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Mental constructions for the learning of the concept of vector space

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Abstract
This study contributes to the literature on linear algebra instruction by designing and researching a teaching sequence based on APOS Theory to introduce engineering students to vector spaces. The sequence offers students multiple opportunities to understand the concept. Another contribution is the evidence that introducing prerequisite concepts—such as equality, sets, and binary operations—before tackling vector space was crucial for grasping the role of proof in determining whether a set is a vector space. The findings confirm that, as other studies have shown, vector space is challenging for students. However, the results demonstrate that students were able to face this challenge. All students showed evidence of developing an understanding of the concept, with nine achieving a clear grasp of vector space and the role of proof by the end of the experience. Additionally, the progress observed—from having difficulties with symbols to successfully proving statements involving unconventional operations—underscores the effectiveness of the teaching approach.

Keywords Vector space · APOS theory · Genetic decomposition · Prerequisite concepts

1 Introduction

The obstacles that arise in teaching and learning linear algebra are related to the nature of its elements: a network of interconnected definitions, axioms, and abstract theorems. This abstract nature of lineal algebra leads to frequent difficulties, especially with the concept of vector space. There is a need for approaches that can help students make sense of the unifying nature of this concept, enabling them to assimilate other concepts, such as subspaces and linear transformations, among others (Dorier & Sierpinska, 2001; Parraguez & Oktaç, 2010).

This paper presents the development and implementation of a teaching strategy to promote vector space understanding among engineering students. Its design is based on APOS theory. The participants had no prior training in mathematical logic or proof processes.

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2 Background

The concept of vector space is both unifying and generalizing (Dorier, 1995). This concept embodies an abstraction of mathematical objects that are already abstract in themselves by considering their shared properties. These features make vector space a central and unifying idea within linear algebra theory and a challenging notion for students to grasp (Dorier & Sierpinska, 2001). To outline the necessary mathematical foundations for learning vector space, we conducted a literature review focused on learning obstacles. These obstacles share common characteristics, allowing us to group them into three categories.

2.1 Learning obstacles attributed to the abstract nature of linear algebra

One major obstacle students face stems from the formalism inherent in linear algebra, linked to the theory's development through axiomatization. This process equipped this branch of mathematics with the means for the generalization and unification of results (Dorier, 1998). Linear algebra consists of many definitions, axioms, and theorems intricately woven together, giving it an abstract and conceptual nature. This characteristic has led researchers to identify specific obstacles:

> Students' difficulties with the formal aspect of the theory of vector spaces are not just a general problem with formalism but mostly a difficulty of understanding the specific use of formalism within the theory of vector spaces and the interpretation of the formal concepts in relation with more intuitive contexts like geometry or systems of linear equations, in which they historically emerged. Various diagnostic studies pointed to a single massive obstacle appearing for all successive generations of students and for nearly all modes of teaching, namely, what these authors termed the obstacle of formalism (Dorier & Sierpinska, 2001, p. 259.)

The obstacle of formalism refers to the difficulties students face when working with formal definitions, theorems, and symbolic manipulations in linear algebra, often without fully grasping their conceptual foundations. In the case of vector spaces, the abstract nature of elements like vectors, linear combinations, and subspaces can cause a disconnect from more intuitive contexts, like geometry or systems of linear equations.

Parraguez and Oktaç (2010) emphasize the importance of formalizing prior concepts (such as sets, functions, and binary operations) as a prerequisite for learning vector space. They conclude that students struggle to develop a coherent understanding of vector spaces without this formalization. This conclusion is based on the idea that learning the concept of vector space requires students to construct schemas for sets, binary operations, and axioms. Parraguez and Oktaç (2010) found that when students fail to formalize these preliminary concepts, they struggle to integrate them into a solid understanding of vector spaces.

This obstacle is also evident in argumentation tasks. When students are asked to prove whether a set is a vector space, they struggle with understanding the nature of proof and the role of counterexamples in the verification process (Mutambara & Bansilal, 2018).

