



## A cyclic basis for closed curve and surface modeling

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### ABSTRACT

We define a cyclic basis for the vectorspace of truncated Fourier series. The basis has several nice properties, such as positivity, summing to 1, that are often required in computer aided design, and that are used by designers in order to control curves by manipulating control points. Our curves have cyclic symmetry, i.e. the control points can be cyclically arranged and the curve does not change when the control points are cyclically permuted. We provide an explicit formula for the elevation of the degree from  $n$  to  $n + r$ ,  $r \geq 1$  and prove that the control polygon of the degree elevated curve converges to the curve itself if  $r$  tends to infinity. Variation diminishing property of the curve is also verified. The proposed basis functions are suitable for the description of closed curves and surfaces with  $C^\infty$  continuity at all of their points.

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### Introduction

Plane curves or space curves are images of continuous functions. By choosing a specific basis in a fixed vectorspace of functions, one can associate a geometric meaning to a coefficient, namely the control point. When the basis consists of positive functions summing to 1, then the curve is in the convex hull of the control points, and the relation between curve and control points is invariant under affine transformations.

Other properties of a basis are also important in design, even though they cannot be formulated so clearly: we want a basis function to have one local maximum, and be as small as possible at points that are sufficiently far away from the maximum. In most cases endpoint interpolation is also required. Normalized totally positive basis with endpoint interpolation property are properties satisfied by a normalized B-basis, which, in some sense, has optimal shape preserving properties (Carnicer and Peña, 1993, 1994). Curves with B-basis can also be described by B-algorithms (cf. Mainar et al., 1999), which is a corner cutting algorithm, and a special case of which is the de Casteljau subdivision algorithm.

Examples for such basis functions are the Bernstein polynomials and its numerous variations. Recently B-bases have been found and studied in spaces including trigonometric functions as well (Zhang, 1996; Mainar et al., 2001).

In this paper, we will devise a basis for the vectorspace

$$\mathcal{V}_n = \langle 1, \cos(u), \sin(u), \dots, \cos(nu), \sin(nu) \rangle$$

of trigonometric polynomials of degree at most  $n$ , i.e. of truncated Fourier series. Peña (1997) proved that this space does not have B-basis on the domain  $[0, \pi]$ , however it has on the interval of length  $< \pi$ , cf. Sánchez-Reyes (1998). Therefore, our proposed cyclic basis is not a B-basis, however it shares the majority of their advantageous properties.

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In order to describe a closed curve by means of a Bernstein-like basis and control points  $\mathbf{d}_0, \mathbf{d}_1, \dots, \mathbf{d}_n$ , the equality  $\mathbf{d}_0 = \mathbf{d}_n$  has to be fulfilled, and  $\mathbf{d}_0$  will be on the curve. This condition guaranties only  $C^0$  continuity at  $\mathbf{d}_0$ , for continuity of order  $k$  at this point the degree of the curve has to be increased to  $n + k$  and its control points  $\mathbf{a}_j$  have to be specified by  $\mathbf{a}_j = \mathbf{d}_{j \bmod (n+1)}$  ( $j = 0, 1, \dots, n + k$ ). In case of B-spline representation the degree does not have to be increased but the control points have to be specified in the same manner.

Instead of endpoint interpolation property, in this paper we intend to parametrize closed curves. Thus, our basis will have cyclic symmetry, i.e. the curve does not change when the control points are cyclically permuted. Further advantage of the cyclic basis is that it does not need redundant control points and all points of the curve are of continuity  $C^\infty$ . This property of this basis is even more advantageous in surface modeling, since to fulfill the geometric constraints of  $C^k$  continuity along the joint of patches is more complicated (e.g. think of the poles of a sphere composed of NURBS patches).

High order continuity is required typically in optical design. Reflector design of vehicle lamps, free-form lens constructions and other ultra-precision optical components naturally requires surfaces of very high order continuity with no singularities. In these areas our method can provide a good alternative to currently used spline surfaces. Moreover, our cyclic basis is useful for designing closed loops with circular symmetry, such as blossoms, ornaments, etc.

### 1. Trigonometric polynomials

Let  $n$  be a positive integer and we consider the vectorspace  $\mathcal{V}_n$ . The class of curves that can be parameterized by such functions (called trigonometric curves) includes numerous classical curves such as limaçon of Pascal, trifolium, epi-cycloid, hypo-cycloid, etc., as special cases. They also arise naturally in numerous areas such as linear differential equations, Fourier analysis, almost periodic functions (under the name of generalized trigonometric polynomials), representation of groups (utilizing its periodicity), electrical circuit analysis (Lissajous curves, as often shown on oscilloscopes), fracture mechanics (as the caustic pattern appearing when a fractured material is shone by a laser beam), etc. The class of trigonometric curves has also been studied under different names (higher cycloid curves, higher planet motions) in Wunderlich (1947), Wunderlich (1950), Pottmann (1984). Paper (Schuster, 2000) proposes a method for the interpolation of odd number of points with closed curves, that are algebraic curves described by a trigonometric parametrization.

The union of all spaces  $\mathcal{V}_n$  is closed under multiplication, because of the elementary trigonometric identities

$$\cos \alpha \cos \beta = \frac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2}, \tag{1}$$

$$\cos \alpha \sin \beta = \frac{\sin(\beta + \alpha) + \sin(\beta - \alpha)}{2}, \tag{2}$$

$$\sin \alpha \sin \beta = \frac{\cos(\alpha + \beta) - \cos(\alpha - \beta)}{-2}.$$

As a consequence, the class of trigonometric curves includes the polynomial Bézier curves (with a different parametrization). It is a subclass of the class of rational Bézier curves, because the curves can be re-parametrized by setting  $u = \arctan(t)/2$ , and the result is a rational parametrization. These curves, called harmonic rational Bézier curves or HRB curves, have been extensively studied in Ge et al. (1997), Sánchez-Reyes (1998).

If  $f$  is an arbitrary periodic and continuous function of bounded variation, then the Fourier expansion of  $f$  converges uniformly to  $f$  by a theorem of Jordan (see Goldberg (1961)). In particular, this holds for piecewise differentiable functions. As a consequence, any closed piecewise differentiable curve can be approximated by trigonometric curves.

In the forthcoming sections we will use persistently the identities

$$1 + \cos \alpha = 2 \cos^2 \frac{\alpha}{2}, \tag{3}$$

$$\sum_{i=0}^n \cos(\varphi + i\alpha) = \frac{\sin \frac{(n+1)\alpha}{2} \cos(\varphi + \frac{n\alpha}{2})}{\sin \frac{\alpha}{2}}, \tag{4}$$

$$\sum_{i=0}^n \sin(\varphi + i\alpha) = \frac{\sin \frac{(n+1)\alpha}{2} \sin(\varphi + \frac{n\alpha}{2})}{\sin \frac{\alpha}{2}} \tag{5}$$

and

$$\cos^{2n} \alpha = \frac{1}{2^{2n}} \binom{2n}{n} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \cos(2(n-k)\alpha). \tag{6}$$

(From hereon, a number in parenthesis above the equality sign indicates that we apply the corresponding trigonometric identity.)

## 2. Basis

We want to achieve that a cyclic shift of control points has the same image curve, with a different parametrization of course. By a result of Hong and Schicho (1998), a re-parametrization of a trigonometric curve is always linear, i.e. a phase change. This implies that all basis functions are fixed if we fix one:

$$C_{i,n}(u) := C_{0,n}\left(u + \frac{2\pi i}{2n+1}\right) \text{ for } i = 1, \dots, 2n.$$

Hence it suffices to specify  $C_{0,n}$ .

When we choose  $C_{0,n}$  as a linear combination of  $1, \cos(u), \dots, \cos(nu)$ , we get symmetric functions. Then the curve will also stay the same when we reverse the order of the control points.

The linear combinations of  $1, \cos(u), \dots, \cos(nu)$  are exactly those trigonometric functions that can be written as polynomials in  $\cos(u)$  of degree at most  $n$ , hence we have

$$C_{0,n}(u) := P_n(\cos u),$$

for some polynomial  $P_n$  of degree at most  $n$ . The polynomial  $P_n$  should be non-negative in the interval  $[-1, 1]$ , in order that  $C_{0,n}$  be positive for all  $u$ . If we want  $C_{0,n}$  to have a unique local maximum at  $u = 0$  (unique up to periodicity), then we need to have a unique maximum of  $P_n|_{[-1,1]}$  at 1. We want to have  $C_{0,n}$  as small as possible “elsewhere”, hence make sure that  $P_n|_{[-1,1]}$  is small elsewhere. It seems to us that the best choice for  $P_n$  is the zeroth Bernstein polynomial of order  $n$  with nodes  $-1, 1$ , properly scaled, i.e. we have

$$P_n := c_n \left(\frac{1+t}{2}\right)^n,$$

where the constant  $c_n$  is yet to be determined.

