ON THE MOTIVE OF THE QUOT SCHEME OF FINITE QUOTIENTS OF A LOCALLY FREE SHEAF

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ABSTRACT. Let X be a smooth variety, E a locally free sheaf on X. We express the generating function of the motives $[\mathrm{Quot}_X(E,n)]$ in terms of the power structure on the Grothendieck ring of varieties. This extends a recent result of Bagnarol, Fantechi and Perroni for curves, and a result of Gusein-Zade, Luengo and Melle-Hernández for Hilbert schemes. We compute this generating function for curves and we express the relative motive $[\mathrm{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}) \to \mathrm{Sym}\,\mathbb{A}^d]$ as a plethystic exponential.

CONTENTS

0. INTRODUCTION

Let X be a smooth quasi-projective variety over $\mathbb{C}$, and let E be a locally free sheaf of rank r on X. The Quot scheme $\mathrm{Quot}_X(E,n)$ parameterises quotients $E \twoheadrightarrow Q$ such that Q is a zero-dimensional sheaf of length n. Recently Bagnarol, Fantechi and Perroni [1] have shown that if C is a smooth proper curve, the class

$$\big[\mathrm{Quot}_C(E,n)\big] \in K_0(\mathrm{Var}_{\mathbb{C}})$$

in the Grothendieck ring of varieties does not depend on E. We use the theory of *power structures* [9] to extend their result to arbitrary dimension. Roughly speaking, a power structure on a ring R is a way of making sense of expressions $A(t)^m$, where $A(t) = 1 + A_1 t + A_2 t^2 + \cdots$ is a power series with coefficients in R and $m \in R$.

For (X,E) as above, we form the generating function

$$\mathsf{Z}_E(t) = \sum_{n \geq 0} \big[\mathrm{Quot}_X(E,n)\big] t^n,$$

and we denote by $\mathsf{P}_{r,n} \in K_0(\mathrm{Var}_{\mathbb{C}})$ the motive of the *punctual Quot scheme*, namely the closed subscheme $P_{r,n} \subset \mathrm{Quot}_X(E,n)$ parameterising quotients that are entirely supported at a single (fixed) point in X.

Our first main result (proved in Theorem 2.3) is the following.

Key words and phrases. Quot schemes, Moduli spaces of sheaves, Grothendieck ring of varieties.

Theorem A. *There is an identity*

$$\mathsf{Z}_E(t)=\left(\sum_{n\geq 0}\mathsf{P}_{r,n}t^n\right)^{[X]}.$$

Since the punctual Quot scheme only depends on r, n and $\dim X$, it follows that the same holds true for the motive of $\operatorname{Quot}_X(E,n)$. Note that this was proved for $r=1$ (the Hilbert scheme case) by Gusein-Zade, Luengo and Melle-Hernández [10].

Our second main result is of *relative* nature and concerns $X=\mathbb{A}^d$. The Quot-to-Chow morphism

$$\operatorname{Quot}_X(E,n)\to\operatorname{Sym}^n X$$

sends a quotient $E\twoheadrightarrow Q$ to the support of Q, viewed as a zero-cycle with multiplicities. We consider the relative motive

$$\mathsf{Z}^{\mathrm{rel}}(\mathbb{A}^d,r)=\sum_{n\geq 0}\left[\operatorname{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r},n)\to\operatorname{Sym}^n\mathbb{A}^d\right]\in K_0(\mathrm{Var}_{\operatorname{Sym}\mathbb{A}^d})$$

over the symmetric product of $\mathbb{A}^d$. We define classes $\Omega_{r,n}\in K_0(\mathrm{Var}_{\mathbb{C}})$ by

$$\sum_{n\geq 0}\mathsf{P}_{r,n}t^n=\operatorname{Exp}\left(\sum_{n>0}\Omega_{r,n}t^n\right)$$

where Exp is the *motivic exponential* (see Section 1.5) induced by the lambda ring structure on $K_0(\mathrm{Var}_{\mathbb{C}})$. For $\mathbb{A}^d$, we refine Theorem A by showing (see Theorem 2.9) that $\mathsf{Z}^{\mathrm{rel}}(\mathbb{A}^d,r)$ is generated on the small diagonal by the absolute motives $\Omega_{r,n}$.

Theorem B. *There is an identity*

$$\mathsf{Z}^{\mathrm{rel}}(\mathbb{A}^d,r)=\operatorname{Exp}_{\cup}\left(\sum_{n>0}\Omega_{r,n}\boxtimes\left[\mathbb{A}^d\xrightarrow{\Delta_n}\operatorname{Sym}^n\mathbb{A}^d\right]\right).$$

See [6, 14] for analogues of this result in the context of motivic Donaldson–Thomas theory and [3] for the calculation of the (absolute) *virtual motive* of $\operatorname{Hilb}^n(\mathbb{A}^3)$.

Finally, our last result (see Section 3.1) is the full "solution" of the motivic theory of the Quot scheme of a smooth curve, which can be summarised by the identities

$$\Omega_{r,n}=\begin{cases}[\mathbb{P}^{r-1}] & \text{if } n=1\\ 0 & \text{if } n>1.\end{cases}$$

Theorem C. *If E is a locally free sheaf on a smooth curve C, there is an identity*

$$\mathsf{Z}_E(t)=\operatorname{Exp}\left(\left[C\times\mathbb{P}^{r-1}\right]t\right).$$

Moreover, in $K_0(\mathrm{Var}_{\operatorname{Sym}\mathbb{A}^1})$ there is an identity

$$\mathsf{Z}^{\mathrm{rel}}(\mathbb{A}^1,r)=\operatorname{Exp}_{\cup}\left(\left[\mathbb{P}^{r-1}\right]\boxtimes\left[\mathbb{A}^1\xrightarrow{\mathrm{id}}\mathbb{A}^1\right]\right).$$

We use the first relation to compute the Hodge–Deligne polynomial of the smooth space $\operatorname{Quot}_C(E,n)$ for a proper curve C (Proposition 3.5). We stress that the formula for Z_E in the proper case was already implicit in the calculation of [1, Prop. 4.5].

In Section 3.3 we discuss the case $r=1$ on a surface, where we find $\Omega_{1,n}=\mathbb{L}^{n-1}$ according to Göttsche's formula [7]. Finally, we conclude by proposing a geometric open problem related to punctual Quot schemes on curves.

We work over the field of complex numbers throughout.

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1. Motivic preliminaries

In this section we recall a few motivic constructions that will be needed later. Most of this material is a simplified version of [6, Section 1], adapted to suit the purposes of this paper.

1.1. The Grothendieck ring of varieties.

Fix a complex scheme S locally of finite type over $\mathbb{C}$. The Grothendieck ring of S-varieties

$$K_0(\mathrm{Var}_S)$$

is the free abelian group generated by isomorphism classes $[X \to S]$ of S-varieties modulo the *scissor relations*, namely the identities

$$\big[X \xrightarrow{f} S\big] = \big[Y \xrightarrow{f|_Y} S\big] + \big[X \setminus Y \xrightarrow{f|_{X\setminus Y}} S\big]$$

imposed whenever $Y \subset X$ is a closed S-subvariety of X. The ring structure is given on generators by fibre product over S,

$$[X \to S] \cdot [Y \to S] = [X \times_S Y \to S]. \tag{1.1}$$

The element

$$\mathbb{L} = [\mathbb{A}^1 \times_{\mathbb{C}} S \to S] \in K_0(\mathrm{Var}_S)$$

is called the *Lefschetz motive* (over S). If S' is another complex scheme, there is an external product

$$K_0(\mathrm{Var}_S) \times K_0(\mathrm{Var}_{S'}) \xrightarrow{\boxtimes} K_0(\mathrm{Var}_{S \times S'}) \tag{1.2}$$

defined on generators by sending $([f\colon X \to S], [g\colon X' \to S']) \mapsto [f \times g\colon X \times X' \to S \times S']$.