2.2 Learning obstacles related to the different semiotic representations of linear algebra notions

Rosso and Barros (2013) identify a learning obstacle where students struggle to recognize the same mathematical object through different semiotic representations. This results in concepts like vectors being perceived as abstract definitions without meaning. Additionally, students have difficulties understanding functions as elements of vector spaces. For instance, Britton and Henderson (2009) show that some engineering students have problems developing a mental picture of a typical vector as an object distinct from its formula $f(x)$ and its graph $y = f(x)$ and considering it as an element in a vector space. This shift demands deeper abstract thinking and moving away from intuitive, concrete manipulations of functions. As a result, understanding concepts like vector spaces often turns into rote memorization of axioms. While students may memorize the axioms, they often struggle to apply them in determining if a set is a vector space (Mutambara & Bansilal, 2018).

Harel (2000) noted that using geometric references to introduce vector spaces can also present obstacles: "when geometry is introduced before the algebraic concepts have been formed, many students view the geometry as the raw material to be studied. As a result, they remain in the restricted world of geometric vectors and do not move up to the general case." (p. 4). Shifting from $\mathbb{R}^n$ to more general vector spaces is challenging, as mental pictures associated with $\mathbb{R}^2$ or $\mathbb{R}^3$ can constitute an obstacle to understanding some of the general results of linear algebra (Gueudet-Chartier, 2004). In $\mathbb{R}^2$ and $\mathbb{R}^3$ students often rely on geometric intuition—visualizing vectors as arrows, linear transformations as rotations or reflections, and spaces as planes or lines. While helpful in two and three dimensions, these visualizations can mislead students when dealing with higher-dimensional spaces where such interpretations are not intuitive.

2.3 Learning obstacles manifested when working with non-traditional vector spaces.

When teaching vector space, examples based on spaces like $\mathbb{R}^n$ are often used as a starting point. Generalizations from these spaces are then applied to introduce standard vector spaces such as matrices, polynomials, or continuous functions. These are the vector spaces typically found in linear algebra textbooks (Andía & Repetto, 2015). However, for some students, working with vector spaces whose elements are not n-tuples of numbers proves difficult. Conventional algorithms are often insufficient when dealing with vectors from non-traditional spaces, such as $(a, 1{,}0)$, $(1{,}0, a)$, and $(1 + a, 1, a)$, in $\mathbb{R}^3$ which involve variables and non-conventional operations (Kú et al., 2008). This didactic obstacle appears when students encounter binary operations that differ from the generalizations of addition and scalar multiplication. For instance, let $\oplus$ be an addition operation defined as $(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 \times y_2)$. Here, instead of component-wise addition, the second component involves multiplication. This operation does not follow the usual vector addition rules, and students might have difficulties understanding or applying properties like commutativity or associativity, as these properties behave differently compared to standard vector addition.

2.4 Proposals to overcome the identified learning obstacles

Various pedagogical recommendations have been proposed to address the learning obstacles associated with vector spaces. Redondo (2001) suggests using analogies to introduce abstract concepts associated with vector spaces. Counterexamples are advised for axiom verification, encouraging students to de-encapsulate mathematical Objects to understand underlying processes (Mutambara & Bansilal, 2018). Diverse vector spaces should be explored in linear algebra courses (Kú et al., 2008), including generalizations of zero vectors, additive inverses, and alternative binary operations to support the development of vector space understanding (Parraguez, 2013). The cognitive construction of $\mathbb{R}^2$ and $\mathbb{R}^3$ vector spaces can also be enhanced by incorporating arithmetic, algebraic, and geometric elements to exemplify key concepts like linear combinations, bases, and homogeneous linear equations (e.g., Rodríguez Jara et al., 2018).