**Theorem 1.** *With the definitions of  $C_{i,n}, C_{0,n}, P_n$  above, we have that  $\sum_{i=0}^{2n} C_{i,n}$  is a constant.*

**Proof.** The sum function  $\sum_{i=0}^{2n} C_{i,n}(u)$  can be expressed for all  $u \in \mathbb{R}$  as

$$\begin{aligned} \sum_{i=0}^{2n} C_{i,n}(u) &= \frac{c_n}{2^n} \sum_{i=0}^{2n} \left(1 + \cos\left(u + \frac{2i\pi}{2n+1}\right)\right)^n \\ &\stackrel{(3)}{=} c_n \sum_{i=0}^{2n} \cos^{2n}\left(\frac{u}{2} + \frac{i\pi}{2n+1}\right) \\ &\stackrel{(6)}{=} c_n \sum_{i=0}^{2n} \left(\frac{1}{2^{2n}} \binom{2n}{n} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \cos\left((n-k)u + \frac{2(n-k)i\pi}{2n+1}\right)\right) \\ &= \frac{2n+1}{2^{2n}} \binom{2n}{n} c_n + \frac{c_n}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \sum_{i=0}^{2n} \cos\left((n-k)u + \frac{2(n-k)i\pi}{2n+1}\right) \\ &\stackrel{(4)}{=} \frac{2n+1}{2^{2n}} \binom{2n}{n} c_n + \frac{c_n}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \frac{\sin((n-k)\pi) \cos\left((n-k)u + \frac{2n(n-k)\pi}{2n+1}\right)}{\sin\frac{(n-k)\pi}{2n+1}} \\ &= \frac{2n+1}{2^{2n}} \binom{2n}{n} c_n. \quad \square \end{aligned}$$

**Corollary 2.** *In order to ensure that these functions form a partition of unity, the constant  $c_n$  must be defined as*

$$c_n = \frac{2^{2n}}{(2n+1)\binom{2n}{n}} = \frac{(2^n n!)^2}{(2n+1)!}, \tag{7}$$

for which the recursion

$$\begin{cases} c_1 = \frac{2}{3}, \\ c_n = \frac{2n}{2n+1} c_{n-1}, \quad n > 1 \end{cases}$$

holds.

**Theorem 3** (Linear independence). *Functions*

$$C_{i,n}(u) = \frac{c_n}{2^n} \left( 1 + \cos\left(u + \frac{2\pi i}{2n+1}\right) \right)^n \quad (i = 0, 1, \dots, 2n)$$

form a basis of vectorspace  $\mathcal{V}_n$ .

**Proof.** Let us introduce the notation  $\lambda_n = \frac{2\pi}{2n+1}$ . Naturally, it is enough to prove that functions

$$(1 + \cos(u + i\lambda_n))^n \quad (i = 0, 1, \dots, 2n) \tag{8}$$

are linearly independent.

Functions (8) are periodic with the period  $2\pi$ , thus we can study the properties of these functions on the interval  $[-2\pi, 0]$ .

By contradiction, suppose that functions (8) are linearly dependent. In this case  $\exists a_0, a_1, \dots, a_{2n} \in \mathbb{R}$  such that  $a_0^2 + a_1^2 + \dots + a_{2n}^2 \neq 0$  and

$$\sum_{i=0}^{2n} a_i (1 + \cos(u + i\lambda_n))^n = 0, \quad \forall u \in [-2\pi, 0]. \tag{9}$$

Substituting the angles  $u_k = -k\lambda_n$  ( $k = 0, 1, \dots, 2n$ ) into Eq. (9), we get a homogeneous linear system for the unknown parameters  $a_0, a_1, \dots, a_{2n}$ . The matrix representation of this system is given by

$$\Lambda_{2n+1} \cdot A = \mathbf{0}_{2n+1,1}, \tag{10}$$

where

$$\Lambda_{2n+1} = \begin{bmatrix} 2^n & (1 + \cos \lambda_n)^n & \dots & (1 + \cos(2n - 1)\lambda_n)^n & (1 + \cos 2n\lambda_n)^n \\ (1 + \cos \lambda_n)^n & 2^n & \ddots & \vdots & (1 + \cos(2n - 1)\lambda_n)^n \\ \vdots & (1 + \cos \lambda_n)^n & \ddots & (1 + \cos \lambda_n)^n & \vdots \\ (1 + \cos(2n - 1)\lambda_n)^n & \vdots & \ddots & 2^n & (1 + \cos \lambda_n)^n \\ (1 + \cos 2n\lambda_n)^n & (1 + \cos(2n - 1)\lambda_n)^n & \dots & (1 + \cos \lambda_n)^n & 2^n \end{bmatrix} \tag{11}$$

is a symmetric band diagonal matrix and  $A = [a_0 \ a_1 \ \dots \ a_{2n-1} \ a_{2n}]^T$ .

We will show that the homogeneous system (10) admits only the trivial solution  $a_0 = a_1 = \dots = a_{2n} = 0$ , that contradicts the linear dependence of the function system (8).

By means of the trigonometric identity  $\cos \alpha = \cos(2\pi - \alpha)$  we obtain that

$$\begin{aligned} \cos(2n - i)\lambda_n &= \cos\left(2\pi - \frac{2(2n - i)\pi}{2n + 1}\right) = \cos\left((i + 1)\frac{2\pi}{2n + 1}\right) \\ &= \cos(i + 1)\lambda_n \quad (i = 0, \dots, 2n - 1). \end{aligned} \tag{12}$$

We introduce the variables

$$x_0 = 2^n, \quad x_i = x_{2n+1-i} = (1 + \cos i\lambda_n)^n \quad (i = 1, 2, \dots, n) \tag{13}$$

and the notations

$$s_j^{\lambda_n} = x_0 + x_1\omega_j + x_2\omega_j^2 + \dots + x_{2n}\omega_j^{2n} \quad (j = 0, 1, \dots, 2n),$$

where  $\omega_j = e^{\frac{2\pi ij}{2n+1}}$  ( $j = 0, 1, \dots, 2n$ ) are the  $(2n + 1)$ th roots of unity.

By the help of equalities (12) and variables (13) the matrix  $\Lambda_{2n+1}$  takes the circulant form

$$\Lambda_{2n+1} = \begin{bmatrix} x_0 & x_1 & \dots & x_{2n} \\ x_{2n} & x_0 & \dots & x_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \dots & x_0 \end{bmatrix},$$

the eigenvalues of which are exactly the real numbers  $s_j^{\lambda_n}$  ( $j = 0, 1, \dots, 2n$ ). Thus, the determinant of  $\Lambda_{2n+1}$  can be expressed as

$$\det \Lambda_{2n+1} = \prod_{j=0}^{2n} s_j^{\lambda_n}. \tag{14}$$

Hereafter, we will show that  $s_j^{\lambda_n} > 0$  ( $j = 0, 1, \dots, 2n$ ). The eigenvalues  $s_j^{\lambda_n}$  ( $j = 0, 1, \dots, 2n$ ) can be written in the form

$$s_j^{\lambda_n} = \sum_{k=0}^{2n} x_k \omega_j^k \stackrel{(3)}{=} 2^n + 2^n \sum_{k=1}^n (\omega_j^k + \omega_j^{2n+1-k}) \cos^{2n} \frac{k\lambda_n}{2}$$

$$= 2^n + 2^{n+1} \sum_{k=1}^n \cos(jk\lambda_n) \cos^{2n} \frac{k\lambda_n}{2}.$$

By simple calculations we get the equality

$$\cos((2n + 1 - j)k\lambda_n) = \cos(jk\lambda_n) \quad (k = 1, 2, \dots, n),$$

thus we have the symmetry

$$s_j^{\lambda_n} = s_{2n+1-j}^{\lambda_n} \quad (j = 1, 2, \dots, n).$$

Now, we give the closed form of the  $j$ th eigenvalue ( $j = 0, 1, 2, \dots, n$ ).

In the case  $j = 0$  we obtain

$$s_0^{\lambda_n} = \frac{\binom{2n}{n}(2n + 1)}{2^n}$$

using trigonometric identities (6), (4) and (2). (Technical details can be found in Róth et al. (2008).)

Applying the sequence of identities (6), (4), (1), (4) and (2) we obtain

$$s_j^{\lambda_n} = \frac{\binom{2n}{n-j}(2n + 1)}{2^n} \quad (j = 1, 2, \dots, n).$$

(Technical details can be found in Róth et al. (2008).)

Thus, the final form of (14) is

$$\det A_{2n+1} = s_0^{\lambda_n} \left( \prod_{j=1}^n s_j^{\lambda_n} \right)^2 > 0,$$

which means that the system (10) has only the trivial solution  $a_0 = a_1 = \dots = a_{2n} = 0$ .  $\square$

**Remark 4.** The function system  $\{C_{i,n}\}_{i=0}^{2n}$  is not a totally positive (or order complete Chebyshev) system. A counterexample can easily be constructed, e.g. for  $n = 2$ .