A morphism $f\colon S \to T$ induces a ring homomorphism $f^*\colon K_0(\mathrm{Var}_T) \to K_0(\mathrm{Var}_S)$ by base change and a $K_0(\mathrm{Var}_T)$-linear map $f_!\colon K_0(\mathrm{Var}_S) \to K_0(\mathrm{Var}_T)$ defined on generators by composition with f.

Definition 1.1. We denote by $S_0(\mathrm{Var}_S)$ the semigroup of *effective* motives, i.e. the semigroup generated by isomorphism classes $[X \to S]$ of complex quasi-projective S-varieties modulo the scissor relations. The product (1.1) turns $S_0(\mathrm{Var}_S)$ into a semiring. There is a natural semiring map $S_0(\mathrm{Var}_S) \to K_0(\mathrm{Var}_S)$, and we say that $\alpha \in K_0(\mathrm{Var}_S)$ is effective if it lies in the image of this map.

1.2. Equivariant motives and the quotient map.

Recall that if S is a scheme with a *good* action by a finite group G (i.e. an action such that every point of S has an affine G-invariant open neighborhood), the quotient S/G exists as a scheme. For instance, finite group actions on quasi-projective varieties are good.

Definition 1.2. Let G be a finite group, S a scheme with good G-action. We denote by $\widetilde{K}_0^G(\mathrm{Var}_S)$ the free abelian group generated by isomorphism classes $[X \to S]$ of G-equivariant S-varieties with good action, modulo the G-equivariant scissor relations. We denote by $K_0^G(\mathrm{Var}_S)$ the quotient of $\widetilde{K}_0^G(\mathrm{Var}_S)$ by the relations

$$[V \to X \to S] = [\mathbb{A}^r_X \to S],$$

where $V \to X$ is a G-equivariant vector bundle of rank r over a G-equivariant S-variety X.

There is a natural ring structure on $\widetilde{K}_0^G(\mathrm{Var}_S)$, where the product of two classes $[X \to S]$ and $[Y \to S]$ is given by taking the diagonal action on $X \times_S Y$. The structures f^*, $f_!$ and $\boxtimes$ naturally extend to the equivariant setting, along with their basic compatibilities. For instance, if $f: S \to T$ (resp. $g: S' \to T'$) is a G-equivariant (resp. G'-equivariant) map, and u, v are equivariant motives over S, S', then

$$(f \times g)_!(u \boxtimes v) = f_! u \boxtimes g_! v \tag{1.3}$$

in the $(G \times G')$-equivariant K-group over $T \times T'$.

One can define a $K_0(\mathrm{Var}_{S/G})$-linear map (cf. [6, Lemma 1.5])

$$\pi_G : \widetilde{K}_0^G(\mathrm{Var}_S) \to K_0(\mathrm{Var}_{S/G}) \tag{1.4}$$

given on generators by taking the orbit space,

$$\pi_G[X \to S] = [X/G \to S/G].$$

This map does not always extend to $K_0^G(\mathrm{Var}_S)$. It does when G acts freely on S, by [5, Lemma 3.2].

1.3. **Lambda ring structures.** Let $n > 0$ be an integer, and let $\mathfrak{S}_n$ be the symmetric group of n elements. By [6, Lemma 1.6], namely the relative version of [3, Lemma 2.4], there exist "n-th power" maps

$$(\cdot)^{\otimes n} : K_0(\mathrm{Var}_S) \to \widetilde{K}_0^{\mathfrak{S}_n}(\mathrm{Var}_{S^n}) \tag{1.5}$$

where $S^n = S \times \cdots \times S$ is endowed with the natural $\mathfrak{S}_n$-action. The power map takes $[f : X \to S]$ to the class of the equivariant function $f^n : X^n \to S^n$. For $A \in K_0(\mathrm{Var}_S)$, consider the classes

$$\pi_{\mathfrak{S}_n}(A^{\otimes n}) \in K_0(\mathrm{Var}_{S^n/\mathfrak{S}_n}).$$

The *lambda ring* operations on $K_0(\mathrm{Var}_{\mathbb{C}})$ are defined by

$$A \mapsto \sigma^n(A) = \pi_{\mathfrak{S}_n}(A^{\otimes n}) \in K_0(\mathrm{Var}_{\mathbb{C}})$$

for effective classes $A \in K_0(\mathrm{Var}_{\mathbb{C}})$, and then taking the unique extension to a lambda ring structure on $K_0(\mathrm{Var}_{\mathbb{C}})$, determined by the relation

$$\sum_{i=0}^{n} \sigma^i([X]-[Y])\sigma^{n-i}[Y] = \sigma^n[X]. \tag{1.6}$$

If S comes with a commutative associative map $\nu: S \times S \to S$, we likewise define

$$\sigma_\nu^n(A) = \overline{\nu}_! \pi_{\mathfrak{S}_n}(A^{\otimes n}) \in K_0(\mathrm{Var}_S)$$

on effective classes $A = [X \to S]$, where $\overline{\nu}$ is the map $S^n/\mathfrak{S}_n \to S$. One then uses the analogue of the relation (1.6) to find a unique set of lambda ring operators σ_ν^n restricting to the previous identity on effective motives.

As a special case, one can consider $(S, \nu) = (\mathbb{N}, +)$, viewed as a symmetric monoid in the category of schemes. We obtain lambda operations $\sigma^n = \sigma^n_+$ on $K_0(\mathrm{Var}_{\mathbb{C}})[\![t]\!]$ via the isomorphism

$$K_0(\mathrm{Var}_{\mathbb{C}})[\![t]\!] \xrightarrow{\sim} K_0(\mathrm{Var}_{\mathbb{N}}) \tag{1.7}$$

defined by sending $\sum_{n\geq 0}[Y_n]t^n \mapsto \left[\coprod_{n\in\mathbb{N}} Y_n \to \{n\}\right]$.

1.4. **Power structures.** The main references for power structures are [9, 10].

Definition 1.3 ([9])**.** A *power structure* on a (semi)ring R is a map

$$\begin{aligned}(1+tR[\![t]\!])\times R &\to 1+tR[\![t]\!]\\ (A(t),m) &\mapsto A(t)^m\end{aligned}$$

satisfying the following conditions:

(1) $A(t)^0 = 1$,
(2) $A(t)^1 = A(t)$,
(3) $(A(t)\cdot B(t))^m = A(t)^m \cdot B(t)^m$,
(4) $A(t)^{m+m'} = A(t)^m \cdot A(t)^{m'}$,
(5) $A(t)^{mm'} = (A(t)^m)^{m'}$,
(6) $(1+t)^m = 1+mt+O(t^2)$,
(7) $A(t)^m\big|_{t\to t^e} = A(t^e)^m$.

