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Research Article

Stability Analysis of an SEIR Model with Treatment

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Abstract. We study the dynamics of a SEIR epidemic model with nonlinear treatment function, that takes into account the limited availability of resources in community. Under some conditions we prove the existence of two possible equilibria: the disease-free equilibrium and the endemic equilibrium. Using Lyapunov's method and Li's geometrical approach, We also show that the reproduction number R_0 is a threshold parameter: the disease-free equilibrium is globally asymptotically stable when the basic reproduction number is less than unity and the unique endemic equilibrium is globally asymptotically stable when the basic reproduction number is greater than this critical value. In the end, we give some concluding remarks concerning the role of treatment on the epidemic propagation.

Keywords: SEIR epidemic model; generalized incidence rates; local and global asymptotic stability; treatment; Lyapunov-LaSalle's principle; geometric approach; compound matrix.

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1. Introduction

Infectious diseases remain to be one of the main sources of deaths for the human beings. The goal of research in epidemiology is to develop vaccines, treatments and intervention strategies for stopping the spread of infectious diseases and hence reducing the deaths.

One important approach to understand transmission mechanisms of infectious diseases is mathematical modeling. In this optic, the differential equations play a crucial role because such equations describe the impact of principal parameters on the spread of diseases. For example, we cite the SIR epidemic model and the SEIR epidemic model which provide good descriptions of infectious diseases (see [5, 11, 16, 17]).

Moreover, the epidemiological models describe the effect of treatment on transmission of infection. The modeling of this effect may be taken into account by introducing a treatment function in an epidemiological model. However, this function changes from one work to another.

In this work, we propose to study the role of treatment on the epidemics transmission. For this, we propose the following SEIR epidemic model with treatment function:

$$\begin{cases} \dfrac{dS}{dt} = A - f(S,I) - dS, \\ \dfrac{dE}{dt} = f(S,I) - (d+\sigma)E, \\ \dfrac{dI}{dt} = \sigma E - (d+\gamma)I - T(I), \\ \dfrac{dR}{dt} = \gamma I - dR + T(I). \end{cases} \tag{1.1}$$

Here $A = dN$ is the recruitment rate, where $N = S + E + I + R$ is the total number of population, S is the number of susceptible individuals, I is the number of infectious individuals, E is the number of exposed individuals, R is the number of recovered individuals, d is the natural death, γ is the recovery rate of the infectious individuals, σ is the rate at which exposed individuals become infectious, $f(S, I)$ is the incidence function, which representing the number of new exposed individuals arising in a host population per unit of time and $T(I)$ is treatment function which describing the number of recovered individuals by treatment per unit of time.

As in [8], We assume that the incidence function $f : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$ is continuously differentiable in the interior of $\mathbb{R}^2_+$ satisfying the following hypotheses.

(H_0): $f(0, I) = f(S, 0) = 0$ for $S,\ I \geq 0$.

(H_1): $f(S, I)$ is a strictly monotone increasing function of $S \geq 0$, for any fixed $I > 0$, and a monotone increasing function of $I \geq 0$, for any fixed $S \geq 0$.

(H_2): $\phi(S, I) = \frac{f(S,I)}{I}$ is bounded and monotone decreasing function of $I > 0$, for any fixed $S \geq 0$, and $K(S) = \lim_{I\to 0^+} \phi(S, I)$ is continuous on $S \geq 0$.

For the treatment function $T : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ we say that it is continuously differentiable and concave satisfying the following hypotheses:

(T_0): $T(0) = 0$.

(T_1): The treatment rate $\frac{T(I)}{I}$ is monotone increasing.

The concavity of the treatment function is explained by limited availability of resources in community. The condition (T_0) is natural: no treatment if there is no infected individuals and the condition (T_1) takes into account the efforts of the concerned community and if necessary the efforts of international organizations to deal with any increase in number of infected individuals.

In (1991, [1]), Anderson and May proposed a treatment function as follows:

$$T(I) = kI \tag{1.2}$$

In (2004, [25]), Wang and Run used the following function

$$T(I) = \begin{cases} 0, & \text{if } I = 0, \\ k, & \text{if } I > 0 \end{cases} \tag{1.3}$$

In (2006, [10, 26]), it is assumed that the treatment function is proportional to the number of infective individuals when it is below the capacity and constant when the number of infective individuals reaches the capacity. Namely, they used the following treatment function

$$T(I) = \begin{cases} kI, & \text{if } 0 \leq I \leq I_0, \\ kI_0, & \text{if } I \geq I_0 \end{cases} \tag{1.4}$$

where I_0 is the infective level at which the health-care system reaches capacity; that is, treatment increases linearly with I before the capacity is reached and is constant afterward. It was shown that the model has bistable endemic equilibria when I_0 is low and backward bifurcation can occur.

In (2008, [22]) Zhang and et al. introduced the following saturated treatment:

$$T(I) = \frac{rI}{1 + kI}, \tag{1.5}$$

where $r > 0$, $k \geq 0$, r. This function approaches a capacity limit as I gets large. In (2012, [23]) Zhou and Fan modified the function (1.5) to

$$T(I) = \frac{\alpha I}{\omega + I}, \tag{1.6}$$

In (2011, [7]), Eckalbar studied an SIR epidemic model with the following quadratic treatment function

$$T(I) = \max\{\gamma I - gI^2; 0\}, \tag{1.7}$$

where $r, g > 0$. The authors supposed that a society's capacity for providing treatment may diminished during a severe epidemic as a critical equipment and supplies are exhausted or as health care workers fall victim to the disease.

In (2012, [4]), Sarah Al-Sheikh studied the model (1.1) with a bilinear incidence rate (i.e., $f(S, I) = \beta SI$) and the treatment function (1.4). The author used Lyapunov's functions and second additive compound matrix to prove that there is a sharp threshold parameter R_0 which completely determines the global dynamics of the system. It is shown that this kind of treatment rate leads to the existence of multiple endemic equilibria where the basic reproduction number plays a big role in determining their stability.

