R_{323}+R_{225}=\int_0^1\left(\frac{d_y}{g}\cdot\left(\frac{d_y}{g}\right)_y d_t+\frac{|d_y|^2}{g^2}d_{yt}\right)\cdot\frac{1}{g}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy$$
$$\le \max_{y\in[0,1]}\left|\frac{d_y}{g}\right|\max_{y\in[0,1]}\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right|\|\sqrt{g}d_t\|_{L^2}\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}$$
$$+\max_{y\in[0,1]}\left|\frac{d_y}{g}\right|^2\left\|\frac{d_{yt}}{\sqrt{g}}\right\|_{L^2}\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}$$
$$\le C\left(1+\left\|\frac{d_{yt}}{\sqrt{g}}\right\|_{L^2}^2\right)\left(1+\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}^2\right), \tag{2.81}$$

where the Holder and Young inequalities are used along with bounds of the director field in hand. Combining all the above estimates with (2.71) and choosing δ small enough, the resultant is written as follows:

$$\frac{1}{2}\frac{d}{dt}\int_0^l\left(\frac{1}{g}\left|\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right|^2\right)dy+\int_0^l\frac{1}{2g}\left|\left(\frac{d_y}{g}\right)_{yt}\right|^2 dy$$
$$\le -2\frac{d}{dt}\int_0^l\frac{1}{g^2}\left(\frac{|d_y|^2}{g^2}d\right)_y\cdot\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y dy$$
$$+C\left(\|F\|_\infty^2+\|\theta\|_\infty^2+\left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty^2+\left\|\frac{d_{yt}}{\sqrt{g}}\right\|_{L^2}^2\right)$$
$$\times\left(1+\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}^2\right). \tag{2.82}$$

Integrating (2.82) over the time interval $[0,t]$, it is not hard to see that

$$\frac{1}{2}\int_0^l\left(\frac{1}{g}\left|\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right|^2\right)dy+\int_0^t\int_0^l\frac{1}{2g}\left|\left(\frac{d_y}{g}\right)_{yt}\right|^2 dy\,ds$$
$$\le C\left(\|F\|_\infty^2+\|\theta\|_\infty^2+\left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty^2+\left\|\frac{d_{yt}}{\sqrt{g}}\right\|_{L^2}^2\right)$$
$$\times\left(1+\left\|\frac{1}{\sqrt{g}}\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right\|_{L^2}^2\right). \tag{2.83}$$

By using Gronwall's inequality and (2.41), we have

$$\int_0^l\frac{1}{g}\left|\left(\frac{1}{g}\left(\frac{d_y}{g}\right)_y\right)_y\right|^2 dy+\int_0^t\int_0^l\frac{1}{g}\left|\left(\frac{d_y}{g}\right)_{yt}\right|^2 dy\,ds\le C. \tag{2.84}$$

Moreover, from (1.8)$_3$ and inequality (2.84), we get

$$\int_0^l \frac{1}{g}|d_{yt}|^2\,dy \leq C. \tag{2.85}$$

Similarly, one can deduce from the director field equation that

$$\int_0^T \int_0^l g|d_{tt}|^2\,dy\,dt \leq C.$$

Combining (2.84) and (2.85), the proof of Lemma 6 is completed. □

Using temperature, one can rewrite (2.57) as

$$\begin{aligned}
&-\frac{R}{c_v}\kappa\partial_y\left(\frac{\partial_y\theta}{g}\right)\\
&\quad= -R\rho_0\partial_t\theta + \frac{R}{c_v}\left(\partial_y uF + \frac{1}{2}\left|\frac{d_y}{g}\right|^2\partial_y u + g\left|\frac{1}{g}\left(\frac{d_y}{g}\right)_y + \frac{1}{g^2}|d_y|^2 d\right|^2\right),\\
&\partial_t F - \frac{\mu}{g}\partial_y\left(\frac{\partial_y F}{\rho_0}\right) = -(R)\frac{\rho_0}{g}\partial_t\theta - \frac{1}{g}\partial_y uF - \frac{d_y}{g}\cdot\frac{d_{yt}}{g}.
\end{aligned} \tag{2.86}$$

