THE EXISTENCE OF INTEGRALS OF DYNAMICAL SYSTEMS LINEAR IN THE VELOCITIES

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1. Introduction

A dynamical system means here a system specified by generalised co-ordinates $q^\alpha(\alpha = 1, 2, \ldots, n)$ and a Lagrangian L which is a quadratic polynomial in the generalised velocities, say

$$L = \tfrac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + a_\alpha\dot{q}^\alpha + a,$$

(with a summation convention). For any ignorable coordinate q^α (that is, a coordinate not entering explicitly in L) there is a corresponding first integral, $\dfrac{\partial L}{\partial \dot{q}^\alpha} = constant$, that is linear in the velocities, and it is usually stated **(1) (2)** that any dynamical system with an integral linear in the velocities has an ignorable coordinate (in the sense that there exists, at least locally, some coordinate-system in which one of the coordinates is ignorable). Consider now a particle moving in a plane under an inverse cube law central force. The equations of motion are

$$\frac{d}{dt}(r^2\dot{\theta}) = 0, \qquad \ddot{r} - r\dot{\theta}^2 + \mu/r^3 = 0,$$

the first having an integral $r^2\dot{\theta} = h$ linear in the velocities because θ is ignorable. The second equation is then

$$\ddot{r} + (\mu - h^2)/r^3 = 0.$$

If $\mu = h^2$ there is a second linear integral but no second ignorable coordinate. This raises doubts about the statements in the books.

It is true that Birkhoff ((**1**) p. 45) distinguishes " conditional integrals " which hold only for a given value of the total energy, but the integral in this example is not a conditional one.

2. The Error in the Usual Proofs

The proof given by Whittaker can be taken as typical: it is given in terms of the Hamiltonian but is easily translated into Lagrangian terms as follows: any integral linear in the velocities is also linear in the generalised momenta $p_\alpha = \dfrac{\partial L}{\partial \dot{q}^\alpha}$ and so has the form $b^\alpha p_\alpha = constant$. By suitable choice of b^α, the right-hand side can be taken as unity. The vector field b^α defines a direction at every point of the configuration space, and by choosing the corresponding field lines as one set of coordinate-lines we can arrange matters so that b^α has the form (1, 0, ..., 0) everywhere in some region. The integral then has the form $p_1 = 1$ so that, by the equations of motion $\frac{\partial L}{\partial q^1} = 0$, and Whittaker concludes that L does not involve q^1 explicitly, i.e. q^1 is now ignorable. The fallacy in this argument is that the equation $\frac{\partial L}{\partial q^1} = 0$ need not be an identity but may hold in virtue of the original assumption $p_1 = 1$.

3. The Corrected Result

The explicit form of the equations of motion is

$$\dot{q}^\beta\{\dot{q}^\alpha_{;\beta} + a^{\alpha\rho}(a_{\rho;\beta} - a_{\beta;\rho})\} - a^{\alpha\rho}a_{;\rho} = 0, \quad \text{.....................(1)}$$

where semi-colons denote covariant differentiation with respect to the metric $a_{\alpha\beta}$ and as usual $a^{\alpha\rho}a_{\rho\beta} = \delta^\alpha_\beta$. If we assume there is just one linear integral

$$b_\alpha \dot{q}^\alpha = 1, \quad \text{...(2)}$$

so that

$$b_\alpha \dot{q}^\beta \dot{q}^\alpha_{;\beta} + \tfrac{1}{2}(b_{\alpha;\beta} + b_{\beta;\alpha})\dot{q}^\alpha \dot{q}^\beta = 0,$$

we find, by eliminating the generalised acceleration in (1), that

$$b_\alpha a^{\alpha\rho}\{(a_{\rho;\beta} - a_{\beta;\rho})\dot{q}^\beta - a^{\alpha\rho}a_{;\rho}\} = \tfrac{1}{2}(b_{\alpha;\beta} + b_{\beta;\alpha})\dot{q}^\alpha \dot{q}^\beta. \quad \text{..............(3)}$$