In (2012, [24]), X. Zhou and J. Cui studied the model (1.1) with $f(S, I) = \beta SI$ and the treatment function (1.6). The backward bifurcation and global dynamics are shown by compound matrix and geometric approach.

In (2012, [10]), Z. Hu et al. analyzed the following SIR epidemic model with nonlinear incidence rate, vertical transmission, and the treatment function (1.4):

$$\begin{cases} \dfrac{dS}{dt} = bm(S + R) - bS - \dfrac{\beta SI}{1 + \alpha I} + p\delta I, \\ \dfrac{dI}{dt} = \dfrac{\beta SI}{1 + \alpha I} + (1 - p)\delta I - \gamma I - \delta I - T(I), \\ \dfrac{dR}{dt} = \delta I - bR + b(1 - m)(S + R) + T(I), \end{cases} \tag{1.8}$$

The authors established that a backward bifurcation occurs when $R_0 < 1$, that is, the disease-free equilibrium coexists with an endemic equilibrium, and system (1.8) has multiple endemic equilibrium when $R_0 > 1$ where a bifurcation diagram displays forward bifurcations. Furthermore, it is shown that when there are two endemic equilibria, one of them is always unstable and the other one is stable under certain conditions. When there are three endemic equilibria, bistable endemic equilibria can occur. The dynamical behaviors of system (1.8) is determined by the existence of limit cycles.

In (2013, [13]), J. Li and N. Cui analyzed an SIRS epidemic model with the treatment function as follows:

$$\begin{cases} \dfrac{dS}{dt} = (1 - p)A - \dfrac{\beta SI}{1 + \alpha I^2} - dS + \gamma R, \\ \dfrac{dI}{dt} = pA - \dfrac{\beta SI}{1 + \alpha I^2} - (d + \mu + \upsilon)I - T(I), \\ \dfrac{dR}{dt} = mI - (d + \gamma)R + T(I). \end{cases} \tag{1.9}$$

The authors investigated the existence of equilibrium and proved the global asymptotical stability of the endemic equilibrium by using Dulac's criteria and Poincare-Bendixson Theorem. Furthermore, they obtained that the model (1.9) undergoes a Hopf bifurcation.

In (2014, [21]), J. Zhang et al. explored the dynamics of the following SEIR epidemic with saturated incidence rate and saturated treatment function:

$$\begin{cases} \dfrac{dS}{dt} = A - \dfrac{\beta SI}{1 + kI} - dS, \\ \dfrac{dE}{dt} = \dfrac{\beta SI}{1 + \alpha I} - (d + \epsilon)E, \\ \dfrac{dI}{dt} = \epsilon E - (d + \mu + \upsilon)I - \dfrac{rI}{1 + kI}, \\ \dfrac{dR}{dt} = \upsilon I - dR + \dfrac{rI}{1 + kI}. \end{cases} \tag{1.10}$$

By means of Lyapunov functional and a geometric approach, the authors established that under the condition $R_0 > 1$, $0 \leq k < k_1$, and $d > r$, the endemic equilibrium of system (1.10) is globally asymptotically stable.

In (2015, [2]), Dubey et al. investigated the impact of awareness programs as well as treatment on an SIR model. They showed that if the exposure to the awareness program is high and adequate treatment is available, then the infection can be eliminated.

In (2015, [3]), Dubey et al. considered the incidence function as Beddington-DeAngelis type and the treatment rate as Holling type II (saturated). They showed that the disease-free equilibrium is locally asymptotically stable when reproduction number is less than one and obtained the global stability of the endemic equilibrium using Lyapunov function. Also, they investigated the existence of Hopf bifurcation by using Andronov-Hopf bifurcation theorem.

In the present article, we extend the model from [3, 4] to include general nonlinear incidence function $f(S, I)$, and we establish the complete global dynamics. Especially, to establish the global stability of the endemic equilibrium, we use a geometrical approach [15] and a Lyapunov function. our results are consistent with those in [3, 4], in the special case $f(S, I) = \beta SI$.

The rest of this paper is organized as follows. In section 2, some preliminary results that will be useful in the rest of this work are established. In section 3, the existence and uniqueness of disease-free equilibrium and endemic equilibrium of the SEIR model (1.1) are given. In section 4, stability and instability of the disease-free equilibrium are discussed. In section 8, local asymptotic stability of the endemic equilibrium is obtained by Hurwitz's criteria. The global stability of the endemic equilibrium is proved by Li's geometrical approach [15] in section 6, and by Lyapunov's method in section 7. Finally, in section 9, a brief discussion is given to conclude this paper.

2. Preliminary Results

In this section we present some preliminary results that will be useful in the rest of this work.

Lemma 2.1. *The equation* $N - \frac{d+\sigma}{\sigma d}[(d+\gamma)I + T(I)] = 0$ *has a unique positive solution* I^0.

Proof. Define the function g on $\mathbb{R}_+$ by $g(I) = N - \frac{d+\sigma}{\sigma d}[(d+\gamma)I + T(I)]$. We have $g(0) = N > 0$ and $g(\frac{\sigma A}{(d+\sigma)(d+\gamma)}) = -\frac{d+\sigma}{\sigma d}T(I) < 0$. Furthermore, by the hypothesis (T_1), we have $g'(I) = -\frac{d+\sigma}{\sigma d}[(d+\gamma) + T'(I)] < 0$. Thus the equation $N - \frac{d+\sigma}{\sigma d}[(d+\gamma)I + T(I)] = 0$ has a unique positive solution. □

Lemma 2.2 (see [8]). *The following statements are logically equivalent.*

(i): *The function* $\frac{T(I)}{I}$ *is monotone increasing function.*

(ii): $\frac{T(I)}{I} - T'(I) \leq 0$.

Lemma 2.3 (see [8]). *The following statements are equivalent:*

(i): *The function* $\frac{f(S,I)}{I}$ *is monotone decreasing function of* $I > 0$, *for any fixed* $S \geq 0$.