For prospective engineers, there are few didactic proposals to strengthen their understanding of vector spaces. One example is the work of Fernández-Cézar et al. (2020), who propose modeling real engineering problems through homogeneous systems of linear equations to introduce the concept of vector space. The underlying idea of this didactic proposal is to counteract the formalism with which linear algebra is usually taught by showing future engineers situations where linear algebra functions as a tool for solving real problems. However, engineering students require a more comprehensive understanding of this concept, allowing them to tackle fundamental mathematical topics for their education, such as differential equations and control theory. These proposals can help counteract learning difficulties associated with vector spaces, such as perceiving the associated concepts as abstract definitions lacking meaning. They can also help students learn to manage unconventional algorithms and non-traditional vector spaces.

Considering that the literature review shows that understanding vector space is challenging for students, we recognized the need to design a didactic approach based on APOS theory. We considered two pedagogical suggestions from the literature: the formalization of prior concepts—such as sets, functions, and binary operations—as a prerequisite for learning the concept of vector space (Parraguez & Oktaç, 2010) and the use of diverse vector spaces and binary operations (Parraguez, 2013; Weller et al., 2002) in the design of a new didactic proposal.

3 Theoretical framework and research questions

This research is based on APOS theory, a constructivist theory developed by Dubinsky (Arnon et al., 2014, Ch. 1). It adapts Piaget's genetic epistemology to the learning of advanced mathematics. It intends to understand the constructions students need to do to learn mathematical concepts and provides tools to design specific activities to address the difficulties reported in the literature and to foster the construction of new knowledge. We considered using this theory as it was designed to foster students' reflection, which is necessary to learn complex abstract concepts such as vector space.

Its main mental structures are Actions, Processes, Objects, and Schemas. According to this theory, when students face a new mathematical concept, they construct new knowledge by reflecting on their previous knowledge. This construction starts with Actions on an already constructed Object. Actions are the structures needed to operate on Objects to transform them. Students using mainly Actions rely on memorized facts or algorithms that can be considered externally driven when working on those Objects. When students reflect on their Actions, describe them, or even reverse the steps performed, without following them one by one, they evidence they have interiorized those Actions into a Process. They can thus reflect on such Actions without the need for external stimuli. This construction can be recognized when students can find the result of problems without performing all the steps or when they can generalize procedures and arguments on them. Processes can be reversed and coordinated with other Processes into a new Process.

When students need to operate on a particular Process, they become aware of it as a whole, perform Actions on it, and encapsulate it into an Object. They can go back from the Object to the Process where it came from by de-encapsulating it. Performing new Actions on it reinitiates the APOS construction cycle. A Schema in APOS theory is an overarching structure defined as a coherent collection of related Actions, Processes, Objects, and other Schemas.

Although this progression is presented as a linear sequence, development does not always follow a straightforward, step-by-step path. Instead, individuals may move between stages as needed by the circumstances.

In APOS theory, students are considered to have developed an Action conception of a concept when they rely in most cases on Actions throughout their mathematical work. Students who give evidence of having constructed mostly Processes are considered to have constructed a Process conception of the studied concept. Students showing an Object conception of a concept are those who evidence the possibility of performing Actions on Objects.

In addition to APOS theory theoretical structures, the use of the theory in research and teaching relies on a crucial component of the theory: the genetic decomposition (GD). It is a hypothetical epistemological model describing the structures and mechanisms involved in the construction of a mathematical concept or topic. It is a theoretical conjecture predicting how a generic student constructs a concept. It is developed by researchers using what is known about the construction of a concept, historical issues, and the teaching experience of researchers. There is no claim that a unique GD exists. Several GDs can coexist but must be experimentally tested and refined if necessary. The GD is used in the design of research instruments, in the design of learning situations, and in data analysis.

The didactic approach used is detailed in the methodology section. It included tasks aimed at helping students grasp the prerequisite concepts needed to understand vector space. This approach led us to consider a related research question:

How did the construction of prerequisite concepts support the construction of the concept of vector space in this course?

After working with the prerequisites, students were presented with the activity sets designed with the GD for vector space. In order to study students learning of vector space, we focused on the question:

Which of the constructions from the GD do students manifest through a didactic approach designed with APOS theory?