Throughout we use the following:

Notation 1.4. Partitions $\alpha \vdash n$ are written as $\alpha = (1^{\alpha_1}\cdots i^{\alpha_i}\cdots s^{\alpha_s})$, meaning that there are α_i parts of size i. In particular we recover $n = \sum_i i\alpha_i$. The *automorphism group* of α is the product of symmetric groups $G_\alpha = \prod_i \mathfrak{S}_{\alpha_i}$.

Example 1.5. If $R = \mathbb{Z}$, $A(t) = 1+\sum_{n>0} A_n t^n \in \mathbb{Z}[\![t]\!]$ and $m \in \mathbb{N}$, the known formula [16, p. 40]

$$A(t)^m = 1+\sum_{n\geq 0}\sum_{\alpha\vdash n}\left(\prod_{i=0}^{\|\alpha\|-1}(m-i)\cdot\frac{\prod_i A_i^{\alpha_i}}{\prod_i \alpha_i!}\right)t^n$$

defines a power structure on $\mathbb{Z}$, where we have set $\|\alpha\| = \sum_i \alpha_i$.

Gusein-Zade, Luengo and Melle-Hernández have proved [9, Thm. 2] that there is a unique power structure

$$(A(t),m)\mapsto A(t)^m$$

on $K_0(\mathrm{Var}_{\mathbb{C}})$ extending the one defined in *loc. cit.* on the semiring $S_0(\mathrm{Var}_{\mathbb{C}})$ of effective motives. The latter is given by the formula

$$A(t)^{[X]} = 1+\sum_{n\geq 0}\sum_{\alpha\vdash n}\pi_{G_\alpha}\left(\left[\prod_i X^{\alpha_i}\setminus\Delta\right]\cdot\prod_i A_i^{\otimes\alpha_i}\right)t^n. \tag{1.8}$$

Here, $\Delta \subset \prod_i X^{\alpha_i}$ is the "big diagonal" (the locus in the product where at least two entries are equal), and the product in big round brackets is a G_α-equivariant motive in $\widetilde{K}_0^{G_\alpha}(\mathrm{Var}_{\mathbb{C}})$, thanks to the power map (1.5).

Remark 1.6. We will not encounter non-effective coefficients in this paper, so we will have direct access to Formula (1.8).

1.5. **Motivic exponential.** The *motivic exponential* is a group isomorphism

$$\mathrm{Exp}\colon t\,K_0(\mathrm{Var}_{\mathbb C})[\![t]\!] \xrightarrow{\sim} 1 + t\,K_0(\mathrm{Var}_{\mathbb C})[\![t]\!],$$

converting sums into products and preserving effectiveness. If $A = \sum_{n>0} A_n t^n$ is an effective power series, one has by definition

$$\mathrm{Exp}\left(\sum_{n>0} A_n t^n\right) = \prod_{n>0} (1-t^n)^{-A_n},$$

and if A and B are effective, one sets

$$\mathrm{Exp}(A-B) = \prod_{n>0}(1-t^n)^{-A_n} \cdot \left(\prod_{n>0}(1-t^n)^{-B_n}\right)^{-1}. \tag{1.9}$$

More generally, if $(S, \nu\colon S\times S \to S)$ is a commutative monoid in the category of schemes, with a submonoid $S_+ \subset S$ such that the induced map $\coprod_{n\geq 1} S_+^{\times n} \to S$ is of finite type, we similarly define

$$\mathrm{Exp}_\nu(A) = \sum_{n\geq 0} \sigma_\nu^n(A)$$

on effective classes, and for A and B two effective classes, we define $\mathrm{Exp}_\nu(A-B)$ by the analogue of (1.9), i.e. by $\mathrm{Exp}_\nu(A)\cdot \mathrm{Exp}_\nu(B)^{-1}$.

1.6. **Motives over symmetric products.** The machinery described so far will be applied to the following situation. For a variety X, we will consider $(\mathrm{Sym}(X), \cup)$, where

$$\mathrm{Sym}(X) = \coprod_{n\geq 0} \mathrm{Sym}^n(X)$$

can be viewed as a monoid via the morphism

$$\mathrm{Sym}(X)\times \mathrm{Sym}(X) \xrightarrow{\cup} \mathrm{Sym}(X)$$

sending two zero-cycles (with multiplicities) on X to their union. The submonoid $\mathrm{Sym}(X)_+ = \coprod_{n>0}\mathrm{Sym}^n(X)$ allows one to construct the map $\mathrm{Exp}_\cup$ as in Section 1.5.

In order to recover a formal power series in $K_0(\mathrm{Var}_{\mathbb C})[\![t]\!]$ from a relative motive over $\mathrm{Sym}(X)$, we consider the operation

$$\#_!\left(\sum_{n\geq 0}[Y_n \to \mathrm{Sym}^n X]\right) = \sum_{n\geq 0}[Y_n]t^n. \tag{1.10}$$

In other words we take the direct image along the "tautological" map $\#\colon \mathrm{Sym}(X)\to \mathbb N$ which collapses $\mathrm{Sym}^n(X)$ onto the point n. In the right hand side of (1.10), we use the isomorphism (1.7) to identify relative motives over $\mathbb N$ and formal power series with coefficients in $K_0(\mathrm{Var}_{\mathbb C})$.

The following result, a special case of [6, Prop. 1.12], will be needed in the proof of Theorem 2.9.

Lemma 1.7. *Let U be a variety and let $\Delta_n\colon U \to \mathrm{Sym}^n U$ be the small diagonal. Let $A = \sum_{n>0} A_n$ be an effective motive over $\mathbb N_{>0}$ and set $B = \mathrm{Exp}(A) = 1 + \sum_{n>0} B_n$. Define*

$$\mathsf Z = \sum_{n\geq 0}\sum_{\alpha\vdash n} \cup_! \pi_{G_\alpha} j_\alpha^* \left(\boxtimes_{i\mid \alpha_i\neq 0} \Delta_{i!}\left([U \xrightarrow{\mathrm{id}} U]\boxtimes B_i\right)^{\otimes \alpha_i}\right) \in K_0(\mathrm{Var}_{\mathrm{Sym}\,U}),$$

where j_α is the G_α-equivariant open immersion $\prod_i \operatorname{Sym}^i(U)^{\alpha_i} \setminus \Delta \hookrightarrow \prod_i \operatorname{Sym}^i(U)^{\alpha_i}$. Then there is an identity

$$\mathsf{Z} = \operatorname{Exp}_{\cup}\left(\sum_{n>0} A_n \boxtimes \left[U \xrightarrow{\Delta_n} \operatorname{Sym}^n U\right]\right)$$

Moreover,

$$\#_! \operatorname{Exp}_{\cup}\left(\sum_{n>0} A_n \boxtimes \left[U \xrightarrow{\Delta_n} \operatorname{Sym}^n U\right]\right) = B^{[U]} \in K_0(\operatorname{Var}_{\mathbb{C}})[\![t]\!].$$

We briefly explain how to read the right hand side of the first equation of the lemma. First of all, we view $\cup$ as a map $\operatorname{Sym}(U)^b \to \operatorname{Sym}(U)$ for any $b > 0$. The map π_{G_α} appearing in the definition of Z sends a G_α-equivariant relative motive over $\prod_i \operatorname{Sym}^i(U)^{\alpha_i} \setminus \Delta$ to a relative motive over $\prod_i \operatorname{Sym}^{i\alpha_i}(U) \setminus \Delta$, therefore we can apply the direct image $\cup_!$ to get a relative motive over $\operatorname{Sym}^n U$, where $n = \sum_i i\alpha_i$.

