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On (semi)topology L-algebras

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Abstract. Here, we define (semi)topological L-algebras and some related results are approved. Then we deduce conditions that mention an L-algebra to be a semi-topological or a topological L-algebra and we check some attributes of them. Chiefly, we display in an L-algebra $\mathfrak{L}$, if $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a semi-topological L-algebra and $\{1\}$ is an open set or $\mathfrak{L}$ is bounded and satisfies the double negation property, then $(\mathfrak{L}, \tau)$ is a topological L-algebra. Finally, we construct a discrete topology on a quotient L-algebra, under suitable conditions. Also, different kinds of topology such as T_0 and Hausdorff are investigated.

1 Introduction

Algebra and topology, two basic areas of mathematics, play complementary roles. Topology studies continuity and convergence and provides a general framework for studying the concept of limit. Algebra studies a variety of operations and provides a basis for algorithms and calculations. In applications, in higher-level areas of mathematics, such as functional analysis, dynamical systems, representation theory, and others, topology and algebra

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naturally come into contact with each other. Many of the most important objects of mathematics exhibit a combination of algebraic and topological structures. Spaces of topological functions and generally linear topological spaces, topological groups and topological fields, transformation groups, topological networks are objects of this kind. Often an algebraic structure and a topology come together naturally. This is when both are determined by the nature of the elements of the considered set. The rules that describe the relationship between topology and algebraic operations are almost always clear, and it is natural that the operations should be continuous, joint continuous, joint, or separate. In the 20th century, many topologists and algebraists have contributed to topological algebra.

L-algebras, which are related to algebraic logic and quantum structures, were introduced by Rump [17]. Many examples shown that L-algebras are very useful. Yang and Rump [19], characterized pseudo-MV-algebras and Bosbach's non-commutative bricks as L-algebras. Wu and Yang [24] proved that orthomodular lattices form a special class of L-algebras in different ways. It was shown that every lattice-ordered effect algebra has an underlying L-algebra structure in Wu et al. [23]. Also, other mathematicians studied the relationship between basic algebras and L-algebras. They proved that a basic algebra which satisfies $(\xi \boxplus \omega') \boxplus (\sigma \boxplus \omega')' = (\xi \boxplus \sigma') \boxplus (\omega \boxplus \sigma')'$, can be converted into an L-algebra. Conversely, if an L-algebra with 0 and some conditions such that it is an involutive bounded lattice can be organized into a basic algebra, it must be a lattice-ordered effect algebra.

In the following, the notion of (semi)topological L-algebras is defined and some related results are achieved. Then conditions that imply an L-algebra to be a semi-topological or a topological L-algebra are investigated and some attitudes of them are checked. Also, it is proved that if $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a semi-topological L-algebra and $\{1\}$ is an open set or $\mathfrak{L}$ is bounded and satisfies the double negation property, then $(\mathfrak{L}, \tau)$ is a topological L-algebra. At last, a discrete topology on quotient L-algebra, under suitable conditions is constructed.

2 Preliminaries

This section lists the known default contents that will be used later.

Definition 2.1. [17] An algebraic structure $(\mathfrak{L}; \twoheadrightarrow, 1)$ of type $(2,0)$ is called an L-algebra if for any $\omega, \sigma, \xi \in \mathfrak{L}$ it satisfies in the next conditions:
(L_1) $\omega \twoheadrightarrow \omega = \omega \twoheadrightarrow 1 = 1$ and $1 \twoheadrightarrow \omega = \omega$,
(L_2) $(\omega \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \xi) = (\sigma \twoheadrightarrow \omega) \twoheadrightarrow (\sigma \twoheadrightarrow \xi)$,
(L_3) if $\omega \twoheadrightarrow \sigma = \sigma \twoheadrightarrow \omega = 1$, then $\omega = \sigma$.

Note. We have to notice that a logical unit is always unique and it is the element of an L-algebra $\mathfrak{L}$.

If the operation $\twoheadrightarrow$ is considered as a logical concept, there is a partial order in $\mathfrak{L}$ given by

$$\omega \leqslant \sigma \text{ iff } \omega \twoheadrightarrow \sigma = 1. \tag{2.1}$$

You can see the proof in [17].

Proposition 2.2. [19] *Let $\mathfrak{L}$ be an L-algebra. Then $\omega \leqslant \sigma$ implies $\xi \twoheadrightarrow \omega \leqslant \xi \twoheadrightarrow \sigma$, for any $\omega, \sigma, \xi \in \mathfrak{L}$.*

Proposition 2.3. [19] *For an L-algebra $\mathfrak{L}$, the following are equivalent:*
(i) $\omega \leqslant \sigma \twoheadrightarrow \omega$,
(ii) *if $\omega \leqslant \sigma$, then $\sigma \twoheadrightarrow \xi \leqslant \omega \twoheadrightarrow \xi$,*
(iii) $((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \xi) \twoheadrightarrow \xi \leqslant ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \xi) \twoheadrightarrow ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \xi)$, *for any $\omega, \sigma, \xi \in \mathfrak{L}$.*

Definition 2.4. [17] (i) An L-algebra $(\mathfrak{L}, \twoheadrightarrow, 1)$ which satisfies in the following condition

$$\omega \twoheadrightarrow (\sigma \twoheadrightarrow \omega) = 1, \qquad (K)$$

for any $\omega, \sigma \in \mathfrak{L}$ is called a *KL-algebra.*
(ii) A CKL-algebra is an L-algebra which satisfies in the following condition

$$\omega \twoheadrightarrow (\sigma \twoheadrightarrow \xi) = \sigma \twoheadrightarrow (\omega \twoheadrightarrow \xi), \qquad (C)$$

for any $\omega, \sigma, \xi \in \mathfrak{L}$.

Note. Clearly, every CKL-algebra is a KL-algebra, since for any $\omega, \sigma \in \mathfrak{L}$, we have

$$\omega \twoheadrightarrow (\sigma \twoheadrightarrow \omega) = \sigma \twoheadrightarrow (\omega \twoheadrightarrow \omega) = \sigma \twoheadrightarrow 1 = 1.$$

Proposition 2.5. [17] *Assume* $(\mathfrak{L}, \twoheadrightarrow, 1)$ *is a* CKL*-algebra. Then for any* $\omega, \sigma, \xi \in \mathfrak{L}$, *we have:*
(i) *if* $\omega \leqslant \sigma$, *then* $\xi \twoheadrightarrow \omega \leqslant \xi \twoheadrightarrow \sigma$,
(ii) $\omega \twoheadrightarrow (\sigma \twoheadrightarrow \omega) = 1$, *i.e.*, $\omega \leqslant \sigma \twoheadrightarrow \omega$,
(iii) $\omega \leqslant (\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma$,
(iv) $\omega \leqslant \sigma \twoheadrightarrow \xi$ *iff* $\sigma \leqslant \omega \twoheadrightarrow \xi$,
(v) *if* $\omega \leqslant \sigma$, *then* $\sigma \twoheadrightarrow \xi \leqslant \omega \twoheadrightarrow \xi$,
(vi) $((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \xi) \twoheadrightarrow \xi \leqslant ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \xi) \twoheadrightarrow ((\sigma \twoheadrightarrow \omega) \twoheadrightarrow \xi)$,
(vii) $\xi \twoheadrightarrow \sigma \leqslant (\sigma \twoheadrightarrow \omega) \twoheadrightarrow (\xi \twoheadrightarrow \omega)$.

Definition 2.6. [17] A non-empty subset $\mathfrak{I}$ of an L-algebra $(\mathfrak{L}, \twoheadrightarrow, 1)$ is called an *ideal of* $\mathfrak{L}$ if it satisfies the following conditions for all $\omega, \sigma \in \mathfrak{I}$,
(I_1) $1 \in \mathfrak{I}$,
(I_2) if $\omega \in \mathfrak{I}$ and $\omega \twoheadrightarrow \sigma \in \mathfrak{I}$, then $\sigma \in \mathfrak{I}$,
(I_3) if $\omega \in \mathfrak{I}$, then $(\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma \in \mathfrak{I}$,
(I_4) if $\omega \in \mathfrak{I}$, then $\sigma \twoheadrightarrow \omega \in \mathfrak{I}$ and $\sigma \twoheadrightarrow (\omega \twoheadrightarrow \sigma) \in \mathfrak{I}$.

An ideal $\mathfrak{I}$ of $\mathfrak{L}$ is called a *proper ideal* if $\mathfrak{I} \neq \mathfrak{L}$. The set of all ideals of $\mathfrak{L}$ is denoted by $\mathcal{I}(\mathfrak{L})$.

Let $\mathfrak{L} \neq \emptyset$ and $\{\circledast_i\}_{i \in I}$ be a family of operations of type 2 on $\mathfrak{L}$ and τ be a topology on $\mathfrak{L}$. Then:
(i) $(\mathfrak{L}, \{\circledast\}_{i \in I}, \tau)$ is a right (left) topological algebra if for any $i \in I$, $(\mathfrak{L}, \circledast_i, \tau)$ is a right (left) topological algebra,
(ii) $(\mathfrak{L}, \{\circledast_i\}_{i \in I}, \tau)$ is a (semi)topological algebra if for all $i \in I$, $(\mathfrak{L}, \circledast_i, \tau)$ is a (semi)topological algebra (see [14, 15]).

Note: In the continuation of this article, $\mathfrak{L}$ is an L-algebra and τ is a topology on $\mathfrak{L}$.

3 (Semi)topological L-algebra

In this section, we introduce the concepts of (semi)topological L-algebra and investigate some related results.

Notation: If $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a (semi)topological algebra, then $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is called a *(semi)topological* L*-algebra*. In addition, we say $(\mathfrak{L}, \tau)$ is a *(semi)topological* L*-algebra* if $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a (semi)topological L-algebra.

Let $\mathfrak{U}, \mathfrak{V} \subseteq \mathfrak{L}$. Then we consider $\mathfrak{U} \twoheadrightarrow \mathfrak{V}$ and $\mathfrak{U} \times \mathfrak{V}$ as follows:

$$\mathfrak{U} \twoheadrightarrow \mathfrak{V} = \{\omega \twoheadrightarrow \sigma \mid \omega \in \mathfrak{U} \text{ and } \sigma \in \mathfrak{V}\} \quad , \quad \mathfrak{U}\times\mathfrak{V} = \{(\omega, \sigma) \mid \omega \in \mathfrak{U} \text{ and } \sigma \in \mathfrak{V}\}.$$

Example 3.1. (i) Obviously, every L-algebra with the discrete topology is a topological L-algebra.
(ii) Let $(\mathfrak{L} = \{\eta, \epsilon, \rho, 1\}, \leqslant)$ be a poset, where $\eta \leqslant \epsilon \leqslant \rho \leqslant 1$. Define the operation $\twoheadrightarrow$ on $\mathfrak{L}$ as follows:

$\twoheadrightarrow$	η	ϵ	ρ	1
η	1	1	1	1
ϵ	η	1	1	1
ρ	η	ϵ	1	1
1	η	ϵ	ρ	1

Then $(\mathfrak{L}, \twoheadrightarrow, 1)$ is an L-algebra and $(\mathfrak{L}, \twoheadrightarrow, \tau)$, where $\tau = \{\emptyset, \{\eta\}, \{\epsilon, \rho, 1\}, \mathfrak{L}\}$ is a topological L-algebra.

Note. Clearly, each topological L-algebra is a semi-topological L-algebra. Next example shows that every semi-topological L-algebra is not a topological L-algebra.

Example 3.2. Consider $(\mathfrak{L} = \{\eta, \epsilon, \rho, 1\}, \leqslant)$ be a poset, where $\eta \leqslant \epsilon \leqslant \rho \leqslant 1$. Define the operation $\twoheadrightarrow$ on $\mathfrak{L}$ as follows:

$\twoheadrightarrow$	η	ϵ	ρ	1
η	1	1	1	1
ϵ	ϵ	1	1	1
ρ	η	ϵ	1	1
1	η	ϵ	ρ	1

Then $(\mathfrak{L}, \twoheadrightarrow, 1)$ is an L-algebra and $(\mathfrak{L}, \twoheadrightarrow, \tau)$, where $\tau = \{\emptyset, \{\rho, 1\}, \{\epsilon, \rho, 1\}, \mathfrak{L}\}$ is a right semi-topological L-algebra but it is not a topological L-algebra, since $\eta \twoheadrightarrow \eta = 1 \in \{\rho, 1\}$ and $\mathfrak{L} \twoheadrightarrow \mathfrak{L} = \mathfrak{L} \nsubseteq \{\rho, 1\}$.