4 Method

Following the APOS-based research methodology (see Fig. 1), we analyze the mental constructions a group of engineering students evidenced while being introduced to the concept of vector space. Students were enrolled in an introductory linear algebra course designed with APOS theory and its associated teaching methodology, the ACE cycle (see Sect. 4.3). The authors devised the GD used in this study

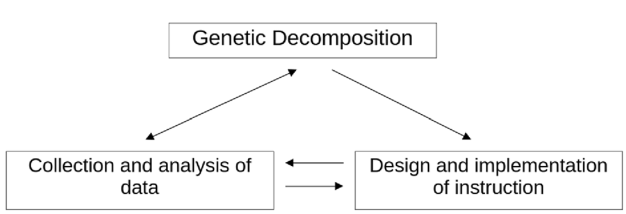

Fig. 1 APOS-based research methodology (Adapted from Arnon et al., 2014)

(Sects. 4.1 and 4.2). It models the necessary constructions in learning the vector space concept. The activities to introduce vector space were designed according to the constructions in GD (Sect. 4.4). The students had not been introduced to concepts from abstract algebra, proof, or more advanced courses prior to this introduction to vector space.

4.1 Vector space prerequisites

Based on the suggestions from previous studies (e.g., Parraguez & Oktaç, 2010), we decided to start by helping students construct the needed prerequisite concepts: equality, set, and binary operations. We used an available GD for set and binary operations (Arnon et al., 2014), and we developed a GD for equality as follows:

The construction of equality starts with Actions to determine the difference between a mathematical definition and a logical proposition. They allow students identifying which propositions need to be verified for truth. Learning about equality involves the Action to compare two mathematical expressions to determine if a statement with an equal sign is true based on a given definition.

When students evaluate propositions and decide if they are true, they interiorize these Actions into a Process where they can check the equality of mathematical expressions without needing an explicit definition. This Process varies depending on the situation in which the definitions are generated. If students can combine two or more correct expressions using equality properties—such as being reflexive, symmetric, and transitive—they show the construction of equality, and give evidence of the encapsulation of the equality Process into an Object.

4.2 Vector space GD

Vector spaces are defined as a set of vectors V together with a field F in which two binary operations that satisfy ten axioms are defined (Lay et al., 2021).

We designed a GD according to this definition (see Fig. 2) using elements from the GD proposed by Parraguez and Oktaç (2010) and Arnon et al. (2014). In this GD, we include mechanisms to integrate the construction of axioms in vector spaces by characterizing them into two groups: first, the axioms involving an equality relation for all vectors (axiom with universal quantifier). Then, the axioms involving an element that satisfies a property (axiom with existence quantifier), like the existence of the neutral element or the additive inverse for a vector.

Constructing sets with binary operations starts with Actions on the elements of a given set, consisting of applying binary operations to elements of the set. When individuals apply different binary operations to elements of several sets, they can reflect on their Actions and interiorize them