### 3. Cyclic curves

We list some geometric properties of the cyclic curve

$$\mathbf{a}_n(u) = \sum_{i=0}^{2n} C_{i,n}(u) \mathbf{d}_i, \quad u \in [-\pi, \pi], \quad n \geq 1, \tag{15}$$

$$C_{i,n}(u) = \frac{c_n}{2^n} \left( 1 + \cos \left( u + \frac{2\pi i}{2n + 1} \right) \right)^n, \tag{16}$$

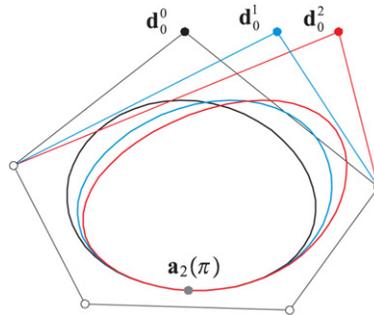
of degree  $n$ , where  $\mathbf{d}_i \in \mathbb{R}^\delta$  ( $\delta \geq 2$ ) are called control points and  $c_n$  is defined by (7). Actually, the domain of (15) can be any interval of length  $2\pi$ , due to the periodicity of the basis functions.

Obviously, the curve is contained in the *convex hull* of its control points. The unique maximum of basis function  $C_{i,n}(u)$  is at  $u = -2\pi i/(2n + 1)$ , therefore control point  $\mathbf{d}_i$  has the most significant effect on the shape of the curve in the neighborhood of the point  $\mathbf{a}_n(-2\pi i/(2n + 1))$ .

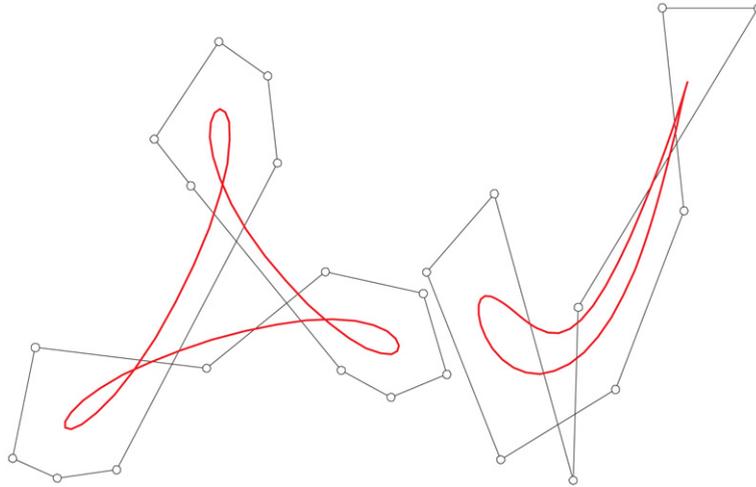
$C_{i,n}(u)$  vanishes at  $u = \pi - 2\pi i/(2n + 1)$ , thus control point  $\mathbf{d}_i$  has no effect on the corresponding point. That is, this point of the curve is invariant under the modification of control point  $\mathbf{d}_i$ . Except this point, control point  $\mathbf{d}_i$  affects all the points of the curve, i.e. the curve is globally controlled. These properties are illustrated in Fig. 1.

However, we emphasize in the next proposition that while control points have a global effect on the shape, this influence dramatically decreases on further parts of the curve, especially for higher values of  $n$ .

**Proposition 5.** *The increase of  $n$  decreases the effect of control points on the shape of the curve.*



**Fig. 1.** Three different curves obtained by the translation of control point  $\mathbf{d}_0$ ; point  $\mathbf{a}_2(\pi)$  is unchanged.



**Fig. 2.** Modeling of complex shapes;  $n = 7$  left hand side,  $n = 4$  right hand side.

**Proof.** Let us consider the basis function

$$C_{0,n}(u) = \frac{c_n}{2^n} (1 + \cos(u))^n, \quad u \in [-\pi, \pi]$$

and a small predefined constant  $\varepsilon \in (0, 1)$ . The angles where function  $C_{0,n}$  equals to  $\varepsilon$  are

$$\pm \arccos\left(2\left(\frac{\varepsilon}{c_n}\right)^{\frac{1}{n}} - 1\right).$$

Due to the properties of the basis functions, inequality  $C_{0,n}(u) \geq \varepsilon$  holds between these two angles. It can be shown easily that

$$\lim_{n \rightarrow \infty} \left(\sqrt[n]{(2n+1) \binom{2n}{n}}\right) = 4$$

and

$$\lim_{n \rightarrow \infty} \sqrt[n]{\varepsilon} = 1.$$

Thus, we have the limits

$$\lim_{n \rightarrow \infty} \pm \arccos\left(2\left(\frac{\varepsilon}{c_n}\right)^{\frac{1}{n}} - 1\right) = \pm \arccos 1 = 0,$$

which means that as degree  $n$  tends to infinity the control point  $\mathbf{d}_0$  has zero influence on the curve.  $\square$

Several test examples show that this decrease of the influence is fast.

Using this curve description one can model various shapes by means of the specification of control points. This is illustrated in Fig. 2.

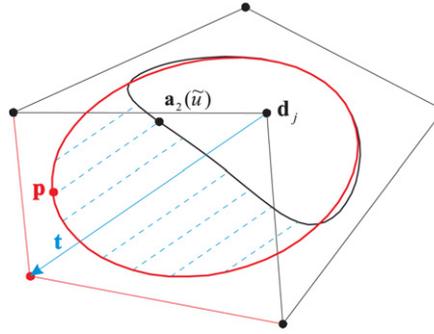


Fig. 3. Constrained shape modification.

Cyclic curves are uniquely determined by their control polygon, therefore we have only one option for their shape modification which is control point repositioning. Like in case of any curve determined by the combination of control points and basis functions, if we translate a control point, points of the curve will move along straight lines parallel to the translation vector (functional translation). This enables us to implement shape modification subject to positional constraint. Assuming that we want to move the point  $\mathbf{a}_n(\tilde{u})$  of the curve to the new location  $\mathbf{p}$  by the translation of control point  $\mathbf{d}_j$ , ( $j \in \{0, 1, \dots, 2n\}$ ), an easy calculation shows that the shift vector is

$$\mathbf{t} = \frac{\mathbf{p} - \mathbf{a}_n(\tilde{u})}{C_{j,n}(\tilde{u})}$$

provided  $C_{j,n}(\tilde{u}) \neq 0$ . Such a shape modification is illustrated in Fig. 3.

### 3.1. Hodograph, derivatives

Curve (15) can be written in the form

$$\begin{aligned} \mathbf{a}_n(u) &= \frac{c_n}{2^n} \sum_{i=0}^{2n} (1 + \cos(u + i\lambda_n))^n \mathbf{d}_i \\ &\stackrel{(3)}{=} c_n \sum_{i=0}^{2n} \cos^{2n} \left( \frac{u}{2} + \frac{i\pi}{2n+1} \right) \mathbf{d}_i \\ &\stackrel{(6)}{=} \frac{\binom{2n}{n} c_n}{2^{2n}} \sum_{i=0}^{2n} \mathbf{d}_i + \frac{c_n}{2^{2n-1}} \sum_{i=0}^{2n} \sum_{k=0}^{n-1} \binom{2n}{k} \cos \left( (n-k)u + \frac{2(n-k)i\pi}{2n+1} \right) \mathbf{d}_i \\ &\stackrel{(7)}{=} \frac{1}{2n+1} \sum_{i=0}^{2n} \mathbf{d}_i + \frac{2}{(2n+1)\binom{2n}{n}} \sum_{i=0}^{2n} \sum_{k=0}^{n-1} \binom{2n}{k} \cos \left( (n-k)u + \frac{2(n-k)i\pi}{2n+1} \right) \mathbf{d}_i, \end{aligned}$$

where  $\frac{1}{2n+1} \sum_{i=0}^{2n} \mathbf{d}_i$  is the center of gravity of the control polygon.

Thus, the hodograph of curve (15) is

$$\frac{d}{du} \mathbf{a}_n(u) = -\frac{2}{(2n+1)\binom{2n}{n}} \sum_{i=0}^{2n} \sum_{k=0}^{n-1} \binom{2n}{k} (n-k) \sin \left( (n-k)u + \frac{2(n-k)i\pi}{2n+1} \right) \mathbf{d}_i.$$

The  $r$ th ( $r \geq 1$ ) order derivatives of curve (15) are given by

$$\frac{d^r}{du^r} \mathbf{a}_n(u) = \frac{2}{(2n+1)\binom{2n}{n}} \sum_{i=0}^{2n} \sum_{k=0}^{n-1} \binom{2n}{k} (n-k)^r \cos \left( (n-k)u + \frac{2(n-k)i\pi}{2n+1} + \frac{r\pi}{2} \right) \mathbf{d}_i.$$

### 3.2. Degree elevation

Assuming that we have a cyclic curve of degree  $n$ , we provide an explicit formula for the control points of that cyclic curve of degree  $n+r$ , ( $r \geq 1$ ) which describes exactly the given curve. Moreover, we show that the control polygon of the degree elevated curve converges to the curve itself as  $r$  tends to infinity.