(ii): $\frac{f(S;I)}{I} - \frac{\partial f(S,I)}{\partial I} \geq 0$.

3. Existence and Uniqueness of Equilibria

The first three equations in system (1.1) do not depend on the fourth equation, and therefore this equation can be omitted without loss of generality. Hence, system (1.1) can be rewritten as

$$\begin{cases} \dfrac{dS}{dt} = A - f(S,I) - dS, \\ \dfrac{dE}{dt} = f(S,I) - (d+\sigma)E, \\ \dfrac{dI}{dt} = \sigma E - (d+\gamma)I - T(I), \end{cases} \tag{3.1}$$

Therefore, we will study (3.1) in the following feasible region:

$$\Omega = \{(S,E,I) \in \mathbb{R}^3_+ : S + E + I \le N\}.$$

We can easily verify that Ω is positively invariant with respect to (3.1).

Let the basic reproduction number

$$R_0 = \frac{\sigma K(N)}{(d+\sigma)(d+\gamma+T'_d(0))},$$

where $T'_d(0)$ is the right derivative at the $I = 0$. The following theorem presents the existence and uniqueness of the possible equilibria.

Theorem 3.1. *System (3.1) always has a disease-free equilibrium $P_0 = (N,0,0)$ which exists for all parameter values. On the other hand, if $R_0 > 1$ then system (3.1) also admits an unique endemic equilibrium: $P_* = (S^*, E^*, I^*)$.*

Proof. A steady state (S,E,I) of model (3.1) satisfying the following system.

$$\begin{cases} A - dS - f(S,I) = 0, \\ f(S,I) - (d+\sigma)E = 0, \\ \sigma E - (d+\gamma)I - T(I) = 0. \end{cases} \tag{3.2}$$

If $I = 0$, then $E = 0$, and $S = N$, therefore the disease-free equilibrium $P_0 = (N,0,0)$ of (3.1) exists for all parameters values.

Furthermore, if $I \neq 0$, then we have

$$\begin{cases} S = N - \dfrac{d+\sigma}{\sigma d}[(d+\gamma)I + T(I)], \\ E = \dfrac{d+\gamma}{\sigma} I + \dfrac{T(I)}{\sigma}, \\ f(S,I) = \dfrac{d+\sigma}{\sigma}[(d+\gamma)I + T(I)]. \end{cases} \tag{3.3}$$

For $S^0 = N - \frac{d+\sigma}{\sigma d}[(d+\gamma)I + T(I)]$, we consider the following function on the interval $[0, I^0]$:

$$h(I) = \phi(S^0, I) - \frac{d+\sigma}{\sigma}\left[(d+\gamma) + \frac{T(I)}{I}\right],$$

where I^0 is the unique solution of the equation $N - \frac{d+\sigma}{\sigma d}[(d+\gamma)I + T(I)] = 0$ (see Lemma 2.1). By (H_1) and (H_2), the function h is continuous and

$$h'(I) = \frac{\partial\phi(S^0,I)}{\partial I} - \frac{(d+\sigma)(d+\gamma+T'(I))}{\sigma d}\frac{\partial\phi(S^0,I)}{\partial S} - \frac{(d+\sigma)}{\sigma}\left(\frac{T(I)}{I(t)}\right)' \le 0,$$

then h is monotone decreasing function of $I > 0$. Furthermore, if $R_0 > 1$, we have

$$\lim_{I\to 0^+} h(I) = \frac{(d+\sigma)(d+\gamma+T'_d(0))}{\sigma}(R_0 - 1) > 0,$$

and

$$h(I^0) = -\frac{(d+\sigma)\left((d+\gamma)+\dfrac{T(I^0)}{I^0}\right)}{\sigma} < 0,$$

where I^0 is a root of the equation $N - \frac{(d+\sigma)((d+\gamma)I+T(I))}{d\sigma} = 0$. Hence, there exist unique endemic equilibrium $P_* = (S^*, E^*, I^*)$ in $\overset{\circ}{\mathrm{T}}$. The proof of Theorem 3.1 is completed. □

4. On the Stability of the Disease-Free Equilibrium

We have the following theorem on the global asymptotic stability of the disease-free equilibrium P_0 of (3.1).

Theorem 4.1. (i): *If $R_0 \leq 1$, then the disease free equilibrium P_0 is globally asymptotically stable.*

(ii): *If $R_0 > 1$, then the disease free equilibrium P_0 is unstable.*

Proof. (i) Define a Lyapunov functional

$$W_0(t) = V_0(t) + U_0(t),$$

where

$$V_0(t) = \int_N^S \left(1 - \frac{K(N)}{K(u)}\right) du,$$

and

$$U_0(t) = E + \frac{d+\sigma}{\sigma} I.$$

We will show that $\frac{dW_0(t)}{dt} \leq 0$ for all $t \geq 0$. We have:

$$\frac{dV_0(t)}{dt} = \mu\left(1 - \frac{K(N)}{K(S)}\right)(N - S) - f(S, I) + \frac{K(N)}{K(S)} f(S, I),$$

and

$$\frac{dU_0(t)}{dt} = f(S, I) - \frac{(d+\sigma)(d+\gamma)}{\sigma} I - \frac{d+\sigma}{\sigma} T(I).$$

Then

$$\frac{dW_0(t)}{dt} = \mu\left(1 - \frac{K(N)}{K(S)}\right)(N - S) + \frac{\phi(S, I)}{K(S)} K(N) I - \frac{(d+\sigma)(d+\gamma)}{\sigma} I - \frac{d+\sigma}{\sigma} T(I).$$

