Faiz Muhammad Khan, Hidayat Ullah Khan*, Safyan Mukhtar, Asghar Khan and Nor Haniza Sarmin

Some Innovative Types of Fuzzy Ideals in *AG*-Groupoids

https://doi.org/10.1515/jisys-2017-0258
Received June 1, 2017; previously published online September 23, 2017.

Abstract: *AG*-groupoids (non-associative structure) are basic structures in Flocks theory. This theory mainly focuses on distance optimization, motion replication, and leadership maintenance with a wide range of applications in physics and biology. In this paper, we define some new types of fuzzy ideals of *AG*-groupoids called (α, β)-fuzzy bi-ideals, (α, β)-fuzzy interior ideals, $(\overline{\beta}, \overline{\alpha})$-fuzzy bi-ideals, and $(\overline{\beta}, \overline{\alpha})$-fuzzy interior ideals, where $\alpha, \beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$ and $\overline{\alpha}, \overline{\beta} \in \{\overline{\in}_\gamma, \overline{q}_\delta, \overline{\in}_\gamma \vee \overline{q}_\delta, \overline{\in}_\gamma \wedge \overline{q}_\delta\}$, with $\alpha \neq \in_\gamma \wedge q_\delta$ and $\overline{\beta} \neq \overline{\in}_\gamma \wedge \overline{q}_\delta$. An important milestone achieved by this paper is providing the connection between classical algebraic structures (ordinary bi-ideals, interior ideals) and new types of fuzzy algebraic structures [$(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals]. Special attention is given to $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals and $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$ -fuzzy bi-ideals.

Keywords: *AG*-groupoids, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals, $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy bi-ideals, $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy interior ideals.

AMS Mathematics Subject Classification: 20M10, 20N99.

1 Introduction

The fuzzification of algebraic structures like semigroups, ordered semigroups, *AG*-groupoids, hemirings, near-rings, and *BCK*/*BCI*-algebra in terms of several types of fuzzy ideals and investigation of new types of fuzzy ideals is a central focus for researchers nowadays. Due to the diverse applications of such characterizations in many applied branches such as control engineering, computer science, Flocks theory, fuzzy coding theory, fuzzy finite state machines, and fuzzy automata, efforts were made by mathematicians to investigate some new types of fuzzy ideals that can fill out the gap in the latest studies of ideal theory. Mordeson et al. [28] presented an up-to-date account of fuzzy sub-semigroups and fuzzy ideals of a semigroup. Kuroki [23] introduced the notion of fuzzy bi-ideals in semigroups. Kehayopulu and Tsingelis applied the fuzzy concept in ordered semigroups and studied some properties of fuzzy left (right) ideals and fuzzy filters in ordered semigroups [15]. Fuzzy implicative and Boolean filters of R_0-algebra were initiated by Liu and Li [24].

The concept of a quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in Refs. [1, 2], played a significant role in generating some different types of fuzzy subgroups. The notion of (α, β)-fuzzy subgroups by using the "belongs to" relation ($\in$) and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup was introduced by Bhakat and Das [1]. In particular, the $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of the Rosenfeld fuzzy subgroup [33]. This concept opened

***Corresponding author: Hidayat Ullah Khan,** Department of Mathematics, University of Malakand at Chakdara, Dir (L), Khyber Pakhtunkhwa 18800, Pakistan, e-mail: hidayatullak@yahoo.com
Faiz Muhammad Khan: Department of Mathematics and Statistics, University of Swat, Khyber Pakhtunkhwa, Pakistan; and Department of Applied Mathematics, School of Natural and Applied Sciences, Northwestern Polytechnical University, Xi'an, China
Safyan Mukhtar: Department of Mathematics, Bacha Khan University Charsadda, Khyber Pakhtunkhwa, Pakistan
Asghar Khan: Department of Mathematics, Abdul Wali Khan University Mardan, Khyber Pakhtunkhwa, Mardan, Pakistan
Nor Haniza Sarmin: Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia

a new dimension for researchers to investigate similar types of generalizations of the existing fuzzy subsystems of other algebraic structures. In algebra, the concept of (α, β)-fuzzy sets was introduced by Davvaz in Ref. [3], where $(\in, \in\vee q)$-fuzzy subnearrings (ideals) of a nearring were initiated and studied. With this objective in view, Ma et al. in Ref. [25] introduced the interval-valued $(\in, \in\vee q)$-fuzzy ideals of pseudo-*MV* algebras and gave some important results of pseudo-*MV* algebras. Jun and Song [11] discussed general forms of fuzzy interior ideals in semigroups; for further details, the readers are referred to Ref. [12]. Kazanci and Yamak introduced the concept of a generalized fuzzy bi-ideal in semigroups [13] and gave some properties of fuzzy bi-ideals in terms of $(\in, \in\vee q)$-fuzzy bi-ideals. Jun et al. [12] gave the concept of a generalized fuzzy bi-ideal in ordered semigroups and characterized regular-ordered semigroups in terms of this notion. Davvaz [3] used the idea of generalized fuzzy sets in hyperstructures and introduced different generalized fuzzy subsystems of hyperstructures. In Ref. [26], Ma et al. introduced the concept of a generalized fuzzy filter of R_0-algebra and provided some properties in terms of this notion. Many other researchers used the idea of generalized fuzzy sets and gave several characterization results in different branches of algebra [1, 3, 11–13, 16, 18–21, 25–27, 34, 35]. The concept of (α, β)-fuzzy interior ideals in ordered semigroups was first introduced by Khan and Shabir in Ref. [16], where some basic properties of (α, β)-fuzzy interior ideals were discussed. Khan et al. [21, 22] characterized ordered semigroups in terms of some new types of fuzzy bi-ideals and fuzzy generalized bi-ideals.

Besides this, the concept of a left almost semigroup (*LA*-semigroup) [14] was first introduced by Kazim and Naseeruddin in 1972. In Ref. [10], the same structure was called a left invertive groupoid. Protic and Stevanovic [32] called it Abel-Grassmann's groupoid, abbreviated as *AG*-groupoid, a groupoid whose elements satisfy the left invertive law: $(ab)c=(cb)a$ for all $a, b, c\in G$. An *AG*-groupoid is the midway structure between a commutative semigroup and a groupoid [30]. It is a useful non-associative structure with a wide range of applications in the theory of flocks [31]. In an *AG*-groupoid, the medial law, $(ab)(cd)=(ac)(bd)$ for all $a, b, c, d\in G$ [14], holds. If there exists an element e in an *AG*-groupoid G such that $ex=x$ for all $x\in G$, then G is called an *AG*-groupoid with left identity e. If an *AG*-groupoid G has right identity, then G is a commutative monoid. If an *AG*-groupoid G contains left identity, then $(ab)(cd)=(dc)(ba)$ holds for all $a, b, c, d\in G$. Also, $a(bc)=b(ac)$ holds for all $a, b, c\in G$.

Recently, Yin and Zhan [36] introduced more general forms of $(\in, \in\vee q)$-fuzzy (implicative, positive implicative and fantastic) filters and $(\overline{\in}, \overline{\in}\vee \overline{q})$-fuzzy (implicative, positive implicative and fantastic) filters of *BL*-algebras and defined $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy (implicative, positive implicative, and fantastic) filters and $(\overline{\in}_\gamma, \overline{\in}_\gamma\vee \overline{q}_\delta)$-fuzzy (implicative, positive implicative, and fantastic) filters of *BL*-algebras, and gave some interesting results in terms of these notions.

In this paper, we generalize the concept of $(\in, \in\vee q)$-fuzzy bi-ideals given in Ref. [18], $(\in, \in\vee q)$-fuzzy left (right) ideals and $(\in, \in\vee q)$-fuzzy interior ideals given in Ref. [19], and define (α, β)-fuzzy bi-ideals, (α, β)-fuzzy interior ideals, $(\overline{\beta}, \overline{\alpha})$-fuzzy bi-ideals, and $(\overline{\beta}, \overline{\alpha})$-fuzzy interior ideals in *AG*-groupoids, where $\alpha, \beta\in\{\in_\gamma, q_\delta, \in_\gamma\vee q_\delta, \in_\gamma\wedge q_\delta\}$ and $\overline{\alpha}, \overline{\beta}\in\{\overline{\in}_\gamma, \overline{q}_\delta, \overline{\in}_\gamma\vee \overline{q}_\delta, \overline{\in}_\gamma\wedge \overline{q}_\delta\}$ with $\alpha\neq\in_\gamma\wedge q_\delta$ and $\overline{\beta}\neq\overline{\in}_\gamma\wedge\overline{q}_\delta$. Special attention is given to $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy bi-ideals, $(\overline{\in}_\gamma, \overline{\in}_\gamma\vee \overline{q}_\delta)$-fuzzy bi-ideals, $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy interior ideals, and $(\overline{\in}_\gamma, \overline{\in}_\gamma\vee \overline{q}_\delta)$-fuzzy bi-ideals. Some interesting results in terms of these new types of notions are also obtained.

2 Preliminaries

The present section contains some fundamental concepts on *AG*-groupoids that are essential for this paper.

Throughout the paper, G denotes an *AG*-groupoid unless otherwise stated.

For subsets A, B of an *AG*-groupoid G, we denote by $AB=\{ab\in G\mid a\in A, b\in B\}$. A non-empty subset A of G is called a left (right) ideal [29] of G if $GA\subseteq A(AG\subseteq A)$. A non-empty subset A of an *AG*-groupoid G is called an *AG*-subgroupoid of G if $A^2\subseteq A$. An *AG*-subgroupoid A of G is called a bi-ideal [29] of G if $(AG)A\subseteq A$. An *AG*-subgroupoid A of G is called an interior ideal [29] of G if $(GA)G\subseteq A$.

Now, we give some fuzzy logic concepts.