2. The motive of the Quot scheme

2.1. **Main characters.** Let X be a smooth quasi-projective variety of dimension d. Let E be a rank r locally free sheaf on X. For a given integer $n \geq 0$, the Quot scheme

$$\operatorname{Quot}_X(E, n)$$

parameterises quotients $E \twoheadrightarrow Q$ such that

$$\dim(\operatorname{Supp} Q) = 0, \quad \chi(Q) = n.$$

The Quot-to-Chow map

$$\sigma_n \colon \operatorname{Quot}_X(E, n) \to \operatorname{Sym}^n X$$

constructed in [8, Section 6] (see also [15, Cor. 7.15] for a modern treatment) takes a quotient $E \twoheadrightarrow Q$ to the zero-cycle (with multiplicities) determined by the set-theoretic support of Q. We define the *punctual Quot scheme* to be the preimage

$$\operatorname{Quot}_X(E, n)_p = \sigma_n^{-1}(n \cdot p)$$

of the cycle $n \cdot p \in \operatorname{Sym}^n X$, where $p \in X$ is a point. This is easily seen to only depend on a formal neighborhood of $p \in X$ (but not on p, X or E). In particular, one has isomorphisms

$$\operatorname{Quot}_X(E, n)_p \cong \operatorname{Quot}_X(\mathscr{O}_X^{\oplus r}, n)_p \cong \operatorname{Quot}_{\mathbb{A}^d}(\mathscr{O}_{\mathbb{A}^d}^{\oplus r}, n)_0 \tag{2.1}$$

where 0 is the origin in $\mathbb{A}^d$. This scheme will be denoted $P_{r,n}$ from now on, and

$$\mathsf{P}_{r,n} = \left[P_{r,n}\right] \in K_0(\operatorname{Var}_{\mathbb{C}})$$

will denote its motive.

We pause for a second to explain how to prove the second isomorphism in (2.1). Using smoothness of X, we can fix étale coordinates around $p \in X$. This means we can find a pair (U, φ) where $p \in U \subset X$ is an open neighborhood and $\varphi \colon U \to \mathbb{A}^d$ is an étale map such that $\varphi(p) = 0 \in \mathbb{A}^d$. As in the proof of [2, Lemma A.1], we can further shrink U until $U \cap \varphi^{-1}(0)$ is the single point p. Then, we consider the open subscheme $W \subset \operatorname{Quot}_U(\mathscr{O}_U^{\oplus r}, n) \subset \operatorname{Quot}_X(\mathscr{O}_X^{\oplus r}, n)$ consisting of quotients $\mathscr{O}_U^{\oplus r} \twoheadrightarrow Q$ such that $\varphi|_{\operatorname{Supp} Q}$ is injective. Note that W contains $\operatorname{Quot}_U(\mathscr{O}_U^{\oplus r}, n)_p = \operatorname{Quot}_X(\mathscr{O}_X^{\oplus r}, n)_p$ as a closed subscheme. By [2, Proposition A.3], sending

$$\left(\mathscr{O}_U^{\oplus r} \twoheadrightarrow Q\right) \mapsto \left(\mathscr{O}_{\mathbb{A}^d}^{\oplus r} \to \varphi_*\varphi^*\mathscr{O}_{\mathbb{A}^d}^{\oplus r} = \varphi_*\mathscr{O}_U^{\oplus r} \twoheadrightarrow \varphi_* Q\right)$$

defines an étale morphism $\Phi\colon W \to \operatorname{Quot}_{\mathbb{A}^d}(\mathscr{O}_{\mathbb{A}^d}^{\oplus r}, n)$. Its restriction

$$\Phi^{-1}\big(\operatorname{Quot}_{\mathbb{A}^d}(\mathscr{O}_{\mathbb{A}^d}^{\oplus r}, n)_0\big) \to \operatorname{Quot}_{\mathbb{A}^d}(\mathscr{O}_{\mathbb{A}^d}^{\oplus r}, n)_0 \tag{2.2}$$

to the punctual Quot scheme of $\mathbb{A}^d$ is étale and bijective, hence an isomorphism. For surjectivity, use that p is the only point in $U \cap \varphi^{-1}(0)$, and for injectivity use that $\varphi|_U$ is an immersion around p, so that $\varphi^*\varphi_* Q \xrightarrow{\sim} Q$ is an isomorphism for all Q supported entirely at p. Finally, again by our choice of U, the source of the morphism (2.2) is naturally identified with $\operatorname{Quot}_U(\mathscr{O}_U^{\oplus r}, n)_p$.

Remark 2.1. The *punctual motives* $\mathsf{P}_{r,n}$ clearly depend on the dimension $d = \dim X$, but we omit d from the notation.

Example 2.2. On a curve (i.e. if $d = 1$), by [1, Prop. 2.6] we have

$$\mathsf{P}_{r,1} = \big[\mathbb{P}^{r-1}\big]. \tag{2.3}$$

2.2. **Absolute motives.** Let X and E be as in the previous section. Define the generating functions

$$\begin{aligned} \mathsf{P}_r(t) &= \sum_{n\geq 0} \mathsf{P}_{r,n} t^n, \\ \mathsf{Z}_E(t) &= \sum_{n\geq 0} \big[\operatorname{Quot}_X(E,n)\big] t^n \end{aligned}$$

in the power series ring $K_0(\mathrm{Var}_{\mathbb{C}})[\![t]\!]$. The following result (namely Theorem A from the Introduction) is the higher rank analogue of the corresponding statement for the Hilbert scheme of points [10, Thm. 1], obtained by setting $r = 1$.

Theorem 2.3. *Let X be a smooth quasi-projective variety. Let E be a rank r locally free sheaf on X. There is an identity*

$$\mathsf{Z}_E(t) = \mathsf{P}_r(t)^{[X]}. \tag{2.4}$$

Proof. For α a partition of n, let $\operatorname{Sym}^\alpha X \subset \operatorname{Sym}^n X$ be the locally closed subvariety parameterising zero-cycles whose support is distributed according to α. We get a motivic decomposition

$$\big[\operatorname{Quot}_X(E,n)\big] = \sum_{\alpha \vdash n} \big[\operatorname{Quot}_X(E,n)_\alpha\big], \tag{2.5}$$

where we have set $\operatorname{Quot}_X(E,n)_\alpha = \sigma_n^{-1}(\operatorname{Sym}^\alpha X)$. By standard arguments (see e.g. [4, Sec. 4] and [13, Sec. 3]), one sees that the deepest stratum of the Quot-to-Chow map

$$\sigma_{(n)}\colon \operatorname{Quot}_X(E,n)_{(n)} \to X$$

is a Zariski locally trivial fibration with fibre $P_{r,n}$. This relies on the local case $X = \mathbb{A}^d$, where one has a global decomposition

$$\operatorname{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)_{(n)} \cong \mathbb{A}^d \times P_{r,n}$$

under which $\sigma_{(n)}$ is identified with the first projection.

