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APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

Numerical solution of second-order hyperbolic telegraph equation via new cubic trigonometric B-splines approach

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Abstract: This paper presents a new approach and methodology to solve the second-order one-dimensional hyperbolic telegraph equation with Dirichlet and Neumann boundary conditions using the cubic trigonometric B-spline collocation method. The usual finite difference scheme is used to discretize the time derivative. The cubic trigonometric B-spline basis functions are utilized as an interpolating function in the space dimension, with a θ weighted scheme. The scheme is shown to be unconditionally stable for a range of θ values using the von Neumann (Fourier) method. Several test problems are presented to confirm the accuracy of the new scheme and to show the performance of trigonometric basis functions. The proposed scheme is also computationally economical and can be used to solve complex problems. The numerical results are found to be in good agreement with known exact solutions and also with earlier studies.

Subjects: Computer Mathematics; Mathematical Modeling; Mathematical Physics



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PUBLIC INTEREST STATEMENT

The trigonometric B-spline functions were used extensively in Computer Aided Geometric Design (CAGD) as tools to generate curves and surfaces. An advantage of these piecewise functions is its local support properties where the functions are said to have support in specific interval. Due to these properties, trigonometric B-splines have been used to generate the numerical solutions of linear and non-linear partial differential equations. In this paper, the cubic trigonometric B-spline basis function is considered. Collocation method based on the proposed basis functions and finite difference approximation are developed to solve the one-dimensional telegraph equation. Trigonometric B-splines are used to interpolate the solution in x-dimension and finite difference approximations are used to discretize the time derivatives. The proposed method has been proved to be unconditionally stable.

Keywords: second-order one-dimensional telegraph equation; cubic trigonometric B-spline basis functions; cubic trigonometric B-spline collocation method; stability

AMS subject classifications: 65D07; 65N30; 65N35; 74S20; 42A10; 41A05; 42A15; 65T40

1. Introduction

1.1. Problem

Consider the second-order one-dimensional hyperbolic telegraph equation (“the telegraph equation”), given by

$$\frac{\partial^2 u}{\partial t^2}(x, t) + 2\alpha \frac{\partial u}{\partial t}(x, t) + \beta^2 u(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) + q(x, t) \quad a \leq x \leq b, \quad t \geq 0 \quad (1)$$

with initial conditions

$$u(x, 0) = g_1(x), \quad u_t(x, 0) = g_2(x), \quad a \leq x \leq b \quad (2)$$

and the following two types of boundary conditions

(1) Dirichlet boundary conditions

$$u(a, t) = f_1(t), \quad u(b, t) = f_2(t), \quad t \geq 0 \quad (3)$$

(2) Neumann boundary conditions

$$u_x(a, t) = w_1(t), \quad u_x(b, t) = w_2(t), \quad t \geq 0 \quad (4)$$

1.2. Applications

The study of electric signal in a transmission line, dispersive wave propagation, pulsating blood flow in arteries and random motion of bugs along a hedge are amongst a host of physical and biological phenomena which can be described by the telegraph Equation (1). Details of the above-mentioned phenomena and other phenomena which can be described by the telegraph Equation (1) can be found in Bohme (1987), Dehghan and Ghesmati (2010), (Mohanty & Jain, 2001a) and Pascal (1986). Clearly, the equation and its solution are of importance in many areas of applications.

1.3. Literature review

Several numerical methods have been developed to solve the telegraph equation subject to Dirichlet boundary conditions and the references are in Mohanty and Jain, (2001a, 2001b), Mohanty, Jain, and Arora (2002), Mohanty (2004) and Mohanty, Jain, and George (1996). In Liu, Liu, and Chen (2009), two semi-discretization methods based on quartic splines function have been developed to solve the telegraph equations. A class of unconditionally stable finite difference schemes constructed with the help of quartic splines functions has been developed by H. W. Liu and L. B. Liu (2009) for the solution of the telegraph equation. Further several numerical methods have been developed by Dehghan and Shokri (2008) and Mohebbi and Dehghan (2008) in collaboration with different authors. These include the thin plate splines radial basis functions (RBF) for the numerical solution of the telegraph equation (Dehghan & Shokri, 2008) and high-order compact finite difference method to solve the telegraph equation (Mohebbi & Dehghan, 2008). Further details on other numerical methods including interpolating scaling functions (Lakestani & Saray, 2010), RBFs (Esmaeilbeigi, Hosseini, & Mohyud-Din, 2011), quartic B-spline collocation method (QuBSM) (Dosti & Nazemi, 2012), cubic B-spline collocation method (CuBSM) (Mittal & Bhatia, 2013; Rashidinia, Jamalzadeh, & Esfahani, 2014) for the solution of the telegraph equation subject to Dirichlet boundary conditions

are in the literature. Thus many numerical methods have been developed to solve the telegraph Equation (1) with Dirichlet boundary conditions.

Some numerical methods have been developed for numerical solution of the telegraph equation with Neumann boundary conditions. These include methods by Dehghan and Ghesmati (2010) who constructed a dual reciprocity boundary integral equation (DRBIE) method in which cubic radial basis function (C-RBF), thin plate spline radial basis function (TPS-RBF) and linear radial basis functions (L-RBF) are utilized for the numerical solution of the telegraph equation with Neumann boundary conditions. L. B. Liu and H. W. Liu (2013) have developed a compact difference unconditionally stable scheme (CDS) to solve the telegraph equation with Neumann boundary conditions. Further, Mittal and Bhatia (2014) have developed a technique based on collocation of cubic B-spline collocation method (CuBSM) for solving the telegraph equation with Neumann boundary conditions.

The trigonometric B-spline collocation method has attracted attention in the literature and has been used for the numerical solutions of several linear and non-linear partial differential equations (Abbas, Majid, Ismail, & Rashid, 2014a, 2014b, 2014c; Zin, Abbas, Majid, & Ismail, 2014; Zin, Majid, Ismail, & Abbas, 2014a, 2014b). The trigonometric B-splines have many geometric properties like local support, smoothness and capability of handling local phenomena. These properties make trigonometric B-spline appropriate to solve linear and non-linear partial differential equations easily and effortlessly. Fyfe (1969) found that the spline method is better than the usual finite difference scheme because it has the flexibility to obtain the solution at any point in the domain with greater accuracy. The trigonometric B-spline produced more accurate results for linear and non-linear initial boundary value problems as compared to traditional B-spline functions (Abd Hamid, Abd Majid, & Md Ismail, 2010; Nikolis, 1995).

In this work, a numerical collocation finite difference technique based on cubic trigonometric B-spline is presented for the solution of telegraph Equation (1) with initial conditions in Equation (2) and different two types of boundary conditions in Equations (3) and (4). Several studies have been carried out as the ordinary B-spline collocation methods to solve the proposed problem subject to different types of boundary conditions but not with cubic trigonometric B-spline collocation method. A usual finite difference scheme is applied to discretize the time derivative while cubic trigonometric B-spline is utilized as an interpolating function in the space dimension. The proposed method is unconditionally stable over $0.5 \leq \theta \leq 1$ and this is proved by von Neumann approach. The feasibility of the method is shown by test problems and the approximated solutions are found to be in good agreement with the exact solutions. The proposed method is superior to C-RBF (Dehghan & Ghesmati, 2010), TPS-RBF (Dehghan & Ghesmati, 2010), L-RBF (Dehghan & Ghesmati, 2010), RBF (Dehghan & Shokri, 2008), QuBSM (Dosti & Nazemi, 2012), CDS (L. B. Liu & H. W. Liu, 2013), CuBSM (Mittal & Bhatia, 2013), 2014) due to smaller storage and CPU time in seconds.