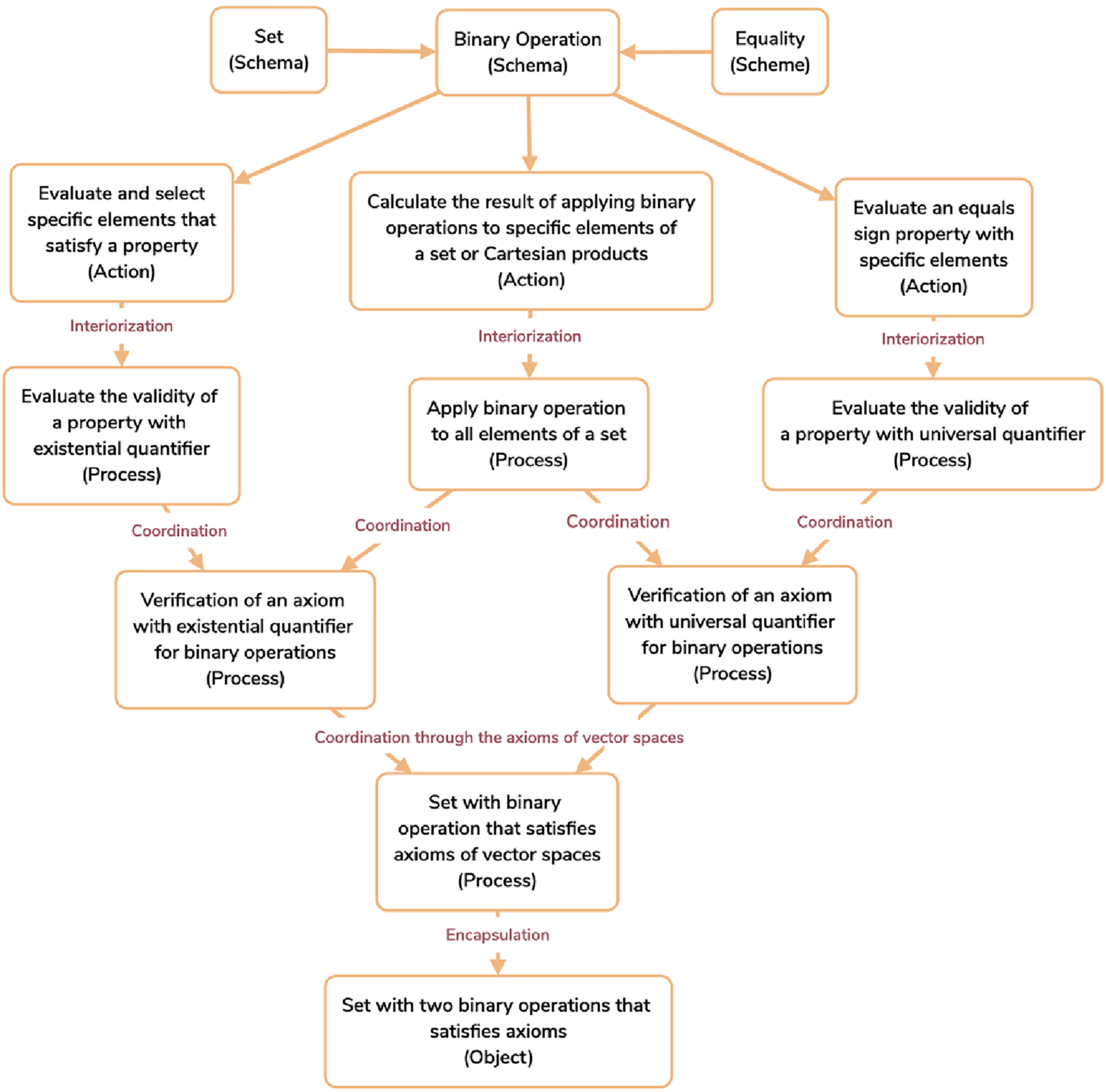

Fig. 2 Genetic decomposition of the concept of vector space

into a Process where these operations are considered functions. When they reflect on elements satisfying different membership conditions together with diverse binary operations, they interiorize them into a Process defined in terms of a membership condition and the binary operation Process defined on a set. These two Processes can be coordinated into a sets with binary operations Process.

Proving that a set of vectors with a binary operation satisfies an axiom with a universal quantifier starts with the student recognizing the axiom as an equality between two expressions related by an equal sign. The student calculates the required binary operations in both members of the equality using different elements of a given set according to the definition of the binary operation and compares the obtained results in terms of their equality. When an individual reflects on these Actions, he/she interiorizes them into a Process and is aware that the equality is valid for all elements in the set. By coordinating this Process with that of sets with binary operations, students construct a new Process where they can verify an axiom with a universal quantifier without needing to check for different elements of the set.