For a fixed partition $\alpha \vdash n$, let

$$V_\alpha \hookrightarrow \prod_i \operatorname{Quot}_X(E,i)^{\alpha_i}$$

be the open subscheme parameterising finite quotients with disjoint supports. By [2, Prop. A.3] (but see also [4, Lemma 4.10] for the Hilbert scheme version), taking the union of points gives an étale map

$$u_\alpha \colon V_\alpha \to \operatorname{Quot}_X(E,n)$$

and we let U_α denote its image. The stratum $\operatorname{Quot}_X(E,n)_\alpha$ sits inside U_α as a closed subscheme. We let the cartesian diagram

$$\begin{tikzcd} Z_\alpha \arrow[r, hook] \arrow[d, "\widetilde{u}_\alpha"'] \arrow[dr, phantom, "\square"] & V_\alpha \arrow[d, "u_\alpha"] \\ \operatorname{Quot}_X(E,n)_\alpha \arrow[r, hook] & U_\alpha \end{tikzcd} \tag{2.6}$$

define the scheme Z_α. The map $\widetilde{u}_\alpha$ is a finite étale cover with Galois group G_α, in particular we have

$$\operatorname{Quot}_X(E,n)_\alpha = Z_\alpha/G_\alpha. \tag{2.7}$$

In fact, Z_α can also be realised as the fibre product

$$\begin{tikzcd} Z_\alpha \arrow[r, hook] \arrow[d, "f_\alpha"'] \arrow[dr, phantom, "\square"] & \prod_i \operatorname{Quot}_X(E,i)^{\alpha_i}_{(i)} \arrow[d] \\ \prod_i X^{\alpha_i}\setminus\Delta \arrow[r, hook] & \prod_i X^{\alpha_i} \end{tikzcd} \tag{2.8}$$

where the bottom open immersion is the complement of the big diagonal and the map f_α is a G_α-equivariant piecewise trivial fibration with fibre $\prod_i P_{r,i}^{\alpha_i}$. This implies the identity

$$[Z_\alpha] = \left[\prod_i X^{\alpha_i}\setminus\Delta\right]\cdot\prod_i \mathsf{P}_{r,i}^{\otimes\alpha_i}$$

in $K_0^{G_\alpha}(\mathrm{Var}_{\mathbb{C}})$. Using (2.7), it follows that

$$\begin{aligned}\left[\operatorname{Quot}_X(E,n)_\alpha\right] &= \pi_{G_\alpha}[Z_\alpha] \\ &= \pi_{G_\alpha}\left(\left[\prod_i X^{\alpha_i}\setminus\Delta\right]\cdot\prod_i \mathsf{P}_{r,i}^{\otimes\alpha_i}\right),\end{aligned}$$

where $\pi_{G_\alpha}\colon K_0^{G_\alpha}(\mathrm{Var}_{\mathbb{C}})\to K_0(\mathrm{Var}_{\mathbb{C}})$ is the quotient map extending (1.4). Since the classes $\mathsf{P}_{r,i}$ are effective, combining the decomposition (2.5) with the power structure formula (1.8) and summing over n proves the result. □

The following is a generalisation of [1, Thm. 4.1] to arbitrary varieties.

Corollary 2.4. *The series $\mathsf{Z}_E(t)$ does not depend on E. In particular, the identity*

$$\left[\operatorname{Quot}_X(E,n)\right] = \left[\operatorname{Quot}_X(\mathscr{O}_X^{\oplus r},n)\right]$$

holds in $K_0(\mathrm{Var}_{\mathbb{C}})$ for all locally free sheaves E of rank r on X.

Definition 2.5. Define absolute classes $\Omega_{r,n}\in K_0(\mathrm{Var}_{\mathbb{C}})$ via

$$\operatorname{Exp}\left(\sum_{n>0}\Omega_{r,n}t^n\right) = \mathsf{P}_r(t). \tag{2.9}$$

Remark 2.6. In terms of the motivic exponential, we can rephrase Equation (2.4) as

$$\mathsf{Z}_E(t) = \operatorname{Exp}\left([X]\sum_{n>0}\Omega_{r,n}t^n\right). \tag{2.10}$$

It is then clear that to determine the series Z_E one has to compute the fully punctual classes $\Omega_{r,n}$. We will do this in the case of curves (for arbitrary r) in Section 3.1, and for surfaces (only for $r=1$) in Section 3.3.

2.3. **Relative motives.** Let (X,E) be as in the previous sections. Consider the relative motive

$$\mathsf{Z}_E^{\mathrm{rel}} = \sum_{n\geq 0}\left[\operatorname{Quot}_X(E,n)\xrightarrow{\sigma_n}\operatorname{Sym}^n X\right] \in K_0(\operatorname{Var}_{\operatorname{Sym}X}).$$

In other words, $\mathsf{Z}_E^{\mathrm{rel}} = [\operatorname{Quot}_X(E)\to \operatorname{Sym}X]$, the class of $\operatorname{Quot}_X(E) = \coprod_n \operatorname{Quot}_X(E,n)$ over $\operatorname{Sym}X$. Note that $\mathsf{Z}_E^{\mathrm{rel}}$ is a refinement of Z_E, in the sense that

$$\#_!\mathsf{Z}_E^{\mathrm{rel}} = \mathsf{Z}_E(t),$$

where $\#_!$ is the operation introduced in (1.10).

We simply write

$$\mathsf{Z}^{\mathrm{rel}}(X,r) = \mathsf{Z}^{\mathrm{rel}}_{\mathscr{O}^{\oplus r}}$$

when $E=\mathscr{O}^{\oplus r}$ is the trivial bundle over X. We will show below (Theorem 2.9) that the relative motive $\mathsf{Z}^{\mathrm{rel}}(\mathbb{A}^d,r)\in K_0(\operatorname{Var}_{\operatorname{Sym}\mathbb{A}^d})$ is generated under $\operatorname{Exp}_\cup$ by the motives $\Omega_{r,n}$ defined in (2.9), extended on the small diagonal

$$\mathbb{A}^d \xrightarrow{\Delta_n} \operatorname{Sym}^n\mathbb{A}^d.$$

Example 2.7. Set $r=1$, $d=1$ (i.e. we consider line bundles on curves). Then $\operatorname{Quot}_X(L,n) = \operatorname{Hilb}^n X = \operatorname{Sym}^n X$ for all line bundles L on X, and

$$\mathsf{Z}^{\mathrm{rel}}(X,1) = \mathsf{Z}^{\mathrm{rel}}_{\mathscr{O}_X} = \left[\operatorname{Sym}X\xrightarrow{\mathrm{id}}\operatorname{Sym}X\right] = \mathbb{1}\in K_0(\operatorname{Var}_{\operatorname{Sym}X}).$$

Pushing this forward via $\#$ yields

$$\mathsf{Z}_{\mathscr{O}_X}(t) = \sum_{n\geq 0}\left[\operatorname{Sym}^n X\right]t^n = \zeta_X(t),$$

the Kapranov *motivic zeta function* of the curve X.

Remark 2.8. By definition of the power structure and of the motivic exponential, one has

$$\zeta_Y(t) = (1-t)^{-[Y]} = \operatorname{Exp}([Y]t),$$

for every variety Y. Moreover, the identities

$$\zeta_Y(\mathbb{L}^s t) = \zeta_{\mathbb{A}^s\times Y}(t) = \operatorname{Exp}(\mathbb{L}^s[Y]t) \tag{2.11}$$

hold in $K_0(\operatorname{Var}_{\mathbb{C}})[\![t]\!]$ for every $s\in\mathbb{N}$.