Let G be an AG-groupoid. By a fuzzy subset μ of an AG-groupoid G, we mean a mapping, $\mu{:}G\to[0, 1]$. We denote by $\mu(G)$ the set of all fuzzy subsets of G. The order relation "$\subseteq$" on $\mu(G)$ is defined as follows:

$$\mu_1 \subseteq \mu_2 \text{ if and only if } \mu_1(x) \le \mu_2(x) \text{ for all } x \in G \text{ and for all } \mu_1, \mu_2 \in \mu(G).$$

A fuzzy subset μ of G is called a fuzzy AG-subgroupoid [17] if $\mu(xy) \ge \mu(x) \wedge \mu(y)$ for all $x, y \in G$. μ is called a fuzzy left (right) ideal of G if $\mu(xy) \ge \mu(y)$ ($\mu(xy) \ge \mu(x)$) for all $x, y \in G$. μ is called a fuzzy ideal of G if it is both a fuzzy left and right ideal of G.

Let G be an AG-groupoid and μ a fuzzy subset of G. Then, μ is called a fuzzy bi-ideal [17] of G if it satisfies the following conditions:

(1) $(\forall x, y \in G)\ (\mu(xy) \ge \mu(x) \wedge \mu(y))$.

(2) $(\forall x, y, z \in G)\ (\mu(xy)z) \ge \mu(x) \wedge \mu(z))$.

A fuzzy subset μ of G is called a fuzzy interior ideal [17] of G if it satisfies the following conditions:

(1) $(\forall x, y \in G)\ (\mu(xy) \ge \mu(x) \wedge \mu(y))$.

(2) $(\forall x, a, y \in G)\ ((\mu(xa)y) \ge \mu(a))$.

Let μ be a fuzzy subset of G and $\varnothing \ne A \subseteq G$, then the characteristic function μ_A of A is defined as

$$\mu_A : G \to [0, 1],\ a \mapsto \mu_A(a) := \begin{cases} 1 \text{ if } a \in A, \\ 0 \text{ if } a \notin A. \end{cases}$$

Let μ and λ be the two fuzzy subsets of AG-groupoid G. Then, the product $\mu \circ \lambda$ is defined by

$$(\mu \circ \lambda)(x) = \{.\ \bigvee_{x=yz} (\mu(y)\lambda(z)) \text{ if } x = yz,\ 0 \quad \text{otherwise}$$

for all $y, z \in G$.

For an AG-groupoid G, the fuzzy subsets "0" and "1" of G are defined as follows:

$$0 : G \to [0,\ 1] \,|\, x \to 0(x) = 0,$$
$$1 : G \to [0,\ 1] \,|\, x \to 1(x) = 1.$$

Clearly, the fuzzy subset 0 (resp. 1) of G is the least (resp. the greatest) element of the AG-groupoid $(\mu(G), \circ)$ [i.e. $0 \le \mu$ and $\mu \le 1$ for every $\mu \in \mu(G)$]. The fuzzy subset 0 is the zero element of $(\mu(G), \circ)$ [i.e. $\mu \circ 0 = 0 \circ \mu = 0$ and $0 \le \mu$ for every $\mu \in \mu(G)$]. Moreover, $\mu_S = 1$ and $\mu_\varnothing = 0$.

2.1 Lemma [18]

Let G be an AG-groupoid and μ a fuzzy subset of G. Then, μ is a fuzzy bi-ideal of G if and only if μ_A is a fuzzy bi-ideal of G.

Let G be an AG-groupoid and μ a fuzzy subset of G. Then, for every $t \in (0, 1]$, the set

$$U(\mu;\ t) := \{x \,|\, x \in G \text{ and } \mu(x) \ge t\}$$

is called a level set of μ with support x and value t.

2.2 Theorem [18]

Let G be an AG-groupoid and μ a fuzzy subset of G. Then, μ is a fuzzy bi-ideal of G if and only if $U(\mu;\ t)(\neq\varnothing)$ is a bi-ideal of G for every $t \in (0, 1]$.

2.3 Theorem [19]

Let G be an AG-groupoid and μ a fuzzy subset of G. Then, μ is a fuzzy interior ideal of G if and only if $U(\mu; t)(\neq\emptyset)$ is an interior ideal of G for every $t\in(0, 1]$.

Let μ be a fuzzy subset of G, then the set of the form

$$\mu(y):=\begin{cases} t(\neq 0) & \text{if } y=x,\\ 0 & \text{otherwise,}\end{cases}$$

is called a fuzzy point with support x and value t and is denoted by $[x; t]$. A fuzzy point $[x; t]$ is said to belong to (resp. quasi-coincidence) with a fuzzy set μ, written as $[x; t]\in\mu$ (resp. $[x; t]q\mu$) if $\mu(x)\geq\lambda$ [resp. $\mu(x)+\lambda>1$]. If $[x; t]\in\mu$ or $[x; t]q\mu$, then we write $[x; t]\in\vee q\mu$. The symbol $\overline{\in\vee q}$ means $\in\vee q$ does not hold.

2.4 Definition

A fuzzy subset μ of G is called an $(\bar{\in}, \bar{\in}\vee\bar{q})$-fuzzy bi-ideal of G if it satisfies the following conditions:
(1) $(\forall x, y\in G)(\forall s, t\in(0, 1])([xy; s\wedge t]\bar{\in}\mu\Rightarrow[x; s]\bar{\in}\vee\bar{q}\mu$ or $[y; t]\bar{\in}\vee\bar{q}\mu)$,
(2) $(\forall x, a, y\in G)(\forall s, t\in(0, 1])([(xa)y; s\wedge t]\bar{\in}\mu\Rightarrow[x; s]\bar{\in}\vee\bar{q}\mu$ or $[y; t]\bar{\in}\vee\bar{q}\mu)$.

2.5 Example

Consider $G=\{a, b, c, d, e\}$ with the following multiplication table:

·	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	e	c	d
d	a	a	d	e	c
e	a	a	c	d	e

Then $(G, \cdot)$ is an *AG*-groupoid. Define a fuzzy subset $\mu:G\to[0, 1]$ as follows:

$$\mu(x)=\begin{cases} 0.8 & \text{if } x=a,\\ 0.6 & \text{if } x\in\{b, c, d, e\}.\end{cases}$$

Then, by Definition 2.4, μ is an $(\bar{\in}, \bar{\in}\vee\bar{q})$-fuzzy bi-ideal of G.

2.6 Theorem

If μ is a fuzzy subset of G, then μ is an $(\bar{\in}, \bar{\in}\vee\bar{q})$-fuzzy bi-ideal of G if and only if
(3) $(\forall x, y\in G)(\mu(xy)\vee 0.5\geq\mu(x)\wedge\mu(y))$,
(4) $(\forall x, a, y\in G)(\mu((xa)y)\vee 0.5\geq\mu(x)\wedge\mu(y))$.

Proof. (1)$\Rightarrow$(3). If there exist $x, y\in G$ such that $\mu(xy)\vee 0.5<t=\mu(x)\wedge\mu(y)$, then $0.5<t\leq 1$, $[xy; t]\bar{\in}\mu$ but $[x; t]\in\mu$ and $[y; t]\in\mu$. By (1), we have $[x; t]\bar{q}\mu$ or $[y; t]\bar{q}\mu$. Then, $(\mu(x)\geq t$ and $t+\mu(x)\leq 1)$ or $(\mu(y)\geq t$ and $t+\mu(y)\leq 1)$, which implies that $t\leq 0.5$, a contradiction.

(3) ⇒ (1). Let $x, y \in G$ and $s, t \in (0, 1]$ be such that $[xy; s \wedge t] \overline{\in} \mu$, then $\mu(xy) < s \wedge t$.

(a) If $\mu(xy) \geq \mu(x) \wedge \mu(y)$, then $\mu(x) \wedge \mu(y) < s \wedge$ t, and consequently, $\mu(x) < s$ or $\mu(y) < t$. It follows that $[x; s] \overline{\in} \mu$ or $[y; t] \overline{\in} \mu$. Thus, $[x; s] \overline{\in} \vee \overline{q}\mu$ or $[y; t] \overline{\in} \vee \overline{q}\mu$.

(b) If $\mu(xy) < \mu(x) \wedge \mu(y)$, then by (3), we have $0.5 \geq \mu(x) \wedge \mu(y)$. Let $[x; s] \overline{\in} \mu$ or $[y; t] \overline{\in} \mu$ then $s \leq \mu(x) \leq 0.5$ or $t \leq \mu(y) \leq 0.5$. It follows that $[x; s]\overline{q}\mu$ or $[y; t]\overline{q}\mu$, and $[x; s] \overline{\in} \vee \overline{q}\mu$ or $[y; t] \overline{\in} \vee \overline{q}\mu$.

(2) ⇒ (4). If there exist $x, a, y \in G$ such that $\mu((xa)y) \vee 0.5 < t = \mu(x) \wedge \mu(y)$, then $0.5 < t \leq 1$, $[(xa)y; t] \overline{\in} \mu$ but $[x; t] \in \mu$ and $[y; t] \in \mu$. By (2), we have $[x; t]\overline{q}\mu$ or $[y; t]\overline{q}\mu$. Then, $(\mu(x) \geq t$ and $t + \mu(x) \leq 1)$ or $(\mu(y) \geq t$ and $t + \mu(y) \leq 1)$, which implies that $t \leq 0.5$, a contradiction.

(4) ⇒ (2). Let $x, a, y \in G$ and $r, t \in (0, 1]$ be such that $[(xa)y; r \wedge t] \overline{\in} \mu$, then $\mu((xa)y) < r \wedge t$.

(a) If $\mu((xa)y) \geq \mu(x) \wedge \mu(y)$, then $\mu(x) \wedge \mu(y) < r \wedge t$, and consequently, $\mu(x) < r$ or $\mu(y) < t$. It follows that $[x; r] \overline{\in} \mu$ or $[y; t] \overline{\in} \mu$. Thus, $[x; r] \overline{\in} \vee \overline{q}\mu$ or $[y; t] \overline{\in} \vee \overline{q}\mu$.