Example 3.3. Set $\mathfrak{L} = [0, 1]$. Define

$$\omega \twoheadrightarrow \sigma = \begin{cases} 1 & \omega \leqslant \sigma \\ \sigma & \omega \succ \sigma \end{cases}$$

Then $(\mathfrak{L}, \twoheadrightarrow, 1)$ is an L-algebra and τ is a topology on $\mathfrak{L}$ with the base $\mathcal{B} = \{[\eta, \epsilon] \cap \mathfrak{L} \mid \eta, \epsilon \in \mathbb{R}\}$. Then $(\mathfrak{L}, \tau)$ is a topological L-algebra.

Proof. We prove $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a topological L-algebra. Assume $\omega \twoheadrightarrow \sigma \in \mathfrak{U} \in \tau$, where $\omega, \sigma \in \mathfrak{L}$. If $\omega \leqslant \sigma$, then $1 = \omega \twoheadrightarrow \sigma \in \mathfrak{U}$. So, $\omega \in [0, \sigma] \subseteq \tau$ and $\sigma \in [\sigma, 1] \in \tau$ and clearly, $[0, \sigma] \twoheadrightarrow [\sigma, 1] \subseteq \mathfrak{U}$. If $\omega > \sigma$, then $\omega \twoheadrightarrow \sigma = \sigma \in \mathfrak{U}$. Thus $\omega \in [\sigma, \omega] \in \tau$, and $\sigma \in [0, \sigma] \cap \mathfrak{U} \in \tau$, and so $[\sigma, \omega] \twoheadrightarrow ([0, \sigma] \cap \mathfrak{U}) = [0, \sigma] \cap \mathfrak{U} \subseteq \mathfrak{U}$. Hence, $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a topological L-algebra. Therefore, $(\mathfrak{L}, \tau)$ is a topological L-algebra. $\square$

Lemma 3.4. *If $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$, then for any $\omega, \sigma, \xi \in \mathfrak{L}$, $\omega \twoheadrightarrow \sigma \in \mathfrak{I}$ and $\sigma \twoheadrightarrow \xi \in \mathfrak{I}$ imply $\omega \twoheadrightarrow \xi \in \mathfrak{I}$.*

Proof. Since $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$ and $\sigma \twoheadrightarrow \xi \in \mathfrak{I}$, by (I_4), $(\sigma \twoheadrightarrow \omega) \twoheadrightarrow (\sigma \twoheadrightarrow \xi) \in \mathfrak{I}$. By (L_2), we have $(\omega \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \xi) \in \mathfrak{I}$. By (I_2) we get $\omega \twoheadrightarrow \xi \in \mathfrak{I}$. $\square$

For an arbitrary element $\eta \in \mathfrak{L}$ and $\emptyset \neq \mathfrak{V} \subseteq \mathfrak{L}$, define the subset

$$\mathfrak{V}(\eta) = \{\omega \in \mathfrak{L} \mid \ \omega \twoheadrightarrow \eta,\ \eta \twoheadrightarrow \omega \in \mathfrak{V}\}.$$

Theorem 3.5. *There is a nontrivial topology τ on $\mathfrak{L}$ such that $(\mathfrak{L}, \tau)$ is a topological L-algebra.*

Proof. Assume

$$\tau = \{\mathfrak{U} \subseteq \mathfrak{L} \mid \text{ for each } \eta \in \mathfrak{U},\ \exists\, \mathfrak{I} \in \mathcal{I}(\mathfrak{L}) \text{ s.t. } \mathfrak{I}(\eta) \subseteq \mathfrak{U}\}.$$

Assume $\{\mathfrak{U}_i \mid i \in \Delta\}$ is a family of members of τ. For any $\omega \in \bigcup_{i\in\Delta} \mathfrak{U}_i$, there exists $j \in \Delta$ such that $\omega \in \mathfrak{U}_j$, and so there is $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$, where $\mathfrak{I}(\omega) \subseteq \mathfrak{U}_j \subseteq \bigcup_{i\in\Delta} \mathfrak{U}_i$. Hence, $\bigcup_{i\in\Delta} \mathfrak{U}_i \in \tau$. In addition, for any $\omega \in \bigcap_{i\in\Delta} \mathfrak{U}_i$ and any $i \in \Delta$, $\omega \in \mathfrak{U}_i$. Then $\mathfrak{I}_i \in \mathcal{I}(\mathfrak{L})$ exists such that $\omega \in \mathfrak{I}_i(\omega) \subseteq \mathfrak{U}_i$. Set $\mathfrak{I} = \bigcap_{i\in\Delta} \mathfrak{I}_i$. Clearly, $\bigcap_{i\in\Delta} \mathfrak{I}_i \in \mathcal{I}(\mathfrak{L})$. Then $\omega \in \mathfrak{I}(\omega) \subseteq \bigcap_{i\in\Delta} \mathfrak{I}_i(\omega) \subseteq \bigcap_{i\in\Delta} \mathfrak{U}_i$. Hence, $\bigcap_{i\in\Delta} \mathfrak{U}_i \in \tau$. Thus, τ is a topology on $\mathfrak{L}$. Now, we prove that the operation $\twoheadrightarrow$ is continuous. For this, assume $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$, $\omega \in \mathfrak{L}$ and $\sigma \in \mathfrak{I}(\omega)$. Hence, $\omega \twoheadrightarrow \sigma \in \mathfrak{I}$ and $\sigma \twoheadrightarrow \omega \in \mathfrak{I}$. If there exists $\xi \in \mathfrak{I}(\sigma)$, then $\xi \twoheadrightarrow \sigma$, $\sigma \twoheadrightarrow \xi \in \mathfrak{I}$, and by Lemma 3.7, $\xi \twoheadrightarrow \omega \in \mathfrak{I}$ and $\omega \twoheadrightarrow \xi \in \mathfrak{I}$. Hence, $\xi \in \mathfrak{I}(\omega)$, and so $\mathfrak{I}(\sigma) \subseteq \mathfrak{I}(\omega)$, thus $\mathfrak{I}(\omega) \in \tau$. Therefore, τ is a nontrivial topology. Suppose $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$ and $\omega, \sigma \in \mathfrak{L}$. Since $\omega \in \mathfrak{I}(\omega)$ and $\sigma \in \mathfrak{I}(\sigma)$, and $\mathfrak{I}(\omega \twoheadrightarrow \sigma) = \mathfrak{I}(\omega) \twoheadrightarrow \mathfrak{I}(\sigma)$, clearly, $\twoheadrightarrow$ is continuous. $\square$

Proposition 3.6. *Suppose τ is as in Theorem* 3.5 *and $\mathfrak{X} \subseteq \mathfrak{L}$. Then:*
(i) *for each $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$, $\mathfrak{I}(\mathfrak{X})$ is an open and closed subset of $\mathfrak{L}$. In addition, each ideal is an open and closed set,*
(ii) $\overline{\mathfrak{X}} = \bigcap\{\mathfrak{I}(\mathfrak{X}) \mid \mathfrak{I} \in \mathcal{I}(\mathfrak{L})\}$.

Proof. (i) Assume $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$, and $\sigma \in \overline{\mathfrak{I}(\mathfrak{X})} = \bigcup_{\omega\in\mathfrak{X}}\mathfrak{I}(\omega)$. Then, $\mathfrak{I}(\sigma) \cap \mathfrak{I}(\mathfrak{X}) \neq \emptyset$. Hence, there is $\omega \in \mathfrak{X}$, such that $\mathfrak{I}(\sigma) = \mathfrak{I}(\omega)$ and so $\sigma \in \mathfrak{I}(\omega) \subseteq \mathfrak{I}(\mathfrak{X})$. Therefore, $\mathfrak{I}(\mathfrak{X})$ is closed. But $\mathfrak{I}(\mathfrak{X})$ is open because it is a union of open sets.
(ii) Suppose $\mathfrak{X} \subseteq \mathfrak{L}$ and $\omega \in \mathfrak{X}$. Since for all $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$, $\omega \twoheadrightarrow \omega = 1 \in \mathfrak{I}$, we have $\omega \in \mathfrak{I}(\omega)$, and so $\omega \in \bigcap\{\mathfrak{I}(\mathfrak{X}) \mid \mathfrak{I} \in \mathcal{I}(\mathfrak{L})\}$. Conversely, consider $\omega \in \bigcap\{\mathfrak{I}(\mathfrak{X}) \mid \mathfrak{I} \in \mathcal{I}(\mathfrak{L})\}$. Then, for all $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$, $\omega \in \mathfrak{I}(\mathfrak{X})$. Since $\mathfrak{I}(\mathfrak{X}) = \bigcup_{\eta\in\mathfrak{X}}\mathfrak{I}(\eta)$, there exists $\epsilon \in \mathfrak{X}$ such that $\omega \in \mathfrak{I}(\epsilon)$. Moreover, since $\omega \twoheadrightarrow \epsilon \in \mathfrak{I}$ and $\epsilon \twoheadrightarrow \omega \in \mathfrak{I}$, we get $\mathfrak{I}(\epsilon) = \mathfrak{I}(\omega)$. Thus, $\epsilon \in \mathfrak{I}(\epsilon) \cap \mathfrak{X}$, since $\mathfrak{I}(\epsilon) = \mathfrak{I}(\omega)$, we have $\epsilon \in \mathfrak{I}(\omega) \cap \mathfrak{X}$. So, $\omega \in \mathfrak{X}$. Therefore, $\bar{\mathfrak{X}} = \bigcap\{\mathfrak{I}(\mathfrak{X}) \mid \mathfrak{I} \in \mathcal{I}(\mathfrak{L})\}$. □

Lemma 3.7. *Let $\mathfrak{L}$ be a CKL-algebra. Then for any $\omega, \sigma, \lambda, \zeta \in \mathfrak{L}$, we have*
(i) $((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \sigma = \omega \twoheadrightarrow \sigma$,
(ii) $(\lambda \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma) \leqslant (\lambda \twoheadrightarrow \zeta) \twoheadrightarrow [(\zeta \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)]$.

Proof. (i) Suppose $\omega, \sigma \in \mathfrak{L}$, then by Proposition 2.5(iii), we have $\omega \twoheadrightarrow \sigma \leq ((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \sigma$. Also, by Proposition 2.5(iii), $\omega \leqslant (\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma$ and by Proposition 2.5(v), we have $((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \sigma \leqslant \omega \twoheadrightarrow \sigma$. Hence, $((\omega \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \sigma = \omega \twoheadrightarrow \sigma$.
(ii) Assume $\omega, \sigma, \lambda, \zeta \in \mathfrak{L}$. Then

$$
\begin{aligned}
&[(\lambda \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)] \twoheadrightarrow [(\lambda \twoheadrightarrow \zeta) \twoheadrightarrow [(\zeta \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)]]\\
&= (\lambda \twoheadrightarrow \zeta) \twoheadrightarrow [((\lambda \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)) \twoheadrightarrow ((\zeta \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma))] && \text{by (C)}\\
&= (\lambda \twoheadrightarrow \zeta) \twoheadrightarrow [(\zeta \twoheadrightarrow \sigma) \twoheadrightarrow (((\lambda \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)) \twoheadrightarrow (\omega \twoheadrightarrow \sigma))] && \text{by (C)}\\
&= (\lambda \twoheadrightarrow \zeta) \twoheadrightarrow [(\zeta \twoheadrightarrow \sigma) \twoheadrightarrow ((\omega \twoheadrightarrow ((\lambda \twoheadrightarrow \sigma) \twoheadrightarrow \sigma)) \twoheadrightarrow (\omega \twoheadrightarrow \sigma))] && \text{by (C)}\\
&= (\lambda \twoheadrightarrow \zeta) \twoheadrightarrow [(\zeta \twoheadrightarrow \sigma) \twoheadrightarrow ((((\lambda \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow (((\lambda \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \sigma))]\\
& && \text{by } (L_2)\\
&= (\lambda \twoheadrightarrow \zeta) \twoheadrightarrow [(\zeta \twoheadrightarrow \sigma) \twoheadrightarrow ((((\lambda \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow (\lambda \twoheadrightarrow \sigma))] && \text{by (i)}\\
&= (\zeta \twoheadrightarrow \sigma) \twoheadrightarrow [(((\lambda \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow ((\lambda \twoheadrightarrow \zeta) \twoheadrightarrow (\lambda \twoheadrightarrow \sigma))] && \text{by (C)}\\
&= (\zeta \twoheadrightarrow \sigma) \twoheadrightarrow [(((\lambda \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow ((\zeta \twoheadrightarrow \lambda) \twoheadrightarrow (\zeta \twoheadrightarrow \sigma))] && \text{by } (L_2)\\
&= (((\lambda \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow [(\zeta \twoheadrightarrow \lambda) \twoheadrightarrow ((\zeta \twoheadrightarrow \sigma) \twoheadrightarrow (\zeta \twoheadrightarrow \sigma))] && \text{by (C)}\\
&= (((\lambda \twoheadrightarrow \sigma) \twoheadrightarrow \sigma) \twoheadrightarrow \omega) \twoheadrightarrow [(\zeta \twoheadrightarrow \lambda) \twoheadrightarrow 1] \qquad \text{by } (L_1)\\
&= 1. \qquad \text{by } (L_1)
\end{aligned}
$$

Therefore, $(\lambda \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma) \leqslant (\lambda \twoheadrightarrow \zeta) \twoheadrightarrow [(\zeta \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)]$. □

In the following example, we show that the condition (C) in Lemma 3.7 is necessary.