**Theorem 6** (Degree elevation). Let  $\mathbf{a}_n(u)$  be the cyclic curve determined by the control polygon

$$\mathbf{D}_n = [\mathbf{d}_i]_{i=0}^{2n} \in \mathcal{M}_{1,2n+1}(\mathbb{R}^\delta) \quad (\delta \geq 2, n \geq 1)$$

and consider the curve

$$\begin{aligned} \mathbf{g}_{n,r}(u) &:= \mathbf{g}_{n,r}(u; \mathbf{D}_n) \\ &= \frac{1}{2n+1} \sum_{i=0}^{2n} \mathbf{d}_i + \frac{\binom{2(n+r)}{n+r} c_n}{2^{2n-1}} \sum_{k=0}^{n-1} \frac{\binom{2n}{r+k}}{\binom{2(n+r)}{r+k}} \sum_{i=0}^{2n} \cos\left((n-k)u + \frac{2(n-k)i\pi}{2n+1}\right) \mathbf{d}_i \\ &= \frac{1}{2n+1} \sum_{i=0}^{2n} \mathbf{d}_i + \frac{2\binom{2(n+r)}{n+r}}{(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} \frac{\binom{2n}{r+k}}{\binom{2(n+r)}{r+k}} \sum_{i=0}^{2n} \cos\left((n-k)u + \frac{2(n-k)i\pi}{2n+1}\right) \mathbf{d}_i, \quad u \in [-2\pi, 0], r \geq 1 \end{aligned}$$

determined by the same sequence of points. Consider also the cyclic curve

$$\mathbf{a}_{n+r}(u) = \frac{c_{n+r}}{2^{n+r}} \sum_{j=0}^{2(n+r)} \left(1 + \cos\left(u + \frac{2j\pi}{2(n+r)+1}\right)\right)^{n+r} \mathbf{d}_j^r, \quad u \in [-2\pi, 0]$$

of degree  $n+r$  determined by the control points

$$\mathbf{d}_j^r = \mathbf{g}_{n,r}\left(-\frac{2j\pi}{2(n+r)+1}\right) \in \mathbb{R}^\delta, \quad j = 0, 1, \dots, 2(n+r). \tag{17}$$

Then the equality

$$\mathbf{a}_n(u) \equiv \mathbf{a}_{n+r}(u), \quad \forall u \in [-2\pi, 0]$$

holds, i.e. the procedure above provides a closed formula for the elevation of the degree from  $n$  to  $n+r$ .

**Proof.** Here we present only an outline of the proof, the details of rather lengthy reformulations can be found in Róth et al. (2008). We perform the following reformulations on the curve  $\mathbf{a}_{n+r}$ .

$$\begin{aligned} \mathbf{a}_{n+r}(u) &= \frac{c_{n+r}}{2^{n+r}} \sum_{j=0}^{2(n+r)} \left(1 + \cos\left(u + \frac{2j\pi}{2(n+r)+1}\right)\right)^{n+r} \mathbf{d}_j^r \\ &\stackrel{(2),(6)}{=} \frac{1}{2n+1} \sum_{i=0}^{2n} \mathbf{d}_i + \mathbf{s}_1(u) + \mathbf{s}_2(u) + \mathbf{s}_3(u), \end{aligned}$$

where

$$\begin{aligned} \mathbf{s}_1(u) &= \frac{c_{n+r}}{2^{2(n+r)-1}} \left( \frac{1}{2n+1} \sum_{i=0}^{2n} \mathbf{d}_i \right) \\ &\cdot \left( \sum_{l=0}^{n+r-1} \binom{2(n+r)}{l} \sum_{j=0}^{2(n+r)} \cos\left((n+r-l)u + \frac{2(n+r-l)j\pi}{2(n+r)+1}\right) \right) \\ &\stackrel{(4)}{=} \mathbf{0}, \end{aligned}$$

$$\begin{aligned} \mathbf{s}_2(u) &= \frac{1}{2(n+r)+1} \frac{\binom{2(n+r)}{n+r} c_n}{2^{2n-1}} \\ &\cdot \sum_{j=0}^{2(n+r)} \sum_{k=0}^{n-1} \frac{\binom{2n}{r+k}}{\binom{2(n+r)}{r+k}} \sum_{i=0}^{2n} \cos\left(2(n-k)\pi \left(\frac{i}{2n+1} - \frac{j}{2(n+r)+1}\right)\right) \mathbf{d}_i \\ &\stackrel{(4),(5)}{=} \mathbf{0} \end{aligned}$$

and

$$\mathbf{s}_3(u) = \mathbf{s}_{31}(u) + \mathbf{s}_{32}(u),$$

where

$$\begin{aligned}
 \mathbf{s}_{31}(u) &= \frac{c_{n+r}}{2^{2(n+r)-1}} \frac{\binom{2(n+r)}{n+r} c_n}{2^{2n-1}} \\
 &\cdot \sum_{j=0}^{2(n+r)} \left( \sum_{l=0}^{r-1} \binom{2(n+r)}{l} \cos\left( (n+r-l)u + \frac{2(n+r-l)j\pi}{2(n+r)+1} \right) \right) \\
 &\cdot \left( \sum_{k=0}^{n-1} \frac{\binom{2n}{r+k}}{\binom{2(n+r)}{r+k}} \sum_{i=0}^{2n} \cos\left( 2(n-k)\pi \left( \frac{i}{2n+1} - \frac{j}{2(n+r)+1} \right) \right) \mathbf{d}_i \right) \\
 &\stackrel{(4),(5)}{=} \mathbf{0}
 \end{aligned}$$

and after an index transformation we obtain

$$\begin{aligned}
 \mathbf{s}_{32}(u) &= \frac{c_{n+r}}{2^{2(n+r)-1}} \frac{\binom{2(n+r)}{n+r} c_n}{2^{2n-1}} \\
 &\cdot \sum_{j=0}^{2(n+r)} \left( \sum_{l=0}^{n-1} \binom{2(n+r)}{r+l} \cos\left( (n-l)u + \frac{2(n-l)j\pi}{2(n+r)+1} \right) \right) \\
 &\cdot \left( \sum_{k=0}^{n-1} \frac{\binom{2n}{r+k}}{\binom{2(n+r)}{r+k}} \sum_{i=0}^{2n} \cos\left( 2(n-k)\pi \left( \frac{i}{2n+1} - \frac{j}{2(n+r)+1} \right) \right) \mathbf{d}_i \right).
 \end{aligned}$$

It can be shown that in the expression above only those terms differ from zero that are independent of  $j$ . This happens only in that subtotal where  $l = k$ . In any other subtotals we can rearrange the sum in such a way that for the innermost summation (with respect to  $j$ ) we can apply trigonometric identity (4) or (5) that results zero. Thus,

$$\mathbf{s}_{32}(u) \stackrel{(4),(5)}{=} \frac{c_n}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \sum_{i=0}^{2n} \cos\left( (n-k)u + \frac{2(n-k)\pi i}{2n+1} \right) \mathbf{d}_i.$$

Therefore,

$$\mathbf{a}_{n+r}(u) = \mathbf{a}_n(u), \quad \forall u \in [-2\pi, 0]. \quad \square$$

**Proposition 7 (Convergence). Equality**

$$\lim_{r \rightarrow \infty} \mathbf{g}_{n,r}(u) = \mathbf{a}_n(u), \quad \forall u \in [-2\pi, 0]$$

holds, i.e. the control polygon  $\mathbf{D}_{n+r} = [\mathbf{d}_i]_{i=0}^{2(n+r)}$  of the degree elevated curve converges to the cyclic curve  $\mathbf{a}_n(u)$ .

**Proof.** Performing the reformulation

$$\begin{aligned}
 \mathbf{a}_n(u) &= \frac{c_n}{2^n} \sum_{i=0}^{2n} \left( 1 + \cos\left( u + \frac{2i\pi}{2n+1} \right) \right)^n \mathbf{d}_i \\
 &= \frac{1}{2n+1} \sum_{i=0}^{2n} \mathbf{d}_i + \frac{c_n}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \sum_{i=0}^{2n} \cos\left( (n-k)u + \frac{2(n-k)i\pi}{2n+1} \right) \mathbf{d}_i
 \end{aligned}$$

and applying the limits

$$\lim_{\substack{r \rightarrow \infty \\ r \geq 1}} \frac{\binom{2(n+r)}{n+r}}{\binom{2(n+r)}{r+k}} = 1, \quad \forall k \in \{0, 1, \dots, n-1\}, \quad n \geq 1$$

we obtain the equality

$$\lim_{r \rightarrow \infty} \mathbf{g}_{n,r}(u) = \mathbf{a}_n(u), \quad \forall u \in [-2\pi, 0]. \quad \square$$

Fig. 4 illustrates this convergence by means of a planar curve of degree three and by a spatial curve of degree four.