(b) If $\mu((xa)y) < \mu(x) \wedge \mu(y)$, then by (4), we have $0.5 \geq \mu(x) \wedge \mu(y)$. Let $[x; r] \overline{\in} \mu$ or $[y; t] \overline{\in} \mu$, then $r \leq \mu(x) 0.5$ or $t \leq \mu(y) \leq 0.5$. It follows that $[x; r]\overline{q}\mu$ or $[y; t]\overline{q}\mu$, and $[x; r] \overline{\in} \vee \overline{q}\mu$ or $[y; t] \overline{\in} \vee \overline{q}\mu$. □

2.7 Definition

A fuzzy subset μ of G is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior ideal of G if it satisfies the following conditions:

(1) $(\forall x, y \in G)(\forall s, t \in (0, 1])([xy; s \wedge t] \overline{\in} \mu \Rightarrow [x; s] \overline{\in} \vee \overline{q}\mu$ or $[y; t] \overline{\in} \vee \overline{q}\mu)$,

(2) $(\forall x, a, y \in G)(\forall s \in (0, 1])([(xa)y; s] \overline{\in} \mu \Rightarrow [a; s] \overline{\in} \vee \overline{q}\mu$.

2.8 Theorem

If μ is a fuzzy subset of G, then μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior ideal of G if and only if

(3) $(\forall x, y \in G)(\mu(xy) \vee 0.5 \geq \mu(x) \wedge \mu(y))$,

(4) $(\forall x, a, y \in G)(\mu((xa)y) \vee 0.5 \geq \mu(a))$.

Proof. (2) ⇒ (4). If there exist $x, a, y \in G$ such that $\mu((xa)y) \vee 0.5 < t = \mu(a)$, then $0.5 < t \leq 1$, $[(xa)y; t] \overline{\in} \mu$ but $[a; t] \in \mu$. By (2), we have $[a; t]\overline{q}\mu$. Then, $(\mu(a) \geq t$ and $t + \mu(a) \leq 1)$, which implies that $t \leq 0.5$, a contradiction.

(4) ⇒ (2). Let $x, a, y \in G$ and $r \in (0, 1]$ be such that $[(xa)y; r] \overline{\in} \mu$, then $\mu((xa)y) < r$.

(a) If $\mu((xa)y) < \mu(a)$, then $\mu(a) < r$, and consequently, $\mu(a) < r$. It follows that $[a; r] \overline{\in} \mu$. Thus, $[a; r] \overline{\in} \vee \overline{q}\mu$.

(b) If $\mu((xa)y) < \mu(a)$, then by (4), we have $0.5 \geq \mu(a)$. Let $[a; r] \overline{\in} \mu$, then $r \leq \mu(a) 0.5$. It follows that $[a; r]\overline{q}\mu$, so $[a; r] \overline{\in} \vee \overline{q}\mu$. The remaining proof follows from Theorem 2.6. □

2.9 Definition

A fuzzy subset μ of G is called an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right) ideal of G if it satisfies the following conditions:

$$(\forall x, y \in G)(\forall s \in (0, 1])([xy; s] \overline{\in} \mu \Rightarrow [y; s] \overline{\in} \vee \overline{q}\mu(\text{resp. } [x; s] \overline{\in} \vee \overline{q}\mu).$$

2.10 Theorem

If μ is a fuzzy subset of G, then μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right) ideal of G if and only if

$$(\forall x, y \in G)(\mu((xy) \vee 0.5 \geq \mu(y)(\text{resp. } \mu(x))).$$

Proof. Suppose that μ is an $(\bar{\in}, \bar{\in}\vee\bar{q})$-fuzzy left ideal. If there exist $x, y\in G$ such that $\mu(xy)\vee 0.5<t=\mu(y)$, then $0.5<t\leq 1$, $[xy; t]\bar{\in}\mu$ but $[y; t]\in\mu$. By Definition 2.9, we have $[a; t]\bar{q}\mu$. Then, $(\mu(a)\geq t$ and $t+\mu(a)\leq 1)$, which implies that $t\leq 0.5$, a contradiction. Let $x, y\in G$ and $s\in(0, 1]$ be such that $[xy; s]\bar{\in}\mu$, then $\mu(xy)<s$.

(a) If $\mu(xy)\geq\mu(y)$, then $\mu(y)<s$. It follows that $[y; s]\bar{\in}\mu$. Thus, $[y; s]\bar{\in}\vee\bar{q}\mu$.
(b) If $\mu(xy)<\mu(y)$, then by hypothesis, we have $0.5\geq\mu(y)$. Let $[y; s]\bar{\in}\mu$, then $s\leq\mu(y)\leq 0.5$. It follows that $[y; s]\bar{q}\mu$, so $[y; a]\bar{\in}\vee\bar{q}\mu$. In a similar way, the case for right ideal can be proved. □

2.11 Corollary

If μ is a fuzzy subset of G, then μ is an $(\bar{\in}, \bar{\in}\vee\bar{q})$-fuzzy bi-ideal of G if and only if

$$(\forall x,\ y\in G)(\mu((xy)\vee 0.5\geq\mu(y)\vee\mu(x)).$$

Proof. The proof follows from Theorem 2.10. □

3 $(\in_\gamma, \in_\gamma\vee q_\delta)$-Fuzzy Bi-ideals of *AG*-Groupoids

Ma et al. [27] investigated new types of fuzzy ideals called $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy (positive implicative, implicative, and commutative) ideals and $(\bar{\in}_\gamma, \bar{\in}_\gamma\vee\bar{q}_\delta)$-fuzzy (positive implicative, implicative, and commutative) ideals of *BCI*-algebras, and provided the classification of *BCI*-algebras in terms of these types of fuzzy ideals. The concept of general forms of a fuzzy interior ideal in an *AG*-groupoid was studied in Ref. [19]. Generalizing the concepts of Refs. [18, 19], here we extend our studies to more general forms of fuzzy bi-deals, fuzzy left (right) ideals, and fuzzy interior ideals in *AG*-groupoids. In this section, some new types of fuzzy bi-ideals called (α, β)-fuzzy bi-ideals of *G* are introduced and some new types of characterization of *AG*-groupoids are investigated.

In what follows, let $\gamma, \delta\in[0, 1]$ be such that $\gamma<\delta$. For a fuzzy point $[x; s]$ and a fuzzy subset μ of *X*, we say that
(1) $[x; r]\in_\gamma\mu$ if $\mu(x)\geq r>\gamma$.
(2) $[x; r]q_\delta\mu$ if $\mu(x)+r>2\delta$.
(3) $[x; r]\in_\gamma\vee q_\delta\mu$ if $[x; r]\in_\gamma\mu$ or $[x; r]q_\delta\mu$.
(4) $[x; r]\in_\gamma\wedge q_\delta\mu$ if $[x; r]\in_\gamma\mu$ and $[x; r]q_\delta\mu$.
(5) $[x; r]\bar{\alpha}\mu$ if $[x; r]\alpha\mu$ does not hold for $\alpha\in\{\in_\gamma, q_\delta, \in_\gamma\vee q_\delta, \in_\gamma\wedge q_\delta\}$.

3.1 Definition

A fuzzy subset μ of *G* is called an (α, β)-fuzzy *AG*-subgroupoid of *G* if it satisfies the condition

$$(\forall x,\ y\in G)(\forall s,\ t\in(\gamma,\ 1])([x;\ s]\alpha\mu,\ [x;\ t]\alpha\mu\rightarrow[xy;\ s\wedge t]\beta\mu).$$

Note that $\alpha, \beta\in\{\in_\gamma, q_\delta, \in_\gamma\vee q_\delta, \in_\gamma\wedge q_\delta\}$ and $\alpha\neq\in_\gamma\wedge q_\delta$. The case $\alpha=\in_\gamma\wedge q_\delta$ is omitted. Because for a fuzzy subset μ such that $\mu(x)<\delta$ for any $x\in G$. In the case $[x; t]\in_\gamma\wedge q_\delta\mu$, we have $\mu(x)\geq t$ and $\mu(x)+t>2\delta$. Therefore, $\mu(x)+\mu(x)>\mu(x)+t>2\delta$, which implies that $2\mu(x)>2\delta$. Thus, $\mu(x)>\delta$, a contradiction. This means that $\{[x; t]\,|\,[x; t]\in_\gamma\wedge q_\delta\mu\}=\varnothing$.

3.2 Theorem

A fuzzy subset μ of G is an $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy AG-subgroupoid of G if and only if

$$(\forall x,\ y\in G)(\gamma,\ \delta\in[0,\ 1])(\mu(xy)\vee\gamma\geq\mu(x)\wedge\mu(y)\wedge\delta).$$

Proof. Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy *AG*-subgroupoid of G. If there exist $x, y \in G$ such that $\mu(xy) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta$. Choose $t \in (\gamma, 1]$ such that $\mu(xy) \vee \gamma < t \leq \mu(x) \wedge \mu(y) \wedge \delta$. Then, $[x; t] \in_\gamma \mu$ and $[y; t] \in_\gamma \mu$ but $\mu(xy) < t$ and $\mu(xy) + t < 2t < 2\delta$, so $[xy; t] \overline{\in_\gamma \vee q_\delta} \mu$, a contradiction. Hence, $\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$ for all $x, y \in G$ and $\gamma, \delta \in [0, 1]$.