Example 3.8. Let $(\mathfrak{L} = \{\eta, \epsilon, \rho, 1\}, \leqslant)$ be a poset, where $\eta, \epsilon, \rho \leqslant 1$. Define the operation $\twoheadrightarrow$ on $\mathfrak{L}$ as follows:

$\twoheadrightarrow$	η	ϵ	ρ	1
η	1	ϵ	ϵ	1
ϵ	ρ	1	ρ	1
ρ	ρ	ρ	1	1
1	η	ϵ	ρ	1

Then $(\mathfrak{L}, \twoheadrightarrow, 1)$ is an L-algebra which is not a CKL-algebra, since

$$\eta \twoheadrightarrow (\epsilon \twoheadrightarrow \rho) = \eta \twoheadrightarrow \rho = \epsilon \neq 1 = \epsilon \twoheadrightarrow \epsilon = \epsilon \twoheadrightarrow (\eta \twoheadrightarrow \rho).$$

Moreover, we can see that Proposition 3.7(i) does not hold, because

$$((\eta \twoheadrightarrow \rho \twoheadrightarrow \rho) \twoheadrightarrow \rho = (\epsilon \twoheadrightarrow \rho) \twoheadrightarrow \rho = \rho \twoheadrightarrow \rho = 1 \neq \epsilon = \eta \twoheadrightarrow \rho,$$

and so Proposition 3.7(ii) does not hold, too.

Theorem 3.9. *Suppose Γ is a family of nonempty subsets of $\mathfrak{L}$, where $\mathfrak{L}$ is a CKL-algebra and Γ is closed under intersection. Let for every $\omega, \sigma \in \mathfrak{L}$ and $\mathfrak{V} \in \Gamma$, where*
(i) *$\omega \in \mathfrak{V}$ and $\omega \leqslant \sigma$ imply $\sigma \in \mathfrak{V}$,*
(ii) *if $\omega \in \mathfrak{V}$, then $\mathfrak{U} \in \Gamma$ exists such that $\mathfrak{U}(\omega) \subseteq \mathfrak{V}$,*
(iii) *$\mathfrak{W} \in \Gamma$ exists where $\mathfrak{W}(\omega) \subseteq \mathfrak{V}$, for any $\omega \in \mathfrak{W}$ or equivalently, $\mathfrak{W}(\mathfrak{W}) \subseteq \mathfrak{V}$.*
Then there is a nontrivial topology τ on $\mathfrak{L}$ such that $(\mathfrak{L}, \tau)$ is a topological L-algebra.

Proof. Clearly, $\mathfrak{I}(\mathfrak{L}) \subseteq \Gamma$. Let

$$\tau = \{\mathfrak{N} \subseteq \mathfrak{L} \mid \ \forall \eta \in \mathfrak{N}, \ \exists\, \mathfrak{V} \in \Gamma \text{ s.t. } \mathfrak{V}(\eta) \subseteq \mathfrak{N}\}.$$

Assume $\{\mathfrak{N}_i \mid i \in I\} \subseteq \tau$. Then, for every $\eta \in \bigcup_{i \in I} \mathfrak{N}_i$, there exist $i \in I$ and $\mathfrak{V} \in \Gamma$ such that $\mathfrak{V}(\eta) \subseteq \mathfrak{N}_i \subseteq \bigcup_{i \in I} \mathfrak{N}_i$. Hence, τ is closed under union. For any $\eta \in \bigcap_{i \in I} \mathfrak{N}_i$ and any $i \in I$, there exists $\mathfrak{V}_i \in \Gamma$ such that $\mathfrak{V}_i(\eta) \subseteq \mathfrak{N}_i$. Put $\mathfrak{V} = \bigcap_{i \in I} \mathfrak{V}_i$, then $\mathfrak{V}(\eta) \subseteq \bigcap_{i \in I} \mathfrak{V}_i(\eta) \subseteq \bigcap_{i \in I} \mathfrak{N}_i$, and so τ is closed under intersection. Hence, τ is a topology on $\mathfrak{L}$. Now, we prove that for each $\mathfrak{V} \in \Gamma$ and $\eta \in \mathfrak{L}$, $\mathfrak{V}(\eta)$ is an open set. Consider $\eta \in \mathfrak{L}$, $\mathfrak{V} \in \Gamma$ and $\omega \in \mathfrak{V}(\eta)$. Then, $\omega \twoheadrightarrow \eta$, $\eta \twoheadrightarrow \omega \in \mathfrak{V}$. By (ii), there exist $\mathfrak{U}_1$ and $\mathfrak{U}_2 \in \Gamma$ such that $\mathfrak{U}_1(\eta \twoheadrightarrow \omega) \subseteq \mathfrak{V}$ and $\mathfrak{U}_2(\omega \twoheadrightarrow \eta) \subseteq \mathfrak{V}$. Put $\mathfrak{W} = \mathfrak{U}_1 \cap \mathfrak{U}_2 \in \Gamma$. If $\sigma \in \mathfrak{W}(\omega)$, then $\omega \twoheadrightarrow \sigma$ and $\sigma \twoheadrightarrow \omega \in \mathfrak{W}$. By Proposition 2.5(vii),

$$\omega \twoheadrightarrow \sigma \leqslant (\sigma \twoheadrightarrow \eta) \twoheadrightarrow (\omega \twoheadrightarrow \eta), \quad \text{and} \quad \sigma \twoheadrightarrow \omega \leqslant (\omega \twoheadrightarrow \eta) \twoheadrightarrow (\sigma \twoheadrightarrow \eta).$$

By (i), $(\sigma \twoheadrightarrow \eta) \twoheadrightarrow (\omega \twoheadrightarrow \eta) \in \mathfrak{W}$, and $(\omega \twoheadrightarrow \eta) \twoheadrightarrow (\sigma \twoheadrightarrow \eta) \in \mathfrak{W}$. Thus,

$$\sigma \twoheadrightarrow \eta \in \mathfrak{W}(\omega \twoheadrightarrow \eta) \subseteq (\mathfrak{U}_1 \cap \mathfrak{U}_2)(\omega \twoheadrightarrow \eta) \subseteq \mathfrak{U}_2(\omega \twoheadrightarrow \eta) \subseteq \mathfrak{V}.$$

Similarly, $\eta \twoheadrightarrow \sigma \in \mathfrak{V}$. Then $\sigma \in \mathfrak{V}(\eta)$, and so $\mathfrak{W}(\omega) \subseteq \mathfrak{V}(\eta)$. Hence, $\mathfrak{V}(\eta)$ is an open set and τ is a nontrivial topology. Clearly, the set $\mathcal{B} = \{\mathfrak{V}(\eta) \mid \mathfrak{V} \in \Gamma,\ \eta \in \mathfrak{L}\}$ is a base for τ. Now, we prove that the operation $\twoheadrightarrow$ is continuous. For this, assume $\omega \twoheadrightarrow \sigma \in \mathfrak{V}(\omega \twoheadrightarrow \sigma)$. By (iii), there is $\mathfrak{W} \in \Gamma$ such that $\mathfrak{W}(\mathfrak{W}) \subseteq \mathfrak{V}$. Let $\lambda \in \mathfrak{W}(\omega)$ and $\zeta \in \mathfrak{W}(\sigma)$. Then $\lambda \twoheadrightarrow \omega$, $\omega \twoheadrightarrow \lambda$, $\zeta \twoheadrightarrow \sigma$ and $\sigma \twoheadrightarrow \zeta \in \mathfrak{W}$. Thus, by Lemma 3.7 and Proposition 2.5(vii), we have,

$$(\zeta \twoheadrightarrow \sigma) \twoheadrightarrow ((\lambda \twoheadrightarrow \zeta) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)) \geq (\lambda \twoheadrightarrow \sigma) \twoheadrightarrow (\omega \twoheadrightarrow \sigma) \geq \omega \twoheadrightarrow \lambda.$$

Since $\mathfrak{W} \in \Gamma$ and $\omega \twoheadrightarrow \lambda \in \mathfrak{W}$, by (i), $(\zeta \twoheadrightarrow \sigma) \twoheadrightarrow ((\lambda \twoheadrightarrow \zeta) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)) \in \mathfrak{W}$. Also, by Proposition 2.5(ii),

$$\zeta \twoheadrightarrow \sigma \leqslant ((\lambda \twoheadrightarrow \zeta) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)) \twoheadrightarrow (\zeta \twoheadrightarrow \sigma).$$

Again, since $\mathfrak{W} \in \Gamma$ and $\zeta \twoheadrightarrow \sigma \in \mathfrak{W}$, by (i),

$$((\lambda \twoheadrightarrow \zeta) \twoheadrightarrow (\omega \twoheadrightarrow \sigma)) \twoheadrightarrow (\zeta \twoheadrightarrow \sigma) \in \mathfrak{W}.$$

Thus, $(\lambda \twoheadrightarrow \zeta) \twoheadrightarrow (\omega \twoheadrightarrow \sigma) \in \mathfrak{W}(\zeta \twoheadrightarrow \sigma) \subseteq \mathfrak{W}(\mathfrak{W}) \subseteq \mathfrak{V}$. This implies $(\lambda \twoheadrightarrow \zeta) \twoheadrightarrow (\omega \twoheadrightarrow \sigma) \in \mathfrak{V}$. Similarly, $(\omega \twoheadrightarrow \sigma) \twoheadrightarrow (\lambda \twoheadrightarrow \zeta) \in \mathfrak{V}$. Hence, $\lambda \twoheadrightarrow \zeta \in \mathfrak{V}(\omega \twoheadrightarrow \sigma)$. Therefore, $\mathfrak{W}(\omega) \twoheadrightarrow \mathfrak{W}(\sigma) \subseteq \mathfrak{V}(\omega \twoheadrightarrow \sigma)$, which implies that the operation $\twoheadrightarrow$ is continuous. $\square$

Theorem 3.10. *Consider τ is a topology on $\mathfrak{L}$ and $f : \mathfrak{L}^3 \to \mathfrak{L}^2$ is defined by $f(\eta, \epsilon, \rho) = (\eta \twoheadrightarrow \epsilon, \epsilon \twoheadrightarrow \rho)$, for all $\eta, \epsilon, \rho \in \mathfrak{L}$. If $\{1\}$ is an open set and f is continuous, then $(\mathfrak{L}, \tau)$ is a topological L-algebra.*