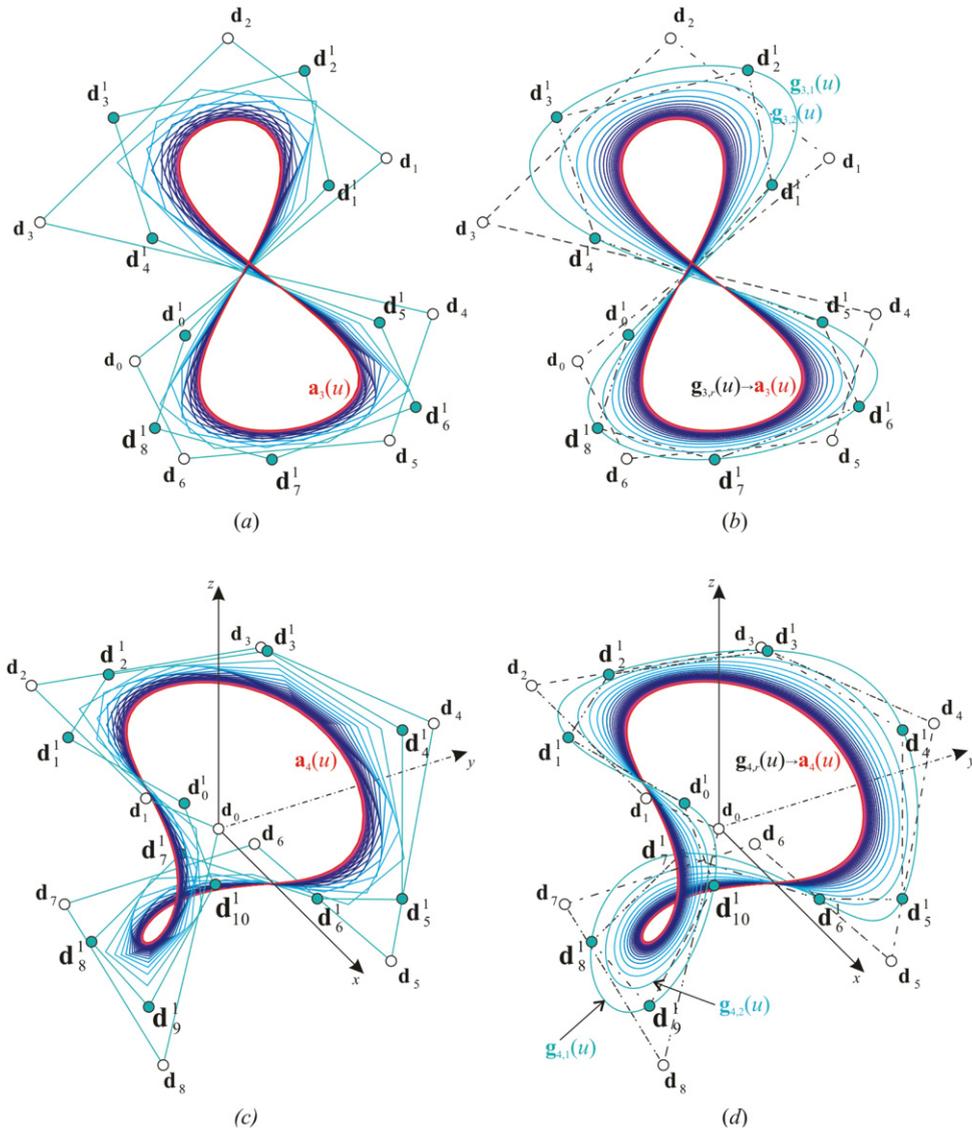


Fig. 4. (a) A plane cyclic curve of degree 3 with its original and degree elevated control polygons. (b) The loci of vertices of the degree elevated control polygons converge to the cyclic curve. (c)–(d) A similar example is provided in case of a space cyclic curve of degree 4.

### 3.3. Variation diminishing

Before starting our results we have to borrow and explain a few concepts and properties that were introduced and proved in Pólya and Schoenberg (1958).

**Definition 8** (Cyclic variations of sign of finite real sequences). If  $d = \{d_i\}_{i=1}^n$  is a finite sequence of real numbers we shall denote by  $v(d)$  the number of variations of sign in the terms of this sequence and the number

$$v_c(d) = \begin{cases} 0, & \text{if all } d_i = 0, \\ v(d_i, d_{i+1}, \dots, d_n, d_1, d_2, \dots, d_{i-1}, d_i), & \text{if } d_i \neq 0 \end{cases}$$

denotes the cyclic variations of sign of sequence  $d$ .

If one thinks of the  $d_i$  as arranged clockwise in cyclic order, it becomes obvious that  $v_c(d)$  does not depend on the particular non-vanishing term  $d_i$  we start with. Notice that  $v_c(d)$  is always an even number.

**Definition 9** (Cyclic variations of sign of real  $2\pi$ -periodic functions). Let  $f(u)$  be a real-valued function of period  $2\pi$ . Let  $\mathcal{U}$  the set of all real sequences  $u = \{u_i\}_{i=1}^n$  that fulfill the condition

$$u_1 < u_2 < \dots < u_n < u_1 + 2\pi. \tag{18}$$

The cyclic variations of sign of  $f(u)$  is defined by the number

$$v_c(f) = \sup_{u \in \mathcal{U}} \{v_c(d): d = \{f(u_i)\}_{i=1}^n\}.$$

Note, the number  $v_c(f)$ , if finite, is even (e.g.,  $v_c(\sin u) = 2$ ,  $v_c(\sin 2u) = 4$ ,  $v_c(|\sin u|) = 0$ ).

**Definition 10** (*Variation diminishing transformations on the circle*). Such a transformation is characterized by a non-negative weight-function, or kernel,  $\Omega(u)$ , of period  $2\pi$ , of bounded variation and normalized by the conditions

$$\frac{1}{2\pi} \int_0^{2\pi} \Omega(u) \, du = 1, \quad \Omega(u) = \frac{1}{2}(\Omega(u + 0) + \Omega(u - 0)).$$

Let  $f(u)$  be an arbitrary  $2\pi$ -periodic real-valued and integrable function. We say that the convolution transformation

$$g(u) = \frac{1}{2\pi} \int_0^{2\pi} \Omega(u - \varphi) f(\varphi) \, d\varphi \tag{19}$$

is variation diminishing provided that the inequality

$$v_c(g) \leq v_c(f)$$

holds for each  $f$ . We mean the same thing if we say that  $\Omega(u)$  is a variation diminishing kernel.

**Definition 11** (*de la Vallée Poussin kernels and means*). The de la Vallée Poussin kernel is defined as

$$\omega_n(u) = \frac{1}{\binom{2n}{n}} \left(2 \cos \frac{u}{2}\right)^{2n}, \quad n \geq 1, \tag{20}$$

the Fourier expansion of which has the simple form

$$\omega_n(u) = \frac{1}{\binom{2n}{n}} \sum_{k=-n}^n \binom{2n}{n+k} e^{iku} = 1 + 2 \sum_{k=1}^n \frac{n!}{(n-k)!} \frac{n!}{(n+k)!} \cos ku.$$

For  $\Omega(u) = \omega_n(u)$  the transformation (19) becomes

$$V_n(u) = \frac{1}{\binom{2n}{n}} \frac{1}{2\pi} \int_0^{2\pi} \left(2 \cos \frac{u - \varphi}{2}\right)^{2n} f(\varphi) \, d\varphi$$

and defines the de la Vallée Poussin mean, or simply  $V$ -mean, of the function  $f(u)$ .

It is easily verified that  $V_n(u)$  is a trigonometric polynomial of an order not exceeding  $n$ , which is readily expressed in terms of the Fourier coefficients of  $f(u)$ . Indeed, if

$$f(u) \approx \sum_{k=-\infty}^{\infty} f_k e^{iku} \quad (f_{-k} = \bar{f}_k), \tag{21}$$

we obtain

$$V_n(u) = \frac{1}{\binom{2n}{n}} \sum_{k=-\infty}^{\infty} \binom{2n}{n+k} f_k e^{iku}$$

by convoluting (20) and (21). In terms of real Fourier series (i.e.,  $2f_k = a_k - ib_k$ )

$$f(u) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos ku + b_k \sin ku)$$

we find

$$V_n(u) = \frac{a_0}{2} + \frac{1}{\binom{2n}{n}} \sum_{k=1}^n \binom{2n}{n+k} (a_k \cos ku + b_k \sin ku).$$

The main results of article (Pólya and Schoenberg, 1958) were the fact that the de Vallée Poussin kernels  $\omega_n$  ( $n \geq 1$ ) possess the cyclic variation diminishing property. However, the authors also demonstrated several lemmas. One of them (see Lemma 3 in Pólya and Schoenberg (1958), p. 300) is very important for us:

**Lemma 12.** Let  $\varphi_1, \varphi_2, \dots, \varphi_m$  ( $m \geq 2$ ) be  $m$  points in counterclockwise order on the circle such that

$$\varphi_1 < \varphi_2 < \dots < \varphi_m < \varphi_1 + 2\pi$$

and let

$$T_n(u) = \sum_{k=1}^m \omega_n(u - \varphi_k) a_k, \quad T_n(u) \not\equiv 0,$$

where at least two among the  $a_k$  do not vanish. Then

$$z_c(T_n) \leq v_c(a),$$

where  $a = \{a_k\}_{k=1}^m$  and the number  $z_c(T_n)$  denotes the number of real zeros of  $T_n(u)$  within a period, including multiplicities.

The following theorem proves that no hyperplane can intersect a cyclic curve more times than its control polygon.

**Theorem 13** (Variation diminishing). Cyclic curve (15) of degree  $n \geq 1$  possesses the cyclic variation diminishing property.