By the hypotheses (H_1) and (H_2) we obtain that

$$\left(1 - \frac{K(N)}{K(S)}\right)(N - S) \leq 0,$$

where strict equality holds if and only if $S = N$ and

$$\frac{\phi(S, I)}{K(S)} \leq 1.$$

Furthermore, It follows from the hypothesis (T_1) that

$$T_d'(0) \leq \frac{T(I)}{I}.$$

Hence

$$\frac{dW_0(t)}{dt} \leq \frac{(d+\sigma)(d+\gamma+T_d'(0))I}{\sigma}(R_0-1).$$

Therefore, $R_0 \leq 1$ ensures that $\frac{dW_0(t)}{dt} \leq 0$ for all $t \geq 0$, where $\frac{dW_0(t)}{dt} = 0$ holds if $(S, E, I) = (N, 0, 0)$. Hence, it follows from system (3.1) that $\{P_0\}$ is the largest invariant set in $\left\{(S, E, I) | \frac{dW_0(t)}{dt} = 0\right\}$. From the Lyapunov-LaSalle asymptotic stability, we obtain that P_0 is globally asymptotically stable. This completes the proof of the claim (i).

(ii) The characteristic equation at the disease-free equilibrium P_0 is given by

$$(\lambda + d)(\lambda^2 + a_1\lambda + a_0) = 0, \tag{4.1}$$

where

$$a_1 = 2d + \sigma + \gamma + T_d'(0),$$

$$a_0 = (d+\sigma)(d+\gamma+T_d'(0))(1-R_0).$$

Since $a_1 > 0$, it is easy to show that (4.1) has a real positive root when $R_0 > 1$. Hence, P_0 is unstable when $R_0 > 1$. This prove the claim (ii). □

5. Local Stability of the Endemic Equilibrium

We have the following theorem on the local asymptotic stability of the endemic equilibrium P^* of (3.1).

Theorem 5.1. *Suppose the hypothesis* (H_0)–(H_2) *and* (T_0)–(T_1) *hold.*

If $R_0 > 1$, *then the endemic equilibrium* P^* *is locally asymptotically stable.*

Proof. Let $x = S - S^*$, $y = E - E^*$ and $z = I - I^*$.

Then by linearizing system (3.1) around P^*, we have

$$\begin{cases} \frac{dx}{dt} = -\left(d + \frac{\partial f(S^*, I^*)}{\partial S}\right)x(t) - \frac{\partial f(S^*, I^*)}{\partial I}z(t), \\ \frac{dy}{dt} = \frac{\partial f(S^*, I^*)}{\partial S}x(t) - (d+\sigma)y(t) + \frac{\partial f(S^*, I^*)}{\partial I}z(t), \\ \frac{dz}{dt} = \sigma y(t) - (d+\gamma+T'(I^*))z(t) \end{cases} \tag{5.1}$$

The characteristic equation associated to system (5.1) is given by

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{5.2}$$

where

$$a_2 = 3d + \sigma + \gamma + \frac{\partial f(S^*, I^*)}{\partial S} + T'(I^*),$$

$$a_1 = (d+\gamma+T'(I^*))(d+\sigma) - \sigma\frac{\partial f(S^*, I^*)}{\partial I} + (2d+\sigma+\gamma+T'(I^*))\left(d + \frac{\partial f(S^*, I^*)}{\partial S}\right),$$

$$a_0 = \left(d + \frac{\partial f(S^*, I^*)}{\partial S}\right)(d+\sigma)(d+\gamma+T'(I^*)) - d\sigma\frac{\partial f(S^*, I^*)}{\partial I}.$$

By using Lemma 2.3 and equations in system (3.1) (second and third), we can easily obtain that

$$\left(d+\gamma+\frac{T(I^*)}{I^*}\right)(d+\sigma)\geq\sigma\frac{\partial f(S^*,I^*)}{\partial I}.$$

Furthermore, it follows from Lemma 2.2 that

$$(d+\gamma+T'(I^*))(d+\sigma)\geq\sigma\frac{\partial f(S^*,I^*)}{\partial I},$$

this inequality leads to

$$a_i>0,\quad i=0,1,2,$$

and

$$a_1a_2-a_0>0.$$

Hence, by the Hurwitz's criterion, we have the local stability of P^* for $R_0>1$. This concludes the proof of Theorem 5.1. □

6. Global Stability of the Endemic Equilibrium by Geometrical Approach

In order to study the global stability of the endemic equilibrium P^*, we use the geometrical approach which is developed in the papers of Smith [19] and Li and Muldowney [15]. We obtain that P^* is globally asymptotically stable when $R_0>1$.

Let $\overset{\circ}{\Omega}$ be the interior of

$$\Omega=\{(S,E,I)\in\mathbb{R}^3_+ : S+E+I\leq N\}.$$

To show the existence of a compact set in $\overset{\circ}{\Omega}$ that is absorbing for (3.1) is equivalent to proving that (3.1) is uniformly persistent, which means that there exists a constant $c>0$ such that every solution (S,E,I) of (3.1) with $(S(0),E(0),I(0))$ in $\overset{\circ}{\Omega}$ satisfies

$$\liminf_{t\to\infty}S(t)\geq c,\quad \liminf_{t\to\infty}E(t)\geq c,\quad \liminf_{t\to\infty}I(t)\geq c \tag{6.1}$$

Here c is independent of initial data in $\overset{\circ}{\Omega}$, see [15]. We can prove the following result.

Proposition 6.1. *The system (3.1) is uniformly persistent if and only if* $R_0>1$.

Proof. By Theorem 4.3 in [9], we can see that uniform persistence of system (3.1) is equivalent to instability of the disease-free equilibrium $P_0=(N,0,0)$. Combine the local stability analysis for this equilibrium in Theorem 4.1 and Theorem 4.3 in [9], we know that system (3.1) is uniformly persistent if and only if $R_0>1$. □

Let $|.|$ denote a vector norm in $\mathbb{R}^n$, $(n\in\mathbb{N})$. The Lozinskiĭ measure of a $\binom{n}{2}\times\binom{n}{2}$ matrix B with respect to the norm $|.|$ is defined as $\mu(B)=\lim_{h\to0^+}\frac{|I_n+hB|-1}{h}$ where I_n is the unit matrix.