1.4. Outlines of current paper

The outline of this paper is as follows: in Section 2, the cubic trigonometric B-spline collocation method is explained. In Section 3, numerical solution of proposed problem (1) is discussed. In Section 4, the stability of proposed method is investigated. In Section 5, the results of numerical experiments are presented and compared with exact solutions and some previous methods. Finally, in Section 6, the conclusion of this study is given.

2. Description of new trigonometric B-spline method

In this approach, the space derivatives are approximated using cubic trigonometric B-spline method (CuTBSM). A mesh Ω which is equally divided by knots x_i into N subintervals $[x_i, x_{i+1}]$, $i = 0, 1, 2, \dots, N - 1$ such that, $\Omega: a = x_0 < x_1 < \dots < x_N = b$ is used. For the telegraph equation (1), an approximate solution using collocation method with cubic trigonometric B-spline is obtained in the form (Abd Hamid et al., 2010; Nikolis, 1995)

$$U(x, t) = \sum_{i=-3}^{N-1} C_i(t) TB_i(x) \tag{5}$$

where $C_i(t)$ are to be calculated for the approximated solutions $u(x, t)$ to the exact solutions $u_{exc}(x, t)$, at the point (x_i, t_j) . A C^2 piecewise cubic trigonometric B-spline basis functions $TB_i(x)$ over the uniform mesh can be defined as (Abbas et al., 2014a, 2014b, 2014c).

$$TB_i(x) = \frac{1}{\omega} \begin{cases} \xi^3(x_i), & x \in [x_i, x_{i+1}] \\ \xi(x_i)(\xi(x_i)\zeta(x_{i+2}) + \zeta(x_{i+3})\xi(x_{i+1})) + \zeta(x_{i+4})\xi^2(x_{i+1}), & x \in [x_{i+1}, x_{i+2}] \\ \zeta(x_{i+4})(\xi(x_{i+1})\zeta(x_{i+3}) + \zeta(x_{i+4})\xi(x_{i+2})) + \xi(x_i)\zeta^2(x_{i+3}), & x \in [x_{i+2}, x_{i+3}] \\ \zeta^3(x_{i+4}), & x \in [x_{i+3}, x_{i+4}] \end{cases} \tag{6}$$

where, $\xi(x_i) = \sin\left(\frac{x-x_i}{2}\right)$, $\zeta(x_i) = \sin\left(\frac{x_i-x}{2}\right)$, $\omega = \sin\left(\frac{h}{2}\right) \sin(h) \sin\left(\frac{3h}{2}\right)$ and $h = (b-a)N$.

The approximations U_i^j at the point (x_i, t_j) over subinterval $[x_i, x_{i+1}]$ can be defined as:

$$U_i^j = \sum_{k=i-3}^{i-1} C_k^j TB_k(x) \tag{7}$$

The values of $TB_i(x)$ and its derivatives at knots are required to obtain the approximate solutions and these derivatives are recorded in Table 1.

$$\text{where } a_1 = \frac{\sin^2\left(\frac{h}{2}\right)}{\sin(h)\sin\left(\frac{3h}{2}\right)}, a_2 = \frac{2}{1+2\cos(h)}, a_3 = -\frac{3}{4\sin\left(\frac{3h}{2}\right)}, a_4 = \frac{3}{4\sin\left(\frac{3h}{2}\right)}, a_5 = \frac{3(1+3\cos(h))}{16\sin^2\left(\frac{h}{2}\right)(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right))},$$

$$a_6 = -\frac{3\cos^2\left(\frac{h}{2}\right)}{2\sin^2\left(\frac{h}{2}\right)(1+2\cos(h))}.$$

From (5) and (6), the values at the knots of U_i^j and their derivatives up to second order are calculated in the terms of time parameters C_i^j as:

$$\begin{cases} U_i^j = a_1 C_{i-3}^j + a_2 C_{i-2}^j + a_1 C_{i-1}^j \\ (U_x)_i^j = a_3 C_{i-3}^j + a_4 C_{i-1}^j \\ (U_{xx})_i^j = a_5 C_{i-3}^j + a_6 C_{i-2}^j + a_5 C_{i-1}^j \end{cases} \tag{8}$$

The Equation (5) and boundary conditions given in (3) and (4) are used to obtain the approximate solution at end points of the mesh as:

$$\begin{cases} U(x_0, t_{j+1}) = a_1 C_{-3} + a_2 C_{-2} + a_1 C_{-1} = f_1(t_{j+1}) \\ U(x_N, t_{j+1}) = a_1 C_{N-3} + a_2 C_{N-2} + a_1 C_{N-1} = f_2(t_{j+1}) \end{cases} \tag{9}$$

and

$$\begin{cases} U_x(x_0, t_{j+1}) = a_3 C_{-3} + a_4 C_{-1} = w_1(t_{j+1}) \\ U_x(x_N, t_{j+1}) = a_3 C_{N-3} + a_4 C_{N-1} = w_2(t_{j+1}) \end{cases} \tag{10}$$

Table 1. Values $TB_i(x)$ and its derivatives

Basis	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
TB_i	0	a_1	a_2	a_1	0
TB_i'	0	a_3	0	a_4	0
TB_i''	0	a_5	a_6	a_5	0

3. Numerical solution of telegraph equation

In this section, a numerical solution of telegraph Equation (1) is obtained using collocation approach based on cubic trigonometric basis functions. The discretization in time derivative is obtained by forward finite difference scheme and θ weighted scheme applied to problem (1) to obtain a tri-diagonal of linear equations. The proposed θ weighted scheme is closely related to the accuracy of the method and numerical stability. A uniform mesh Ω with grid points (x_i, t_j) to discretize the grid region $\Delta = [a, b] \times [0, T]$ with $x_i = a + ih, i = 0, 1, 2, \dots, N$ and $t_j = j\Delta t, j = 0, 1, 2, 3, \dots, M$, is used $T = M\Delta t$. The quantities h and Δt are mesh space size and time step size, respectively. Using θ weighted technique, the approximations for the solutions of telegraph Equation (1) at t_{j+1} th time level can be given by as (Abbas et al., 2014b)

$$(U_{tt})_i^j + 2\alpha(U_t)_i^j = \theta g_i^{j+1} + (1 - \theta)g_i^j + q(x_i, t_j) \quad \theta \in [0, 1] \quad (11)$$

where $g_i^j = (U_{xx})_i^j - \beta^2 U_i^j$ and the subscripts j and $j + 1$ are successive time levels, $j = 0, 1, 2, \dots, M$. Using the central finite difference discretization of the time derivatives and rearranging the Equation (11), we obtain

$$\frac{U_i^{j+1} - 2U_i^j + U_i^{j-1}}{\Delta t^2} + 2\alpha \frac{U_i^{j+1} - U_i^j}{\Delta t} = (1 - \theta)g_i^j + \theta g_i^{j+1} + q(x_i, t_j) \quad (12)$$