**Proof.** Consider the equation

$$H(x_1, x_2, \dots, x_\delta) = h_1 x_1 + h_2 x_2 + \dots + h_\delta x_\delta = 0$$

of a hyperplane in  $\mathbb{R}^\delta$ , where  $\sum_{i=1}^\delta h_i^2 = 1$ . Let

$$d_i = H(\mathbf{d}_i) = H(d_{i1}, d_{i2}, \dots, d_{i\delta}) \quad (i = 0, 1, \dots, 2n)$$

and consider the function

$$h(u) = H\left(\sum_{i=0}^{2n} C_{i,n}(u) \mathbf{d}_i\right) = \sum_{i=0}^{2n} C_{i,n}(u) H(\mathbf{d}_i) = \sum_{i=0}^{2n} C_{i,n}(u) d_i.$$

Thus, it is sufficient to show that for all real sequences  $d = \{d_i\}_{i=0}^{2n}$  (where at least two among the  $d_i$  do not vanish) the inequality

$$v_c(h) \leq v_c(d)$$

holds.

Observe that the basis function  $C_{i,n}(u)$  ( $i = 0, 1, \dots, 2n$ ) can be written in the form

$$\begin{aligned} C_{i,n}(u) &= \frac{c_n}{2^n} \left(1 + \cos\left(u + \frac{2i\pi}{2n+1}\right)\right)^n \\ &= \frac{2^{2n}}{2^n(2n+1)\binom{2n}{n}} 2^n \cos^{2n}\left(\frac{u}{2} + \frac{i\pi}{2n+1}\right) \\ &= \frac{1}{2n+1} \frac{1}{\binom{2n}{n}} \left(2 \cos\left(\frac{u}{2} + \frac{i\pi}{2n+1}\right)\right)^{2n} \\ &= \frac{1}{2n+1} \omega_n\left(u + \frac{2i\pi}{2n+1}\right). \end{aligned}$$

Thus, for arbitrary but fixed real sequence  $d = \{d_i\}_{i=0}^{2n}$  (where at least two among the  $d_i$  do not vanish)

$$h(u) = \sum_{i=0}^{2n} C_{i,n}(u) d_i = \frac{1}{2n+1} \sum_{i=0}^{2n} \omega_n\left(u + \frac{2i\pi}{2n+1}\right) d_i.$$

Now, using Lemma 12 (with settings  $m = 2n + 1$ ,  $\varphi_{m+1-k} = -\frac{2(k-1)\pi}{2n+1}$ ,  $a_{m+1-k} = d_{k-1}$ ,  $k = 1, 2, \dots, m$ ,  $a = \{a_k\}_{k=1}^m$ ) we conclude that

$$\begin{aligned} v_c(h) &\leq z_c(h) = z_c\left(\frac{1}{2n+1} \sum_{i=0}^{2n} \omega_n\left(u + \frac{2i\pi}{2n+1}\right) d_i\right) \\ &= z_c\left(\frac{1}{2n+1} \sum_{k=1}^m \omega_n(u - \varphi_k) a_k\right) \\ &= z_c\left(\sum_{k=1}^m \omega_n(u - \varphi_k) a_k\right) \\ &\leq v_c(a) = v_c(d) \end{aligned}$$

which completes the proof.  $\square$

**Corollary 14** (Convexity preserving). *If the control polygon  $\{d_i\}_{i=0}^{2n}$  of a plane cyclic curve (15) is convex then the curve itself is also convex, i.e. it forms the boundary of a convex region in the plane.*

**Remark 15** ((Weak) Descartes system). The function system  $C = \{C_{i,n}\}_{i=0}^{2n}$  is a (weak) Descartes system due to the facts that  $\sum_{i=0}^{2n} C_{i,n}(u) \equiv 1$ ,  $\forall u \in [\varphi, \varphi + 2\pi]$ ,  $\varphi \in \mathbb{R}$  and system  $C$  has the cyclic variation diminishing property (cf. Proposition 5.9 in Carnicer (1999), pp. 39). However, we have to emphasize that system  $C$  is not a totally positive (or order complete Chebyshev) system, cf. Remark 4.

### 3.4. Ellipses and circles

If  $n = 1$  we obtain an ellipse. It means that we can specify a complete circle or ellipse by just three control points, while using NURBS description we need at least six control points for  $C^2$  continuity at joints.

**Theorem 16.** *If  $n = 1$  the ellipse touches the sides of the control triangle at its midpoints, and the center of the ellipse is the centroid of the triangle (cf. Fig. 5), i.e. the ellipse is the Steiner inellipse of the control triangle.*

**Proof.** Using the notation  $u_j = \pi(1 - 2j/3)$ , simple computation shows that

$$C_{i,1}(u_j) = \begin{cases} 0 & \text{if } i = j, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Thus, the ellipse contains the midpoints of the sides, and at these points the sides must be tangential to the ellipse due to the convex hull property.

Let us consider an affine transformation that transforms the ellipse to a circle. In our case this transformation takes the tangential control triangle to a regular triangle that is tangential to the circle. The center of the inscribed circle of a regular triangle coincides with the centroid of the triangle. Both the centroid and the center of conics are affine invariants, therefore the proof is completed.  $\square$

By means of Theorem 16 we can easily construct (compute) a control triangle for a given ellipse. Natural questions are: How many different control triangles has an ellipse (we do not consider control triangles with cyclically permuted vertices different)? What is the locus of vertices of control triangles?

To answer these questions we consider the affinity we used in the proof of Theorem 16. It is obvious that there is a one to one correspondence between the tangential regular triangles of the circle and the control triangles of the ellipse. The

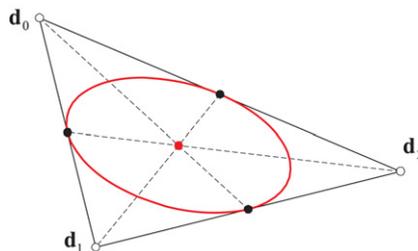


Fig. 5. The center of the ellipse is the centroid of the control triangle.

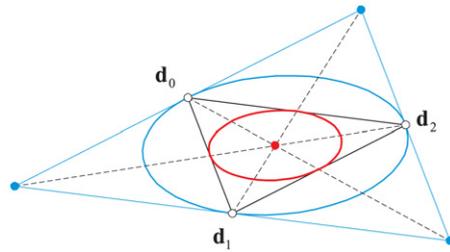


Fig. 6. The locus of control points of different representations with its control polygon.

tangential regular triangles of the circle form a one-parameter family (the family parameter is the rotation angle about the center), and their vertices are on a circle that is concentric with the circle under consideration. Applying the inverse of the affine transformation above, we obtain a one-parameter family of control triangles of the given ellipse the vertices of which are on an ellipse that is concentric with the given one. Moreover, the ellipse of the control points can be obtained by a double scaling from the given ellipse (cf. Fig. 6). Therefore, this ellipse is the Steiner circumellipse of the control triangle.

In what follows, we show that circles and ellipses can also be obtained for higher  $n$  values with special arrangements of control points. We will make use of the following lemma.

**Lemma 17.** *Trigonometric identities*

$$\sum_{i=0}^{2n} (1 + \cos(\gamma + (i - k)\lambda_n))^n \cos i\lambda_n = \frac{\binom{2n}{n-1} (2n + 1) \cos(\gamma - k\lambda_n)}{2^n}, \tag{22}$$

$$\sum_{i=0}^{2n} (1 + \cos(\gamma + (i - k)\lambda_n))^n \sin i\lambda_n = -\frac{\binom{2n}{n-1} (2n + 1) \sin(\gamma - k\lambda_n)}{2^n} \tag{23}$$

hold  $\forall \gamma \in \mathbb{R}$  ( $k = 0, 1, \dots, 2n$ ).

**Proof.** Identity (22) can be proved as follows.

$$\begin{aligned} & \sum_{i=0}^{2n} (1 + \cos(\gamma + (i - k)\lambda_n))^n \cos i\lambda_n \\ & \stackrel{(3)}{=} 2^n \sum_{i=0}^{2n} \cos^{2n} \left( \frac{\gamma}{2} + \frac{(i - k)\pi}{2n + 1} \right) \cos \frac{2i\pi}{2n + 1} \\ & \stackrel{(6)}{=} \frac{1}{2^n} \binom{2n}{n} \sum_{i=0}^{2n} \cos \frac{2i\pi}{2n + 1} + \frac{1}{2^{n-1}} \sum_{l=0}^{n-1} \binom{2n}{l} \sum_{i=0}^{2n} \cos \left( (n - l)\gamma + \frac{2(n - l)(i - k)\pi}{2n + 1} \right) \cos \frac{2i\pi}{2n + 1} \\ & \stackrel{(4)}{=} \frac{1}{2^{n-1}} \sum_{l=0}^{n-1} \binom{2n}{l} \sum_{i=0}^{2n} \cos \left( (n - l)\gamma + \frac{2(n - l)(i - k)\pi}{2n + 1} \right) \cos \frac{2i\pi}{2n + 1} \\ & \stackrel{(1)}{=} \frac{1}{2^n} \sum_{l=0}^{n-1} \binom{2n}{l} \left( \sum_{i=0}^{2n} \cos \left( (n - l)\gamma - \frac{2(n - l)k\pi}{2n + 1} + \frac{2(n - l - 1)i\pi}{2n + 1} \right) \right. \\ & \quad \left. + \sum_{i=0}^{2n} \cos \left( (n - l)\gamma - \frac{2(n - l)k\pi}{2n + 1} + \frac{2(n - l + 1)i\pi}{2n + 1} \right) \right) \\ & \stackrel{(4)}{=} \frac{1}{2^n} \left( (2n + 1) \binom{2n}{n - 1} \cos \left( \gamma - \frac{2k\pi}{2n + 1} \right) + \sum_{l=0}^{n-2} \binom{2n}{l} \frac{\sin((n - l - 1)\pi) \cos \left( (n - l)\gamma - \frac{2(n - l)k\pi}{2n + 1} + \frac{2n(n - l - 1)\pi}{2n + 1} \right)}{\sin \frac{(n - l - 1)\pi}{2n + 1}} \right. \\ & \quad \left. + \sum_{l=0}^{n-1} \binom{2n}{l} \frac{\sin((n - l + 1)\pi) \cos \left( (n - l)\gamma - \frac{2(n - l)k\pi}{2n + 1} + \frac{2n(n - l + 1)\pi}{2n + 1} \right)}{\sin \frac{(n - l + 1)\pi}{2n + 1}} \right) \\ & = \frac{\binom{2n}{n-1} (2n + 1)}{2^n} \cos(\gamma - k\lambda_n). \end{aligned}$$