We state our main result in the following theorem.

Theorem 6.2. *Assume that* $R_0>1$. *Then the unique endemic equilibrium* P^* *is globally asymptotically stable in* $\overset{\circ}{\Omega}$.

Proof. By Proposition 6.1, when $R_0 > 1$, there exists a compact set K in $\mathring{\Omega}$ that is absorbing for (3.1). The proof of the Theorem consists of choosing a suitable vector norm in $\mathbb{R}^3$ and a 3×3 matrix-valued function $A(x)$ so that

$$\overline{q}_2 := \limsup_{t\to\infty} \sup_{x_0\in K} \frac{1}{t}\int_0^t \mu_1(B(x(s,x_0)))ds < 0 \tag{6.2}$$

where $\mu(B) = \lim_{h\to 0^+} \frac{|I+hB|-1}{h}$, $B = A_g A^{-1} + AJ^{[2]}A^{-1}$, $x = (S, E, I)$ and $g(x)$ denote the vector field of (3.1), i.e. $\frac{dx(t)}{dt} = g(x)$. The Jacobian matrix $J = \frac{\partial g}{\partial x}$ associated with a general solution $x(t)$ of (3.1) is given by:

$$J = \begin{pmatrix} -d - \dfrac{\partial f(S,I)}{\partial S} & 0 & -\dfrac{\partial f(S,I)}{\partial I} \\ \dfrac{\partial f(S,I)}{\partial S} & -d-\sigma & \dfrac{\partial f(S,I)}{\partial I} \\ 0 & \sigma & -d-\gamma-T'(I) \end{pmatrix}$$

The second additive compound matrix $J^{[2]}$ of the Jacobian matrix J is given by

$$J^{[2]} = \begin{pmatrix} -2d-\sigma-\dfrac{\partial f(S,I)}{\partial S} & \dfrac{\partial f(S,I)}{\partial I} & \dfrac{\partial f(S,I)}{\partial I} \\ \sigma & -2d-\gamma-\dfrac{\partial f(S,I)}{\partial S} - T'(I) & 0 \\ 0 & \dfrac{\partial f(S,I)}{\partial S} & -2d-\gamma-\sigma-T'(I) \end{pmatrix}$$

Set the function $A(x) = A(S, E, I) = \mathrm{diag}\{1, \frac{E}{I}, \frac{E}{I}\}$. Then,

$$A_g A^{-1} = \mathrm{diag}\left\{0, \frac{E'}{E} - \frac{I'}{I}, \frac{E'}{E} - \frac{I'}{I}\right\},$$

where the matrix A_g is obtained by replacing each entry a_{ij} of $A(x)$ by its derivative in the direction of g. The matrix $B = A_g A^{-1} + AJ^{[2]}A^{-1}$ can be written in the following block form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{6.3}$$

where $B_{11} = -2d - \sigma - \frac{\partial f(S,I)}{\partial S}$,

$$B_{12} = \frac{I}{E}\begin{pmatrix} \dfrac{\partial f(S,I)}{\partial I} & \dfrac{\partial f(S,I)}{\partial I} \end{pmatrix}, \qquad B_{21} = \begin{pmatrix} \dfrac{\sigma E}{I} \\ 0 \end{pmatrix}$$

$$B_{22} = \begin{pmatrix} \dfrac{E'}{E} - \dfrac{I'}{I} - 2d - \gamma - \dfrac{\partial f(S,I)}{\partial S} - T'(I) & 0 \\ \dfrac{\partial f(S,I)}{\partial S} & \dfrac{E'}{E} - \dfrac{I'}{I} - 2d - \sigma - \gamma - T'(I) \end{pmatrix}$$

Let (u, v, w) denote the vectors in $\mathbb{R}^3$, we select a norm in $\mathbb{R}^3$ as

$$|(u, v, w)| = \max\{|u|, |v| + |w|\},$$

and let $\overline{\mu}_1$ denote the Lozinskiĭ measure with respect to this norm. Using the method of estimating $\overline{\mu}_1$ in ([18]), we have

$$\overline{\mu}_1(B) \leq \sup(g_1, g_2) \tag{6.4}$$

where

$$g_1 = \mu_1(B_{11}) + |B_{12}| \tag{6.5}$$

$$g_2 = \mu_1(B_{22}) + |B_{21}|, \tag{6.6}$$

$|B_{12}|, |B_{21}|$ are matrix norms with respect to the l_1 vector norm. We have

$$\mu_1(B_{11}) = -2d - \sigma - \frac{\partial f(S,I)}{\partial S}, \tag{6.7}$$

$$|B_{12}| = \frac{I}{E}\frac{\partial f(S,I)}{\partial I}, \tag{6.8}$$

$$|B_{21}| = \frac{\sigma E}{I}. \tag{6.9}$$

To calculate $\mu_1(B_{22})$, we add the absolute value of the off-diagonal one in each column of B_{22}, and then take the maximum of two sums, (see [6]), we obtain

$$\mu_1(B_{22}) = \frac{E'}{E} - \frac{I'}{I} - 2d - \gamma - \frac{\partial f(S,I)}{\partial S} \tag{6.10}$$

$$g_1 = -2d - \sigma - \frac{\partial f(S,I)}{\partial S} + \left(\frac{\partial f(S,I)}{\partial I}\right)\frac{I}{E}, \tag{6.11}$$

$$g_2 = \frac{\sigma E}{I} + \frac{E'}{E} - \frac{I'}{I} + \frac{\sigma E}{I} - 2d - \gamma - T'(I) + \max\{0; -\sigma\} \tag{6.12}$$