Multiplying the above equation by $g\partial_t F$, integrating the resultant over $(0,l)$, and integrating by parts, it follows from the Hölder and Young inequalities, Corollary 3, and Corollary 1 that

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}\int_0^l\left|\frac{\partial_y F}{\sqrt{\rho_0}}\right|^2 dy + \int_0^l g|\partial_t F|^2\,dy\\
&\quad= -R\int_0^l \rho_0\partial_t\theta\partial_t F\,dy + \int_0^l \partial_y uF\partial_t F\,dy - \int_0^l d_y\cdot\frac{d_{yt}}{g}\partial_t F\,dy\\
&\quad\leq \frac{1}{2}\|\sqrt{g}\partial_t F\|_2^2 + C\left(\|\sqrt{\rho_0}\partial_t\theta\|_2^2 + \|\partial_y v\|_2^2\|F\|_\infty^2 + \left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty\left\|\frac{d_{yt}}{g}\right\|_2^2\right)\\
&\quad\leq \frac{1}{2}\|\sqrt{g}\partial_t F\|_2^2 + C\left(\|\sqrt{\rho_0}\partial_t\theta\|_2^2 + \|F\|_\infty^2 + \left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty\right),
\end{aligned}$$

thus

$$\mu\frac{d}{dt}\left\|\frac{\partial_y F}{\sqrt{\rho_0}}\right\|_2^2 + \|\sqrt{g}\partial_t F\|_2^2 \leq C\left(\|\sqrt{\rho_0}\partial_t\theta\|_2^2 + \|F\|_\infty^2 + \left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty\right) \tag{2.87}$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T.

Multiplying (2.87) by $2A_3$ and summing the resultant with Lemma 6 and (2.68), one obtains

$$\begin{aligned}
&\frac{d}{dt}\left(c_v\|\sqrt{\rho_0}\partial_t\theta\|_2^2 + 2A_3\mu\left\|\frac{\partial_y F}{\sqrt{\rho_0}}\right\|_2^2\right) + \kappa\left\|\frac{\partial_y\partial_t\theta}{\sqrt{g}}\right\|_2^2 + A_3\|\sqrt{g}\partial_t F\|_2^2\\
&\quad\leq C\left(\|\theta\|_\infty^2 + \|F\|_\infty^2 + \left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty\right)\left(\|\sqrt{\rho_0}\partial_t\theta\|_2^2 + 1\right)
\end{aligned}$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, N_1, N_2, and T. Applying the Gronwall inequality to the above inequality, by Corollary 3, and using (2.60), the conclusion of Proposition 7 is completed. □

Proposition 8 *Given $T \in (0,\infty)$. It holds that*

$$\sup_{0\le t\le T}\left(\left\|\left(\partial_y^2 g, \partial_y^2 u, \partial_y\theta, \partial_y^2\theta\right)\right\|_2^2 + \|\theta\|_\infty\right) + \int_0^T \left\|\left(\partial_y^3 u, \partial_y\partial_t u, \partial_y^3\theta\right)\right\|_2^2 \le C$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, and T, where

$$\mathcal{N}_3 := \left\|\rho_0''\right\|_2 + \|h_1\|_2 + \|h_2\|_2,$$

and m_1, $\mathcal{N}_1$, and $\mathcal{N}_2$ are the numbers in Propositions 1, 2, and 4, respectively.