The Equation (12) yields it as

$$(1 + 2\alpha k)U_i^{j+1} - k^2\theta g_i^{j+1} = 2(1 + \alpha k)U_i^j + k^2(1 - \theta)g_i^j - U_i^{j-1} + k^2q(x_i, t_j) \quad (13)$$

where $k = \Delta t$ is the time step. It is noted that the system becomes an explicit scheme when $\theta = 0$, a fully implicit scheme when $\theta = 1$, and a Crank–Nicolson scheme when $\theta = 1/2$ (Abbas et al., 2014a, 2014b). Hence, (13) becomes,

$$\left\{ \begin{array}{l} (1 + 2\alpha k)U_i^{j+1} - k^2\theta \left((U_{xx})_i^{j+1} - \beta^2 U_i^{j+1} \right) \\ = 2(1 + \alpha k)U_i^j + k^2(1 - \theta) \left((U_{xx})_i^j - \beta^2 U_i^j \right) - U_i^{j-1} + k^2q(x_i, t_j) \end{array} \right. \quad (14)$$

The initial condition (2) is substituted into last term of Equation (14) for computing C^1 .

By central difference approximation,

$$U_i^{-1} = U_i^1 - 2k g_2(x_i) \quad (15)$$

After that, the system thus obtained for $j \geq 1$ on simplifying (14) after using (8) consists of $N + 1$ linear equations in $N + 3$ unknowns $C^{j+1} = (C_{-3}^{j+1}, C_{-2}^{j+1}, C_{-1}^{j+1}, \dots, C_{N-1}^{j+1})$ at the time level $t = t_{j+1}$. The boundary conditions given in Equations (9) or (10) are used for two additional linear equations to obtain a unique solution of the resulting system. Thus, the system becomes a matrix system of dimension $(N + 3) \times (N + 3)$ which is a tri-diagonal system that can be solved by the Thomas Algorithm (Burdern & Faires, 2004; Hoffman, 1992; Iyengar & Jain, 2009; Rosenberg, 1969; Sastry, 2009).

3.1. Initial state

After the initial vectors C^0 have been computed from the initial conditions, the approximate solutions U_i^{j+1} at a particular time level can be calculated repeatedly by solving the recurrence relation (14) (Abbas et al., 2014a, 2014b). C^0 can be obtained from the initial and boundary values of the derivatives of the initial condition as follows (Abbas et al., 2014b).

$$\begin{cases} (U_i^0)_x = g'_1(x_i), & i = 0 \\ U_i^0 = g_1(x_i), & i = 0, 1, 2, \dots, N \\ (U_i^0)_x = g'_1(x_i), & i = N \end{cases} \quad (16)$$

Thus the Equations (16) yield a $(N + 3) \times (N + 3)$ matrix system for which the solution can be computed by the use of the Thomas algorithm.

4. Stability of proposed method

In this section, the von Neumann stability method is applied to investigate the stability of the proposed scheme. Such an approach has been used by many researchers (Abbas et al., 2014a, 2014b, 2014c; Siddiqi & Arshed, 2013). Substituting the approximate solution $U(x, t)$, their derivatives at the knots with $q(x, t) = 0$ (Strikwerda, 2004, chapter 9), into Equation (14) yields a difference equation with variables C_m given by

$$\begin{cases} \left((1 + 2\alpha k + k^2\theta\beta^2)a_1 - k^2\theta a_5 \right) C_{m-3}^{j+1} + \left((1 + 2\alpha k + k^2\theta\beta^2)a_2 - k^2\theta a_6 \right) C_{m-2}^{j+1} \\ + \left((1 + 2\alpha k + k^2\theta\beta^2)a_1 - k^2\theta a_5 \right) C_{m-1}^{j+1} \\ = \left((2 + 2\alpha k - (1 - \theta)k^2\beta^2)a_1 + (1 - \theta)k^2 a_5 \right) C_{m-3}^j \\ + \left((2 + 2\alpha k - (1 - \theta)k^2\beta^2)a_2 + (1 - \theta)k^2 a_6 \right) C_{m-2}^j \\ + \left((2 + 2\alpha k - (1 - \theta)k^2\beta^2)a_1 + (1 - \theta)k^2 a_5 \right) C_{m-1}^j \\ - \left(a_1 C_{m-3}^{j-1} + a_2 C_{m-2}^{j-1} + a_1 C_{m-1}^{j-1} \right) \end{cases} \quad (17)$$

Simplifying it leads to

$$w_1 C_{m-3}^{j+1} + w_2 C_{m-2}^{j+1} + w_1 C_{m-1}^{j+1} = w_3 C_{m-3}^j + w_4 C_{m-2}^j + w_3 C_{m-1}^j - a_1 C_{m-3}^{j-1} - a_2 C_{m-2}^{j-1} - a_1 C_{m-1}^{j-1} \quad (18)$$

where

$$\begin{aligned} w_1 &= (1 + 2\alpha k + k^2\theta\beta^2)a_1 - k^2\theta a_5, \\ w_2 &= (1 + 2\alpha k + k^2\theta\beta^2)a_2 - k^2\theta a_6, \\ w_3 &= (2 + 2\alpha k - (1 - \theta)k^2\beta^2)a_1 + (1 - \theta)k^2 a_5, \\ w_4 &= (2 + 2\alpha k - (1 - \theta)k^2\beta^2)a_2 + (1 - \theta)k^2 a_6 \end{aligned} \quad (19)$$

Now on inserting the trial solutions (one Fourier mode out of the full solution) at a given point x_m , $C_m^j = \delta^j \exp(im \eta h)$ into Equation (18) and rearranging the equations, η is the mode number, h is the element size and $i^2 = -1$, we obtain

$$\begin{cases} w_1 \delta^{j+1} e^{i\eta(m-3)h} + w_2 \delta^{j+1} e^{i\eta(m-2)h} + w_1 \delta^{j+1} e^{i\eta(m-1)h} \\ = w_3 \delta^j e^{i\eta(m-3)h} + w_4 \delta^j e^{i\eta(m-2)h} + w_3 \delta^j e^{i\eta(m-1)h} \\ - a_1 \delta^{j-1} e^{i\eta(m-3)h} + a_2 \delta^{j-1} e^{i\eta(m-2)h} + a_1 \delta^{j-1} e^{i\eta(m-1)h} \end{cases} \quad (20)$$

Dividing Equation (20) by $\delta^{j-1} e^{i\eta(m-2)h}$ and rearranging, we obtain

$$\delta^2 (w_2 + 2w_1 \cos(\eta h)) - \delta (w_4 + 2w_3 \cos(\eta h)) + (a_2 + 2a_1 \cos(\eta h)) = 0 \quad (21)$$