Rewriting the second and the third equations in (3.1), we obtain respectively,

$$\frac{E'}{E} = \frac{f(S,I)}{E} - (d + \sigma) \tag{6.13}$$

$$\frac{I'}{I} = \frac{\sigma E}{I} - (d + \gamma) - \frac{T(I)}{I} \tag{6.14}$$

Substituting (6.13) into (6.11) and (6.14) into (6.12), respectively, we have

$$g_1 = \frac{E'}{E} - d - \frac{\partial f(S,I)}{\partial S} + \frac{\partial f(S,I)}{\partial I}\frac{I}{E} - \frac{f(S,I)}{E} \tag{6.15}$$

$$g_2 = \frac{E'}{E} - d + \frac{T(I)}{I} - T'(I) \tag{6.16}$$

Using the inequality in Lemma (2.2), we have

$$g_1 \leq \frac{E'}{E} - d \tag{6.17}$$

Furthermore, if the treatment function $T(I)$ satisfies the hypothesis (T_1), then by applying Lemma (2.3), we obtain

$$g_2 \leq \frac{E'}{E} - d \tag{6.18}$$

Then, we have

$$\mu_1(B) \leq \frac{E'}{E} - d \tag{6.19}$$

Since (3.1) is uniformly persistent when $R_0 > 1$, there exists $c > 0$ and $t_0 > 0$ such that $t > t_0$ implies

$$c \leq E(t) \leq N, \quad \text{and} \quad c \leq I(t) \leq N$$

for all $(S(0), E(0), I(0)) \in K$.

And for $t > t_0$ we have

$$\frac{1}{t}\int_0^t \overline{\mu}_1(B)ds \leq \frac{1}{t}\int_0^{t_0} \overline{\mu}_1(B)ds + \frac{1}{t}\log\frac{E(t)}{E(t_0)} - d\frac{t-t_0}{t} \leq \frac{-d}{2}, \tag{6.20}$$

for all $(S(0), E(0), I(0)) \in K$, which implies

$$\overline{q}_2 < 0 \tag{6.21}$$

This concludes the proof. □

7. Global Stability of the Endemic Equilibrium by Lyapunov's Method

In this section, we give a proof of the global asymptotic stability of the endemic equilibrium P_* for $R_0 > 1$.

Theorem 7.1. *Assume that the hypothesis* (H_0)–(H_2) *and* (T_0)–(T_1) *hold. If* $R_0 > 1$, *then the endemic equilibrium* P_* *of system (3.1) is the only equilibrium and is globally asymptotically stable.*

Proof. We define the following Lyapunov functional.

$$\begin{aligned} V(t) &= \int_{S^*}^{S}\left(1 - \frac{f(S^*, I^*)}{f(u, I^*)}\right)du \\ &\quad + \frac{d+\sigma}{\sigma}\left(I - I^* - I^*\ln\frac{I}{I^*}\right) + E - E^* - E^*\ln\frac{E}{E^*} \end{aligned} \tag{7.1}$$

The time derivative of the function $V(t)$ along the positive solution of system (3.1) becomes

$$\begin{aligned} \frac{dV(t)}{dt} &= \left(1 - \frac{f(S^*, I^*)}{f(S, I^*)}\right)(A - f(S, I) - dS) \\ &\quad + \frac{d+\sigma}{\sigma}\left(1 - \frac{I^*}{I}\right)(\sigma E - (d+\gamma)I - T(I)) \\ &\quad + \left(1 - \frac{E^*}{E(t)}\right)(f(S, I) - (d+\sigma)E) \end{aligned}$$

Using the relations $A = dS^* + f(S^*, I^*)$, $(d+\sigma)E^* = f(S^*, I^*)$, and $\sigma E^* = (d+\gamma)I^* + T(I^*)$ we have

$$\begin{aligned}
\frac{dV(t)}{dt} &= d(S^* - S)\left(1 - \frac{f(S^*, I^*)}{f(S, I^*)}\right) \\
&\quad + f(S^*, I^*)\left(2 + \frac{f(S, I)}{f(S, I^*)} - \frac{f(S^*, I^*)}{f(S, I^*)} - \frac{I^* E}{I E^*} - \frac{(d+\gamma)I + T}{\sigma E^*} + \frac{(d+\gamma)I^*}{\sigma E^*}\right. \\
&\quad \left. + \frac{T I^*}{\sigma E^* I} - \frac{E^* f(S, I)}{E f(S^*), I^*}\right) \\
&= d(S^* - S)\left(1 - \frac{f(S^*, I^*)}{f(S, I^*)}\right) \\
&\quad + f(S^*, I^*)\left(-1 - \frac{I}{I^*} + \frac{f(S, I)}{f(S, I^*)} + \frac{I}{I^*}\frac{f(S, I^*)}{f(S, I)}\right) \\
&\quad + f(S^*, I^*)\left(4 - \frac{I^* E}{I E^*} - \frac{f(S^*, I^*)}{f(S, I^*)} - \frac{I}{I^*}\frac{f(S, I^*)}{f(S, I)} - \frac{E^*}{E}\frac{f(S, I)}{f(S^*, I^*)}\right) \\
&\quad + \frac{I}{I^*} - \frac{T(I^*)}{\sigma E^*} - \frac{T(I)}{\sigma E^*} - \frac{(d+\gamma)I}{\sigma E^*} + \frac{I^* T(I)}{\sigma E^* I}
\end{aligned}$$

From hypothesis (H_1), we have

$$\left(1 - \frac{f(S^*, I^*)}{f(S, I^*)}\right)(S^* - S) \leq 0,$$

and from hypothesis (H_1) and (H_2), we have

$$-1 - \frac{I}{I^*} + \frac{f(S, I)}{f(S, I^*)} + \frac{I}{I^*}\frac{f(S, I^*)}{f(S, I)} = \left(\frac{I}{I^*} - \frac{f(S, I)}{f(S, I^*)}\right)\left(\frac{f(S, I^*)}{f(S, I)} - 1\right) \leq 0.$$