Proof. Let $\eta \in \mathfrak{L}$ and $f_\eta(\epsilon) = f(\eta, \epsilon, \eta) = (\eta \twoheadrightarrow \epsilon, \epsilon \twoheadrightarrow \eta)$. Since f is continuous, f_η is continuous. Now, since $\{1\}$ is open, $\{1\} \times \{1\}$ is open in $\mathfrak{L}^2$. In addition,

$$\begin{aligned} f_\eta^{-1}(1,1) &= \{\epsilon \in \mathfrak{L} \mid \ f_\eta(\epsilon) = (1,1)\} = \{\epsilon \in \mathfrak{L} \mid (\eta \twoheadrightarrow \epsilon, \epsilon \twoheadrightarrow \eta) = (1,1)\} \\ &= \{\epsilon \in \mathfrak{L} \mid \eta \twoheadrightarrow \epsilon = 1, \epsilon \twoheadrightarrow \eta = 1\} = \{\epsilon \in \mathfrak{L} \mid \epsilon = \eta\} \\ &= \{\eta\}. \end{aligned}$$

Hence, $\{\eta\}$ is an open set and τ is a discrete topology. Therefore, $(\mathfrak{L}, \tau)$ is a topological L-algebra. □

Theorem 3.11. *Consider $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a semi-topological L-algebra. If $\{1\}$ is an open set, then $(\mathfrak{L}, \tau)$ is a topological L-algebra.*

Proof. Suppose $\{1\}$ is an open set and $\omega \in \mathfrak{L}$. Since $(\mathfrak{L}, \tau)$ is a semi-topological L-algebra and $\omega \twoheadrightarrow \omega = 1 \in \{1\}$, there is an open sets $\mathfrak{U}$ such that $\omega \in \mathfrak{U}$, $\omega \twoheadrightarrow \mathfrak{U} = 1$ and $\mathfrak{U} \twoheadrightarrow \omega = \{1\}$, which implies $\mathfrak{U} = \{\omega\}$. Because, for any $\sigma \in \mathfrak{U}$, $\omega \twoheadrightarrow \sigma = 1$ and $\sigma \twoheadrightarrow \omega = 1$, since $\mathfrak{L}$ is an L-algebra, $\omega = \sigma$. Hence, τ is a discrete topology on $\mathfrak{L}$, and so $(\mathfrak{L}, \tau)$ is a topological L-algebra. □

Lemma 3.12. *Every ideal of $\mathfrak{L}$ is upset.*

Proof. Assume $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$, $\omega \in \mathfrak{I}$ and $\sigma \in \mathfrak{L}$ such that $\omega \leqslant \sigma$. By (I_1), $\omega \twoheadrightarrow \sigma = 1 \in \mathfrak{I}$. Since $\omega \in \mathfrak{I}$, by (I_2) we get $\sigma \in \mathfrak{I}$. Hence, $\mathfrak{I}$ is upset. □

Proposition 3.13. *Assume $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a topological L-algebra and $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$. Then:*
(i) *if 1 is an interior point of $\mathfrak{I}$, then $\mathfrak{I}$ is an open set,*
(ii) *if $\mathfrak{I}$ is an open set, then $\mathfrak{I}$ is closed,*
(iii) *if $\mathfrak{L}$ is connected, then $\mathfrak{L}$ has no open proper ideal.*

Proof. Consider $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a topological L-algebra and $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$.
(i) Suppose $\omega \in \mathfrak{I}$. Since 1 is an interior point of $\mathfrak{I}$, there exists $\mathfrak{U} \in \tau$ such that $\omega \twoheadrightarrow \omega = 1 \in \mathfrak{U} \subseteq \mathfrak{I}$. Since the operation $\twoheadrightarrow$ is continuous, there exists $\mathfrak{V} \in \tau$ such that $\omega \in \mathfrak{V}$ and $\mathfrak{V} \twoheadrightarrow \mathfrak{V} \subseteq \mathfrak{I}$. Now, for all $\sigma \in \mathfrak{V}$, we have $\omega \twoheadrightarrow \sigma \in \mathfrak{V} \twoheadrightarrow \mathfrak{V} \subseteq \mathfrak{I}$, and so $\omega \twoheadrightarrow \sigma \in \mathfrak{I}$. Since $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$ and $\omega \in \mathfrak{I}$, by Lemma 3.12, $\sigma \in \mathfrak{I}$. Thus, $\sigma \in \mathfrak{V} \subseteq \mathfrak{I}$, which implies that $\mathfrak{I}$ is an open set.
(ii) Let $\mathfrak{I}$ be an open set. We prove that $\mathfrak{I}$ is closed. For this, we show $\mathfrak{I}^c$

is an open set. Consider $\omega \in \mathfrak{I}^c$. Then $\omega \notin \mathfrak{I}$. Since $\omega \twoheadrightarrow \omega = 1 \in \mathfrak{I} \in \tau$ and the operation $\twoheadrightarrow$ is continuous, there exists $\mathfrak{U} \in \tau$ such that $\omega \in \mathfrak{U}$ and $\mathfrak{U} \twoheadrightarrow \mathfrak{U} \subseteq \mathfrak{I}$. Now, we prove $\mathfrak{U} \subseteq \mathfrak{I}^c$. For this, assume $\mathfrak{U} \cap \mathfrak{I} \neq \emptyset$. Then there is $\sigma \in \mathfrak{U} \cap \mathfrak{I}$ such that $\sigma \twoheadrightarrow \mathfrak{U} \subseteq \mathfrak{I}$. So, for all $\xi \in \mathfrak{U}$, $\sigma \twoheadrightarrow \xi \in \mathfrak{I}$. Since $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$, by (I_2), $\xi \in \mathfrak{I}$, and so $\mathfrak{U} \subseteq \mathfrak{I}$. Thus, $\omega \in \mathfrak{I}$, which is a contradiction. Then $\mathfrak{U} \cap \mathfrak{I} = \emptyset$. Hence, $\omega \in \mathfrak{U} \subseteq \mathfrak{I}^c$ shows that $\mathfrak{I}^c$ is an open set, and so $\mathfrak{I}$ is closed.
(iii) Suppose $\mathfrak{I}$ is an open ideal of $\mathfrak{L}$. Then by (ii), $\mathfrak{I}$ is closed. Since $\mathfrak{L}$ is connected, we have $\mathfrak{L} = \mathfrak{I}$. □

A topological space $\mathfrak{L}$ is called *totally disconnected*, if every connected subset $\mathfrak{X} \subseteq \mathfrak{L}$ is either empty or a singleton. A subset $\mathfrak{X}$ of $\mathfrak{L}$ is called a *component subspace*, if it is the maximal connected subspace (see [14, 15]).

Proposition 3.14. *Consider $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a semi-topological L-algebra. Then $\mathfrak{L}$ is totally disconnected iff every its connected subset containing 1 consists just 1.*

Proof. Suppose $\mathfrak{L}$ is totally disconnected and $\mathfrak{X} \subseteq \mathfrak{L}$ is a connected of 1. So, $\mathfrak{X} = \{1\}$. Conversely, assume $\mathfrak{P}$ is a connected subset of $\mathfrak{L}$ and $\omega \in \mathfrak{P}$. Then $1 \in (\mathfrak{P} \twoheadrightarrow \omega) \cap (\omega \twoheadrightarrow \mathfrak{P})$. Since $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a semi-topological L-algebra and $\mathfrak{P}$ is connected, obviously, $\omega \twoheadrightarrow \mathfrak{P}$ and $\mathfrak{P} \twoheadrightarrow \omega$ are connected. By assumption, $\mathfrak{P} \twoheadrightarrow \omega = \{1\}$ and $\omega \twoheadrightarrow \mathfrak{P} = \{1\}$ and so $\mathfrak{P} = \{\omega\}$. Therefore, $\mathfrak{L}$ is totally disconnected. □

For an L-algebra, a binary relation $\sim$ is a congruence relation on $\mathfrak{L}$ if it is an equivalence relation such that for any $\omega, \sigma, \xi \in \mathfrak{L}$,

$$\omega \sim \sigma \Leftrightarrow (\xi \twoheadrightarrow \omega) \sim (\xi \twoheadrightarrow \sigma) \text{ and } (\omega \twoheadrightarrow \xi) \sim (\sigma \twoheadrightarrow \xi).$$

Theorem 3.15. [17] *Let $(\mathfrak{L}, \twoheadrightarrow, 1)$ be an L-algebra. Then each $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$ of $\mathfrak{L}$ defines a congruence relation on $\mathfrak{L}$, for any $\omega, \sigma \in \mathfrak{L}$, where*

$$\omega \sim \sigma \;\Leftrightarrow\; \omega \twoheadrightarrow \sigma,\ \sigma \twoheadrightarrow \omega \in \mathfrak{I}.$$

Conversely, every congruence relation $\sim$ defines an ideal $\mathfrak{I} = \{\omega \in \mathfrak{L} \mid \omega \sim 1\}$.

Since $\sim$ is a congruence relation on $\mathfrak{L}$, assume $\mathfrak{L}/\mathfrak{I} = \{[\omega] \mid \omega \in \mathfrak{L}\}$, where $[\omega] = \{\sigma \in \mathfrak{L} \mid \omega \sim \sigma\}$. Then the binary relation $\leqslant$ on $\mathfrak{L}/\mathfrak{I}$ defined by:

$$[\omega] \leqslant [\sigma] \text{ iff } \omega \twoheadrightarrow \sigma \in \mathfrak{I},$$

is a partial order on $\mathfrak{L}/\mathfrak{I}$. Thus $(\mathfrak{L}/\mathfrak{I}, \twoheadrightarrow, [1])$ is an L-algebra, where for any $\omega, \sigma \in \mathfrak{L}$, $[1] = \mathfrak{I}$ and $[\omega] \twoheadrightarrow [\sigma] = [\omega \twoheadrightarrow \sigma]$.

Suppose $\mathfrak{L}$ is an L-algebra and $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$. Clearly, $\mathfrak{L}/\mathfrak{I}$ is a quotient L-algebra and $\pi_{\mathfrak{I}} : \mathfrak{L} \to \mathfrak{L}/\mathfrak{I}$ is a canonical epimorphism. Consider τ is a topology on $\mathfrak{L}$ and $\mathfrak{U}$ is a subset of $\mathfrak{L}/\mathfrak{I}$. Then we say $\mathfrak{U}$ is an open subset of $\mathfrak{L}/\mathfrak{I}$ iff $\pi_{\mathfrak{I}}^{-1}(\mathfrak{U})$ is an open subset of $\mathfrak{L}$. Now, if we consider $\bar{\tau} = \{\mathfrak{U} \subseteq \mathfrak{L}/\mathfrak{I} \mid \pi_{\mathfrak{I}}^{-1}(\mathfrak{U}) \in \tau\}$, then obviously, $\bar{\tau}$ is a topology on $\mathfrak{L}/\mathfrak{I}$. This topology on $\mathfrak{L}/\mathfrak{I}$ is called *the quotient topology induced by* $\pi_{\mathfrak{I}}$. Obviously, it is the largest topology on $\mathfrak{L}/\mathfrak{I}$ making $\pi_{\mathfrak{I}}$ continuous.