The wave number is given as:

$$\eta = \frac{2\pi}{\lambda} \quad (22)$$

where λ is the wave length. Let

$$N = \frac{\lambda}{h}$$

which represents the number of grid interval over one wavelength. Then the Equation (22) can be rearranged to the form (Strikwerda, 2004)

$$\varphi = \eta h = \frac{2\pi}{N} \tag{23}$$

where $\varphi = \eta h$ is dimensionless wave number. As the shortest waves represented at the considered grid points have wavelength $2h$, whereas the longest ones tend to infinity, then $2 \leq N \leq \infty$ implies that $0 \leq \varphi \leq \pi$ (Strikwerda, 2004). Let

$$\begin{aligned} A &= w_2 + 2w_1 \cos(\varphi) \\ B &= w_4 + 2w_3 \cos(\varphi) \\ C &= a_2 + 2a_1 \cos(\varphi) \end{aligned}$$

Then the Equation (21) yields

$$A\delta^2 - B\delta + C = 0 \tag{24}$$

Applying the Routh–Hurwitz criterion (Siddiqi & Arshed, 2013) on Equation (24), the necessary and sufficient conditions for Equation (14) to be unconditionally stable as follows: Consider the transformation $\delta = \frac{1+\xi}{1-\xi}$ and simplifying the Equation (14) becomes as

$$(A + B + C)\xi^2 + 2(A - C)\xi + (A - B + C) = 0 \tag{25}$$

The unconditionally stability condition $|\xi| \leq 1$ under the following necessary and sufficient conditions

$$A + B + C \geq 0, A - C \geq 0, A - B + C \geq 0 \tag{26}$$

$$\begin{aligned} A + B + C &= (w_2 + w_4 + a_2) + 2(w_1 + w_3 + a_1) \cos(\varphi) \\ A - B + C &= (w_2 - w_4 + a_2) + 2(w_1 - w_3 + a_1) \cos(\varphi) \\ A - C &= (w_2 - a_2) + 2(w_1 - a_1) \cos(\varphi) \end{aligned} \tag{27}$$

Since φ ranges from 0 to π , then inequalities (26) can be verify for its extreme values only (Strikwerda, 2004). Setting $\varphi = \pi$, the values of $w_i, i = 1, 2, 3, 4$ and $a_i, i = 1, 2$, it can be easily proved that

$$A + B + C = \left(16(1 + k\alpha) + 2k^2(3 + 4\beta^2)(-1 + 2\theta) \sin^2\left(\frac{h}{4}\right) \right) \cos^2 \text{ec}^2\left(\frac{h}{4}\right) \geq 0 \tag{28}$$

The inequality given in Equation (28) satisfy if $-1 + 2\theta \geq 0 \Rightarrow \theta \geq \frac{1}{2}$.

$$A - B + C = k^2 \cos^2 \text{ec}^2\left(\frac{h}{4}\right) \left(6 \cos^2\left(\frac{h}{4}\right) + 8\beta^2 \sin^2\left(\frac{h}{4}\right) \right) \geq 0 \tag{29}$$

$$A - C = k \cos^2 \text{ec}^2\left(\frac{h}{2}\right) \left(6k\theta \cos^2\left(\frac{h}{4}\right) + 2(8\alpha + 4k\beta^2\theta) \sin^2\left(\frac{h}{4}\right) \right) \geq 0 \tag{30}$$

Thus the proposed scheme for telegraph equation is unconditionally stable in the region $0.5 \leq \theta \leq 1$ without any restriction on grid size and time step size but h should be chosen in such a way that the accuracy of the scheme is not degraded.

5. Numerical experiments

This section presents some numerical results of the hyperbolic telegraph equation (1) with initial (2) and boundary conditions (3) or (4). To test the accuracy of proposed method, several numerical experiments for different values of α and β are given in this section with L_∞ , L_2 and root mean square (RMS) errors are calculated by

$$L_\infty = \|u_{\text{exc}} - U_N\|_\infty = \max_j |u_j - (U_N)_j|$$

$$L_2 = \|u_{\text{exc}} - U_N\|_2 = \sqrt{h \sum_{j=0}^N |u_j - (U_N)_j|^2}$$

$$\text{RMS} = \sqrt{\frac{\sum_{j=0}^N |u_j - (U_N)_j|^2}{N + 1}}$$

We compare the numerical solutions obtained by cubic trigonometric B-spline collocation method for telegraph equation (1) with known exact solutions and those numerical methods in the literature. We carry out (14) by the proposed method and Intel@Core™ i5-2410M CPU@2.30 GHz with 8GB RAM and 64-bit operating system (Windows 7). The numerical implementation is carried out in Mathematica 9. Numerical results are computed by cubic trigonometric B-spline collocation method for the telegraph equation (1) at different time levels with smaller storage and CPU time which are tabulated in different Tables. All Figures are drawn at the value of weighting parameter $\theta = 0.5$.

Example 1 Consider the following particular case of Equation (1) in the domain $[0, \pi]$ with $\alpha = 2$, $\beta = \sqrt{2}$ (Dehghan & Shokri, 2008; Mittal & Bhatia, 2013)

$$\frac{\partial^2 u}{\partial t^2}(x, t) + 4 \frac{\partial u}{\partial t}(x, t) + 2u(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) + q(x, t) \quad 0 \leq x \leq \pi, \quad t \geq 0$$

subject to the following initial and boundary conditions

$$\begin{cases} u(x, t = 0) = \sin(x), & \frac{\partial u}{\partial t}(x, t = 0) = -\sin(x) \\ u(x = 0, t) = 0, & u(x = \pi, t) = 0 \end{cases}$$

where $q(x, t) = -2e^{-t} \sin(x)$. The exact solution of this problem is $u_{\text{exc}}(x, t) = e^{-t} \sin(x)$.

The proposed method is applied to calculate the numerical solutions of the telegraph equation (1)–(2) with $h = 0.02$, $\Delta t = 0.0001$ at different time levels. The absolute errors (L_∞) and relative error (L_2) at weighting parameter $\theta = 0.5$, different time levels and also CPU time in second, are reported in Table 2. It can be concluded that our results are more accurate as compared to results obtained by Dehghan and Shokri (2008) and Mittal and Bhatia (2013). In Table 3 and Figure 1, we report the absolute errors, relative errors and RMS for $h = 0.02$, $\Delta t = 0.01$ at different time levels