Proof Combining (i) of Proposition 6 and Proposition 7, we get

$$\sup_{0\le t\le T}\|\partial_y\theta\|_2^2 \le C\sup_{0\le t\le T}\left(1 + \|\sqrt{\rho_0}\partial_t\theta\|_2^2\right) \le C\left(1 + \|h_1\|_2^2 + \|h_2\|_2^2\right) \tag{2.88}$$

and thus, by (i) of Proposition 2 and Corollary 3, we have that

$$\sup_{0\le t\le T}\|\theta\|_\infty \le C\sup_{0\le t\le T}\left(\|\partial_y\theta\|_2 + 1\right) \le C\left(1 + \|h_1\|_2 + \|h_2\|_2\right) \tag{2.89}$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T. Using (2.88)–(2.89), it follows from the Hölder inequality and Corollaries 1–3 that

$$\begin{aligned}
\|\partial_y p\|_2 &= R\left\|\frac{\rho_0'}{g}\theta + \frac{\rho_0}{g}\partial_y\theta - \frac{\rho_0}{g^2}\partial_y g\theta\right\|_2 \\
&\le C\left(\left\|\rho_0'\right\|_2\|\theta\|_\infty + \|\rho_0\|_\infty\|\partial_y\theta\|_2 + \|\rho_0\|_\infty\|\partial_y g\|_2\|\theta\|_\infty\right) \\
&\le C\left(1 + \|h_1\|_2 + \|h_2\|_2\right)
\end{aligned} \tag{2.90}$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T.

Noticing that $\partial_y u = \frac{1}{\mu}(gF + R\rho_0\theta + \frac{g}{2}|\frac{d_y}{g}|^2)$ and using $(1.8)_1$, one has

$$\partial_t\partial_y u = \frac{1}{\mu}\left(\partial_y uF + g\partial_t F + R\rho_0\partial_t\theta + d_y\cdot\frac{d_{yt}}{g} - \frac{1}{2}\left|\frac{d_y}{g}\right|^2 u_y\right),$$

$$\partial_y^2 u = \frac{1}{\mu}\left(\partial_y gF + g\partial_y F + R\rho_0'\theta + R\rho_0\partial_y\theta + d_y\cdot\left(\frac{d_y}{g}\right)_y + \frac{1}{2}\left|\frac{d_y}{g}\right|^2 g_y\right),$$

and thus, by the Hölder and Sobolev embedding inequalities and using (2.88)–(2.89), it follows from Corollaries 3–1 and Proposition 7 that

$$\begin{aligned}
&\int_0^T \|\partial_t\partial_y u\|_2^2\,dt \\
&\quad\le C\int_0^T\left(\|\partial_y u\|_2^2\|F\|_\infty^2 + \|g\|_\infty^2\|\partial_t F\|_2^2 + \|\sqrt{\rho_0}\partial_t\theta\|_2^2 + \|d_y\|_\infty^2\left\|\frac{d_{yt}}{g}\right\|_2^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+ C\int_0^T \left(\left\| \left| \frac{d_y}{g} \right|^2 \right\|_\infty^2 \|u_y\|_2^2 \right) dt \\
&\le C\int_0^T \left(\|\partial_y u\|_2^2 \left(\|F\|_{H^1}^2 + \left\| \left| \frac{d_y}{g} \right|^2 \right\|_\infty^2 \right) + \|g\|_\infty^2 \|\partial_t F\|_2^2 + \|\sqrt{\rho_0}\partial_t \theta\|_2^2 \right) \qquad (2.91) \\
&+ C\int_0^T \left(\|d_y\|_\infty^2 \left\| \frac{d_{yt}}{g} \right\|_2^2 \right) dt \\
&\le C\left(1 + \|h_1\|_2^2 + \|h_2\|_2^2\right) \qquad (2.92)
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{0\le t\le T} \left\| \partial_y^2 u \right\|_2^2 &\le C \sup_{0\le t\le T} \left(\|\partial_y g\|_2^2 \|F\|_\infty^2 + \|g\|_\infty^2 \|\partial_y F\|_2^2 + \left\| \rho_0' \right\|_2^2 \|\theta\|_\infty^2 + \|\partial_y \theta\|_2^2 \right) \\
&\le C \sup_{0\le t\le T} \left[\|\partial_y g\|_2^2 \left(\|F\|_{H^1}^2 + \left\| \left| \frac{d_y}{g} \right|^2 \right\|_\infty^2 \right) + \|g\|_\infty^2 \|\partial_y F\|_2^2 + \|\theta\|_\infty^2 \right. \\
&\quad \left. + \|\partial_y \theta\|_2^2 + C\|d_y\|_\infty^2 \left\| \left(\frac{d_y}{g} \right)_y \right\|_2^2 \right] \\
&\le C\left(1 + \|h_1\|_2^2 + \|h_2\|_2^2\right) \qquad (2.93)
\end{aligned}
$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T. In the above inequality the constraint condition $|d|^2 = 1$ is used, respectively.