The proof of identity (23) is analogous to that of identity (22), the difference is only in the applied identities. Instead of identities (3), (6), (4), (1), (4) identities (3), (6), (5), (2), (5) have to be applied, respectively.  $\square$

**Theorem 18.** Curve (15) is a circle if and only if its control points are the vertices of a  $2n + 1$  sided regular polygon. The radius  $r_n$  of the circle is

$$r_n = R \frac{n}{n + 1}, \tag{24}$$

where  $R$  is the radius of the circumscribed circle of the polygon.

**Proof.** At first we prove that if the control points are the vertices of a regular  $2n + 1$  sided polygon curve (15) is a circle. Let us assume that the center of the polygon is the origin and its control points are  $\mathbf{d}_i = [R \cos i\lambda_n \ R \sin i\lambda_n]^T$  ( $i = 0, 1, \dots, 2n$ ), where  $\lambda_n = 2\pi / (2n + 1)$ .

The coordinate functions of curve (15) determined by these control points are

$$d_x(u) = \frac{Rc_n}{2^n} \sum_{i=0}^{2n} (1 + \cos(u + i\lambda_n))^n \cos i\lambda_n, \tag{25}$$

$$d_y(u) = \frac{Rc_n}{2^n} \sum_{i=0}^{2n} (1 + \cos(u + i\lambda_n))^n \sin i\lambda_n. \tag{26}$$

Introducing the notation

$$r_n = d_x(0) = \frac{Rc_n}{2^n} \sum_{i=0}^{2n} (1 + \cos(i\lambda_n))^n \cos i\lambda_n, \tag{27}$$

we show that

$$\begin{bmatrix} d_x(u) \\ d_y(u) \end{bmatrix} = \begin{bmatrix} r_n \cos u \\ -r_n \sin u \end{bmatrix}, \quad \forall u \in [0, 2\pi].$$

Application of the case  $\gamma = k = 0$  of (22) in Lemma 17 for the right hand side of (27) yields

$$r_n = \frac{Rc_n \binom{2n}{n-1} (2n + 1)}{2^{2n}}$$

from which we obtain (24) considering Eq. (7) of Corollary 2.

Applying the case  $\gamma = u, k = 0$  of (22) in Lemma 17 for the right hand side of (25), and the case  $\gamma = k = 0$  of (22) for the right hand side of (27), we obtain the equality

$$d_x(u) = r_n \cos u.$$

Applying the case  $\gamma = u, k = 0$  of (23) in Lemma 17 for the right hand side of (26), and the case  $\gamma = k = 0$  of (22) for the right hand side of (27), we obtain the equality

$$d_y(u) = -r_n \sin u.$$

Note, that

$$\lim_{n \rightarrow \infty} r_n = R \lim_{n \rightarrow \infty} \frac{n}{n + 1} = R,$$

i.e. the control polygon converges to the circle as  $n$  tends to infinity.

Now, we show that, if the curve (15) describes a circle then its control points form the vertices of a  $2n + 1$  sided regular polygon, and the relation between the radius  $r_n$  of the circle and the radius  $R$  of the circumscribed circle of the control polygon is  $r_n = Rn / (n + 1)$ .

We assume that the center of the circle is the origin, control points  $\mathbf{d}_i$  have the coordinates  $\mathbf{d}_i = [x_i \ y_i]^T$  ( $i = 0, 1, \dots, 2n$ ) and the circle is  $\mathbf{c}(u) = [r \cos u \ r \sin u]^T, \forall u \in [-2\pi, 0]$ .

Control points  $\mathbf{d}_i$  ( $i = 0, 1, \dots, 2n$ ) have to satisfy the conditions

$$\begin{cases} \frac{c_n}{2^n} \sum_{i=0}^{2n} (1 + \cos(u + i\lambda_n))^n x_i = r \cos u, \\ \frac{c_n}{2^n} \sum_{i=0}^{2n} (1 + \cos(u + i\lambda_n))^n y_i = r \sin u, \end{cases} \quad \forall u \in [-2\pi, 0]. \tag{28}$$

Let us substitute  $u_k = -k\lambda_n$  ( $k = 0, 1, \dots, 2n$ ) into Eqs. (28). We get linear systems

$$\begin{aligned} M_{2n+1}X &= C, \\ M_{2n+1}Y &= S, \end{aligned} \tag{29}$$

where  $M_{2n+1} = \frac{c_n}{2^n} \Lambda_{2n+1}$ ,  $X = [x_0 \ x_1 \ \dots \ x_{2n}]^T$ ,  $Y = [y_0 \ y_1 \ \dots \ y_{2n}]^T$ ,  $C = [r \ r \cos \lambda_n \ \dots \ r \cos 2n\lambda_n]^T$ ,  $S = [0 \ r \sin \lambda_n \ \dots \ r \sin 2n\lambda_n]^T$ . Matrix  $\Lambda_{2n+1}$  is specified in the proof of Theorem 3. We have seen that  $\det \Lambda_{2n+1} \neq 0$ , thus the determinant of matrix  $M_{2n+1}$  is

$$\det M_{2n+1} = \left(\frac{c_n}{2^n}\right)^{2n+1} \det \Lambda_{2n+1} \neq 0.$$

Therefore, the systems (29) admit unique solutions. We will prove that these unique solutions are

$$x_i = \frac{n+1}{n} r \cos i\lambda_n$$

and

$$y_i = \frac{n+1}{n} r \sin i\lambda_n.$$

At first, we prove that

$$\frac{c_n}{2^n} \frac{n+1}{n} \sum_{i=0}^{2n} (1 + \cos(i-k)\lambda_n)^n \cos i\lambda_n = \cos k\lambda_n \quad (k = 0, 1, \dots, 2n). \tag{30}$$

Applying the case  $\gamma = 0$  of (22) in Lemma 17 for the left hand side of (30), we obtain

$$\frac{c_n(n+1)}{n2^{2n}} \binom{2n}{n-1} (2n+1) \cos k\lambda_n.$$

The usage of formula (7) of Corollary 2 yields Eq. (30).

Analogously, applying the case of  $\gamma = 0$  of (23) and (7) we obtain equalities

$$\frac{c_n}{2^n} \frac{n+1}{n} \sum_{i=0}^{2n} (1 + \cos(i-k)\lambda_n)^n \sin i\lambda_n = \sin k\lambda_n \quad (k = 0, 1, \dots, 2n). \quad \square$$

**Corollary 19.** *Curve (15) is an ellipse if and only if its control points can be obtained from the vertices of a  $2n + 1$  sided regular polygon by an affine transformation.*

#### 4. Cyclic surfaces

Using basis functions (16) we can define tensor product surfaces of degree  $(n, m)$  ( $n \geq 1, m \geq 1$ ) as well in the form

$$\mathbf{s}_{n,m}(u, v) = \sum_{i=0}^{2n} \sum_{j=0}^{2m} \mathbf{d}_{ij} C_{i,n}(u) C_{j,m}(v), \quad u, v \in [-\pi, \pi] \tag{31}$$

where points  $\mathbf{d}_{ij}$  are called control points that form the control net of the surface, and functions  $C_{i,n}(u)$  and  $C_{j,m}(v)$  are defined by (16).

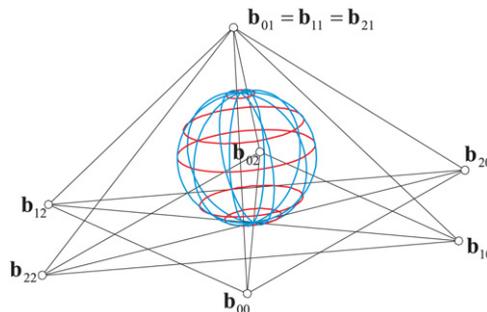


Fig. 7. A sphere with its control net.