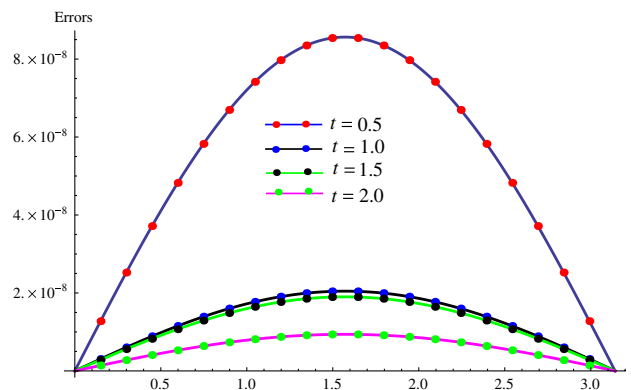
Table 2. Relative errors, maximum errors and CPU time of Example 1 with $\Delta t = 0.0001$, $h = 0.02$

t	CuTBSM			CPU (s)	CuBSM (Mittal & Bhatia, 2013)		CPU (s)	RBF (Dehghan & Shokri, 2008)		CPU (s)
	L_2	L_∞	θ		L_2	L_∞		L_2	L_∞	
0.5	4.45E-08	3.55E-08	0.56	2.89	2.33E-06	1.86E-06	3.04	7.95E-05	8.37E-06	5
1.0	5.02E-09	4.01E-09	0.56	4.15	4.36E-06	3.48E-06	4.89	1.45E-04	1.57E-05	12
1.5	8.36E-10	6.66E-10	0.56	5.01	4.78E-06	3.82E-06	5.27	1.59E-04	1.74E-05	19
2.0	8.06E-10	6.42E-10	0.56	7.04	4.27E-06	3.40E-06	7.53	1.42E-04	1.58E-05	28

Table 3. Relative errors, maximum errors, RMS, values of θ and CPU time of Example 1 with $\Delta t = 0.01, h = 0.02$

t	CuTBSM			θ	CPU (s)	CuBSM (Mittal & Bhatia, 2013)		
	L_2	L_∞	RMS			L_2	L_∞	CPU (s)
0.5	1.07E-07	8.56E-08	6.03E-08	0.66500	0.32	2.33E-06	1.86E-06	0.51
1.0	2.56E-08	2.04E-08	1.44E-08	0.66615	0.45	4.36E-06	3.48E-06	0.59
1.5	2.38E-08	1.90E-08	1.33E-08	0.66649	0.56	4.78E-06	3.82E-06	0.63
2.0	1.17E-08	9.38E-09	6.61E-09	0.66664	0.65	4.27E-06	3.40E-06	0.68

Figure 1. Error graph of Example 1 at different time levels with $h = 0.02, \Delta t = 0.01$.



with CPU time with different values of weighting parameter θ due to the purpose of comparison with existing methods. The numerical results of this problem are in good agreement with exact solution and are more accurate than cubic B-spline collocation method (Mittal & Bhatia, 2013). Figure 2 depicts the graphs of comparison between exact and numerical solutions at time levels $t = 1, 2, 3$ with $h = 0.02, \Delta t = 0.01$. Figure 3 shows the space-time graph of exact and approximate solutions at $t = 3$ with $h = 0.02, \Delta t = 0.01$.

Example 2 In this problem, we consider the telegraph equation (1) in the domain $[0, 2]$ with $\alpha = 10, \beta = 5$ (Dosti & Nazemi, 2012; Mittal & Bhatia, 2013)

$$\frac{\partial^2 u}{\partial t^2}(x, t) + 20 \frac{\partial u}{\partial t}(x, t) + 25u(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) + q(x, t) \quad 0 \leq x \leq 2, t \geq 0$$

subject to the following initial and boundary conditions

$$\begin{cases} u(x, 0) = \tan(x/2), & \frac{\partial u}{\partial t}(x, 0) = \frac{1}{2}(1 + \tan^2(x/2)) \\ u(0, t) = \tan(t/2), & u(2, t) = \tan((2+t)/2) \end{cases}$$

and function $q(x, t) = \alpha(1 + \tan^2((x+t)/2)) + \beta^2 \tan((x+t)/2)$. The exact solution of this equation is $u_{\text{exc}}(x, t) = \tan((x+t)/2)$.

In this problem, we take $L = 2, h = 0.02$ and two values of time step size $k = 0.0001$ and $k = 0.001$ due to the purpose of comparison with existing methods. In Table 4, we report the absolute errors and relative errors of this problem using present method at different time levels and different values of weighting parameter θ . In Table 5, we also recorded the absolute errors and relative errors at different time levels for $h = 0.001, k = 0.001$ and concluded that our results are more

Figure 2. Comparison of numerical and exact solution of Example 1 at different time levels with $h = 0.02, \Delta t = 0.01$.

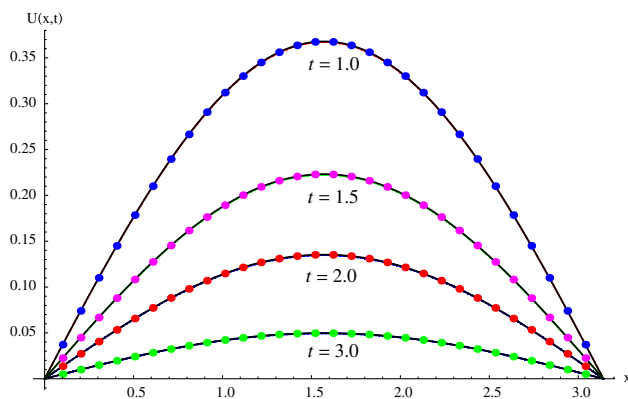


Figure 3. Space-time graphs of Example 1 at $T = 3$ with $h = 0.02, \Delta t = 0.01$.

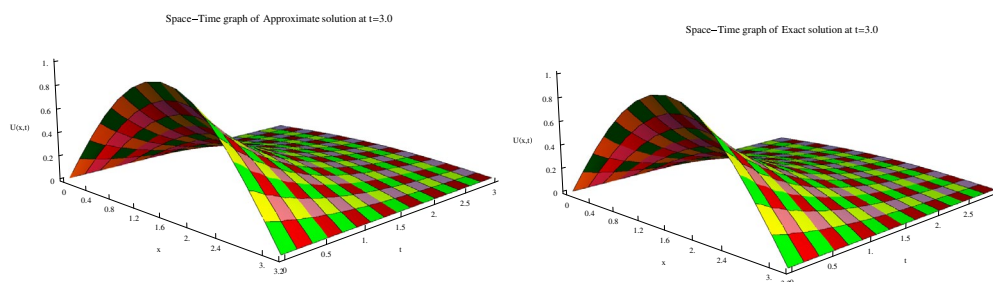


Table 4. L_2, L_∞ errors of Example 2 at different time levels with $h = 0.02$

Method	t	$L_2(k = 0.0001)$	$L_\infty(k = 0.0001)$	$L_2(k = 0.001)$	$L_\infty(k = 0.001)$
CuTBSM	0.2	2.13E-05	3.35E-05	1.85E-04	2.79E-04
	0.4	5.25E-05	7.92E-05	4.43E-04	6.35E-04
	0.6	1.02E-04	1.63E-04	7.96E-04	1.17E-03
	0.8	2.39E-04	4.65E-04	1.47E-03	2.40E-03
	1.0	1.73E-03	5.59E-03	4.19E-03	9.51E-03
CuBSM (Mittal & Bhatia, 2013)	0.2	5.03E-05	3.48E-05	1.88E-04	2.63E-04
	0.4	9.52E-05	5.34E-05	4.89E-04	7.00E-04
	0.6	2.20E-04	9.47E-04	9.49E-04	1.49E-03
	0.8	7.83E-04	1.88E-04	1.87E-03	3.41E-03
	1.0	7.92E-03	5.87E-04	5.10E-03	1.12E-02

Table 5. Relative errors and maximum errors of Example 2 with $\Delta t = h = 0.001$

t	CuTBSM		CuBSM (Mittal & Bhatia, 2013)		QuBSM (Dosti & Nazemi, 2012)
	L_2	L_∞	L_2	L_∞	L_∞
0.2	1.82E-04	2.72E-04	2.18E-04	3.61E-04	2.77E-04
0.4	4.34E-04	6.17E-04	5.66E-04	1.03E-04	7.08E-04
0.6	7.71E-04	1.12E-03	1.15E-04	2.59E-03	1.39E-03
0.8	1.37E-03	2.18E-03	2.60E-03	7.62E-03	3.09E-03
1.0	2.99E-03	5.63E-03	1.03E-02	4.66E-02	1.34E-03

Figure 4. Numerical and exact solutions of Example 2 at different time levels with $h = 0.02, \Delta t = 0.001$.