Conversely, assume that $\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$ for all $x, y \in G$ and $\gamma, \delta[0, 1]$. Let $[x; s] \in_\gamma \mu$ and $[y; t] \in_\gamma \mu$ for some $s, t \in (\gamma, 1]$, then $\mu(x) \geq s > \gamma$ and $\mu(y) \geq t > \gamma$ and by hypothesis,

$$\begin{aligned} \mu(xy) \vee \gamma &\geq \mu(x) \wedge \mu(y) \wedge \delta \\ &\geq s \wedge t \wedge \delta \\ &= \begin{cases} s \wedge t & \text{if } s \wedge t \leq \delta \\ \delta & \text{if } \delta < s \wedge t \end{cases}. \end{aligned}$$

That is, $\mu(xy) \vee \gamma \geq s \wedge t$ but $s \wedge t > \gamma$, therefore $\mu(xy) \geq s \wedge t > \gamma$ and hence $[xy; s \wedge t] \in_\gamma \mu$ or $\mu(xy) \vee \gamma \geq \delta$ but $\gamma < \delta$, therefore $\mu(xy) \geq \delta$, thus $\mu(xy) + s \wedge t \geq \delta + s \wedge t > 2\delta$, that is $[xy; s \wedge t] q_\delta \mu$ and hence $[xy; s \wedge t] \in_\gamma \vee q_\delta \mu$. Consequently, μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy *AG*-subgroupoid of G. □

3.3 Definition

A fuzzy subset μ of G is called an (α, β)-fuzzy bi-ideal of G if it satisfies the following conditions:
(1) $(\forall x, y \in G)(\forall s, t \in (\gamma, 1])([x; s]\alpha\mu, [x; t]\alpha\mu \rightarrow [xy; s \wedge t]\beta\mu)$,
(2) $(\forall x, a, y \in G)(\forall s, t \in (\gamma, 1])([x; s]\alpha\mu, [y; t]\alpha\mu \rightarrow [(xa)y; s \wedge t]\beta\mu)$.

3.4 Example

Let G be an *AG*-groupoid as shown in Example 2.5. Then $\{a\}$, $\{a, c, d, e\}$ and G are bi-ideals of G. Define a fuzzy subset $\mu : S \rightarrow [0, 1]$ as follows:

$$\mu(x) = \begin{cases} 0.8 & \text{if } x = a, \\ 0.6 & \text{if } x = b, \\ 0.2 & \text{if } x = c, \\ 0.3 & \text{if } x = d, \\ 0.4 & \text{if } x = e. \end{cases}$$

Then, μ is an $(\in_{0.1}, \in_{0.1} \vee q_{0.2})$-fuzzy bi-ideal of G.

3.5 Theorem

Let μ be a fuzzy subset of G. Then, the following conditions are equivalent:
(1) *μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G.*
(2) *μ satisfies the following conditions:*
(2.1) $(\forall x, y \in G)(\forall \gamma, \delta \in [0, 1])(\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta)$,
(2.2) $(\forall x, a, y \in G)(\forall \gamma, \delta \in [0, 1))(\mu((xa)y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta)$.

Proof. Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G and consider $(\mu(xy) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta)$ for some $x, y \in G$ and γ, $\delta \in [0, 1]$, then $(\mu(xy) \vee \gamma < t \leq \mu(x) \wedge \mu(y) \wedge \delta)$ for some $t \in (\gamma, 1]$. From this, we observe that $[x; t] \in_\gamma \mu$ and $[y; t] \in_\gamma \mu$ but $\mu(xy) < t$ and $\mu(xy) + t < 2t < 2\delta$, so $[xy; t] \overline{\in_\gamma \vee q_\delta} \mu$, a contradiction. Hence, $\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$ for all $x, y \in G$ and $\gamma, \delta \in [0, 1]$. Next, $\mu((xa)y) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta$ for some $x, a, y \in G$ and $\gamma, \delta \in [0, 1]$; then there exist some $t \in (\gamma, 1]$ such that $\mu((xa)y) \vee \gamma < t \leq \mu(x) \wedge \mu(y) \wedge \delta$. This shows that $[x; t] \in_{\gamma} \mu$ and $[y; t] \in_{\gamma} \mu$ but $\mu((xa)y) < t$ and $\mu((xa)y) + t < 2t < 2\delta$, so $[(xa)y; \lambda] \overline{\in_{\gamma} \vee q_{\delta}} \mu$, a contradiction. Therefore, $\mu((xa)y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$ for all x, a, $y \in G$ and $\gamma, \delta \in [0, 1]$.

Conversely, let μ satisfy conditions (2.1) and (2.2) and consider $[x; s] \in_{\gamma} \mu$, $[y; t] \in_{\gamma} \mu$ for some $x, y \in G$ and $s, t \in (\gamma, 1]$. Then, by (2.1):

$$\begin{aligned} \mu(xy) \vee \gamma &\geq \mu(x) \wedge \mu(y) \wedge \delta \\ &\geq s \wedge t \wedge \delta \\ &= \begin{cases} s \wedge t & \text{if } s \wedge t \leq \delta \\ \delta & \text{if } \delta < s \wedge t \end{cases}. \end{aligned}$$

This shows that $[xy; s \wedge t] \in_{\gamma} \vee q_{\delta} \mu$, and by (2.2),

$$\begin{aligned} \mu((xa)y) \vee \gamma &\geq \mu(x) \wedge \mu(y) \wedge \delta \\ &\geq s \wedge t \wedge \delta \\ &= \begin{cases} s \wedge t & \text{if } s \wedge t \leq \delta \\ \delta & \text{if } \delta < s \wedge t \end{cases}. \end{aligned}$$

It follows from Theorem 3.2 that $[(xa)y; s \wedge t] \in_{\gamma} \vee q_{\delta} \mu$. Consequently, μ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal of G. □

3.6 Proposition

Let $2\delta = 1 + \gamma$ and A be a non-empty subset of G. Then, A is an AG-subgroupoid of G if and only if the fuzzy subset μ of G defined as

$$\mu(x) = \begin{cases} \geq \delta & \text{if } x \in A, \\ \gamma & \text{if } x \in G \setminus A, \end{cases}$$

is an $(\alpha, \in_{\gamma} \vee q_{\delta})$-fuzzy *AG*-subgroupoid of G.

Proof. The proof follows from Ref. [20]. □

3.7 Theorem

Let μ be a fuzzy subset of G. Then, the following conditions are equivalent:

(1) *μ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy AG-subgroupoid of G.*

(2) *$U(\mu; t)(\neq \emptyset)$ is an AG-subgroupoid of G for all $t \in (\gamma, \delta]$.*

Proof. Suppose that μ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy *AG*-subgroupoid of G and $x, y \in U(\mu; t)$ for some $t \in (\gamma, \delta]$. Then, $\mu(x) \geq t$ and $\mu(y) \geq t$ and by hypothesis

$$\begin{aligned} \mu(xy) \vee \gamma &\geq \mu(x) \wedge \mu(y) \wedge \delta \\ &\geq t \wedge t \wedge \delta \\ &= t (\text{as } t \in (\gamma, \delta]). \end{aligned}$$

Hence, $\mu(xy) \geq t > \gamma$; therefore, $\mu(xy) \in U(\mu; t)$. Thus, $U(\mu; t)$ is an *AG*-subgroupoid of G.

Conversely, assume that $U(\mu; t)(\neq \emptyset)$ is an *AG*-subgroupoid of G for all $t \in (\gamma, \delta]$. If there exist $x, y \in G$ such that $\mu(xy) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta$, then $\mu(xy) \vee \gamma < t \leq \mu(x) \wedge \mu(y) \wedge \delta$ for some $t \in (\gamma, 1]$. From this, we see that $x, y \in U(\mu; t)$ but $xy \notin U(\mu; t)$, a contradiction. Hence, $\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$ for all $x, y \in G$ and $\gamma, \delta \in [0, 1]$. □

The proof of Proposition 3.8 is straightforward and is omitted.

3.8 Proposition

Let $2\delta = 1+\gamma$ and A be a non-empty subset of G. Then, A is a bi-ideal of G if and only if the fuzzy subset μ of G defined as

$$\mu(x) = \begin{cases} \geq \delta & \text{if } x \in A, \\ \gamma & \text{if } x \in G \setminus A, \end{cases}$$

is an $(\alpha, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal of G.

Proof. The proof follows from Ref. [20]. □

3.9 Theorem

Let μ be a fuzzy subset of G. Then, the following conditions are equivalent:
(1) *μ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal of G.*
(2) *$U(\mu; t)(\neq \emptyset)$ is a bi-ideal of G for all $t \in (\gamma, \delta]$.*

Proof. Suppose that μ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal of G and let $x, a, y \in G$ be such that $x, y \in U(\mu; t)$ for some $t \in (\gamma, \delta]$. Then, $\mu(x) \geq t$ and $\mu(y) \geq t$ and by hypothesis

$$\begin{aligned} \mu((xa)y) \vee \gamma &\geq \mu(x) \wedge \mu(y) \wedge \delta) \\ &\geq t \wedge t \wedge \delta \\ &= t \text{ (as } t \in (\gamma, \delta]). \end{aligned}$$

Hence, $\mu((xa)y) \geq t > \gamma$. This shows that $(xa)y \in U(\mu; t)$ and hence $U(\mu; t)(\neq \emptyset)$ is a bi-ideal of G for all $x, a, y \in G$ and $t \in (\gamma, \delta]$.

Conversely, assume that $U(\mu; t)(\neq \emptyset)$ is a bi-ideal of G for all $t \in (\gamma, \delta]$. If there exist $x, a, y \in G$ such that $\mu((xa)y) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta$, then choose $t \in (\gamma, \delta]$ such that $\mu((xa)y) \vee \gamma < t \leq \mu(x) \wedge \mu(y) \wedge \delta$. From here, we have $x, y \in U(\mu; t)$ but $(xa)y \notin U(\mu; t)$, a contradiction. Hence, $\mu((xa)y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$ for all $x, a, y \in G$ and $t \in (\gamma, \delta]$. The remaining proof is a consequence of Theorem 3.7. Therefore, μ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal of G. □

3.10 Lemma

A non-empty subset A of G is a bi-ideal if and only if the characteristic function μ_A of A is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy bi-ideal of G.