Using (2.86), we have

$$
\begin{aligned}
\partial_y^2 \theta &= g\partial_y \left(\frac{\partial_y \theta}{g} \right) + \partial_y g \frac{\partial_y \theta}{g} \\
&= \frac{g}{\kappa} \left(c_v \rho_0 \partial_t \theta - \partial_y u F - \frac{1}{2} \left| \frac{d_y}{g} \right|^2 \partial_y u - g \left| \frac{1}{g} \left(\frac{d_y}{g} \right)_y + \frac{1}{g^2} |d_y|^2 d \right|^2 \right) \\
&\quad + \partial_y g \frac{\partial_y \theta}{g},
\end{aligned}
$$

and thus, by the Hölder, Young, and Gagliardo–Nirenberg inequalities and (2.88), it follows from Corollaries 3–1 and Proposition 7 that

$$
\begin{aligned}
\left\| \partial_y^2 \theta \right\|_2 &\le C\left(\|\sqrt{\rho_0}\partial_t \theta\|_2 + \|\partial_y u\|_2 \left(\|F\|_\infty + \left\| \left| \frac{d_y}{g} \right|^2 \right\|_\infty \right) + \|g\|_\infty \|d_t\|_2^2 \right. \\
&\quad \left. + \|\partial_y g\|_2 \|\partial_y \theta\|_\infty \right) \\
&\le C\left(1 + \|g_0\|_2 + \|h_0\|_2 + \|F\|_{H^1} + \left\| \left| \frac{d_y}{g} \right|^2 \right\|_\infty + \|\partial_y \theta\|_2^{\frac{1}{2}} \left\| \partial_y^2 \theta \right\|_2^{\frac{1}{2}} \right) \\
&\le \frac{1}{2} \left\| \partial_y^2 \theta \right\|_2 + C\left(1 + \|h_1\|_2 + \|h_2\|_2\right), \qquad (2.94)
\end{aligned}
$$

which gives

$$
\sup_{0\le t\le T} \left\| \partial_y^2 \theta \right\|_2^2 \le C\left(1 + \|h_1\|_2^2 + \|h_2\|_2^2\right) \qquad (2.95)
$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, and T.

By calculations, one deduces

$$\begin{aligned}\partial_y^2 p &= R\partial_y^2\left(\frac{\rho_0}{g}\theta\right)\\ &= R\left[\rho_0''\frac{\theta}{g} + 2\rho_0'\partial_y\left(\frac{\theta}{g}\right) + \rho_0\partial_y^2\left(\frac{\theta}{g}\right)\right]\\ &= R\left[\rho_0''\frac{\theta}{g} + 2\rho_0'\left(\frac{\partial_y\theta}{g} - \frac{\partial_y g}{g^2}\theta\right)\right.\\ &\quad\left. + \rho_0\left(\frac{\partial_y^2\theta}{g} - \frac{2}{g^2}\partial_y g\partial_y\theta + 2\frac{(\partial_y g)^2}{g^3}\theta - \frac{\partial_y^2 g}{g^2}\theta\right)\right].\end{aligned}$$