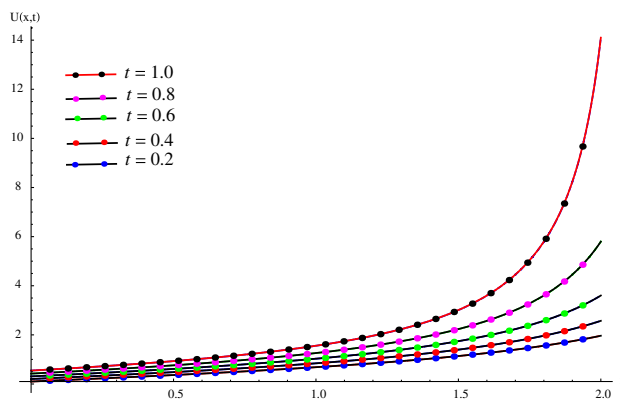
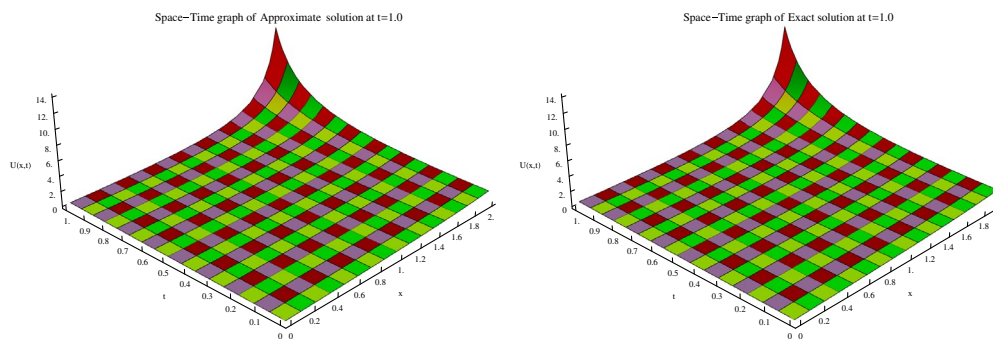


Figure 5. 3D solutions plot of Example 2 at $t = 1.0$ with $h = 0.02$.



accurate than Dosti and Nazemi (2012) and Mittal and Bhatia (2013). Figure 4 illustrates the comparison of exact solution with approximate solution of this problem at various time levels and different values. In Figure 5, we show the space-time graph of approximate and exact solutions at time $t = 1.0$.

Example 3 We consider the telegraph equation (1) in the domain $[0, 1]$ with $\alpha = 0.5, \beta = 1.0$ (Dehghan & Shokri, 2008; Mittal & Bhatia, 2013)

$$\frac{\partial^2 u}{\partial t^2}(x, t) + \frac{\partial u}{\partial t}(x, t) + u(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) + q(x, t)$$

subject to the following initial and boundary conditions

$$\left\{ \begin{aligned} u(x, 0) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 0 \quad 0 \leq x \leq 1 \quad \& \quad u(0, t) = 0, \quad u(1, t) = 0 \quad t \geq 0 \end{aligned} \right.$$

and

$$q(x, t) = (2 - 2t + t^2)(x - x^2)e^{-t} + 2t^2e^{-t}. \text{ The exact solution of this problem is } u_{\text{exc}}(x, t) = (x - x^2)t^2e^{-t}.$$

The absolute errors, relative errors and CPU time in seconds is shown in Table 6 with $\Delta t = 0.001, h = 0.01$. Numerical results are compared with the obtained results in Dehghan and Shokri (2008) and Mittal and Bhatia (2013). It can be concluded that the numerical solutions obtained by our method are good in comparison with Dehghan and Shokri (2008) and Mittal and Bhatia (2013). The graph of exact and numerical solutions at $t = 1, 2, 3, 4, 5$ is shown in Figure 6 and the space-time graph of solutions up to $t = 5$ is presented in Figure 7.

Table 6. Relative errors, maximum errors and CPU time of Example 3 with $\Delta t = 0.001$, $h = 0.01$

t	CuTBSM			CuBSM (Mittal & Bhatia, 2013)			RBF (Dehghan & Shokri, 2008)		
	L_2	L_∞	CPU (s)	L_2	L_∞	CPU (s)	L_2	L_∞	CPU (s)
1.0	6.31E-05	8.76E-05	0.34	4.55E-05	5.91E-05	0.43	1.44E-04	1.85E-06	0
2.0	2.34E-05	3.29E-05	0.57	1.43E-05	1.78E-05	0.77	8.08E-05	1.07E-05	0
3.0	4.62E-06	5.90E-06	1.05	6.42E-06	1.43E-05	1.15	1.29E-04	1.82E-05	1
4.0	2.19E-05	3.04E-05	1.11	8.92E-06	1.35E-05	1.29	1.18E-04	1.65E-05	1
5.0	5.18E-06	6.92E-06	1.26	3.01E-06	5.20E-06	1.46	7.55E-05	1.05E-05	2

Figure 6. Comparison of numerical and exact solutions of Example 3 at different time levels with $h = 0.01$, $\Delta t = 0.001$.

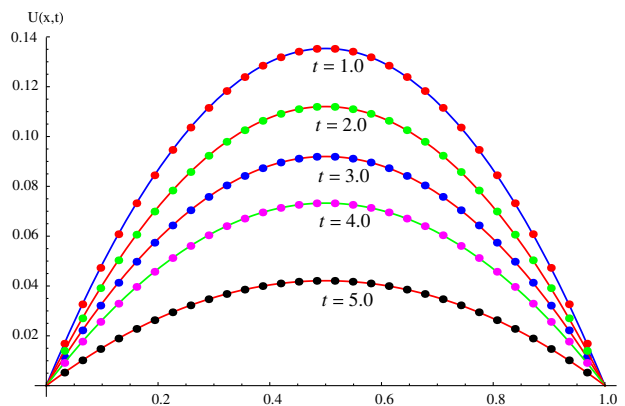
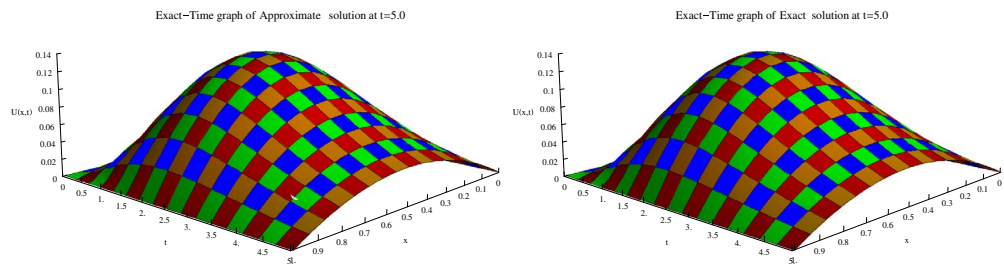


Figure 7. Space-time surface plot of solutions for Example 3 at $t = 5.0$ with $h = 0.01$.