Theorem 3.16. *Let $\mathfrak{L}$ be an L-algebra and $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$. If $(\mathfrak{L}, \tau)$ is a (semi)topological L-algebra and $\pi_{\mathfrak{I}}$ is an open map, then $(\mathfrak{L}/\mathfrak{I}, \bar{\tau})$ is a (semi)topological L-algebra.*

Proof. Assume $(\mathfrak{L}, \tau)$ is a topological L-algebra, and $[\omega] \twoheadrightarrow [\sigma] \in \mathfrak{V} \in \bar{\tau}$, for $[\omega], [\sigma] \in \mathfrak{L}/\mathfrak{I}$. Then $[\omega \twoheadrightarrow \sigma] \in \mathfrak{V}$. Since $\pi_{\mathfrak{I}}$ is continuous, $\omega \twoheadrightarrow \sigma \in \pi_{\mathfrak{I}}^{-1}(\mathfrak{V}) \in \tau$. Since $(\mathfrak{L}, \tau)$ is a topological L-algebra, there exist $\mathfrak{U}, \mathfrak{W} \in \tau$ such that $\omega \in \mathfrak{U}$, $\sigma \in \mathfrak{W}$ and $\omega \twoheadrightarrow \sigma \in \mathfrak{U} \twoheadrightarrow \mathfrak{W} \subseteq \pi_{\mathfrak{I}}^{-1}(\mathfrak{V})$. Since $\pi_{\mathfrak{I}}$ is an open map, $\pi_{\mathfrak{I}}(\mathfrak{U})$ and $\pi_{\mathfrak{I}}(\mathfrak{W})$ are in $\bar{\tau}$, $[\omega] \in \pi_{\mathfrak{I}}(\mathfrak{U})$, $[\sigma] \in \pi_{\mathfrak{I}}(\mathfrak{W})$ and $[\omega] \twoheadrightarrow [\sigma] \in \pi_{\mathfrak{I}}(\mathfrak{U}) \twoheadrightarrow \pi_{\mathfrak{I}}(\mathfrak{W}) \subseteq \mathfrak{V}$. Therefore, $(\mathfrak{L}/\mathfrak{I}, \bar{\tau})$ is a (semi)topological L-algebra. □

Proposition 3.17. *Suppose $(\mathfrak{L}, \tau)$ is a topological L-algebra and $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$. Then:*

(i) $\dfrac{\mathfrak{L}}{\mathfrak{I}}$ *has a discrete topology iff $\mathfrak{I}$ is open,*

(ii) *if $(\mathfrak{L}, \tau)$ is a compact topological L-algebra, then $\mathfrak{L}/\mathfrak{I}$ is a discrete finite topological L-algebra iff $\mathfrak{I}$ is open.*

Proof. (i) Since $\mathfrak{L}/\mathfrak{I}$ has a discrete topology, every single set such as $\{[\omega]\}$ is open, for any $\omega \in \mathfrak{L}$. Since $1 \in \mathfrak{L}$, $\{[1]\}$ is open. Since $\{[1]\} = \mathfrak{I}$, $\mathfrak{I}$ is open.

Conversely, if $\mathfrak{I}$ is an open set, then $\{[1]\}$ is an open set, too. Since $\mathfrak{L}/\mathfrak{I}$ is an L-algebra, by Theorem 3.6, $\mathfrak{L}/\mathfrak{I}$ has a discrete topology.

(ii) Suppose $\mathfrak{L}$ is compact. Since π is a continuous epimorphism, $\pi(\mathfrak{L}) = \mathfrak{L}/\mathfrak{I}$ is compact. Consider $\mathfrak{I}$ is open. Then by (i), $\mathfrak{L}/\mathfrak{I}$ has a discrete topology and so every singleton subset is open. In addition, since $\mathfrak{L}/\mathfrak{I}$ is compact, $\mathfrak{L}/\mathfrak{I}$ is equal to union of finite open subsets. Thus $\mathfrak{L}/\mathfrak{I}$ is finite. The converse, by (i) is clear. $\square$

Definition 3.18. [14, 15] Assume $(\mathfrak{X}, \tau)$ is a topological space and $\omega \in \mathfrak{X}$. A *local basis at* ω is a set $\mathfrak{N}$ of open neighborhoods of ω such that for all $\mathfrak{U} \in \tau$ if $\omega \in \mathfrak{U}$, then there exists $\mathfrak{V} \in \mathfrak{N}$ such that $\omega \in \mathfrak{V} \subseteq \mathfrak{U}$.

Lemma 3.19. *Consider $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$. If τ is a topology on $\mathfrak{L}$ and $\bar{\tau}$ is the quotient topology on $\mathfrak{L}/\mathfrak{I}$, then for each $\omega \in \mathfrak{L}$, $\pi_{\mathfrak{I}}^{-1}(\pi_{\mathfrak{I}}(\omega)) = \omega$. In addition, if $\mathfrak{V} \in \bar{\tau}$, then there exists $\mathfrak{U} \in \tau$ such that $\pi_{\mathfrak{I}}(\mathfrak{U}) = \mathfrak{V}$.*

Theorem 3.20. *Suppose $(\mathfrak{L}, \tau)$ is a semi-topological L-algebra and $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$. Then*

$$\mathcal{B} = \{\pi(\mathfrak{U} \twoheadrightarrow \omega) \mid \mathfrak{U} \in \tau, 1 \in \mathfrak{U}, \omega \in \mathfrak{L}\},$$

is a local base of the space $\mathfrak{L}/\mathfrak{I}$ at the point $[\omega] \in \mathfrak{L}/\mathfrak{I}$, and the map $\pi : \mathfrak{L} \to \mathfrak{L}/\mathfrak{I}$ is open.

Proof. Assume $\mathfrak{U} \in \tau$. Since $1 \in \mathfrak{U}$, clearly $\omega \in \mathfrak{U} \twoheadrightarrow \omega$, for all $\omega \in \mathfrak{L}$. Thus, $[\omega] \in \pi(\mathfrak{U} \twoheadrightarrow \omega)$. Now, let $[\omega] \in \mathfrak{L}/\mathfrak{I}$. Then there exists $\mathfrak{W} \in \bar{\tau}$ such that $[\omega] \in \mathfrak{W}$. Since $\mathfrak{W}$ is open and π is continuous, we have $\omega \in \pi^{-1}(\mathfrak{W}) = \mathfrak{N}$. On the other hand, by (L_1), $\omega = 1 \twoheadrightarrow \omega \in \mathfrak{N}$. Since $\twoheadrightarrow$ is continuous, there exists $\mathfrak{U} \in \tau$ such that $1 \in \mathfrak{U}$ and $\omega \in \mathfrak{U} \twoheadrightarrow \omega \subseteq \mathfrak{N}$. Thus, $[\omega] \in \pi(\mathfrak{U} \twoheadrightarrow \omega) \subseteq \pi(\mathfrak{N}) = \pi(\pi^{-1}(\mathfrak{W})) = \mathfrak{W}$, and so $\pi^{-1}(\pi(\mathfrak{U} \twoheadrightarrow \omega)) \subseteq \mathfrak{N}$. By Lemma 3.19, $\pi^{-1}(\pi(\mathfrak{U} \twoheadrightarrow \omega)) = (\mathfrak{U} \twoheadrightarrow \omega) \subseteq \mathfrak{N}$. Thus, $\pi(\mathfrak{U} \twoheadrightarrow \omega) \subseteq \mathfrak{W}$. Hence, $\mathcal{B}$ is a local basis. By definition of quotient topology, and Lemma 3.19, $\pi(\mathfrak{U} \twoheadrightarrow \omega) = [(\mathfrak{U} \twoheadrightarrow \omega)] = \bigcup_{\sigma \in \mathfrak{U} \twoheadrightarrow \omega} [\sigma]$. Since $\bigcup_{\sigma \in \mathfrak{U} \twoheadrightarrow \omega} [\sigma]$ is open in $\mathfrak{L}$ and π is continuous, we get $\pi(\mathfrak{U} \twoheadrightarrow \omega)$ is open in τ. Therefore, π is open. $\square$

Theorem 3.21. *Assume $\mathfrak{L}$ is an L-algebra and $\mathcal{I}$ is a family of ideals which is closed under intersections. Then there exists a topology τ on $\mathfrak{L}$ such that $(\mathfrak{L}, \tau)$ is a topological L-algebra.*

Proof. Define

$$\tau = \{\mathfrak{U} \subseteq \mathfrak{L} \mid \forall\ \omega \in \mathfrak{U},\ \exists \mathfrak{I} \in \mathcal{I}(\mathfrak{L}) \text{ such that } [\omega] \subseteq \mathfrak{U}\}.$$

For any $\omega \in \mathfrak{L}$ and $\mathfrak{I} \in \mathcal{I}$, clearly, $[\omega] \in \tau$, because if $\sigma \in [\omega]$, then from $\sigma \in [\sigma]$ we get that $[\sigma] = [\omega]$. Obviously, τ is a topology on $\mathfrak{L}$. We prove that the operation $\twoheadrightarrow$ is continuous. For this, suppose $\omega \twoheadrightarrow \sigma \in \mathfrak{U} \in \tau$. Then for some $\mathfrak{I} \in \mathcal{I}$, $[\omega \twoheadrightarrow \sigma] \subseteq \mathfrak{U}$, and so $[\omega] \twoheadrightarrow [\sigma] \subseteq \mathfrak{U}$. Since $[\omega]$ and $[\sigma]$ are two open neighborhoods of ω and σ, respectively, such that $[\omega] \twoheadrightarrow [\sigma] \subseteq [\omega \twoheadrightarrow \sigma] \subseteq \mathfrak{U}$, we get $\twoheadrightarrow$ is continuous. Therefore, $(\mathfrak{L}, \tau)$ is a topological L-algebra. □

Theorem 3.22. *Consider $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a topological L-algebra such that, for any $\emptyset \neq \mathfrak{U} \in \tau$, $1 \in \mathfrak{U}$ and $\eta \notin \mathfrak{L}$. Suppose $\mathfrak{L}_\eta = \mathfrak{L} \cup \{\eta\}$. Then there exists a topology τ_η on $\mathfrak{L}_\eta$ such that $(\mathfrak{L}_\eta, \tau_\eta)$ is a topological L-algebra.*

Proof. Define the operation $\rightsquigarrow$ on $\mathfrak{L}_\eta$ as follows:

$$\omega \rightsquigarrow \sigma = \begin{cases} \omega \twoheadrightarrow \sigma & \text{if } \omega, \sigma \in \mathfrak{L} \\ \eta & \text{if } \omega \in \mathfrak{L},\ \sigma = \eta \\ 1 & \text{if } \omega = \eta,\ \sigma \in \mathfrak{L}_\eta \end{cases}$$

At first, we prove that $(\mathfrak{L}_\eta, \rightsquigarrow, 1)$ is an L-algebra. For this, let $\omega, \sigma, \xi \in \mathfrak{L}_\eta$. We have the following cases:
(L_1):
Case 1. If $\omega \in \mathfrak{L}$, then $\omega \rightsquigarrow \omega = \omega \rightsquigarrow 1 = 1$, and $1 \rightsquigarrow \omega = \omega$.
Case 2. If $\omega = \eta$, then for any $\sigma \in \mathfrak{L}_\eta$, $\eta \rightsquigarrow \sigma = 1$ and so if $\sigma = \eta$ or $\sigma = 1$, then $\eta \rightsquigarrow \eta = \eta \rightsquigarrow 1 = 1$. Now, if $\omega = 1$ and $\sigma = \eta$, then $1 \rightsquigarrow \eta = \eta$. Hence, (L_1) holds.
(L_2): Let $\omega, \sigma, \xi \in \mathfrak{L}_\eta$. We have the following cases:
Case 1. If $\omega, \sigma, \xi \in \mathfrak{L}$, then since $(\mathfrak{L}, \twoheadrightarrow, 1)$ is an L-algebra, clearly (L_2) holds.
Case 2. If $\omega = \eta$ and $\sigma, \xi \in \mathfrak{L}$, then

$$1 = 1 \rightsquigarrow 1 = (\eta \rightsquigarrow \sigma) \rightsquigarrow (\eta \rightsquigarrow \xi) = (\sigma \rightsquigarrow \eta) \rightsquigarrow (\sigma \rightsquigarrow \xi) = \eta \rightsquigarrow (\sigma \rightsquigarrow \xi) = 1.$$

Case 3. If $\sigma = \eta$ and $\omega, \xi \in \mathfrak{L}$, then similar to Case 2, (L_2) holds.
Case 4. If $\xi = \eta$ and $\omega, \sigma \in \mathfrak{L}$, then

$$\begin{aligned} \eta &= (\omega \rightsquigarrow \sigma) \rightsquigarrow \eta = (\omega \rightsquigarrow \sigma) \rightsquigarrow (\omega \rightsquigarrow \eta) \\ &= (\sigma \rightsquigarrow \omega) \rightsquigarrow (\sigma \rightsquigarrow \eta) = (\sigma \rightsquigarrow \omega) \rightsquigarrow \eta = \eta. \end{aligned}$$

Case 5. If $\omega = \sigma = \eta$ and $\xi \in \mathfrak{L}$, then

$$1 = \eta \rightsquigarrow \xi = 1 \rightsquigarrow (\eta \rightsquigarrow \xi) = (\eta \rightsquigarrow \eta) \rightsquigarrow (\eta \rightsquigarrow \xi) = (\eta \rightsquigarrow \eta) \rightsquigarrow (\eta \rightsquigarrow \xi) = 1.$$