Furthermore, since

$$\ln\left(\frac{I^* E}{I E^*}\right) + \ln\left(\frac{f(S^*, I^*)}{f(S, I^*)}\right) + \ln\left(\frac{I}{I^*}\frac{f(S, I^*)}{f(S, I)}\right) + \ln\left(\frac{E^*}{E}\frac{f(S, I)}{f(S^*, I^*)}\right) = 0,$$

then, we can easily obtain that

$$\begin{aligned}
4 - \frac{I^* E}{I(t) E^*} &- \frac{f(S^*, I^*)}{f(S, I^*)} - \frac{I}{I^*}\frac{f(S, I^*)}{f(S, I)} - \frac{E^*}{E}\frac{f(S, I)}{f(S^*, I^*)} \\
&= g\left(\frac{I^* E}{I E^*}\right) + g\left(\frac{f(S^*, I^*)}{f(S, I^*)}\right) + g\left(\frac{I}{I^*}\frac{f(S, I^*)}{f(S, I)}\right) + g\left(\frac{E^*}{E}\frac{f(S, I)}{f(S^*, I^*)}\right)
\end{aligned}$$

where $g(x) = 1 - x + \ln(x)$.

Since the function g is always non-positive for any $x > 0$, and $g(x) = 0$ if and only if $x = 1$, we obtain

$$4 - \frac{I^* E}{I E^*} - \frac{f(S^*, I^*)}{f(S, I^*)} - \frac{I}{I^*}\frac{f(S, I^*)}{f(S, I)} - \frac{E^*}{E}\frac{f(S, I)}{f(S^*, I^*)} \leq 0.$$

Finally, from the hypothesis (T_1), we have

$$\frac{I}{I^*} - \frac{T(I^*)}{\sigma E^*} - \frac{T(I)}{\sigma E^*} - \frac{(d+\gamma)I}{\sigma E^*} + \frac{I^* T(I)}{\sigma E^* I} = \frac{(I - I^*)}{\sigma E^*}\left(\frac{T(I^*)}{I^*} - \frac{T(I)}{I}\right) \leq 0.$$

Thus $\frac{dV(t)}{dt}$ is non-positive. Consequently the functional $V(t)$ satisfies all the conditions of Theorem 5.3 of Kuang [12]. This concludes the proof of Theorem (7.1). □

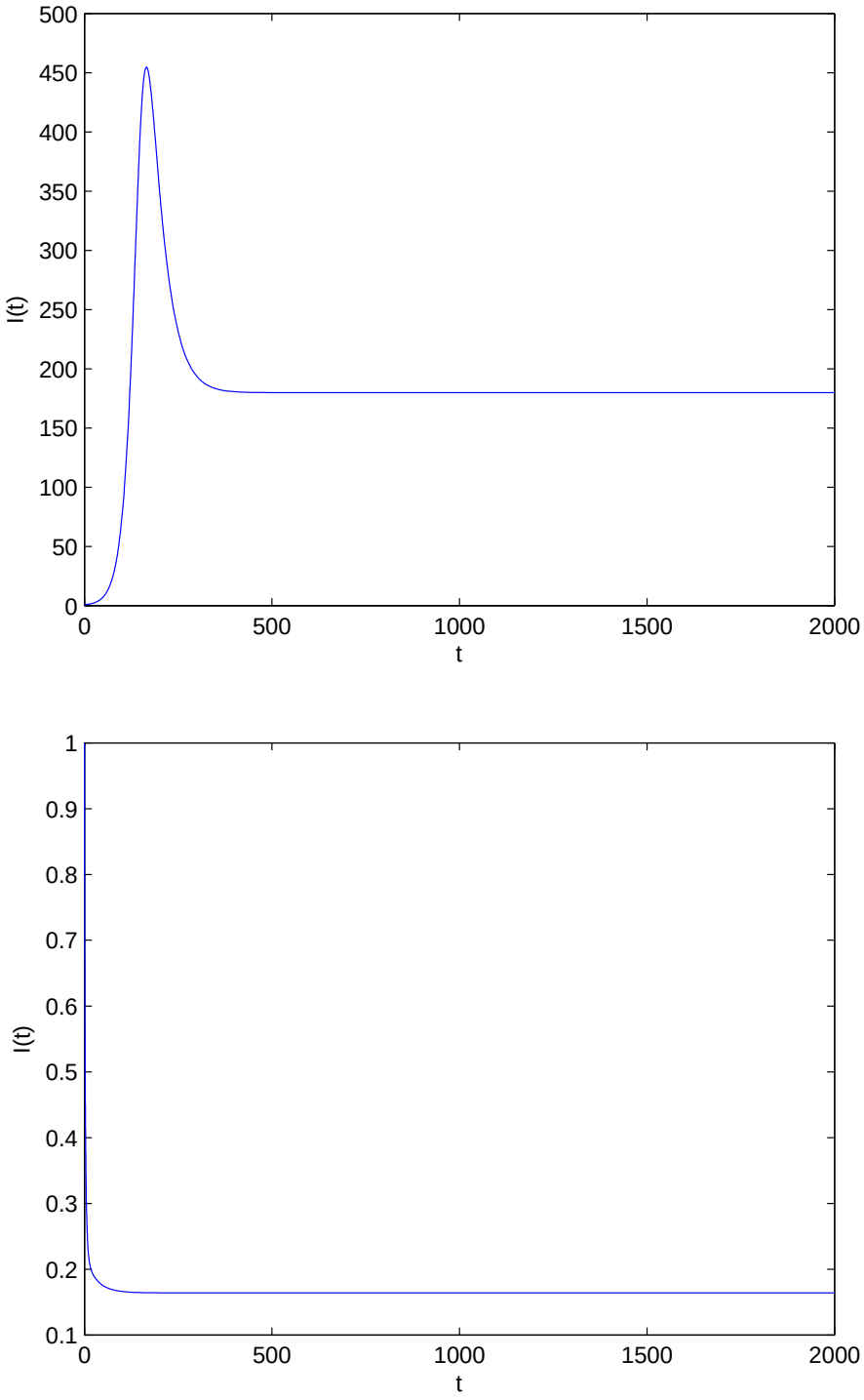

Figure 1: The effect of a treatment on the evolution of infectious individuals: without treatment (left) and with treatment (right).