Therefore, by the Hölder and Sobolev embedding inequalities, using (2.88), (2.89), and (2.95), it follows from Corollary 3 and Corollary 1 that

$$\begin{aligned}\left\|\partial_y^2 p\right\|_2 &\le C\big[\left\|\rho_0''\right\|_2\|\theta\|_\infty + 2\left\|\rho_0'\right\|_\infty\big(\|\partial_y\theta\|_2 + \|\partial_y g\|_2\|\theta\|_\infty\big)\\ &\qquad + \|\rho_0\|_\infty\big(\left\|\partial_y^2\theta\right\|_2 + 2\|\partial_y g\|_\infty\|\partial_y\theta\|_2 + 2\|\partial_y g\|_\infty\|\partial_y g\|_2\|\theta\|_\infty\big)\\ &\qquad + \|\rho_0\|_\infty\left\|\partial_y^2 g\right\|_2\|\theta\|_\infty\big]\\ &\le C\big(1 + \|h_1\|_2 + \|h_2\|_2 + \left\|\partial_y^2 g\right\|_2\big)\end{aligned}\tag{2.96}$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, and T.

Using (2.86) and $(1.8)_1$, we deduce

$$\begin{aligned}\partial_y^3\theta &= \partial_y^2\left(\frac{\partial_y\theta}{g}\right)g + 2\partial_y\left(\frac{\partial_y\theta}{g}\right)\partial_y g + \frac{\partial_y\theta}{g}\partial_y^2 g\\ &= \frac{g}{\kappa}\left[c_v\big(\rho_0\partial_t\partial_y\theta + \rho_0'\partial_t\theta\big) - \partial_y u\partial_y F - \partial_y^2 u F - \frac{1}{2}\left|\frac{d_y}{g}\right|^2\partial_y^2 u - \partial_y u\frac{d_y}{g}\cdot\left(\frac{d_y}{g}\right)_y\right.\\ &\quad\left. - 2g d_t\cdot d_{ty} - |d_t|^2 g_y\right]\\ &\quad + \frac{2}{\kappa}\partial_y g\left(c_v\rho_0\partial_t\theta - \partial_y u F - \frac{1}{2}\left|\frac{d_y}{g}\right|^2\partial_y u - g|d_t|^2\right) + \frac{\partial_y\theta}{g}\partial_y^2 g.\end{aligned}$$

Therefore, by the Hölder and Sobolev embedding inequalities, using (2.88), (2.90), (2.93), (2.95), (2.96), Corollary 3, Corollary 1, and (ii) of Propositions 6, 7, it follows

$$\begin{aligned}\left\|\partial_y^3\theta\right\|_2 &\le C\left[\|\partial_y\partial_t\theta\|_2 + \left\|\rho_0'\right\|_2\|\partial_t\theta\|_\infty + \|\partial_y u\|_\infty\left(\|\partial_y F\|_2 + \left\|\left(\frac{d_y}{g}\right)_y\right\|_2\right)\right.\\ &\qquad + \left\|\partial_y^2 u\right\|_2\left(\|F\|_\infty + \left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty\right) + \|d_t\|_\infty\|d_{ty}\|_2 + \|d_t\|_2^2\|g_y\|_\infty\\ &\qquad + \|\partial_y g\|_\infty\left(\|\sqrt{\rho_0}\partial_t\theta\|_2 + \|\partial_y u\|_2\left(\|F\|_\infty + \left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty\right) + \|d_t\|_2^2\right)\\ &\qquad\left. + \|\partial_y\theta\|_\infty\left\|\partial_y^2 g\right\|_2\right]\\ &\le C\left[\|\partial_y\partial_t\theta\|_2 + \|\sqrt{\rho_0}\partial_t\theta\|_2 + \|\partial_y u\|_{H^1}\left(\|\partial_y F\|_2 + \left\|\left(\frac{d_y}{g}\right)_y\right\|_2\right)\right.\end{aligned}$$