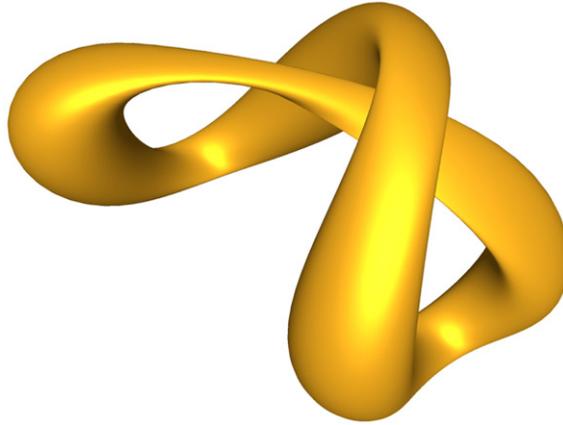


Fig. 8. A free-form closed surface with  $n = 7$ ,  $m = 9$ .

This type of surfaces is suitable for the description of closed surfaces (surfaces without boundary curve) with the advantage of singularity free parametrization, i.e. all points of the surfaces are of  $C^\infty$  continuity. A sphere, e.g. can be described by 9 control points 3 of which are coincident (cf. Fig. 7). In Fig. 8 there is a free-form closed surface with  $n = 7$ ,  $m = 9$ .

Control point  $\mathbf{d}_{ij}$  ( $i = 0, 1, \dots, 2n$ ;  $j = 0, 1, \dots, 2m$ ) has no effect on isoparametric lines

$$\mathbf{s}_{n,m}\left(\pi - \frac{2i\pi}{2n+1}, v\right) \quad \text{and} \quad \mathbf{s}_{n,m}\left(u, \pi - \frac{2j\pi}{2m+1}\right)$$

since  $C_{i,n}\left(\pi - \frac{2i\pi}{2n+1}\right) = C_{j,m}\left(\pi - \frac{2j\pi}{2m+1}\right) = 0$ . Except these two isoparametric lines control point  $\mathbf{d}_{ij}$  affects the position of all points of the surface.

An advantage of our closed  $C^\infty$  continuous surface modeling method is that it is based on only the position of control points and the resulting surface is described as a single surface instead of several patches that have to fulfill heavy geometric constraints in order to achieve  $C^k$  continuity ( $k \geq 1$ , finite) along the joints of patches.

#### 4.1. Degree elevation

Naturally, the degree elevation of the cyclic surface (31) is based on the closed formula (17), i.e. on the degree elevation of cyclic curves. For an arbitrary control net

$$\mathbf{D}_{n,m} = [\mathbf{d}_{ij}]_{i=0, j=0}^{2n, 2m} \in \mathcal{M}_{2n+1, 2m+1}(\mathbb{R}^3)$$

let us introduce the notations

$$\mathbf{D}_{n,m,i,*} = [\mathbf{d}_{i0}, \mathbf{d}_{i1}, \dots, \mathbf{d}_{i,2m}], \quad i = 0, 1, \dots, 2n,$$

$$\mathbf{D}_{n,m,*,j} = [\mathbf{d}_{0j}, \mathbf{d}_{1j}, \dots, \mathbf{d}_{2n,j}]^T, \quad j = 0, 1, \dots, 2m.$$

In order to construct a control net

$$\mathbf{D}_{n+p,m+q} = [\mathbf{d}_{i,j}^{p,q}]_{i=0, j=0}^{2(n+p), 2(m+q)} \in \mathcal{M}_{2(n+p)+1, 2(m+q)+1}(\mathbb{R}^3), \quad p \geq 1, \quad q \geq 1$$

that fulfills the condition

$$\mathbf{s}_{n+p, m+q}(u, v) = \sum_{i=0}^{2(n+p)} \sum_{j=0}^{2(m+q)} \mathbf{d}_{i,j}^{p,q} C_{i,n+p}(u) C_{j,m+q}(v) \equiv \mathbf{s}_{n,m}(u, v), \quad \forall (u, v) \in [-2\pi, 0] \times [-2\pi, 0]$$

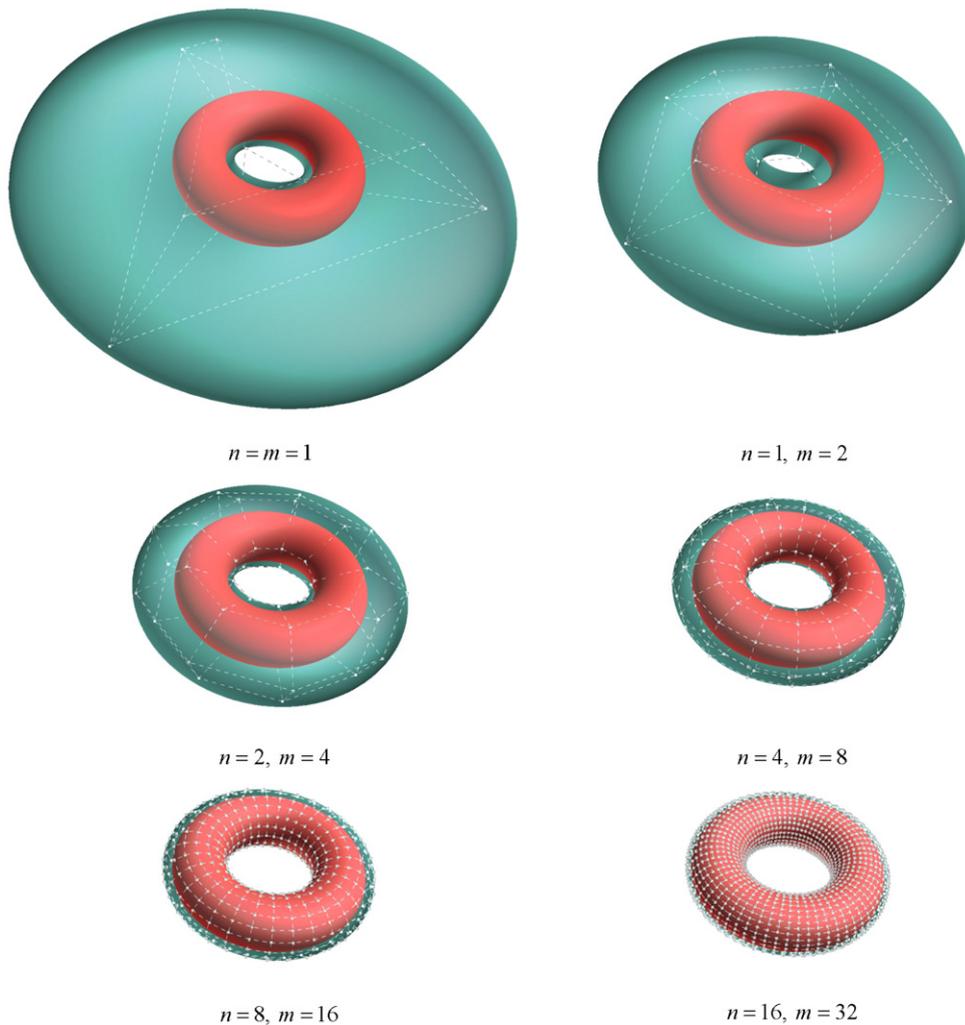
at first we have to calculate the intermediate control net

$$\mathbf{D}_{n,m+q} = \left[ \mathbf{g}_{m,q} \left( -\frac{2j\pi}{2(m+q)+1}; \mathbf{D}_{n,m,i,*} \right) \right]_{i=0, j=0}^{2n, 2(m+q)};$$

then the final one

$$\mathbf{D}_{n+p,m+q} = \left[ \mathbf{g}_{n,p} \left( -\frac{2i\pi}{2(n+p)+1}; \mathbf{D}_{n,m+q,*,j} \right) \right]_{i=0, j=0}^{2(n+p), 2(m+q)}.$$

The convergence property of cyclic curves is inherited by cyclic surfaces, i.e. the locus of control net  $\mathbf{D}_{n+p,m+q}$  converges to the surface  $\mathbf{s}_{n,m}(u, v)$  as parameters  $p$  and  $q$  tend to infinity. Fig. 9 depicts this convergence in case of the exact description of a torus.



**Fig. 9.** Exact description of a torus; transparent surfaces correspond to the loci of control nets that converge to the torus as degrees  $n \geq 1$  and  $m \geq 1$  tend to infinity.

### 5. Conclusions

We defined a cyclic basis in the vectorspace

$$\mathcal{V}_n = \langle 1, \cos(u), \sin(u), \dots, \cos(nu), \sin(nu) \rangle$$

of trigonometric polynomials of degree at most  $n$ . By means of these functions we proposed a curve description method using which one can model closed curves by control points.

This type of curve has properties such as singularity free parametrization (all points of the curve are of the same continuity properties), convex hull, cyclic symmetry, closure for the affine transformation of its control points, global controllability and variation diminishing. We provided a closed formula for the degree elevation from  $n$  to  $n+r$ ,  $r \geq 1$  and proved that the control polygon of the degree elevated curve converges to the curve when  $r \rightarrow \infty$ .

We discussed the special case  $n = 1$ , that results either an ellipse or a circle, in detail. We also described tensor product surfaces by means of these basis functions.

Specification of control point configurations that result famous curves such as epi-cycloid, hypo-cycloid, Lissajous curve, torus knots etc., is to be done in the future.

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