Example 4 Consider the telegraph equation (1) in the domain $[0, 1]$ and $\alpha = 6$, $\beta = 2$ (Dosti & Nazemi, 2012; Mittal & Bhatia, 2013)

$$\frac{\partial^2 u}{\partial t^2}(x, t) + 12 \frac{\partial u}{\partial t}(x, t) + 4u(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) + q(x, t)$$

with following initial and boundary conditions

$$\begin{cases} u(x, 0) = \sin(x), & \frac{\partial u}{\partial t}(x, 0) = 0 & 0 \leq x \leq 1 \\ u(0, t) = 0, & u(1, t) = \cos(t) \sin(1) & t \geq 0 \end{cases}$$

and $q(x, t) = -2\alpha \sin(t) \sin(x) + \beta^2 \cos(t) \sin(x)$. The exact solution of this problem is $u(x, t) = \cos(t) \sin(x)$.

The efficiency can be noted from Table 7 using L_2, L_∞ and RMS errors with $\Delta t = 0.0001$, $h = 0.01$. In Table 8, we also reported the absolute errors and relative errors at different time levels for

Table 7. Relative errors, maximum errors and RMS errors of Example 4 with $\Delta t = 0.001$, $h = 0.01$

t	CuTBSM			CuBSM (Mittal & Bhatia, 2013)		
	L_2	L_∞	RMS	L_2	L_∞	RMS
0.2	2.96E-06	4.63E-06	2.94E-06	2.69E-06	5.24E-06	2.67E-06
0.4	6.77E-06	1.01E-05	6.73E-06	5.61E-06	8.61E-06	5.59E-06
0.6	9.81E-06	1.42E-05	9.76E-06	9.75E-06	1.25E-05	9.70E-06
0.8	1.20E-05	1.71E-05	1.19E-05	1.38E-05	2.03E-05	1.37E-05
1.0	1.34E-05	1.90E-05	1.33E-05	1.73E-05	2.75E-05	1.72E-06

Table 8. L_2 and L_∞ errors of Example 4 at different time levels with $k = 0.001$

Method	t	$L_2(h = 0.01)$	$L_\infty(h = 0.01)$	$L_2(h = 0.005)$	$L_\infty(h = 0.005)$
CuTBSM	0.2	2.92E-05	4.56E-05	2.92E-05	4.56E-05
	0.4	6.69E-05	9.99E-05	6.68E-05	9.98E-05
	0.6	9.70E-05	1.40E-04	9.69E-05	1.40E-04
	0.8	1.19E-04	1.70E-04	1.18E-04	1.70E-04
	1.0	1.33E-04	1.88E-04	1.88E-04	1.88E-04
CuBSM (Mittal & Bhatia, 2013)	0.2	3.67E-05	7.91E-05	3.43E-05	6.82E-05
	0.4	8.90E-05	1.60E-04	8.57E-05	1.49E-04
	0.6	1.37E-04	2.34E-04	1.33E-04	2.24E-04
	0.8	1.79E-04	2.98E-04	1.75E-04	2.89E-04
	1.0	2.13E-04	3.51E-04	2.09E-04	3.43E-04
QuBSM (Dosti & Nazemi, 2012)	0.2	-	-	-	2.43E-05
	0.4	-	-	-	7.93E-05
	0.6	-	-	-	1.21E-04
	0.8	-	-	-	1.49E-04
	1.0	-	-	-	1.65E-04

different values h with $\Delta t = 0.001$ and the numerical results are compared with those of Dosti and Nazemi (2012) and Mittal and Bhatia (2013). We found that our numerical results are comparable to that of QuBSM (Dosti & Nazemi, 2012) and CuBSM (Mittal & Bhatia, 2013) in terms of L_2 , L_∞ errors. Figure 8 presents the comparison of numerical and exact solutions for different time levels with $\Delta t = 0.001$, $h = 0.01$. The space-time graph of numerical and exact solutions at $t = 1.0$ is presented in Figure 9.

Example 5 Consider the following particular case of second-order one-dimensional equation (1) over the region $[0, 2\pi] \times [0, 3]$ with $\alpha = 4$, $\beta = 2$ (Dehghan & Ghesmati, 2010; L. B. Liu & H. W. Liu, 2013; Mittal & Bhatia, 2014)

Figure 8. Comparison of numerical and exact solutions of Example 4 at different time levels with $h = 0.01$, $\Delta t = 0.001$.

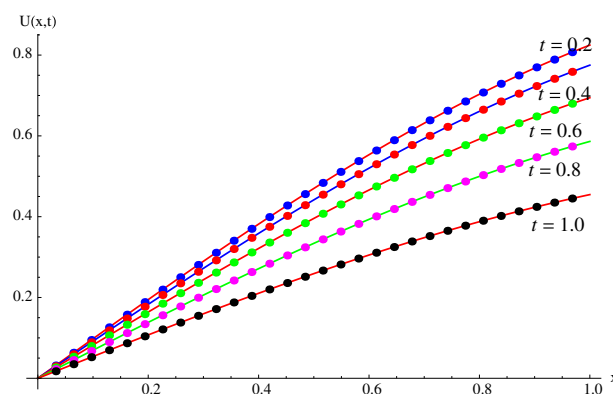
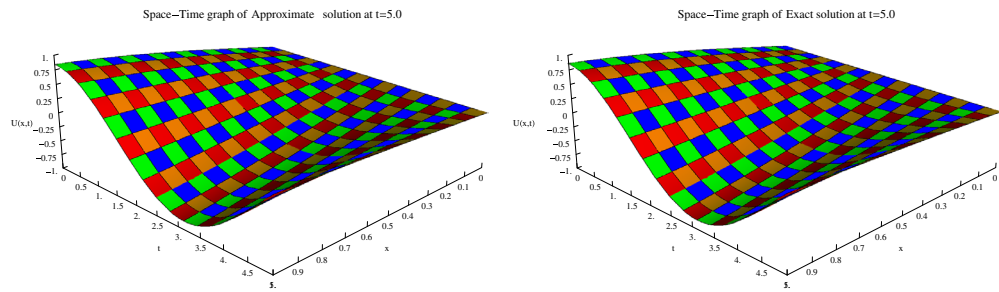


Figure 9. 3D plot of approximate and exact solutions for Example 4 at $t = 5.0$ with $h = 0.01$.



$$\frac{\partial^2 u}{\partial t^2}(x, t) + 8 \frac{\partial u}{\partial t}(x, t) + 4u(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t) + q(x, t) \quad 0 \leq x \leq 2\pi, t \geq 0$$

subject to the following initial and Neumann boundary conditions

$$\begin{cases} u(x, t = 0) = \sin(x), & \frac{\partial u}{\partial t}(x, t = 0) = -\sin(x) \\ u_x(0, t) = e^{-t}, & u_x(2\pi, t) = e^{-t} \end{cases}$$

where $q(x, t) = -2e^{-t} \sin(x)$. The exact solution of this problem is $u_{\text{exc}}(x, t) = e^{-t} \sin(x)$.