$$
\begin{aligned}
&+ \left\|\partial_y^2 u\right\|_2\left(\|F\|_{H^1} + \left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty\right) + + \|\partial_y\theta\|_{H^1}\left\|\partial_y^2 g\right\|_2 + \|d_{ty}\|_2 \\
&+ \|\partial_y g\|_{H^1}\left(\|\sqrt{\rho_0}\partial_t\theta\|_2 + \|\partial_y u\|_2\left(\|F\|_{H^1} + \left\|\left|\frac{d_y}{g}\right|^2\right\|_\infty\right) + \|d_t\|_2^2\right)\Bigg] \\
\le{}& C\left(1 + \|h_1\|_2 + \|h_2\|_2 + \|\partial_y\partial_t\theta\|_2 + \left\|\partial_y^2 g\right\|_2\right) \\
\le{}& C\left(1 + \|h_1\|_2 + \|h_2\|_2\right)\left(1 + \|\partial_y\partial_t\theta\|_2 + \left\|\partial_y^2 g\right\|_2\right)
\end{aligned}
\tag{2.97}
$$

and

$$
\begin{aligned}
\partial_y^3 u ={}& \partial_y^2\left(\frac{\partial_y u}{g}\right)g + 2\partial_y\left(\frac{\partial_y u}{g}\right)\partial_y g + \frac{\partial_y u}{g}\partial_y^2 g \\
={}& \frac{g}{\mu}\left(\rho_0'\partial_t u + \rho_0\partial_t\partial_y u + \partial_y^2 p + \frac{d_y}{g}\cdot\left(\frac{d_y}{g}\right)_y\right) \\
&+ \frac{2}{\mu}\partial_y g\left(\rho_0\partial_t u + \partial_y p + \frac{1}{2}\left|\frac{d_y}{g}\right|^2\right) + \frac{\partial_y u}{g}\partial_y^2 g,
\end{aligned}
$$

which is estimated as

$$
\begin{aligned}
\left\|\partial_y^3 u\right\|_2 \le{}& C\Bigg[\left(\|\partial_t u\|_\infty + \|\partial_y\partial_t u\|_2 + \left\|\partial_y^2 p\right\|_2 + \left\|\left(\frac{d_y}{g}\right)_y\right\|_2\right) \\
&+ \|\partial_y g\|_\infty\left(\|\sqrt{\rho_0}\partial_t u\|_2 + \|\partial_y p\|_2 + \left\|\frac{|d_y|}{\sqrt{g}}\right\|_2^2\right) + \|\partial_y u\|_\infty\left\|\partial_y^2 g\right\|_2\Bigg] \\
\le{}& C\Big[\|\partial_y\partial_t u\|_2 + \|\sqrt{\rho_0}\partial_t u\|_2 + 1 + \|h_1\|_2 + \|h_2\|_2 + \left\|\partial_y^2 g\right\|_2 \\
&+ \|\partial_y g\|_{H^1}\left(1 + \|h_1\|_2 + \|h_2\|_2\right) + \|\partial_y u\|_{H^1}\left\|\partial_y^2 g\right\|_2\Big] \\
\le{}& C\left(1 + \|h_1\|_2 + \|h_2\|_2\right)\left(1 + \|\partial_y\partial_t u\|_2 + \left\|\partial_y^2 g\right\|_2\right)
\end{aligned}
\tag{2.98}
$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, and T.