We now prove Theorem B from the Introduction.

Before we begin, let us observe that for a morphism of varieties $f\colon S\to T$ and an integer $n>0$, there is a commutative diagram

$$\begin{tikzcd} K_0(\operatorname{Var}_S) \arrow[r, "f_!"] \arrow[d, "(\cdot)^{\otimes n}"'] \arrow[dr, phantom, "\circlearrowleft"] & K_0(\operatorname{Var}_T) \arrow[d, "(\cdot)^{\otimes n}"] \\ \widetilde{K}_0^{\mathfrak{S}_n}(\operatorname{Var}_{S^n}) \arrow[r, "f_!^n"'] & \widetilde{K}_0^{\mathfrak{S}_n}(\operatorname{Var}_{T^n}) \end{tikzcd} \tag{2.12}$$

where $(\cdot)^{\otimes n}$ is the power map (1.5).

Theorem 2.9. *There is an identity*

$$\mathsf{Z}^{\mathrm{rel}}(\mathbb{A}^d, r) = \operatorname{Exp}_{\cup}\left(\sum_{n>0} \Omega_{r,n} \boxtimes \left[\mathbb{A}^d \xrightarrow{\Delta_n} \operatorname{Sym}^n \mathbb{A}^d\right]\right) \in K_0(\mathrm{Var}_{\operatorname{Sym}\mathbb{A}^d}).$$

Proof. For a partition $\alpha \vdash n$, set $Q_\alpha^n = \operatorname{Quot}_{\mathbb{A}^d}(\mathscr{O}^{\oplus r}, n)_\alpha$. One has a decomposition

$$\mathsf{Z}^{\mathrm{rel}}(\mathbb{A}^d, r) = \sum_{n\geq 0}\sum_{\alpha\vdash n}\left[Q_\alpha^n \to \operatorname{Sym}^n \mathbb{A}^d\right].$$

Let us consider the G_α-equivariant cartesian diagram

$$\begin{tikzcd}
Z_\alpha \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_i Q_{(i)}^{\alpha_i} \arrow[d] \\
\prod_i (\mathbb{A}^d)^{\alpha_i} \setminus \Delta \arrow[r, hook, "\iota_\alpha"] \arrow[d, hook, "\Delta"] \arrow[dr, phantom, "\square"] & \prod_i (\mathbb{A}^d)^{\alpha_i} \arrow[d, hook, "\Delta"] \\
\prod_i \operatorname{Sym}^i(\mathbb{A}^d)^{\alpha_i} \setminus \Delta \arrow[r, hook, "j_\alpha"] & \prod_i \operatorname{Sym}^i(\mathbb{A}^d)^{\alpha_i}
\end{tikzcd} \tag{2.13}$$

where the top square is Diagram (2.8), the horizontal maps are open immersions (the complements of the big diagonals) and the vertical inclusions are products of small diagonals. We have a base change identity

$$j_\alpha^*\Delta_! = \Delta_!\iota_\alpha^*. \tag{2.14}$$

On the deepest stratum, we have a commutative diagram

$$\begin{tikzcd}
Q_{(n)}^n \arrow[r, "\sim"] \arrow[d, "\sigma_{(n)}"'] & \mathbb{A}^d \times P_{r,n} \arrow[dl, "\mathrm{pr}_1"] \\
\mathbb{A}^d &
\end{tikzcd}$$

inducing an identity

$$\left[Q_{(n)}^n \to \mathbb{A}^d\right] = \left[\mathbb{A}^d \xrightarrow{\mathrm{id}} \mathbb{A}^d\right] \boxtimes \mathsf{P}_{r,n} \in K_0(\mathrm{Var}_{\mathbb{A}^d}). \tag{2.15}$$

For a general partition α of n, consider the equivariant motives

$$\left[Q_{(i)}^i \to \mathbb{A}^d\right]^{\otimes\alpha_i} \in \widetilde{K}_0^{\mathfrak{S}_{\alpha_i}}\left(\mathrm{Var}_{(\mathbb{A}^d)^{\alpha_i}}\right).$$

If ι_α is as in Diagram (2.13), one has

$$\Delta_!\left[Z_\alpha \to \prod_i (\mathbb{A}^d)^{\alpha_i} \setminus \Delta\right] = \Delta_!\iota_\alpha^*\left(\boxtimes_{i|\alpha_i\neq 0}\left[Q_{(i)}^i \to \mathbb{A}^d\right]^{\otimes\alpha_i}\right) \in \widetilde{K}_0^{G_\alpha}\left(\mathrm{Var}_{\prod_i \operatorname{Sym}^i(\mathbb{A}^d)^{\alpha_i}\setminus\Delta}\right).$$

Applying the quotient map π_{G_α} to the last identity, followed by the pushforward along the union of points map, we obtain

$$
\begin{aligned}
\left[Q_\alpha^n \to \operatorname{Sym}^n \mathbb{A}^d\right] &= \cup_! \pi_{G_\alpha} \Delta_! \iota_\alpha^* \left(\boxtimes_{i|\alpha_i \neq 0} \left[Q_{(i)}^i \to \mathbb{A}^d\right]^{\otimes \alpha_i} \right) \\
&= \cup_! \pi_{G_\alpha} j_\alpha^* \Delta_! \left(\boxtimes_{i|\alpha_i \neq 0} \left[Q_{(i)}^i \to \mathbb{A}^d\right]^{\otimes \alpha_i} \right) && \text{by (2.14)} \\
&= \cup_! \pi_{G_\alpha} j_\alpha^* \left(\boxtimes_{i|\alpha_i \neq 0} (\Delta_i^{\alpha_i})_! \left[Q_{(i)}^i \to \mathbb{A}^d\right]^{\otimes \alpha_i} \right) && \text{by (1.3)} \\
&= \cup_! \pi_{G_\alpha} j_\alpha^* \left(\boxtimes_{i|\alpha_i \neq 0} \left(\Delta_{i\,!}\left[Q_{(i)}^i \to \mathbb{A}^d\right]\right)^{\otimes \alpha_i} \right) && \text{by (2.12)} \\
&= \cup_! \pi_{G_\alpha} j_\alpha^* \left(\boxtimes_{i|\alpha_i \neq 0} \Delta_{i\,!} \left(\left[\mathbb{A}^d \xrightarrow{\mathrm{id}} \mathbb{A}^d\right] \boxtimes \mathsf{P}_{r,i} \right)^{\otimes \alpha_i} \right) && \text{by (2.15)}
\end{aligned}
$$

so that summing these classes over all partitions of integers and noting that $\Omega_{r,n}$ are effective (because $\mathsf{P}_{r,n}$ are effective) yields precisely

$$
\operatorname{Exp}_\cup \left(\sum_{n>0} \Omega_{r,n} \boxtimes \left[\mathbb{A}^d \xrightarrow{\Delta_n} \operatorname{Sym}^n \mathbb{A}^d\right] \right)
$$

by an application of Lemma 1.7. □

Remark 2.10. By the last part of Lemma 1.7, the theorem implies the formula

$$
\mathsf{Z}_{\mathscr{O}^{\oplus r}}(t) = \mathsf{P}_r(t)^{\mathbb{L}^d}
$$

of Theorem 2.3 for $X = \mathbb{A}^d$.