The usual result follows if we equate to zero the quadratic terms, since this gives Killing's equation

$$b_{\alpha;\beta} + b_{\beta;\alpha} = 0,$$

which is well-known to be the condition for an ignorable coordinate (Eisenhart (3), p. 234). But all we can really claim is that (2) can be used to eliminate the quadratic terms from (3), so that

$$b_{\alpha;\beta} + b_{\beta;\alpha} = 2\theta b_\alpha b_\beta. \quad \text{................................(4)}$$

We must then be able to eliminate linear terms and terms independent of the velocities as well, so that

$$b^\rho(a_{\rho;\beta} - a_{\beta;\rho}) = \phi b_\beta, \quad \text{................................(5)}$$

and

$$b^\rho a_{;\rho} = \phi - \theta, \quad \text{.....................................(6)}$$

where $b^\rho = a^{\rho\mu}b_\mu$ as usual. Conditions (4), (5), (6) are those for an integral linear in the velocities. (We shall show in § **4** that systems exist for which these conditions have solutions).

Suppose now that (4), (5), (6) have a solution b_α; and consider the expression $u = b_\alpha \dot{q}^\alpha$. Then

$$\dot{u} = \dot{q}^\beta\{b_{\alpha;\beta}\dot{q}^\alpha + b_\alpha \dot{q}^\alpha_{;\beta}\}$$
$$= \tfrac{1}{2}(b_{\alpha;\beta}+b_{\beta;\alpha})\dot{q}^\alpha\dot{q}^\beta - b^\rho\{(a_{\rho;\beta}-a_{\beta;\rho})\dot{q}^\beta - a_{;\rho}\}$$

by (1). Using (4), (5), (6) gives

$$\dot{u} = (u-1)\{\theta(u+1)-\phi\}$$

which has the singular solution $u = 1$ unless $\theta = \phi = 0$, the usual case. We can therefore state the correct theorem in the form:

If a dynamical system has a single first integral linear in the velocities, then either the system has an ignorable coordinate or else the integral is a singular solution.

The general situation may be much more complicated, however, as the example at the beginning of this paper shows. If one first integral linear in the velocities exists another may result by using the first to eliminate some of the quadratic terms. The situation seems too complex to repay analysis.

4. Systems with Linear Integrals

There are many ways of making the conditions (4), (5), (6) integrable, as may be seen by considering the special case in which $a_\rho = 0$, If there is a solution we can, as in § **2**, make it $b^\alpha = (1, 0, \ldots, 0)$ so that $b_\alpha = a_{\alpha 1}$ and (4) becomes

$$a_{\alpha 1,\beta}+a_{\beta 1,\alpha}-2\begin{Bmatrix}\rho\\ \alpha\beta\end{Bmatrix}a_{\rho 1} \equiv a_{\alpha\beta,1} = 2\theta a_{\alpha 1}a_{\beta 1}, \quad\ldots\ldots\ldots\ldots(4')$$

the only remaining condition being

$$a_{;1} = -\theta. \quad\ldots\ldots\ldots\ldots(6')$$

These combine into

$$a_{\alpha\beta,1}+2a_{;1}a_{\alpha 1}a_{\beta 1} = 0.$$

One set of solutions of (7) consists of diagonal $a_{\alpha\beta}$ such that

$$a_{\alpha\beta\ 1} = 0 \text{ unless } \alpha = \beta = 1,$$
$$\frac{\partial}{\partial q^1}\left(\frac{1}{a_{11}}-2a\right) = 0.$$

The two-dimensional case of this solution is equivalent to

$$L = \tfrac{1}{2}\{A(x, y)\dot{x}^2+\dot{y}^2\}+\frac{1}{2A}+\phi(y),$$

which is of some interest since the quadratic form $A(x, y)dx^2+dy^2$ can represent any surface. Hence for a particle constrained to move on an arbitrary surface there are always fields for which the motion has an integral linear in the velocities.

REFERENCES

(1) G. D. Birkhoff, *Dynamical Systems* (New York, 1927), p. 44.

(2) E. T. Whittaker, *Analytical Dynamics*, Third Edition (Cambridge, 1927), § 150.

(3) L. P. Eisenhart, *Riemannian Geometry* (Princeton, 1926).

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