Combining (2.97) with (2.98) and using (2.91), one obtains

$$
\begin{aligned}
\int_0^t \left\|\left(\partial_y^3 u, \partial_y^3\theta\right)\right\|_2^2 d\tau \le{}& C\left(1 + \|h_1\|_2^2 + \|h_2\|_2^2\right)\int_0^t\left(1 + \|\partial_y\partial_t u\|_2^2 + \left\|\partial_y^2 g\right\|_2^2\right)d\tau \\
\le{}& C\left(1 + \|h_1\|_2^2 + \|h_2\|_2^2\right)^2\left(1 + \int_0^t\left\|\partial_y^2 g\right\|_2^2 d\tau\right)
\end{aligned}
\tag{2.99}
$$

for any $t \in [0, T]$, where C is a positive constant depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, and T. Using $(1.8)_1$, one gets $g = 1 + \int_0^t \partial_y u\, d\tau$, and thus it follows from the Hölder inequality that

$$
\left\|\partial_y^2 g\right\|_2^2(t) = \left(\left\|\int_0^t \partial_y^3 u d\tau\right\|_2\right)^2 \le \left(\int_0^t\left\|\partial_y^3 u\right\|_2 d\tau\right)^2 \le t\int_0^t\left\|\partial_y^3 u\right\|_2^2 d\tau.
$$

Combining this with (2.99) and applying the Gronwall inequality, one obtains

$$
\int_0^T\left(\left\|\partial_y^3 u\right\|_2^2 + \left\|\partial_y^3\theta\right\|_2^2\right)dt \le C,
$$

and, further, that

$$\sup_{0\leq t\leq C}\left\|\partial_y^2 g\right\|_2^2\leq C$$

for a positive constant C depending only on R, c_v, μ, κ, m_1, $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, and T. □

3 Proof of Theorem 1.1

Proof of Theorem 1.1 For $\epsilon\in(0,1)$, denote

$$\rho_{0\epsilon}=\rho_0+\epsilon,\qquad \theta_{0\epsilon}=\theta_0+\epsilon.$$

By a similar procedure as in [50], for each $\epsilon\in(0,1)$, there is a unique global strong solution $(g_\epsilon,u_\epsilon,\frac{d_{\epsilon y}}{g_\epsilon},\theta_\epsilon)$ to system (1.8), with ρ_0 replaced by $\rho_{0\epsilon}$, subject to the initial and boundary conditions

$$(g_\epsilon,v_\epsilon,d_{\epsilon y},\theta_\epsilon)|_{t=0}=(1,u_0,d_{0\epsilon y},\theta_{0\epsilon}).$$

It follows that there are two positive constants, independent of $\epsilon\in(0,1)$, such that

$$\inf_{0\leq t\leq T}\inf_{y\in(0,l)}g_\epsilon(y,t)\geq C \tag{3.1}$$

and

$$\sup_{0\leq t\leq T}\left\|\left(g_\epsilon,u_\epsilon,\frac{d_{\epsilon y}}{g_\epsilon},\theta_\epsilon\right)\right\|_{H^2}^2 + \int_0^T\left(\|\partial_t g_\epsilon\|_{H^2}^2+\left\|(u_\epsilon,d_\epsilon,\theta_\epsilon)\right\|_{H^3}^2+\left\|(\partial_t u_\epsilon,\partial_t d_\epsilon,\partial_t\theta_\epsilon)\right\|_{H^1}^2\right)dt\leq C \tag{3.2}$$

for any $\epsilon\in(0,1)$.

Thanks to (3.2), by the Banach–Alaoglu theorem, and using Cantor's diagonal arguments, there is a subsequence, still represented by $(g_\epsilon,u_\epsilon,d_y,\theta_\epsilon)$, such that, for limit $\epsilon\to 0$, (g,u,d_y,θ) is a global strong solution to system (1.8) satisfying the regularities given in Theorem 1.1. □

4 Conclusion

The global strong solution is established to the one-dimensional full compressible nematic liquid crystal flow problem with constant coefficients of viscosity and heat conductivity. The a priori estimates are constructed in the presence of vacuum by using Lagrangian flow map coordinates, effective viscous flux, and some density-weighted Sobolev type inequalities. Moreover, higher order estimates for the director field are obtained by using the constraint $|d|^2=1$. It is possible to extend our results when coefficients depend on temperature and density or on both, which is the matter of future work.

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Declarations

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