Case 6. If $\omega = \xi = \eta$ and $\sigma \in \mathfrak{L}$, then

$$1 = (\eta \rightsquigarrow \sigma) \rightsquigarrow (\eta \rightsquigarrow \eta) = (\sigma \rightsquigarrow \eta) \rightsquigarrow (\sigma \rightsquigarrow \eta) = 1.$$

Case 7. If $\sigma = \xi = \eta$ and $\omega \in \mathfrak{L}$, then

$$1 = \eta \rightsquigarrow (\omega \rightsquigarrow \xi) = (\omega \rightsquigarrow \eta) \rightsquigarrow (\omega \rightsquigarrow \xi) = (\eta \rightsquigarrow \omega) \rightsquigarrow (\eta \rightsquigarrow \xi) = 1.$$

Case 8. If $\omega = \sigma = \xi = \eta$, then clearly (L_2) holds.
Hence, in all above cases, (L_2) holds.
(L_3): If $\omega \rightsquigarrow \sigma = \sigma \rightsquigarrow \omega = 1$, for $\omega, \sigma \in \mathfrak{L}_\eta$, then we have the following cases:
Case 1. If $\omega, \sigma \in \mathfrak{L}$, then since $(\mathfrak{L}, \twoheadrightarrow, 1)$ is an L-algebra, clearly (L_3) holds.
Case 2. Since $\omega \rightsquigarrow \sigma = 1$, we get $\omega = \eta$ and $\sigma \in \mathfrak{L}_\eta$. Also, from $\sigma \rightsquigarrow \omega = 1$, we get $\sigma = \eta$ and $\omega \in \mathfrak{L}_\eta$. Hence, $\omega = \sigma = \eta$, and so (L_3) holds.
Therefore, $(\mathfrak{L}_\eta, \rightsquigarrow, 1)$ is an L-algebra.

In addition, it is clear that,

$$\tau_\eta = \{\mathfrak{U} \cup \{\eta\} \mid \mathfrak{U} \in \tau\} \cup \{\emptyset\},$$

is a topology on $\mathfrak{L}_\eta$. Now, we show $(\mathfrak{L}_\eta, \tau_\eta)$ is a topological L-algebra. For this, we prove $\twoheadrightarrow$ is continuous. Let $\omega \rightsquigarrow \sigma \in \mathfrak{U} \cup \{\eta\}$. In the following cases, we find two sets $\mathfrak{V}, \mathfrak{W} \in \tau_\eta$ such that $\omega \in \mathfrak{V}$, $\sigma \in \mathfrak{W}$ and $\mathfrak{V} \rightsquigarrow \mathfrak{W} \subseteq \mathfrak{U} \cup \{\eta\}$.
Case 1. If $\omega, \sigma \in \mathfrak{L}$, then $\omega \rightsquigarrow \sigma = \omega \twoheadrightarrow \sigma \in \mathfrak{U}$. Since $\twoheadrightarrow$ is continuous, there exist $\mathfrak{V}, \mathfrak{W} \in \tau$ such that $\omega \in \mathfrak{V}$, $\sigma \in \mathfrak{W}$ and $\omega \twoheadrightarrow \sigma \in \mathfrak{V} \twoheadrightarrow \mathfrak{W} \subseteq \mathfrak{U}$. If $\xi_1 \in \mathfrak{V} \cup \{\eta\}$ and $\xi_2 \in \mathfrak{W} \cup \{\eta\}$, since, for any $\mathfrak{U} \in \tau$, $1 \in \mathfrak{U}$, then $\xi_1 \rightsquigarrow \xi_2 \in \{\xi_1 \twoheadrightarrow \xi_2, \eta, 1\} \subseteq \mathfrak{U} \cup \{\eta\}$. Hence, $\mathfrak{V} \cup \{\eta\} \rightsquigarrow \mathfrak{W} \cup \{\eta\} \subseteq \mathfrak{U} \cup \{\eta\}$.
Case 2. If $\omega = \eta$ and $\sigma \in \mathfrak{L}$, then $\omega = \eta \in \{\eta\} \in \tau_\eta$, $\sigma \in \mathfrak{L}_\eta \in \tau_\eta$ and $\{\eta\} \rightsquigarrow \mathfrak{L}_\eta = \{1\} \subseteq \mathfrak{U} \cup \{\eta\}$.
Case 3. If $\omega \in \mathfrak{L}$ and $\sigma = \eta$, then $\omega \in \mathfrak{L}_\eta \in \tau_\eta$, $\sigma = \eta \in \{\eta\} \in \tau_\eta$ and $\omega \twoheadrightarrow \sigma \in \mathfrak{L}_\eta \twoheadrightarrow \{\eta\} \subseteq \{\eta, 1\} \subseteq \mathfrak{U} \cup \{\eta\}$.
Case 4. If $\omega = \sigma = \eta$, then $\omega = \sigma = \eta \in \{\eta\} \in \tau_\eta$ and $\{\eta\} \rightsquigarrow \{\eta\} = \{1\} \in \mathfrak{U} \cup \{\eta\}$. Therefore, $(\mathfrak{L}_\eta, \tau_\eta)$ is a topological L-algebra. □

Theorem 3.23. *For any $n > 2$ there exists a topological L-algebra of order n.*

Proof. Suppose $\mathfrak{L}$ is an L-algebra of order $n > 1$. Clearly, $\tau = \{\mathfrak{L}, \emptyset\}$ is a topology on $\mathfrak{L}$, and so $(\mathfrak{L}, \tau)$ is a topological L-algebra. Now, suppose $\omega \notin \mathfrak{L}$. Define $\mathfrak{L}_\omega = \mathfrak{L} \cup \{\omega\}$. Then by Theorem 3.22, there exists the operation $\twoheadrightarrow$ and topology τ_ω on $\mathfrak{L}_\omega$ such that $(\mathfrak{L}_\omega, \tau_\omega)$ is a topological L-algebra. Since $\tau_\omega = \{\emptyset, \{\omega\}, \mathfrak{L}_\omega\}$, obviously, τ_ω is a non-trivial topology on $\mathfrak{L}_\omega$. $\square$

Theorem 3.24. *For any countable set $\mathfrak{L}$ such that $1 \in \mathfrak{L}$, there exists a topological L-algebra on $\mathfrak{L}$.*

Proof. Consider $\mathfrak{L} = \{\omega_0 = 1, \omega_1, \omega_2, \cdots\}$ as a countable subset and define the operation $\twoheadrightarrow$ on $\mathfrak{L}$ as follows:

$$\omega_i \twoheadrightarrow \omega_j = \begin{cases} 1 & \text{if } i \geq j \\ \omega_j & \text{if } i < j \end{cases} \qquad \text{and} \qquad \omega_i \leqslant \omega_j \text{ iff } \omega_i \twoheadrightarrow \omega_j = 1.$$

Similar to the proof of Theorem 3.22, we can see that $(\mathfrak{L}, \twoheadrightarrow, 1)$ is an L-algebra. The set $\mathfrak{I}_n = \{1, \omega_1, \cdots, \omega_n\} \in \mathcal{I}(\mathfrak{L})$, for any $n \geq 1$. Let $\mathcal{B} = \{\mathfrak{I}_n \mid n \geq 1\}$. By Theorem 3.21, there is a non-trivial topology τ on $\mathfrak{L}$ such that $(\mathfrak{L}, \tau)$ is a topological L-algebra. $\square$

Theorem 3.25. *Let $(\mathfrak{L}, \tau)$ be a topological L-algebra and α be a cardinal number. If $\mid \mathfrak{L} \mid \leq \alpha$, then there exists a topological L-algebra $(\mathcal{B}, \mathfrak{U})$ such that $\mid \mathcal{B} \mid \geq \alpha$, $1 \in \mathfrak{U} \in \mathcal{U}$ and $\mathfrak{L}$ is a sub-algebra of $\mathcal{B}$.*

Proof. Suppose

$\Gamma = \{(\mathfrak{H}, \dashrightarrow, 1, \mathcal{U}) \mid (\mathfrak{H}, \dashrightarrow, 1, \mathcal{U})$ is a topological L-algebra, $\mathfrak{L} \subseteq \mathfrak{H}$, and $\dashrightarrow|_{\mathfrak{L}} = \rightarrow\}$.

The following relation is a partial order on Γ:

$$(\mathfrak{H}, \dashrightarrow, 1, \mathcal{U}) \leqslant (\mathfrak{K}, \looparrowright, 1, \mathcal{V}) \Leftrightarrow \mathfrak{H} \subseteq \mathfrak{K}, \looparrowright|_{\mathfrak{H}} = \dashrightarrow,\ \mathcal{U} \subseteq \mathcal{V}.$$

Assume $\sum = \{(\mathfrak{H}_i, \dashrightarrow_i, 1, \mathcal{U}_i) \mid i \in I\}$ is a chain in Γ. Put $\mathfrak{H} = \bigcup_{i \in I} \mathfrak{H}_i$ and $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i$. If $\omega, \sigma \in \mathfrak{H}$, since $\sum$ is a chain, then for some $i \in I$, $\omega, \sigma \in \mathfrak{H}_i$. Define $\omega \dashrightarrow \sigma = \omega \dashrightarrow_i \sigma$. We prove $\dashrightarrow$ is an operation on $\mathfrak{H}$. Suppose $\omega, \sigma \in \mathfrak{H}_i \cup \mathfrak{H}_j$. Since $\sum$ is a chain, $\mathfrak{H}_i \subseteq \mathfrak{H}_j$ or $\mathfrak{H}_j \subseteq \mathfrak{H}_i$. Without the lost of generality, assume that $\mathfrak{H}_i \subseteq \mathfrak{H}_j$. Then $\dashrightarrow_j|_{\mathfrak{H}_i} = \dashrightarrow_i$. So, $\omega \dashrightarrow \sigma = \omega \dashrightarrow_i \sigma$. Thus, $\dashrightarrow$ is an operation on $\mathfrak{H}$. Now, it is easy to see that $(\mathfrak{H}, \dashrightarrow, 1)$ is an L-algebra, where $\dashrightarrow|_{\mathfrak{L}} = \twoheadrightarrow$.

On the other hand, since $\sum$ is a chain, $\mathcal{U}$ is a topology on $\mathfrak{H}$. We show

$(\mathfrak{H}, \dashrightarrow, 1, \mathcal{U})$ is a topological L-algebra. Assume $\omega \dashrightarrow \sigma \in \mathfrak{U} \in \mathcal{U}$. Then there exists an $i \in I$ such that $\omega \dashrightarrow \sigma = \omega \dashrightarrow_i \sigma \in \mathfrak{U} \in \mathcal{U}_i$. Since $\dashrightarrow_i$ is continuous in $(\mathfrak{H}_i, \mathcal{U}_i)$, there are $\mathfrak{V}, \mathfrak{W} \in \mathfrak{U}_i$ such that $\omega \in \mathfrak{V}$, $\sigma \in \mathfrak{W}$ and $\mathfrak{V} \dashrightarrow_i \mathfrak{W} \subseteq \mathfrak{U}$. This proves that the operation $\dashrightarrow$ is continuous in $(\mathfrak{H}, \mathcal{U})$. Thus, $(\mathfrak{H}, \dashrightarrow, 1, \mathcal{U})$ is an upper bound for $\sum$. By Zorn's Lemma, Γ has a maximal element. Suppose $(\mathcal{B}, \rightsquigarrow, 1, \mathcal{U})$ is a maximal element of Γ. We prove that $\mid \mathcal{B} \mid \geq \alpha$. If $\mid \mathcal{B} \mid < \alpha$, then for some non-empty set $\mathfrak{P}$, $\mid \mathcal{B} \cup \mathfrak{P} \mid = \alpha$. Take $\eta \in \mathfrak{P} - \mathcal{B}$ and put $\mathfrak{H} = \mathcal{B} \cup \{\eta\}$. Then by Theorem 3.22, $\mathfrak{H}$ with the following operations is an L-algebra:

$$\omega \curvearrowright \sigma = \begin{cases} \omega \rightsquigarrow \sigma & \text{if } \omega \in \mathcal{B},\ \sigma \in \mathcal{B} \\ \eta & \text{if } \omega \in \mathcal{B},\ \sigma = \eta \\ 1 & \text{if } \omega = \eta,\ \sigma \in \mathfrak{H} \end{cases}$$