Proof. The proof is straightforward. □

3.11 Definition

A fuzzy subset μ of G is called an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy left (resp. right) ideal of G if the following condition holds:

$$(\forall x, y \in S)(\forall \gamma, \delta \in [0, 1])([y; t] \in_\gamma \mu \rightarrow [xy; t] \in_\gamma \vee q_\delta \mu \text{ (resp. } [yx; t] \in_\gamma \vee q_\delta \mu).$$

3.12 Theorem

Let $2\delta = 1 + \gamma$ and A be a non-empty subset of G. Then, A is a left (resp. right) ideal of G if and only if the fuzzy subset μ of G defined as

$$\mu(x) = \begin{cases} \geq \delta & \text{if } x \in A, \\ \gamma & \text{if } x \in G \setminus A, \end{cases}$$

is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right) ideal of G.

Proof. To prove this, we consider the following three cases:

Case 1. Consider A is a left ideal of G. Let $x, y \in G$ and $t \in (\gamma, 1]$ be such that $[y; t] \in_\gamma \mu$, then $y \in A$. As A is a left ideal of G, therefore $xy \in A$, that is $\mu(xy) \geq \delta$. If $t \leq \delta$, then $\mu(xy) \geq t > \gamma$ and so $[xy; t] \in_\gamma \mu$. If $t > \delta$, then $\mu(xy) + t > \delta + t > 2\delta$ and hence $[xy; t] q_\delta \mu$. Thus, $[xy; t] \in_\gamma \vee q_\delta \mu$. This shows that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of G.

Case 2. Let $x, y \in G$ and $t \in (\gamma, 1]$ be such that $[y; t] q_\delta \mu$, then $y \in A$. Therefore, $xy \in A$, as A is a left ideal of G, that is $\mu(xy) \geq \delta$. If $t \leq \delta$, then $\mu(xy) \geq t > \gamma$ and so $[xy; t] \in_\gamma \mu$. If $t > \delta$, then $\mu(xy) + t > \delta + t > 2\delta$, and hence $[xy; t] q_\delta \mu$. Thus, $[xy; t] \in_\gamma \vee q_\delta \mu$. This shows that μ is a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy left ideal of G.

Case 3. Let $x, y \in G$ and $t \in (\gamma, 1]$ be such that $[y; t] \in_\gamma \vee q_\delta \mu$, follows from case 1 and case 2. □

3.13 Proposition

Every $(\in_\gamma, \in_\gamma)$-fuzzy bi-ideal of G is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G.

Proof. It is straightforward, as for $[x; t] \in_\gamma \mu$, we have $[x; t] \in_\gamma \vee q_\delta \mu$ for all $x \in G$ and $t \in (\gamma, 1]$. □

The converse of Proposition 3.13 is not true in general, as shown in the following example.

3.14 Example

Consider an *AG*-groupoid G and a fuzzy subset μ as defined in Example 2.5. Then, μ is an $(\in_{0.1}, \in_{0.1} \vee q_{0.2})$-fuzzy bi-ideal of G but μ is not an $(\in_{0.1}, \in_{0.1})$-fuzzy bi-ideal of G, as $[d; 0.28] \in_{0.1} \mu$ $[e; 0.38] \in_{0.1} \mu$ and but $[(dc)e; 0.28 \wedge 0.38] = [c; 0.28] \overline{\in}_{0.1} \mu$.

3.15 Remark

A fuzzy subset μ of an *AG*-groupoid G is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G if and only if it satisfies conditions (2.1), and (2.2) of Theorem 3.5.

3.16 Remark

Every fuzzy bi-ideal of an *AG*-groupoid G is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G. However, the converse is not true in general.

3.17 Example

Consider the *AG*-groupoid as given in Example 2.5 and define a fuzzy subset μ as follows:

$$\mu(a)=0.8,\ \mu(b)=0.6,\ \mu(c)=0.2,\ \mu(d)=0.3,\ \mu(e)=0.4.$$

Then, μ is an $(\in_{0.1}, \in_{0.1}\vee q_{0.2})$-fuzzy bi-ideal of G. However,

i) μ is not an $(\in_{0.1}, q_{0.2})$-fuzzy bi-ideal of G, as $[d; 0.18]\in_{0.1}\mu$ and $[e; 0.20]\in_{0.1}\mu$ but

$$[(dc)e;\ 0.18\wedge 0.20]=[c;\ 0.18]\overline{q}_{0.2}\mu.$$

ii) μ is not an $(q_{0.2}, \in_{0.1})$-fuzzy bi-ideal of G, as $[c; 0.28]\in_{0.1}\mu$ and $[d; 0.38]\in_{0.1}\mu$ but

$$[(cd); 0.28\wedge 0.38]=[c;\ 0.28]\overline{\in}_{0.1}\ \mu.$$

iii) μ is not an $(\in_{0.1}\vee q_{0.2}, q_{0.2})$-fuzzy bi-ideal of G, as $[e; 0.18]\in_{0.1}\vee q_{0.2}\mu$ and $[d; 0.28]\in_{0.1}\vee q_{0.2}\mu$ but

$$[(ec);\ 0.18\wedge 0.20]=[c;\ 0.18]\overline{q}_{0.2}\mu.$$

Thus, μ is not an (α, β)-fuzzy bi-ideal of G, for every $\alpha, \beta\in\{\in_{0.1}, q_{0.2}, \in_{0.1}\vee q_{0.2}\}$.

3.18 Proposition

If $\{\mu_i\}_{i\in I}$ is a family of $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy bi-ideals of an AG-groupoid G, then $\bigcap_{i\in I}\mu_i$ is an $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy bi-ideal of G.

Proof. Let $\{\mu_i\}_{i\in I}$ be a family of $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy bi-ideals of G. Let $x, y\in G$. Then

$$\begin{aligned}\Big(\bigcap_{i\in I}\mu_i\Big)(xy)\vee\gamma &= \lim_{i\in I}\mu_i(xy)\vee\gamma \ge \lim_{i\in I}(\mu_i(x)\wedge\mu_i(y)\wedge\delta)\\ &=(\lim_{i\in I}(\mu_i(x)\wedge\delta)\wedge\lim_{i\in I}(\mu_i(y)\wedge\delta))\\ &=\Big(\bigcap\lim_{i\in I}\mu_i\Big)(x)\wedge\Big(\bigcap\lim_{i\in I}\mu_i\Big)(y)\wedge\delta.\end{aligned}$$

Let $x, a, y\in G$. Then

$$\begin{aligned}\Big(\bigcap\lim_{i\in I}\mu_i\Big)((xa)y)\vee\gamma &= \lim_{i\in I}\mu_i((xa)y)\vee\gamma \ge \lim_{i\in I}(\mu_i(x)\wedge\mu_i(y)\wedge\delta)\\ &=(\lim_{i\in I}(\mu_i(x)\wedge\delta)\wedge\lim_{i\in I}(\mu_i(y)\wedge\delta))\\ &=\Big(\bigcap\lim_{i\in I}\mu_i\Big)(x)\wedge\Big(\bigcap\lim_{i\in I}\mu_i\Big)(y)\wedge\delta.\end{aligned}$$

Thus $\bigcap\lim_{i\in I}\mu_i$ is an $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy bi-ideal of G. □

3.19 Lemma

The intersection of any family $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy left (resp. right) ideals of an AG-groupoid G is an $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy left (resp. right) ideal of G.

Proof. We consider the case for a left ideal. The case for a right ideal can be proved similarly. Let $\{\mu_i\}_{i\in I}$ be a family of $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy left ideals of G and $x, y\in G$. Then, $(\lim_{i\in I}\mu_i)(xy)=\lim_{i\in I}(\mu_i(xy))$. As each μ_i is an $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy left ideal of G, we have $\mu_i(xy)\vee\gamma\ge\mu_i(y)\wedge\delta$ for all $i\in I$. Thus

$$\begin{aligned}(\lim_{i\in I}\mu_i)(xy)\vee\gamma &= \lim_{i\in I}(\mu_i(xy)\vee\gamma)\\ &\geq \lim_{i\in I}(\mu_i(y)\wedge\delta)\\ &= (\lim_{i\in I}\mu_i(y))\wedge\delta\\ &= (\lim_{i\in I}\mu_i)(y)\wedge\delta.\end{aligned}$$

Hence, $\lim_{i\in I}\mu_i$ is an $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy left ideal of G. □

3.20 Lemma

The union of any family $(\in_\gamma, \in_\gamma\vee q_\delta)$*-fuzzy left (resp. right) ideals of an AG-groupoid G is an* $(\in_\gamma, \in_\gamma\vee q_\delta)$*-fuzzy left (resp. right) ideal of G.*

For a detailed study of (α, β)*-fuzzy ideals, the readers are referred to Refs.* [4–9, 34, 35] *to properly understand the characteristics and properties of this type of ideal theory.*

4 $(\in_\gamma, \in_\gamma\vee q_\delta)$-Fuzzy Interior Ideals

In this section, some new types of fuzzy interior ideals of *AG*-groupoids are introduced. An important milestone achieved by this section is providing a connection between ordinary interior ideals and fuzzy interior ideals of type $(\in_\gamma, \in_\gamma\vee q_\delta)$ of *AG*-groupoids.

4.1 Definition

A fuzzy subset μ of an *AG*-groupoid G is called an $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy interior ideal of G if it satisfies the following conditions:

(1) $(\forall x, y\in G)(\forall s, t\in(\gamma, 1])([x; s]\in_\gamma\mu, [y; t]\in_\gamma\mu\rightarrow[xy; s\wedge t]\in_\gamma\vee q_\delta\mu)$,
(2) $(\forall x, a, y\in G)(\forall s\in(\gamma, 1])([a; s]\in_\gamma\mu\rightarrow[(xa)y; s]\in_\gamma\vee q_\delta\mu)$.