2.4. **Related work on more general Quot schemes.** The theory developed so far relies crucially on the locally free assumption on E. Indeed, the isomorphisms (2.1) fail even if E is, say, reflexive but not locally free. However, the geometry of the Quot scheme can be interesting also in the non-locally free case. For instance, the Quot scheme of finite quotients of the ideal sheaf $\mathscr{I}_C \subset \mathscr{O}_Y$ of a smooth curve in a 3-fold Y has been studied in [13], where essential local triviality statements on the Quot-to-Chow morphism were proved (see e.g. Corollary 3.2 in *loc. cit.*). Moreover, in [12, Thm. 2.1] it is proven that $\operatorname{Quot}_Y(\mathscr{I}_C, n)$ appears as the typical (scheme-theoretic) fibre of the Hilbert–Chow morphism $\operatorname{Hilb}(Y) \to \operatorname{Chow}(Y)$ in a neighborhood of the cycle of the smooth curve C. (This holds in all dimensions, not just 3-folds.) This was used to prove the C-local DT/PT correspondence for Calabi–Yau 3-folds [12, Thm. 1.1]. The (virtual) motivic theory of $\operatorname{Quot}_Y(\mathscr{I}_C, n)$ was developed in [6].

The enumerative geometry of $\operatorname{Quot}_X(E, n)$, for E a sheaf of homological dimension at most one on a 3-fold, was studied in [2] and related to the local Pandharipande–Thomas theory of X. The Appendix in *loc. cit.* develops the abstract theory comparing various Quot schemes of smooth quasi-projective varieties, and implicitly shows that the singularities of $\operatorname{Quot}_X(E, n)$ only depend on n and $\dim X$. The (virtual) motivic theory in the locally free case for 3-folds was developed in [14], along with a construction of a virtual fundamental class on the Quot scheme.

3. Calculations: curves and surfaces

In this section we compute the fully punctual motives

$$
\Omega_{r,n}
$$

in the case of curves, for all $r>0$ and $n>0$, and in the case of surfaces for $r=1$ and all $n>0$.

3.1. **The class of the Quot scheme on a curve.** We fix a locally free sheaf E of rank r on a smooth quasi-projective curve C.

Lemma 3.1. *On a curve, we have*

$$\Omega_{r,1}=\left[\mathbb{P}^{r-1}\right].$$

Proof. By the properties of the power structure, one has

$$\begin{aligned}\mathsf{P}_r(t) &= \prod_{n\geq 1}(1-t^n)^{-\Omega_{r,n}} \\ &= \prod_{n\geq 1}(1-t)^{-\Omega_{r,n}}\Big|_{t\to t^n} \\ &= \prod_{n\geq 1}(1+\Omega_{r,n}t+\cdots)\Big|_{t\to t^n} \\ &= \prod_{n\geq 1}(1+\Omega_{r,n}t^n+\cdots),\end{aligned}$$

which immediately implies

$$\Omega_{r,1}=\mathsf{P}_{r,1}.$$

On the other hand, the equality $\mathsf{P}_{r,1}=[\mathbb{P}^{r-1}]$ holds by (2.3). ◻

We now reformulate (and generalise to the quasi-projective case) the main formula proved in [1, Prop. 4.5]. The following is Theorem C from the Introduction.

Theorem 3.2. *There is an identity*

$$\mathsf{Z}_E(t)=\operatorname{Exp}\left(\left[C\times\mathbb{P}^{r-1}\right]t\right) \tag{3.1}$$

in $K_0(\mathrm{Var}_{\mathbb{C}})[\![t]\!]$. *Moreover, in* $K_0(\mathrm{Var}_{\operatorname{Sym}\mathbb{A}^1})$ *there is an identity*

$$\mathsf{Z}^{\mathrm{rel}}(\mathbb{A}^1,r)=\operatorname{Exp}_{\cup}\left(\left[\mathbb{P}^{r-1}\right]\boxtimes\left[\mathbb{A}^1\xrightarrow{\mathrm{id}}\mathbb{A}^1\right]\right).$$

Proof. By [1, Prop. 4.5], for projective C one has

$$\left[\operatorname{Quot}_C(E,n)\right]=\sum_{n_1+\cdots+n_r=n}\left[\operatorname{Sym}^{n_1}C\right]\cdots\left[\operatorname{Sym}^{n_r}C\right]\cdot\mathbb{L}^{\sum_{i=0}^{r-1}(i-1)n_i}. \tag{3.2}$$

and it is clear that the generating function $\mathsf{Z}_E(t)$ of these motives can be expanded as a product of shifted motivic zeta functions. More precisely, one has

$$\begin{aligned}\sum_{n\geq 0}\left[\operatorname{Quot}_C(E,n)\right]t^n &= \prod_{i=1}^{r}\zeta_C(\mathbb{L}^{i-1}t) \\ &= \prod_{i=1}^{r}\operatorname{Exp}([C]\mathbb{L}^{i-i}\cdot t) \\ &= \operatorname{Exp}\left([C]\sum_{i=1}^{r}\mathbb{L}^{i-i}\cdot\right) \\ &= \operatorname{Exp}\left(\left[C\times\mathbb{P}^{r-1}\right]t\right),\end{aligned}$$

where the second equality follows by (2.11). So the statement is true when C is projective. In this case, comparing (3.1) with Equation (2.10) and using the injectivity of Exp, we obtain the identities

$$[C]\cdot\Omega_{r,n} = \begin{cases} [C]\cdot[\mathbb{P}^{r-1}] & \text{if } n=1 \\ 0 & \text{if } n>1. \end{cases} \tag{3.3}$$

By Equation (2.10), to prove the statement on an arbitrary C it is enough to show that

$$\Omega_{r,n} = \begin{cases} [\mathbb{P}^{r-1}] & \text{if } n=1 \\ 0 & \text{if } n>1. \end{cases} \tag{3.4}$$

By Lemma 3.1, we already know that $\Omega_{r,1}=[\mathbb{P}^{r-1}]$. Finally, the equation $\Omega_{r,n}=0$ holds for $n>1$ because $\Omega_{r,n}$ is effective. Indeed, write $\Omega_{r,n}=[Y]$ for a variety Y, so that $0=[C]\cdot\Omega_{r,n}=[C\times Y]$. But the class of a variety vanishes if and only if the variety is empty, and this happens if and only if $Y=\emptyset$.

To prove the last assertion, it is enough to combine Theorem 2.9 with the relations (3.4). □

Remark 3.3. The formula (3.2) is proved in [1] over a field k of arbitrary charcateristic.