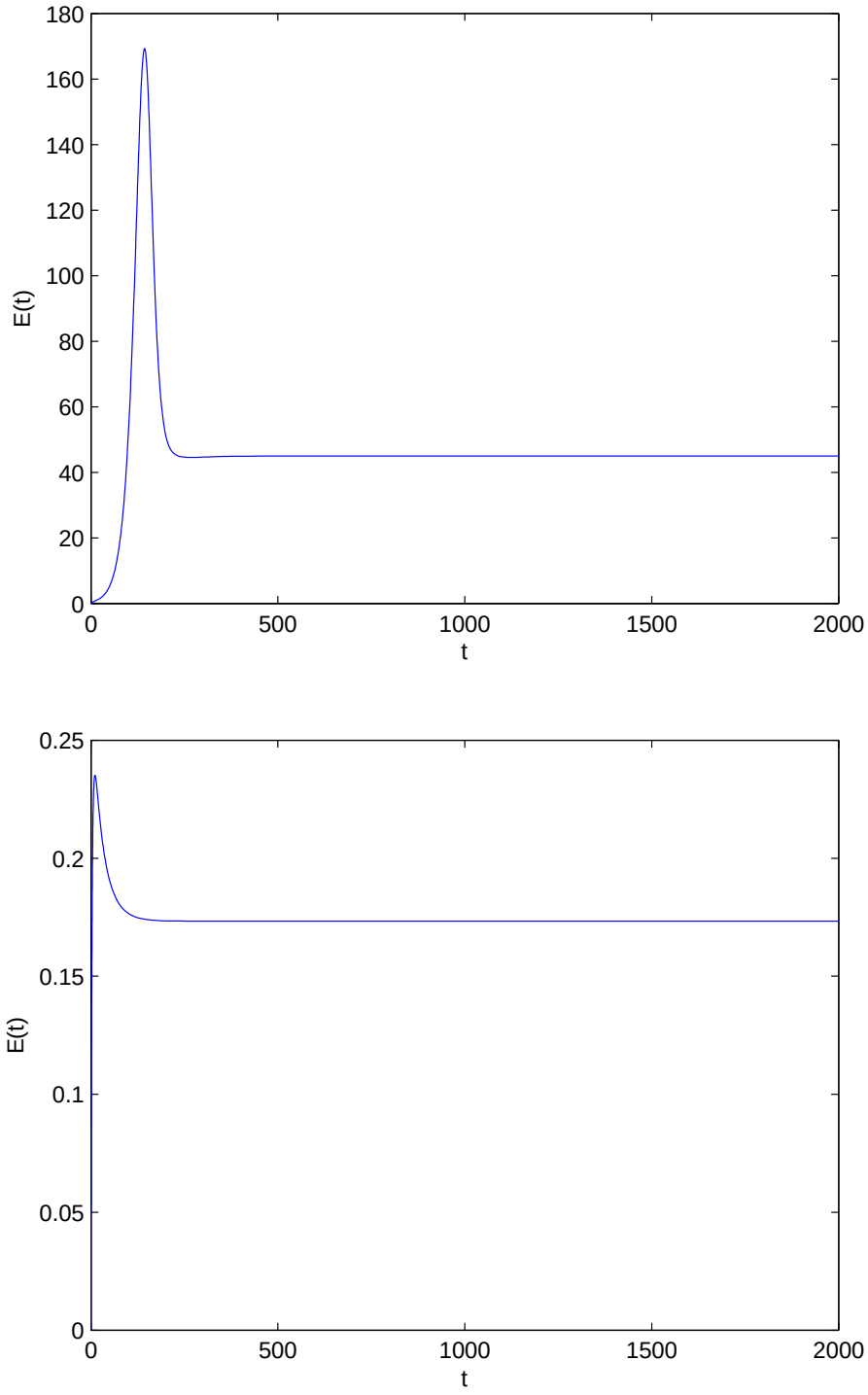

Figure 2: The effect of a treatment on the evolution of exposed individuals: without treatment (left) and with treatment (right).

8. Numerical Simulations

In this section, we give a numerical simulations supporting the theoretical analysis given in the previous sections. Consider the following incidence functions:

$$f(S, I) = \frac{\beta SI}{1 + \alpha_1 S + \alpha_2 I},$$

and

$$T(I) = \frac{r_1 I^2}{1 + r_2 I}.$$

Let's compare the principal results of SEIR model (system (1.1) with treatment) and SEIRI model (system (1.1) without treatment) by a numerical illustration.

We take the following parameters:

$$\alpha_1 = 0.9, \quad \alpha_2 = 0.9, \quad \beta = 0.1, \quad \mu = 0.005, \quad \gamma = 0.02, \quad r_1 = 0.5, \quad r_2 = 0.1, \quad \sigma = 1/10.$$

In Figures 1 and 2, the SEIR epidemic model (with treatment) and the SEIRI epidemic model (without treatment) generate the same global asymptotic proprieties. However, we find that the treatment decreases the number of exposed and infectious individuals at equilibrium (see Figure 1 and Figure 2).

9. Discussion

In (2014, [21]), Zhang studied the model (1.10) with saturated incidence rate and saturated treatment function. The author used a geometric approach to prove that there is a sharp threshold parameter R_0 (the basic reproduction number) which completely determines the global dynamics of the endemic equilibrium under the restriction $0 \leq k < k_1$, and $d > r$. In this work, we presented an extension of Zhang's paper [21] by choosing a generalized nonlinear incidence function and generalized treatment function. Our main contribution is to show that the endemic equilibrium is globally asymptotically stable without any restriction on the parameter values and depend only on the property of treatment function. Further, we show that Lyapunov's method and Li's geometrical approach give the same results.

Competing Interests

The authors declare no competing interests.

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