4.2 Theorem

Let $2\delta=1+\gamma$ *and A be a non-empty subset of G. Then, A is an interior ideal of G if and only if the fuzzy subset* μ *of G defined as*

$$\mu(x)=\begin{cases}\geq\delta & \text{if } x\in A,\\ \gamma & \text{if } x\in G\setminus A,\end{cases}$$

is an $(\alpha, \in_\gamma\vee q_\delta)$*-fuzzy interior ideal of G.*

Proof. We consider the following three cases:

Case 1: Consider A is an interior ideal of G. Let $x, a, y\in G$ and $[a; t]\in_\gamma\mu$ for some $t\in(\gamma, 1]$, then $a\in A$. As A is an interior ideal of G, therefore $xay\in A$, that is $\mu(xay)\geq\delta$. If $t\leq\delta$, then $\mu(xay)\geq t$ and so $[xay; t]\in_\gamma\mu$. If $t>\delta$, then $\mu(xay)+t>\delta+t>2\delta$ and hence $[xay; t]q_\delta\mu$. Thus, $[xay; t]\in_\gamma\vee q_\delta\mu$. This shows that μ is an $(\in_\gamma, \in_\gamma\vee q_\delta)$-fuzzy interior ideal of G.

Case 2: Let $x, a, y \in G$ and $t \in (\gamma, 1]$ such that $[a; t]q_\delta \mu$, then $a \in A$. As A is an interior ideal of G, therefore $xay \in A$ and hence $\mu(xay) \geq \delta$. If $t \leq \delta$, then $\mu(xay) \geq t$ and so $[xay; t] \in_\gamma \mu$. If $t > \delta$, then $\mu(xay) + t > \delta + t > 2\delta$ and hence $[xay; t]q_\delta \mu$. Thus, $[xay; t] \in_\gamma \vee q_\delta \mu$. This shows that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of G.

Case 3: Let $x, a, y \in G$ and $t \in (\gamma, 1]$ be such that $[a; t] \in_\gamma \vee q_\delta \mu$ follows from case 1 and case 2. The remaining proof follows directly from Proposition 3.8. □

4.3 Lemma

A non-empty subset A of G is an interior ideal if and only if the characteristic function μ_A of A is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G.

Proof. The proof follows from Lemma 3.10. □

4.4 Theorem

Suppose that μ is a fuzzy subset of G. Then, the following conditions are equivalent:

(1) *μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G.*

(2) *μ satisfies the following conditions:*

(2.1) $(\forall x, y \in G)(\forall \gamma, \delta \in [0, 1])(\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta)$,

(2.2) $(\forall x, a, y \in G)(\forall \gamma, \delta \in [0, 1))(\mu((xa)y) \vee \gamma \geq \mu(a) \wedge \delta)$.

Proof. Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G. Suppose that $(\mu(xy) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta)$ for some $x, y \in G$ and $\gamma, \delta \in [0, 1]$, then $(\mu(xy) \vee \gamma < t \leq \mu(x) \wedge \mu(y) \wedge \delta)$ for some $t \in (\gamma, 1]$. From this, we observe that $[x; t] \in_\gamma \mu$ and $[y; t] \in_\gamma \mu$ but $\mu(xy) < t$ and $\mu(xy) + t < 2t \leq 2\delta$, so $[xy; t]\overline{\in_\gamma \vee q_\delta}\mu$, a contradiction. Hence, $\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$ for all $x, y \in G$ and $\gamma, \delta \in [0, 1]$. Next, $\mu((xa)y) \vee \gamma < \mu(a) \wedge \delta$ for some $x, a, y \in G$ and $\gamma, \delta \in [0, 1]$, then there exist some $t \in (\gamma, 1]$ such that $\mu((xa)y) \vee \gamma < t \leq \mu(a) \wedge \delta$. This shows that $[a; t] \in_\gamma \mu$ but $\mu((xa)y) \geq t$ and $\mu((xa)y) + t < 2t \leq 2\delta$, so $[(xa)y; t]\overline{\in_\gamma \vee q_\delta}\mu$, a contradiction. This implies $\mu((xa)y) \vee \gamma \geq \mu(a) \wedge \delta$ for all $x, a, y \in G$ and $\gamma, \delta \in [0, 1]$.

Conversely, let μ satisfy conditions (2.1) and (2.2) and consider $[x; s] \in_\gamma \mu$, $[y; t] \in_\gamma \mu$ for some $x, y \in G$ and $s, t \in (\gamma, 1]$. Then, by (2.1)

$$\begin{aligned} \mu(xy) \vee \gamma &\geq \mu(x) \wedge \mu(y) \wedge \delta \\ &\geq s \wedge t \wedge \delta \\ &= \begin{cases} s \wedge t & \text{if } s \wedge t \leq \delta \\ \delta & \text{if } \delta < s \wedge t \end{cases}. \end{aligned}$$

This shows that $[xy; s \wedge t] \in_\gamma \vee q_\delta \mu$. Next, we let $x, a, y \in G$ such that $[a; s] \in_\gamma \mu$ and by (2.2)

$$\begin{aligned} \mu((xa)y) \vee \gamma &\geq \mu(a) \wedge \delta \\ &\geq s \wedge \delta \\ &= \begin{cases} s \text{ if } s \leq \delta \\ \delta \text{ if } \delta < s \end{cases}. \end{aligned}$$

From this, we can say that $[xy; s \wedge t] \in_\gamma \vee q_\delta \mu$. Consequently, μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G. □

4.5 Theorem

Let μ be a fuzzy subset of G. Then, the following conditions are equivalent:
(1) *μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G.*
(2) *$U(\mu; t)(\neq\emptyset)$ is an interior ideal of G for all $t \in (\gamma, \delta]$.*

Proof. Suppose that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G and let $x, a, y \in G$ be such that $a \in U(\mu; t)$ for some $t \in (\gamma, \delta]$. Then, $\mu(a) \geq t$ and by hypothesis

$$\begin{aligned}\mu((xa)y) \vee \gamma &\geq \mu(a) \wedge \delta)\\ &\geq t \wedge \delta\\ &= t(\text{as } t \in (\gamma, \delta]).\end{aligned}$$

Hence, $\mu((xa)y) \geq t > \gamma$. This shows that $(xa)y \in U(\mu; t)$, and hence $U(\mu; t)(\neq\emptyset)$ is an interior of G for all $x, a, y \in G$ and $t \in (\gamma, \delta]$.

Conversely, assume that $U(\mu; t)(\neq\emptyset)$ is an interior of G for all $t \in (\gamma, \delta]$. If there exist $x, a, y \in G$ such that $\mu((xa)y) \vee \gamma < \mu(a) \wedge \delta$, then choose $t \in (\gamma, \delta]$ such that $\mu((xa)y) \vee \gamma < t \leq \mu(a) \wedge \delta$. From here, we have $a \in U(\mu; t)$ but $(xa)y \notin U(\mu; t)$, a contradiction. Hence, $\mu((xa)y) \vee \gamma \geq \mu(a) \wedge \delta$ for all $x, a, y \in G$ and $t \in (\gamma, \delta]$. The remaining proof is a consequence of Theorem 3.9. Therefore, μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G. □

4.6 Proposition

Every $(\in_\gamma, \in_\gamma)$-fuzzy interior ideal of G is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G.

Proof. The proof follows from Proposition 3.13. □

4.7 Remark

A fuzzy subset μ of an *AG*-groupoid G is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G if and only if it satisfies conditions (2.1) and (2.2) of Theorem 4.4.

4.8 Proposition

Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G.

Proof. Let μ be an $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G, then consider $x, y \in G$ and $s, t \in (\gamma, 1]$ such that $[x; s] \in_\gamma \vee q_\delta \mu$, $[y; t] \in_\gamma \vee q_\delta \mu$. Therefore, $[x; s] \in_\gamma \vee q\delta\mu$, $[y; t] \in_\gamma \vee q_\delta \mu$ implies $[x; s] \in_\gamma \mu$ or $[x; s] q_\delta \mu$ and $[y; t] \in_\gamma \mu$ or $[y; t] q_\delta \mu$, which leads to $[x; s] \in_\gamma \mu$, $[y; t] \in_\gamma \mu$. Hence, by Definition 4.1, it implies that $[xy; s \wedge t] \in_\gamma \vee q_\delta \mu)$. Also, suppose $x, a, y \in G$ and $s \in (\gamma, 1]$ such that $[a; s] \in_\gamma \vee q_\delta \mu$. Therefore, $[a; s] \in_\gamma \vee q_\delta \mu$, implies $[a; s] \in_\gamma \mu$ or $[a; s] q_\delta \mu$, which leads to $[x; s] \in_\gamma \mu$. Therefore, Definition 4.1 implies that $[xay; s] \in_\gamma \vee q_\delta \mu)$. Thus, μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G.

Note that, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of G is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal. Similarly, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of G. In the following propositions, we provide the conditions under which the converses of the above statements are true. □

4.9 Proposition

If G is an intra-regular AG-groupoid, then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of G.

Proof. Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of G. Let $a, b \in G$. As G is intra-regular, there exists $x, y \in G$ such that $a=(xa^2)y$. Thus,

$$\begin{aligned}\mu(ab)\vee\gamma &= \mu(((xa^2)y)b)\vee\gamma\\ &= \mu((by)(xa^2))\vee\gamma\\ &= \mu((by)(x(aa)))\vee\gamma\\ &= \mu((by)(a(xa))\vee\gamma\\ &= \mu((by)a(xa)))\vee\gamma\\ &\geq \mu(a)\wedge\delta.\end{aligned}$$

Similarly, we can show that $\mu(ab)\vee\gamma \geq \mu(b)\wedge\delta$, and hence μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of G. □

4.10 Proposition

If G is an intra-regular AG-groupoid, then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G.

Proof. Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of G. Let $a, b \in G$. As G is intra-regular, there exists x, $y \in G$ such that $a=(xa^2)y$. Thus,

$$\begin{aligned}\mu(ab)\vee\gamma &= \mu(((xa^2)y)b)\vee\gamma = \mu(((xa^2)(ey))b)\vee\gamma\\ &= \mu(((ye)(a^2x)b)\vee\gamma = \mu((a^2((ye)x))b)\vee\gamma\\ &= \mu(((aa)((ye)x))b)\vee\gamma = \mu(((x(ye))(aa))b)\vee\gamma\\ &= \mu((a((x(ye)a))b)\vee\gamma \geq \mu(a)\wedge\mu(b)\wedge\delta.\end{aligned}$$

Thus, μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy *AG*-subgroupoid of G and consequently, μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G.