The proposed method is applied to calculate the numerical solutions of telegraph equation (1)–(2) and (4) at $t = 1, 2, 3$ with $\Delta t = 0.01$ and different values of h . The absolute errors, relative errors and RMS errors at different values of weighting parameter θ and also CPU time in second, are reported in Table 9. It can be concluded that our results are more accurate as compared to results

Table 9. L_2, L_∞ and RMS errors and CPU(s) of Example 5 at different time levels with $k = 0.01$

Method	h	t	θ	L_2	L_∞	RMS	CPU (s)
CuTBSM	0.05	1.0	0.797420	1.36E-06	7.42E-07	5.43E-07	0.22
	0.02		0.800670	1.36E-06	7.39E-07	5.41E-07	0.51
	0.05	2.0	0.797420	3.04E-07	1.93E-07	1.20E-07	0.45
	0.02		0.800670	2.96E-07	1.89E-07	1.18E-07	1.03
	0.05	3.0	0.797420	8.33E-08	7.16E-08	3.31E-08	0.67
	0.02		0.800670	8.14E-08	6.87E-08	3.24E-08	1.55
	0.01		0.801147	8.13E-08	6.57E-08	3.21E-08	2.90
C-RBF (Dehghan & Ghesmati, 2010)	0.05	3.0	-	-	-	7.12E-05	-
	0.02		-	-	-	1.71E-05	-
	0.01		-	-	-	8.22E-05	-
TPS-RBF (Dehghan & Ghesmati, 2010)	0.05	3.0	-	-	-	9.01E-05	-
	0.02		-	-	-	2.94E-05	-
	0.01		-	-	-	8.99E-06	-
L-RBF (Dehghan & Ghesmati, 2010)	0.05	3.0	-	-	-	3.01E-04	-
	0.02		-	-	-	7.13E-05	-
	0.01		-	-	-	4.32E-04	-
CuBSM (Mittal & Bhatia, 2014)	0.05	1.0	-	5.11E-04	4.95E-04	2.03E-04	-
	0.02		-	1.78E-04	1.67E-04	7.11E-05	-
	0.05	2.0	-	3.20E-04	2.45E-04	1.27E-04	-
	0.02		-	1.53E-04	1.07E-04	6.11E-04	-
	0.05	3.0	-	1.99E-04	1.34E-04	7.85E-04	0.70
	0.02		-	1.07E-04	6.61E-05	4.26E-05	1.70
	0.01		-	-	3.76E-05	3.10	
CDS (L. B. Liu & H. W. Liu, 2013)	0.05	3.0	-	-	-	6.04E-07	-
	0.02		-	-	-	6.01E-07	-
	0.01		-	-	-	6.00E-07	-

Figure 10. Error graph of Example 5 at different values of h at $t = 3.0$ with $\Delta t = 0.01$.

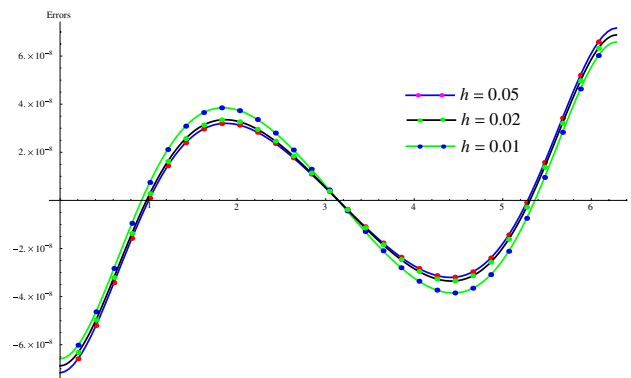
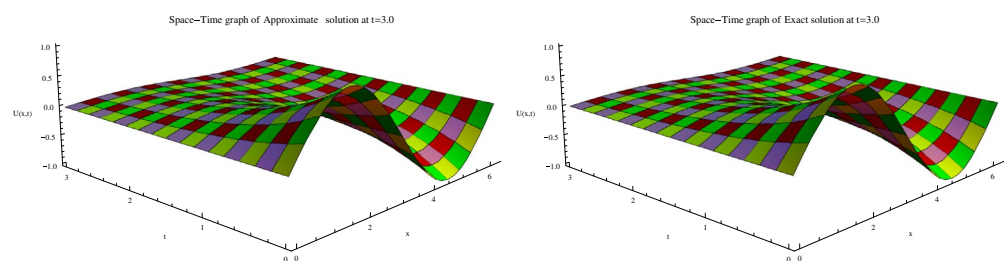


Figure 11. Space-time graph of solutions for Example 5 up to $t = 3.0$ with $k = 0.01$ and $h = 0.05$.



obtained by three RBFs schemes such as Cubic RBF (CRBF) (Dehghan & Ghesmati, 2010), Thin Plate Spline RBF (TPS-RBF) (Dehghan & Ghesmati, 2010), Linear RBF (L-RBF) (Dehghan & Ghesmati, 2010), CDS (L. B. Liu & H. W. Liu, 2013) and CuBSM (Mittal & Bhatia, 2014). Figure 10 depicts the errors of proposed method at different values of h . The numerical results of this problem are also in good agreement with exact solution. Figure 11 shows the space-time graph of approximate and exact solutions at $t = 3$ with $h = 0.05$, $\Delta t = 0.01$.

6. Conclusion

This paper has investigated the application of cubic trigonometric B-spline collocation method to find the numerical solution of the telegraph equation with initial condition and Dirichlet as well as Neumann's type boundary conditions. A usual finite difference approach is used to discretize the time derivatives. The cubic trigonometric B-spline is used for interpolating the solutions at each time. The numerical results shown in Tables 2–9 and Figures 1–11 indicate the reliability of results obtained. The obtained solution to the telegraph equation for various time levels has been compared with the exact solution and existing methods by calculating L_∞ , L_2 and RMS errors. The comparison indicated improved accuracy compared to C-RBF (Dehghan & Ghesmati, 2010), TPS-RBF (Dehghan & Ghesmati, 2010), L-RBF (Dehghan & Ghesmati, 2010), RBF (Dehghan & Shokri, 2008), QuBSM (Dosti & Nazemi, 2012), CDS (L. B. Liu & H. W. Liu, 2013), CuBSM (Mittal & Bhatia, 2013, 2014).

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