The set $\mathcal{D} = \mathcal{U} \cup \{\{a\}\}$ is a sub-base for a topology $\mathcal{V}$ on $\mathfrak{H}$. Similar to the proof of Theorem 3.22, $(\mathfrak{H}, \mathcal{V})$ is a topological L-algebra. But $(\mathfrak{H}, \curvearrowright, 1, \mathcal{V})$ is a member of Γ that $(\mathcal{B}, \rightsquigarrow, 1, \mathcal{U}) < (\mathfrak{H}, \curvearrowright, 1, \mathcal{V})$, which is a contradiction. Therefore, $\mid \mathcal{B} \mid \geq \alpha$ and $\mathfrak{L}$ is a sub-algebra of $\mathcal{B}$. □

Theorem 3.26. *If α is an infinite cardinal number, then there is a topological L-algebra of order α.*

Proof. Suppose $\mathfrak{X}$ is a set of cardinality α and $1 \in \mathfrak{X}$. Consider $\mathfrak{L} = \{\omega_0 = 1, \omega_1, \omega_2, \cdots\}$ as a countable subset of $\mathfrak{X}$ such that $0 \notin \mathfrak{L}$. Similar to Theorem 3.24, define the operation $\twoheadrightarrow$ and $\leqslant$ on $\mathfrak{L}$ as follows,

$$\omega_i \twoheadrightarrow \omega_j = \begin{cases} 1 & \text{if } i \geq j \\ \omega_j & \text{if } i < j \end{cases} \qquad \text{and} \qquad \omega_i \leqslant \omega_j \ \text{ iff } \omega_i \rightarrow \omega_j = 1.$$

Similar to the proof of Theorem 3.22, we can see that $(\mathfrak{L}, \twoheadrightarrow, 1)$ is an L-algebra. Then the set $\mathfrak{I}_n = \{1, \omega_1, \cdots, \omega_n\} \in \mathcal{I}(\mathfrak{L})$, for any $n \geq 1$. Assume $\mathcal{B} = \{\mathfrak{I}_n \mid n \geq 1\}$. By Theorem 3.21, there is a non-trivial topology τ on $\mathfrak{L}$ such that $(\mathfrak{L}, \tau)$ is a topological L-algebra. Now, define the binary operation

$\rightsquigarrow$ on $\mathfrak{X}$ as follows,

$$\omega \rightsquigarrow \sigma = \begin{cases} \omega \twoheadrightarrow \sigma & \text{if } \omega \in \mathfrak{L},\ \sigma \in \mathfrak{L} \\ \sigma & \text{if } \omega \in \mathfrak{L},\ \sigma \notin \mathfrak{L} \\ 1 & \text{if } \omega \notin \mathfrak{L},\ \sigma \in \mathfrak{L} \\ 1 & \text{if } \omega,\ \sigma \notin \mathfrak{L},\ \omega = \sigma \\ 1 & \text{if } \omega, \sigma \notin \mathfrak{L} \cup \{1\},\ \omega \neq \sigma \\ 1 & \text{if } \omega = 0,\ \sigma \notin \mathfrak{L} \cup \{0\} \\ 0 & \text{if } \omega \notin \mathfrak{L} \cup \{0\},\ \sigma = 0 \end{cases}$$

Then similar to the proof of Theorem 3.22, we can see that $(\mathfrak{X}, \rightsquigarrow, 0, 1)$ is a bounded L-algebra of order α and the set $C = \tau \cup \{\{\omega\} \mid \omega \notin \mathfrak{L}\}$ is a sub-base for a topology $\mathcal{U}$ on $\mathfrak{X}$. Since $\{1\} \notin \mathcal{U}$, $\mathcal{U}$ is a non-trivial topology on $\mathfrak{X}$. In the following cases we will show that $(\mathfrak{X}, \rightsquigarrow, \mathcal{U})$ is a topological L-algebra. For this, consider $\omega \rightsquigarrow \sigma \in \mathfrak{U}$. In the following cases, we find two sets $\mathfrak{V}, \mathfrak{W} \in \mathcal{U}$ such that $\omega \in \mathfrak{V}$, $\sigma \in \mathfrak{W}$ and $\mathfrak{V} \rightsquigarrow \mathfrak{W} \subseteq \mathfrak{U}$.

Case 1. If $\omega, \sigma \in \mathfrak{L}$, then $\omega \rightsquigarrow \sigma = \omega \twoheadrightarrow \sigma \in \mathfrak{U} \in \tau$. Since $\twoheadrightarrow$ is continuous in $(\mathfrak{L}, \tau)$, there are $\mathfrak{V}, \mathfrak{W} \in \tau$ containing ω, σ, respectively, such that $\mathfrak{V} \twoheadrightarrow \mathfrak{W} \subseteq \mathfrak{U}$. Hence, $\mathfrak{V} \rightsquigarrow \mathfrak{W} \subseteq \mathfrak{U}$, which implies $\rightsquigarrow$ is continuous in $(\mathfrak{X}, \mathcal{U})$.

Case 2. If $\omega \in \mathfrak{L}$ and $\sigma \notin \mathfrak{L}$, then $\omega \rightsquigarrow \sigma = \{\sigma\} \subseteq \mathfrak{U}$. Thus, $\mathfrak{L}$ and $\{\sigma\}$ are two elements of $\mathcal{U}$ such that $\omega \in \mathfrak{L}$, $\sigma \in \{\sigma\}$ and $\omega \rightsquigarrow \sigma = \{\sigma\}$, and so $\mathfrak{L} \rightsquigarrow \{\sigma\} = \{\sigma\} \subseteq \mathfrak{U}$.

Case 3. If $\omega \notin \mathfrak{L}$ and $\sigma \in \mathfrak{L}$, then $\omega \rightsquigarrow \sigma = \{1\} \subseteq \mathfrak{U}$. Now, $\{\omega\}$ and $\mathfrak{L}$, both, belong to $\mathcal{U}$ and $\omega \in \{\omega\}$, $\sigma \in \mathfrak{L}$ and $\{\omega\} \rightsquigarrow \mathfrak{L} = \{1\} \subseteq \mathfrak{U}$.

Case 4. If $\omega, \sigma \notin \mathfrak{L}$ and $\omega = \sigma$, then $\omega \rightsquigarrow \sigma = \{1\} \subseteq \mathfrak{U}$. Then $\{\omega\}$ is an open set in $\mathcal{U}$ which contains ω and $\{\omega\} \rightsquigarrow \{\omega\} = \{1\} \subseteq \mathfrak{U}$.

Case 5. If $\omega, \sigma \notin \mathfrak{L} \cup \{0\}$ and $\omega \neq \sigma$, then $\omega \rightsquigarrow \sigma = \{1\} \subseteq \mathfrak{U}$. Then $\{\omega\}$ and $\{\sigma\}$ are two open sets in $\mathcal{U}$ which contains ω, σ, respectively, and $\{\omega\} \rightsquigarrow \{\sigma\} = \{1\} \subseteq \mathfrak{U}$.

Case 6. If $\omega = 0$ and $\sigma \notin \mathfrak{L} \cup \{0\}$, then $\omega \rightsquigarrow \sigma = \{1\} \subseteq \mathfrak{U}$. Then $\{0\}$ and $\{\sigma\}$ are two open sets in $\mathcal{U}$ which contains ω, σ, respectively, and $\{\omega\} \rightsquigarrow \{\sigma\} = \{1\} \subseteq \mathfrak{U}$.

Case 7. If $\omega \notin \mathfrak{L} \cup \{0\}$ and $\sigma = 0$, then $\omega \rightsquigarrow \sigma = \{0\} \subseteq \mathfrak{U}$. Then $\{\omega\}$ and $\{0\}$ are two open sets in $\mathcal{U}$ which contains ω, σ, respectively, and $\{\omega\} \rightsquigarrow \{\sigma\} = \{0\} \subseteq \mathfrak{U}$.

These cases prove $(\mathfrak{X}, \rightsquigarrow, \mathcal{U})$ is a topological L-algebra. Therefore, there is a topological L-algebra of order α. $\square$

Theorem 3.27. *Assume $(\mathfrak{L}, \tau)$ is a topological L-algebra and $\mathfrak{U}$ is an open neighborhood of 1. If for any $\omega \in \mathfrak{L}$, $\mathfrak{U} \twoheadrightarrow \omega$ is an open neighborhood of ω, then $(\mathfrak{L}, \tau)$ is a T_0-space.*

Proof. Consider $\omega, \sigma \in \mathfrak{L}$ and $\omega \neq \sigma$. Then $\mathfrak{U} \twoheadrightarrow \omega \in \tau$ and $\mathfrak{U} \twoheadrightarrow \sigma \in \tau$. If $\omega \in \mathfrak{U} \twoheadrightarrow \sigma$ and $\sigma \in \mathfrak{U} \twoheadrightarrow \omega$. Since $1 \in \mathfrak{U}$, we have $\omega \in 1 \twoheadrightarrow \sigma \subseteq \mathfrak{U} \twoheadrightarrow \sigma$, and so $\omega \in \{\sigma\}$. Similarity, $\sigma \in \{\omega\}$, and so $\omega = \sigma$ in both cases, which is a contradiction. Therefore, $(\mathfrak{L}, \tau)$ is a T_0-space. $\square$

Theorem 3.28. *Let $(\mathfrak{L}, \twoheadrightarrow, \tau)$ be a topological L-algebra. Then $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a T_0-space iff for any $1 \neq \omega \in \mathfrak{L}$, there exists $\mathfrak{U} \in \tau$ such that $\omega \in \mathfrak{U}$ and $1 \notin \mathfrak{U}$.*

Proof. Consider $\omega, \sigma \in \mathfrak{L}$ and $\omega \neq \sigma$. Then $\omega \twoheadrightarrow \sigma \neq 1$ or $\sigma \twoheadrightarrow \omega \neq 1$. Without the lost of generality, suppose $\omega \twoheadrightarrow \sigma \neq 1$. Then there exists $\mathfrak{U} \in \tau$ such that $\omega \twoheadrightarrow \sigma \in \mathfrak{U}$ and $1 \notin \mathfrak{U}$. By assumption, since $\twoheadrightarrow$ is continuous, there are $\mathfrak{V}, \mathfrak{W} \in \tau$ such that $\omega \in \mathfrak{V}$, $\sigma \in \mathfrak{W}$ and $\mathfrak{V} \twoheadrightarrow \mathfrak{W} \subseteq \mathfrak{U}$. If $\omega \in \mathfrak{W}$, then $1 = \omega \twoheadrightarrow \omega \in \mathfrak{V} \twoheadrightarrow \mathfrak{W} \subseteq \mathfrak{U}$, which is a contradiction. So, $\omega \notin \mathfrak{W}$. Hence, $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a T_0-space. The proof of converse is clear. $\square$

Corollary 3.29. *If α is an infinite cardinal number, then there is a T_0 topological L-algebra of order α, which it's topology is non-trivial.*

Proof. Assume $(\mathfrak{L}, \twoheadrightarrow, \tau)$ and $(\mathfrak{X}, \rightsquigarrow, \mathcal{U})$ are two topological L-algebras in Theorem 3.26. Clearly, $\mathcal{U}$ is non-trivial. Let $\omega \in \mathfrak{X} - \{1\}$. If $\omega \in \mathfrak{L}$, then for some $n \geq 1$, $\omega \notin \mathfrak{I}_n$. Hence, $\omega \in [\omega]_{\mathfrak{I}_n} \in \mathcal{U}$ and $1 \notin [\omega]_{\mathfrak{I}_n}$. If $\omega \notin \mathfrak{L}$, then $\omega \in \{\omega\} \in \mathcal{U}$ and $1 \notin \{\omega\}$. Now, by Theorem 3.28, $(\mathfrak{X}, \rightsquigarrow, \mathcal{U})$ is a T_0 topological L-algebra of order α. $\square$