In Propositions 4.9 and 4.10, if we take G as regular, then the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideals, and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals coincide. □

5 $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-Fuzzy Bi-(interior) Ideals

In this section, we define $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy bi-ideals, $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy interior ideals of *AG*-groupoids, and investigate some important properties of *AG*-groupoids based on these new types of fuzzy bi-ideals and fuzzy interior ideals. It is important to note that level subsets are used to link ordinary ideals and fuzzy ideals of type $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$. If μ is a fuzzy subset of G and $J=\{t \mid t\in(0, 1]$ and $U(\mu; t)$ is a level subset of μ, then μ will be a fuzzy bi-ideal (resp. an interior ideal) depending on the level subset, whether it is empty or bi-ideal (resp. interior ideal). Further, if μ is a fuzzy subset and $J=\{t \mid t\in(0, 1]$ and $U(\mu; t)$ is an empty set or a bi-ideal (resp. interior ideal) of $G\}$, then we consider the following questions:

(a) If $J=(\delta, 1]$, what type of fuzzy bi-ideal (fuzzy interior ideal) of G will μ be?
(b) If $J=(\gamma, \delta]$, $(\gamma, \delta\in(0, 1])$, will μ be a kind of fuzzy bi-ideal (resp. fuzzy interior ideal) of G or not?

In the following, we give the answers to these questions.

5.1 Definition

A fuzzy subset μ of G is called an $(\overline{\beta}, \overline{\alpha})$-fuzzy bi-ideal of G if it satisfies the following conditions:

(1) $(\forall x, y \in G)(\forall s, t \in (\gamma, 1])([xy; s \wedge t]\overline{\beta}\mu \rightarrow [x; s]\overline{\alpha}\mu$ or $[y; t]\overline{\alpha}\mu)$,

(2) $(\forall x, a, y \in G)(\forall s, t \in (\gamma, 1])([(xa)y; s \wedge t]\overline{\beta}\mu \rightarrow [x; s]\overline{\alpha}\mu$ or $[y; t]\overline{\alpha}\mu)$.

5.2 Theorem

If μ is a fuzzy subset μ of G, then the following conditions are equivalent:

(1) *μ is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy bi-ideal of G.*

(2) *μ satisfies the following conditions:*

 (i) $(\forall x, y \in G)(\mu(xy) \vee \delta \geq \mu(x) \wedge \mu(y))$,

 (ii) $(\forall x, a, z \in G)(\mu((xa)z) \vee \delta \geq \mu(x) \wedge \mu(z))$.

Proof. Let μ be an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy bi-ideal of G. If there exists $x, y \in G$ such that $\mu(xy) \vee \delta < \mu(x) \wedge \mu(y)$, then $\mu(xy) \vee \delta < t \leq \mu(x) \wedge \mu(y)$ for some $t \in (\gamma, 1]$, we see that $[xy; t]\,\overline{\in}_\gamma \mu$ but $[x; t] \in_\gamma \mu$ and $[y; t] \in_\gamma \mu$, a contradiction, and hence $\mu(xy) \vee \delta \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$. Next, we suppose that there exist $a, b, c \in G$ such that $\mu((ab)c) \vee \delta < \mu(a) \wedge \mu(c)$, then $\mu((ab)c) \vee \delta < s \leq \mu(a) \wedge \mu(c)$ for some $s \in (\gamma, 1]$ shows that $[(ab)c; t]\,\overline{\in}_\gamma \mu$ but $[a; t] \in_\gamma \mu$ and $[c; t] \in_\gamma \mu$, a contradiction; therefore, $\mu((xa)z) \vee \delta \geq \mu(x) \wedge \mu(z)$ for all $x, a, z \in G$.

Conversely, suppose μ satisfies conditions (i) and (ii). On the contrary, assume that for $x, y \in G$, $[xy; s \wedge t]\,\overline{\in}_\gamma \mu$ such that $[x; s]\overline{\overline{\in}_\gamma \vee \overline{q}_\delta}\mu$ and $[y; t]\overline{\overline{\in}_\gamma \vee \overline{q}_\delta}\mu$. Hence, $\mu(xy) < s \wedge t$, $\mu(x) \geq s$, $\mu(x) + s > 2\delta$, $\mu(y) \geq t$, and $\mu(y) + t > 2\delta$. We claim that $\mu(x) > \delta$ and $\mu(y) > \delta$. This is because if $\mu(x) > \delta$ and $\mu(y) > \delta$, then $s \leq \mu(x) < \delta$ implies that $s < \delta$, and similarly $t < \delta$. Thus, $\mu(x) + s \leq \delta + s \leq \delta + \delta = 2\delta$, which leads to $[x; s]\overline{q}_\delta \mu$. Hence, $[x; s]\,\overline{\in}_\gamma \vee \overline{q}_\delta \mu$, a contradiction to our assumption. Therefore, $\mu(x) > \delta$ and $\mu(y) > \delta$. Hence

$$\begin{aligned} \mu(x) \wedge \mu(y) &> (s \wedge t) \vee \delta \\ &> \mu(xy) \vee \delta. \end{aligned}$$

Thus, a contradiction to (i). This is due to our wrong supposition. Therefore, for $x, y \in G$ such that $[xy; s \wedge t]\,\overline{\in}_\gamma \mu$, then it implies that $[x; s]\,\overline{\in}_\gamma \vee \overline{q}_\delta \mu$ or $[y; t]\,\overline{\in}_\gamma \vee \overline{q}_\delta \mu$. Similarly, for $x, a, y \in G$, $[(xa)y; s \wedge t]\,\overline{\in}_\gamma \mu$ such that $[x; s]\overline{\overline{\in}_\gamma \vee \overline{q}_\delta}\mu$ and $[y; t]\overline{\overline{\in}_\gamma \vee \overline{q}_\delta}\mu$. Therefore, $\mu((xa)y) < s \wedge t$, $\mu(x) \geq s$, $\mu(x) + s > 2\delta$, $\mu(y) \geq t$ and $\mu(y) + t > 2\delta$. We claim that $\mu(x) > \delta$ and $\mu(y) > \delta$. This is because if $\mu(x) > \delta$ and $\mu(x) > \delta$, then $s \leq \mu(x) < \delta$ implies that $s < \delta$, and similarly $t < \delta$. Hence, $\mu(x) + s \leq \delta + s \leq \delta + \delta = 2\delta$ implies that $[x; s]\overline{q}_\delta \mu$. Hence, $[x; s]\,\overline{\in}_\gamma \vee \overline{q}_\delta \mu$, a contradiction. Therefore, $\mu(x) > \delta$ and $\mu(y) > \delta$. Hence

$$\begin{aligned} \mu(x) \wedge \mu(y) &> (s \wedge t) \vee \delta \\ &> \mu((xa)y) \vee \delta. \end{aligned}$$

Thus, a contradiction to (ii). Therefore, for $x, y \in G$ such that $[(xa)y; s \wedge t]\,\overline{\in}_\gamma \mu$, then it implies that $[x; s]\,\overline{\in}_\gamma \vee \overline{q}_\delta \mu$ or $[y; t]\,\overline{\in}_\gamma \vee \overline{q}_\delta \mu$. Hence, μ is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy bi-ideal of G. □

5.3 Definition

A fuzzy subset μ of G is called an $(\overline{\beta}, \overline{\alpha})$-fuzzy interior ideal of G if it satisfies the following conditions:

$$(\forall x, y \in G)(\forall s, t \in (\gamma, 1])([xy; s \wedge t]\overline{\beta}\mu \rightarrow [x; s]\overline{\alpha}\mu \text{ or } [y; t]\overline{\alpha}\mu),$$

$$(\forall x, a, y \in G)(\forall s \in (\gamma, 1])([(xa)y; s]\overline{\beta}\mu \rightarrow [a; s]\overline{\alpha}\mu).$$

5.4 Theorem

If μ is a fuzzy subset μ of G, then the following conditions are equivalent:
(1) *μ is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy interior ideal of G.*
(2) *μ satisfies the following conditions:*
   (i) $(\forall x, y \in G)(\mu(xy) \vee \delta \geq \mu(x) \wedge \mu(y))$,
   (ii) $(\forall x, a, z \in G)(\mu((xa)z) \vee \delta \geq \mu(a))$.

Proof. It is an immediate consequence of Theorem 5.2. □

5.5 Lemma

If μ is a fuzzy subset of G, then the following are equivalent:
(1) *$U(\mu; t)$ is a bi-ideal of G for all $t \in (\delta, 1]$.*
(2) *μ is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy bi-ideal.*

Proof. Assume that $U(\mu; t)$ is a bi-ideal of G for all $t \in (\delta, 1]$. If there exist $x, y \in G$ such that $\mu(xy) \vee \delta < \mu(x) \wedge \mu(y) = t_1$, then $t_1 \in (\delta, 1]$, $x, y \in U(\mu; t_1)$. However, $\mu(xy) < t_1$ implies $xy \notin U(\mu; t_1)$, a contradiction. Hence, $\mu(xy) \vee \delta \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$.

If there exist $x, y, z \in G$ such that $\mu((xy)z) \vee \delta < \mu(x) \wedge \mu(z) = t_2$, then $t_2 \in (\delta, 1]$, $x, z \in U(\mu; t_2)$. However, $\mu((xy)z) < t_2$ implies $(xy)z \notin U(\mu; t_2)$, a contradiction. Hence, $\mu((xy)z) \vee \delta \geq \mu(x) \wedge \mu(z)$ for all $x, y, z \in G$.