Remark 3.4. By Equation (3.4), the generating function of the punctual motives can be computed as

$$\mathsf{P}_r(t) = \operatorname{Exp}([\mathbb{P}^{r-1}]t) = \zeta_{\mathbb{P}^{r-1}}(t) = \prod_{i=0}^{r-1}\frac{1}{1-\mathbb{L}^i t}. \tag{3.5}$$

3.2. **The Hodge numbers of the Quot scheme on a curve.** The Hodge–Deligne polynomial (also called the E-polynomial) of a smooth complex projective variety Y is given by

$$\mathsf{E}(Y;u,v) = \sum_{p,q}(-1)^{p+q}h^{p,q}(Y)u^p v^q$$

where $h^{p,q}(Y)=\dim_{\mathbb{C}} H^q(Y,\Omega_Y^p)$ are the Hodge numbers of Y. For instance, one has

$$\begin{aligned} \mathsf{E}(\mathbb{P}^{r-1};u,v) &= \sum_{i=0}^{r-1} u^i v^i, \\ \mathsf{E}(C;u,v) &= 1-gu-gv+uv, \end{aligned} \tag{3.6}$$

where C is a smooth projective curve of genus g. Sending $[Y]\mapsto \mathsf{E}(Y;u,v)$ defines a motivic measure

$$K_0(\mathrm{Var}_{\mathbb{C}}) \xrightarrow{\mathsf{E}} \mathbb{Z}[u,v]$$

which is in fact a homomorphism of rings with power structure. The power structure on the polynomial ring $\mathbb{Z}[u,v]$ is determined by the formula

$$(1-t)^{-f(u,v)} = \prod_{i,j}\left(1-u^i v^j t\right)^{-p_{ij}},$$

where we have written $f(u,v)=\sum_{i,j} p_{ij}u^j v^j$ for integers p_{ij}. This implies (cf. [10, Prop. 4]) the basic relation

$$\mathsf{E}\left((1-t)^{-[Y]}\right) = (1-t)^{-\mathsf{E}(Y;u,v)}. \tag{3.7}$$

Let C be a smooth projective curve of genus g, and let E be a rank r locally free sheaf on C. We compute the generating function

$$\mathsf{E}_r(C,t)=\sum_{n\geq 0}\mathsf{E}(\mathrm{Quot}_C(E,n);u,v)t^n.$$

We already know this series does not depend on E.

Proposition 3.5. *There is an identity*

$$\mathsf{E}_r(C,t)=\prod_{i=0}^{r-1}\frac{(1-u^iv^{i+1}t)^g(1-u^{i+1}v^it)^g}{(1-u^iv^it)(1-u^{i+1}v^{i+1}t)} \tag{3.8}$$

in the ring $\mathbb{Z}[u,v][\![t]\!]$.

Proof. We have

$$\begin{aligned}
\mathsf{E}_r(C,t) &= \mathsf{E}\big(\mathrm{Exp}\big([C\times\mathbb{P}^{r-1}]t\big)\big) && \text{by (3.1)}\\
&= \mathsf{E}\Big((1-t)^{-[C\times\mathbb{P}^{r-1}]}\Big) && \text{by definition of Exp}\\
&= (1-t)^{-\mathsf{E}([C\times\mathbb{P}^{r-1}])} && \text{by (3.7)}\\
&= (1-t)^{-(1-gu-gv+uv)\sum_{i=0}^{r-1}u^iv^i}.
\end{aligned}$$

We have used that E is a ring homomorphism and the identities (3.6) in the last step. The result now follows from direct computation and by definition of the power structure on $\mathbb{Z}[u,v]$. □

Remark 3.6. Setting $u=v$ in Formula (3.8) one recovers the generating function of (signed) Poincaré polynomials computed in [1, Remark 4.6], namely

$$\sum_{n\geq 0}P(\mathrm{Quot}_C(F,n),-u)t^n=\prod_{i=0}^{r-1}\frac{(1-u^{2i+1}t)^{2g}}{(1-u^{2i}t)(1-u^{2i+2}t)}.$$

3.3. **The Hilbert scheme of points on a surface.** Let S be a smooth quasi-projective surface, and set $r=1$, so that $\mathrm{Quot}_S(L,n)=\mathrm{Hilb}^n S$ for every line bundle L. We know by Formula (2.10) that

$$\mathsf{Z}_{\mathscr{O}_S}(t)=\mathrm{Exp}\left([S]\sum_{n>0}\Omega_{1,n}t^n\right).$$

On the other hand, by Göttsche's formula [7],

$$\mathsf{Z}_{\mathscr{O}_S}(t)=\mathrm{Exp}\left(\frac{[S]t}{1-\mathbb{L}t}\right)=\mathrm{Exp}\left([S]\sum_{n>0}\mathbb{L}^{n-1}t^n\right).$$

By the injectivity of Exp, we conclude that on a surface S one has

$$(\Omega_{1,n}-\mathbb{L}^{n-1})[S]=0.$$

However, this relation holds universally for *every* quasi-projective surface, in particular for $S=\mathbb{P}^2$ and $S=\mathbb{A}^1\times\mathbb{P}^1$. Therefore

$$(\Omega_{1,n}-\mathbb{L}^{n-1})(1+\mathbb{L}+\mathbb{L}^2)=0=(\Omega_{1,n}-\mathbb{L}^{n-1})(\mathbb{L}+\mathbb{L}^2),$$

showing that

$$\Omega_{1,n}=\mathbb{L}^{n-1},\quad n>0. \tag{3.9}$$

In particular, we recover the known generating function of the motives of punctual Hilbert schemes, given by the formula

$$\mathsf{P}_1(t) = \sum_{n\geq 0} \left[\mathrm{Hilb}^n(\mathbb{A}^2)_0\right] t^n = \prod_{n\geq 1} \left(1-\mathbb{L}^{n-1}t^n\right)^{-1}. \tag{3.10}$$

Finally, we obtain the following relative statement.

Theorem 3.7. *There is an identity*

$$\sum_{n\geq 0}\left[\mathrm{Hilb}^n \mathbb{A}^2 \xrightarrow{\sigma_n} \mathrm{Sym}^n \mathbb{A}^2\right] = \mathrm{Exp}_{\cup}\left(\sum_{n>0} \mathbb{L}^{n-1} \boxtimes \left[\mathbb{A}^2 \xrightarrow{\Delta_n} \mathrm{Sym}^n \mathbb{A}^2\right]\right)$$

in $K_0(\mathrm{Var}_{\mathrm{Sym}\,\mathbb{A}^2})$.

Proof. Combine Theorem 2.9 with Equation (3.9). □

Remark 3.8. The relation (3.10) was already proved in [9], and it was exploited in [11] to provide a motivic check of the classification of modules of length 3 and 4 over the polynomial ring $k[x,y]$.

4. A MOTIVIC-TO-GEOMETRIC OPEN PROBLEM

Let C be a smooth quasi-projective curve. The punctual Quot scheme

$$P_{r,n} \subset \mathrm{Quot}_C(\mathscr{O}^{\oplus r}, n)$$

parameterises quotients $\mathscr{O}^{\oplus r} \twoheadrightarrow Q$ entirely supported at a single (fixed) point $p \in C$. As proved in [1, Prop. 2.6], one has

$$P_{r,1} = \mathbb{P}^{r-1}.$$

How can one describe $P_{r,n}$ for $n > 1$? The relation

$$\mathsf{P}_r(t) = \mathrm{Exp}\left(\left[\mathbb{P}^{r-1}\right]t\right) = \zeta_{\mathbb{P}^{r-1}}(t)$$

established in Equation (3.5) translates into the motivic identity

$$\left[P_{r,n}\right] = \left[\mathrm{Sym}^n \mathbb{P}^{r-1}\right] = \left[\mathrm{Sym}^n P_{r,1}\right]. \tag{4.1}$$

It thus makes sense to ask the following:

Question 4.1. What is the geometric meaning of the relation (4.1)? Can one geometrically compare the schemes $P_{r,n}$ and $\mathrm{Sym}^n P_{r,1}$?

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