Theorem 3.30. *Let $(\mathfrak{L}, \twoheadrightarrow, \tau)$ be a topological L-algebra. Then $(\mathfrak{L}, \tau)$ is a T_1-space iff it is a T_0-space.*

Proof. Consider $(\mathfrak{L}, \tau)$ is a T_0-space and $\omega \neq \sigma$. Then $\omega \twoheadrightarrow \sigma \neq 1$ or $\sigma \twoheadrightarrow \omega \neq 1$. Without the lost of generality, suppose $\omega \twoheadrightarrow \sigma \neq 1$. Then there exists $\mathfrak{U} \in \tau$ such that $\omega \twoheadrightarrow \sigma \in \mathfrak{U}$ and $1 \notin \mathfrak{U}$ or $\omega \twoheadrightarrow \sigma \notin \mathfrak{U}$ and $1 \in \mathfrak{U}$. First assume $\omega \twoheadrightarrow \sigma \in \mathfrak{U}$ and $1 \notin \mathfrak{U}$. Since $\twoheadrightarrow$ is continuous, there are $\mathfrak{V}, \mathfrak{W} \in \tau$ such that $\omega \in \mathfrak{V}$, $\sigma \in \mathfrak{W}$ and $\mathfrak{V} \twoheadrightarrow \mathfrak{W} \subseteq \mathfrak{U}$. If $\omega \in \mathfrak{W}$, then $1 = \omega \twoheadrightarrow \omega \in \mathfrak{V} \twoheadrightarrow \mathfrak{W} \subseteq \mathfrak{U}$, which is a contradiction. Similarly, $\sigma \notin \mathfrak{V}$. Now, if $1 \in \mathfrak{U}$ and $\omega \twoheadrightarrow \sigma \notin \mathfrak{U}$, then since $1 = \omega \twoheadrightarrow \omega = \sigma \twoheadrightarrow \sigma \in \mathfrak{U}$, there are $\mathfrak{N}, \mathfrak{M} \in \tau$ such that $\omega \in \mathfrak{N}$ and $\sigma \in \mathfrak{M}$ such that $\mathfrak{N} \twoheadrightarrow \mathfrak{N} \subseteq \mathfrak{U}$ and $\mathfrak{M} \twoheadrightarrow \mathfrak{M} \subseteq \mathfrak{U}$. If $\sigma \in \mathfrak{N}$, then $\omega \twoheadrightarrow \sigma \in \mathfrak{N} \twoheadrightarrow \mathfrak{N} \subseteq \mathfrak{U}$, which is a contradiction. Similarly, $\omega \notin \mathfrak{M}$. Therefore, $(\mathfrak{L}, \tau)$ is a T_1-space. The proof of converse is clear. □

Corollary 3.31. *If α is an infinite cardinal number, then there is a T_1 topological L-algebra of order α which it's topology is non-trivial.*

Proof. By Corollary 3.29 and Theorem 3.30, the proof is clear. □

Theorem 3.32. *Suppose $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a topological L-algebra. Then the following statements are equivalent:*
(i) $(\mathfrak{L}, \twoheadrightarrow, \tau)$ *is Hausdorff.*
(ii) $\{1\}$ *is closed.*
(iii) *for any $1 \neq \omega \in \mathfrak{L}$, there exist two open sets $\mathfrak{U}$ and $\mathfrak{V}$ of 1 and ω, respectively, such that $\mathfrak{U} \cap \mathfrak{V} = \emptyset$.*
(iv) $(\mathfrak{L}, \twoheadrightarrow, \tau)$ *is a T_1-space.*

Proof. (i $\Rightarrow$ ii) Since $\mathfrak{L}$ is Hausdorff, obviously, $\{1\}$ is closed.
(ii $\Rightarrow$ iii) Let $\{1\}$ be closed and $\omega \neq 1$. Then $1 \twoheadrightarrow \omega = \omega \in \mathfrak{L} - \{1\} \in \tau$. Since $\twoheadrightarrow$ is continuous, there exist two open neighborhoods $\mathfrak{U}$ and $\mathfrak{V}$ of 1 and ω, respectively, such that $\mathfrak{U} \twoheadrightarrow \mathfrak{V} \subseteq \mathfrak{L} - \{1\}$. If $\xi \in \mathfrak{U} \cap \mathfrak{V}$, then $1 = \xi \twoheadrightarrow \xi \in \mathfrak{U} \twoheadrightarrow \mathfrak{V} \subseteq \mathfrak{L} - \{1\}$, which is a contradiction. Therefore, $\mathfrak{U} \cap \mathfrak{V} = \emptyset$.
(iii $\Rightarrow$ iv) Assume $\omega, \sigma \in \mathfrak{L}$ and $\omega \neq \sigma$. Then $\omega \twoheadrightarrow \sigma \neq 1$ or $\sigma \twoheadrightarrow \omega \neq 1$. Without the lost of generality, suppose $\omega \twoheadrightarrow \sigma \neq 1$. By (iii), there exist two disjoint open sets $\mathfrak{U}$ and $\mathfrak{V}$ which contain $\omega \twoheadrightarrow \sigma$ and 1, respectively. Since $\twoheadrightarrow$ is continuous, there are $\mathfrak{N}, \mathfrak{M} \in \tau$ such that $\omega \in \mathfrak{N}$, $\sigma \in \mathfrak{M}$ and $\mathfrak{N} \twoheadrightarrow \mathfrak{M} \subseteq \mathfrak{U}$. If $\omega \in \mathfrak{M}$, then $1 = \omega \twoheadrightarrow \omega \in \mathfrak{N} \twoheadrightarrow \mathfrak{M} \subseteq \mathfrak{U}$, which is a contradiction. So, $\omega \notin \mathfrak{M}$. Similarly, $\sigma \notin \mathfrak{N}$. Hence, $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a T_1-space.
(*iv* $\Rightarrow$ *i*) Consider $\omega, \sigma \in \mathfrak{L}$ and $\omega \neq \sigma$. Then $\omega \twoheadrightarrow \sigma \neq 1$ or $\sigma \twoheadrightarrow \omega \neq 1$. Without the lost of generality, suppose $\omega \twoheadrightarrow \sigma \neq 1$. Since τ is a T_1-space, there exist two open neighborhoods $\mathfrak{U}$ and $\mathfrak{V}$ of $\omega \twoheadrightarrow \sigma$ and 1, respectively, such that $1 \notin \mathfrak{U}$ and $\omega \twoheadrightarrow \sigma \notin \mathfrak{V}$. Since $\twoheadrightarrow$ is continuous, there exist $\mathfrak{N}, \mathfrak{M} \in \tau$ such that $\omega \in \mathfrak{N}$, $\sigma \in \mathfrak{M}$ and $\mathfrak{N} \twoheadrightarrow \mathfrak{M} \subseteq \mathfrak{U}$. If $\xi \in \mathfrak{N} \cap \mathfrak{M}$, then $1 = \xi \twoheadrightarrow \xi \in \mathfrak{U}$, which is a contradiction, and so $\mathfrak{N} \cap \mathfrak{M} = \emptyset$. By the similar way, other case is clear. Therefore, $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is Hausdorff. □

Corollary 3.33. *If α is an infinite cardinal number, then there is a Hausdorff topological L-algebra of order α, which it's topology is non-trivial.*

Proof. By Corollary 3.31 and Theorem 4.2, the proof is clear. $\square$

Suppose $\mathfrak{L}$ is an L-algebra and $\mathfrak{I}$ is a proper ideal of $\mathfrak{L}$. Define $\sum = \{\mathfrak{U} \in \mathcal{I}(\mathfrak{L}) \mid \exists\ \mathfrak{I} \in \mathcal{I}(\mathfrak{L})$ such that $\mathfrak{I} \subseteq \mathfrak{U}\}$ and $f : \sum \hookrightarrow \mathcal{I}(\mathfrak{L}/\mathfrak{I})$ is a map such that $f(\mathfrak{U}) = \bar{\mathfrak{U}}$, for all $\mathfrak{U} \in \sum$. Clearly, f is a one to one corresponding among $\sum$ and $\mathcal{I}(\mathfrak{L}/\mathfrak{I})$.

Proposition 3.34. *Consider $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a topological L-algebra, $\mathfrak{I} \in \mathcal{I}(\mathfrak{L})$ and $\bar{\tau}$ is a quotient topology on $\mathfrak{L}/\mathfrak{I}$. If $\pi_{\mathfrak{I}} : \mathfrak{L} \hookrightarrow \mathfrak{L}/\mathfrak{I}$ is an open map, then*
(i) *$\mathfrak{I}$ is open iff $(\mathfrak{L}/\mathfrak{I}, \bar{\tau})$ is discrete.*
(ii) *$\mathfrak{I}$ is closed iff $(\mathfrak{L}/\mathfrak{I}, \twoheadrightarrow, \bar{\tau})$ is Hausdorff.*

Proof. (i) Suppose $\mathfrak{I}$ is open. Since $\pi_{\mathfrak{I}} : \mathfrak{L} \hookrightarrow \mathfrak{L}/\mathfrak{I}$ is an open map, the set $\pi_{\mathfrak{I}}(\mathfrak{I}) = [1]$ belongs to $\bar{\tau}$. Since $(\mathfrak{L}/\mathfrak{I}, \twoheadrightarrow, \bar{\tau})$ is a topological L-algebra, by Theorem 3.20, $(\mathfrak{L}/\mathfrak{I}, \bar{\tau})$ is discrete. Conversely, suppose $(\mathfrak{L}/\mathfrak{I}, \bar{\tau})$ is discrete. Then $[1]$ is an open set. Since $\pi_{\mathfrak{I}} : \mathfrak{L} \hookrightarrow \mathfrak{L}/\mathfrak{I}$ is continuous, $\mathfrak{I} = {\pi_{\mathfrak{I}}}^{-1}([1]) \in \tau$.
(ii) $(\Rightarrow)$ By assumption, $\mathfrak{I}$ is closed, then $\mathfrak{I}^c$ is open. Thus, for any $\omega, \sigma \in \mathfrak{L}$, if $\omega \twoheadrightarrow \sigma \in \mathfrak{I}^c$, then there are two open neighborhoods $\mathfrak{U}$ and $\mathfrak{V}$ of ω and σ, respectively, such that $\mathfrak{U} \twoheadrightarrow \mathfrak{V} \subseteq \mathfrak{I}^c$, because $\twoheadrightarrow$ is continuous. Also, since π is open, so $\pi(\mathfrak{U})$ and $\pi(\mathfrak{V})$ are two open neighborhoods of $[\omega]$ and $[\sigma]$, respectively, such that $\pi(\mathfrak{U}) \twoheadrightarrow \pi(\mathfrak{V}) \subseteq \pi(\mathfrak{U} \twoheadrightarrow \mathfrak{V}) \subseteq \pi(\mathfrak{I}^c)$. If $[\xi] \in \pi(\mathfrak{U}) \cap \pi(\mathfrak{V})$, then $[1] = [\xi] \twoheadrightarrow [\xi] \in \pi(\mathfrak{U}) \twoheadrightarrow \pi(\mathfrak{V}) \subseteq \pi(\mathfrak{I}^c)$, which is a contradiction. Therefore, $(\mathfrak{L}/\mathfrak{I}, \twoheadrightarrow, \bar{\tau})$ is Hausdorff.
$(\Leftarrow)$ Since $\mathfrak{L}/\mathfrak{I}$ is Hausdorff, the set $\{[1]\}$ is closed in $\mathfrak{L}/\mathfrak{I}$, and so $\mathfrak{I} = \pi^{-1}([1])$ is closed in $\mathfrak{L}$. $\square$

4 Conclusions and future works

In this article, the concept of (semi)topological L-algebras is introduces and some related results are approved. Then the conditions that imply an L-algebra be a semitopological or a topological L-algebra is investigated and some properties of them are studied. Specially, it is shown that if $(\mathfrak{L}, \twoheadrightarrow, \tau)$ is a semitopological L-algebra and $\{1\}$ is an open set or $\mathfrak{L}$ is bounded and satisfies the double negation property, then $(\mathfrak{L}, \tau)$ is a topological L-algebra.

Finally, a discrete topology on quotient L-algebra, under suitable conditions, is constructed.

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