Conversely, suppose that for $x, y \in U(\mu; t)$. By Theorem 5.2, we get

$$\mu(xy) \vee \delta \geq \mu(x) \wedge \mu(y) \geq t,$$

and so $\mu(xy) \geq t$. It follows that $xy \in U(\mu; t)$. Let $x, z \in U(\mu; t)$, then $\mu(x) \geq t$ and $\mu(z) \geq t$. By Theorem 5.2, we get

$$\mu((xy)z) \vee \delta \geq \mu(x) \wedge \mu(z) \geq t,$$

and so $\mu((xy)z) \geq t$. It follows that $(xy)z \in U(\mu; t)$. Thus, $U(\mu; t)$ is a bi-ideal of G for all $t \in (\delta, 1]$. □

5.6 Lemma

If μ is a fuzzy subset of G, then the following are equivalent:
(1) *$U(\mu, t)$ is an interior ideal of G for all $t \in (\delta, 1]$.*
(2) *μ is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy interior ideal.*

Proof. This follows directly from Lemma 5.5. □

Next, we discuss fuzzy bi-ideals with thresholds and fuzzy interior ideals with thresholds.

5.7 Definition

Let $\gamma, \delta \in [0, 1]$ and $\gamma < \delta$, then a fuzzy subset μ of G is called a fuzzy bi-ideal with thresholds $(\gamma, \delta]$ of G if it satisfies the following conditions:
(1) $(\forall x, y \in G)(\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta)$,
(2) $(\forall x, y, z \in G)(\mu((xy)z) \vee \gamma \geq \mu(x) \wedge \mu(z) \wedge \delta)$.

5.8 Theorem

A fuzzy subset μ of G is a fuzzy bi-ideal with thresholds $(\gamma, \delta]$ of G if and only if $U(\mu; t)(\neq\emptyset)$ is a bi-ideal of G for all $\gamma < t \leq \delta$.

5.9 Definition

Let $\gamma, \delta \in (0, 1]$ and $\gamma < \delta$, then a fuzzy subset μ of G is called a fuzzy interior ideal with thresholds $(\gamma, \delta]$ of G if it satisfies the following conditions:
(1) $(\forall x, y \in G)(\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta)$,
(2) $(\forall x, y, z \in G)(\mu((xy)z) \vee \gamma \geq \mu(a) \wedge \delta)$.

5.10 Theorem

A fuzzy subset μ of G is a fuzzy interior ideal with thresholds $(\gamma, \delta]$ of G if and only if $U(\mu; t)(\neq\emptyset)$ is an interior ideal of G for all $\gamma < t \leq \delta$.

6 Concluding Remarks

In science and technology, the use of algebraic structures plays an unavoidable role. For instance, semigroups are basic structures in computer science, control engineering, etc. Moreover, ordered semigroups are used in fuzzy automata, formal languages, coding theory, etc. Also, *AG*-groupoids are used in flocks (physics, biology) theory. Due to these diverse applications, algebraic structures, especially *AG*-groupoids, and related structures are presently a central focus for researchers. In this paper, we studied generalized fuzzy bi-ideals (resp. ideals and interior ideals) and provided different characterization theorems of *AG*-groupoids in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-(interior) ideals and $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy bi-(interior) ideals. The characteristic function and level subset of μ are used to show the connection between ordinary bi-ideals (resp. interior ideals) and fuzzy bi-ideals (resp. fuzzy interior ideals) of type $(\in_\gamma, \in_\gamma \vee q_\delta)$. In particular, if $J = \{t \mid t \in (0, 1]$ and $U(\mu; t)$ is an empty set or a bi-ideal (resp. an ideal or an interior ideal) of G, we discussed what kind of fuzzy bi-ideals (resp. fuzzy ideals or fuzzy interior ideals) of G will μ be.

Bibliography

[1] S. K. Bhakat and P. Das, $(\in, \in \vee q)$-Fuzzy subgroups, *Fuzzy Sets Syst.* **80** (1996), 359–368.
[2] S. K. Bhakat and P. Das, Fuzzy subrings and ideals redefined, *Fuzzy Sets Syst.* **81** (1996), 83–393.
[3] B. Davvaz, $(\in, \in \vee q)$-fuzzy subnear-rings and ideals, *Soft Comput.* **10** (2006), 206–211.
[4] Y. Feng and B. Li, Intuitionistic (λ, μ)-fuzzy sets in gamma-semigroup, *J. Inequal. Appl.* **2013** (2013), 107.
[5] Y. Feng and C. Li, (λ, μ)-Fuzzy linear subspaces, *J. Inequal. Appl.* **2013** (2013), 369.
[6] Y. Feng, Q. Zing and H. Duan, On (λ, μ)-fuzzy subhyper relation, *Ital. J. Pure Appl. Math.* **30** (2016), 79–86.
[7] Y. Feng, D. Tu and H. Li, Interval-valued fuzzy hypergraph and interval-valued fuzzy hyperoperations, *Ital. J. Pure Appl. Math.* **36** (2016), 1–12.
[8] Y. Feng, C. Li and T. Huang, Periodically multiple state-jumps impulsive control systems with impulse time windows, *Neurocomputing* **193** (2016), 7–13.
[9] Y. Feng, J. Yu and C. Li, Linear impulsive control system with impulse time windows, *J. Vibrat. Control* **23** (2017), 111–118.
[10] P. Holgate, Groupoids satisfying a simple invertive law, *Math. Stud.* **1–4** (1992), 101–106.
[11] Y. B. Jun and S. Z. Song, Generalized fuzzy interior ideals in semigroups, *Inform. Sci.* **176** (2006), 3079–3093.
[12] Y. B. Jun, A. Khan and M. Shabir, Ordered semigroups characterized by their $(\in, \in \vee q)$-fuzzy bi-ideals, *Bull. Malays. Math. Sci. Soc.* **32** (2009), 391–408.

[13] O. Kazanci and S. Yamak, Generalized fuzzy bi-ideals of semigroup, *Soft Comput*. **12** (2008), 1119–1124.
[14] M. A. Kazim and M. Naseeruddin, On LA-semigroups, *Alig. Bull. Math*. **2** (1972), 1–7.
[15] N. Kehayopulu and M. Tsingelis, Fuzzy sets in ordered groupoids, *Semigroup Forum* **65** (2002), 128–132.
[16] A. Khan and M. Shabir, (α, β)-Fuzzy interior ideals in ordered semigroups, *Lobachev. J. Math*. **30** (2009), 30–39.
[17] M. Khan, M. Nouman and A. Khan, *Fuzzy Abel Grassmann's Groupoids*, arXiv:0904.007vl [math. GR], pp. 9, 2009.
[18] A. Khan, M. Shabir and Y. B. Jun, Generalized fuzzy Abel Grassmann's groupoids, *Int. J. Fuzzy Syst*. **12** (2010), 340–349.
[19] A. Khan, Y. B. Jun and T. Mahmood, Generalized fuzzy interior ideals of Abel Grassmann's groupoids, *Int. J. Math. Math. Sci*. **2010** (2010), article ID 838392, 14 pages.
[20] F. M. Khan, N. H. Sarmin and A. Khan, Characterization of ordered semigroups by $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals, *Lobachev. J. Math*. **32** (2011), 278–288.
[21] A. Khan, Y. B. Jun, N. H. Sarmin and F. M. Khan, Ordered semigroups characterized by $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals, *Neural Comput. Appl*. **21** (2012), S121–S132.
[22] A. Khan, N. H. Sarmin, B. Davvaz and F. M. Khan, New types of fuzzy bi-ideals in ordered semigroups, *Neural Comput. Appl*. **21** (2012), S295–S305.
[23] N. Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, *Fuzzy Sets Syst*. **5** (1981), 203–215.
[24] L. Liu and K. Li, Fuzzy implicative and Boolean filters of R_0-algebras, *Inform. Sci*. **171** (2005), 61–71.
[25] X. Ma, J. Zhan and Y. B. Jun, Interval valued $(\in, \in \vee q)$-fuzzy ideals of pseudo-MV algebras, *Int. J. Fuzzy Syst*. **10** (2008), 84–91.
[26] X. Ma, J. Zhan and Y. B. Jun, On $(\in, \in \vee q)$-fuzzy filters of R_0-algebras, *Math. Log. Quart*. **55** (2009), 493–508.
[27] X. Ma, J. Zhan and Y. B. Jun, Some kind of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of BCI-algebras, *Comput. Math. Appl*. **61** (2011), 1005–1015.
[28] J. N. Mordeson, D. S. Malik and N. Kuroki, Fuzzy semigroups, in: *Studies in Fuzziness and Soft Computing*, vol. 131, Springer-Verlag, Berlin, 2003.
[29] Q. Mushtaq and M. Khan, Ideals in *AG*-band and AG^*-groupoids, *Quasigroups Relat. Syst*. **14** (2006), 207–215.
[30] Q. Mushtaq and S. M. Yousuf, On LA-semigroups defined by a commutative inverse semigroup, *Math. Bech*. **40** (1988), 59–62.
[31] M. Naseeruddin, *Some Studies in Almost Semigroups and Flocks*, PhD Thesis, Aligarh Muslim University, Aligarh, India, 1970.
[32] P. V. Protic and N. Stevanovic, AG-test and some general properties of Abel-Grassmann's groupoids, *Pure Math. Appl*. **4** (1995), 371–383.
[33] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl*. **35** (1971), 512–517.
[34] M. Shabir and T. Mahmood, Characterization of hemirings by $(\in, \in \vee q_k)$-fuzzy ideals, *Comput. Math. Appl*. **61** (2011), 1059–1078.
[35] M. Shabir, Y. Nawaz and T. Mahmood, Characterization of hemirings by $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals, *Neural Comput. Appl*. **21** (2012), S93–S103.
[36] Y. Yin and J. Zhan, New types of fuzzy filters of *BL*-algebras, *Comput. Math. Appl*. **60** (2010), 2115–2125.