The construction of the "property is valid" Process enables students to discuss the existence of a specific element that satisfies the axiom. When this Process is coordinated with that of the existence quantifier, a new Process is constructed where there is no need to list all the elements of the set to see the existence of an element that satisfies an axiom.

The two previous axiom-verification Processes are coordinated with the Processes corresponding to the axioms that define the vector space into a set with two binary operations that satisfy the axioms defining the vector space Process. When students construct this Process, they have interiorized the need to verify all the axioms included in the vector space definition and can prove their validity in different sets.

Finally, when students can consider a set verifying all these axioms as a whole and can perform new Actions on it—such as identifying if a set V with a field F and two operations defines a vector space as mentioned in Parraguez and Oktaç (2010) or creating functions that map between vector spaces—they encapsulate the vector space Process into an Object. Figure 2 presents a diagram describing a GD of vector space.

In this study, we do not focus on the vector space Schema, which involves progressing through more abstract levels of understanding (Parraguez & Oktaç, 2012). As such, we focus on concrete cases and foundational aspects.

4.3 Design and implementation of instruction: the ACE cycle

The didactic experience was conducted with 20 engineering students enrolled in a first linear algebra course at a Mexican university. These students were unfamiliar with operations different from those traditionally defined for $\mathbb{R}^n$.

Based on the GDs, twenty tasks, including sets and unusual binary operations, were developed (e.g., Sect. 4.4). They were organized into six activity sets, each with three to four tasks.

The first three activity sets focused on building the prerequisite concepts, equality, set and binary operation, through tasks requiring proving vector specific space axioms. The last three activities focused on tasks to prove theoretical elements related to vector space. These tasks asked students to determine whether different sets and binary operations were or not a vector space.

Students were organized into four-member teams to work on the designed activity sets over six sessions. Due to the COVID-19 all sessions took place online and followed the ACE cycle from APOS theory (Arnon et al., 2014) including: teamwork in Activities (A), whole Class discussion (C), and Exercises (E).

This cycle was implemented: The teacher worked each week with one of the six Activity sets (A) designed to help students do the mental constructions suggested in the GDs (See Sects. 4.1 and 4.2). Each team worked collaboratively on the tasks (A). The teacher held one hour of weekly online whole-class sessions (C) with each team for students to present their task work. The teacher encouraged students to reflect on their tasks' responses. Depending on students' answers, the teacher clarified their thinking. When needed, the teacher provided hints to foster students' progress in understanding the concepts. The main objective of this discussion was to help students make the constructions suggested in the GDs. Tasks not solved in class sessions were assigned as Exercises (E) to work at home to reinforce the activities and the classroom discussion. The teacher was available online to answer student's questions about them.

4.4 Designed tasks and their analysis with the GDs

The design of the tasks enabled students to engage with various binary operations, sets, and vector spaces, enhancing their comprehension. Students applied these concepts across various contexts, such as coordinate vectors, matrices, polynomials, and finite fields. The tasks were not merely about computing but also involved analysis, reflection, and argumentation.

In this section, we present four tasks and their analysis in terms of their relation to the constructions called for by the GDs. They illustrate the activities the students engaged in and help to contextualize the results discussed in Sect. 5.

The following task was part of activity set two to promote the construction of set:

Task 1

Consider the set $\mathbb{Z}_p = \{0,1,2,\ldots,p-1\}$ and define $V = \left(\mathbb{Z}_p\right)^n$ as the set of n-tuples of numbers belonging to set $\mathbb{Z}_p$.

(a) Show examples of elements from V. How many elements are in set $V = \left(\mathbb{Z}_5\right)^2$?
(b) What are the differences between $W = \left(\mathbb{Z}_5\right)^3$ and V?
(c) If $\mathbb{Z}_7 = \{0,1,2,3,4,5,6\}$ and $U = \left(\mathbb{Z}_7\right)^2$. What are the differences and similarities between U, V and W?

A priori analysis

In this task, students could calculate the cardinality of different $\left(\mathbb{Z}_p\right)^n$ sets by reflecting on the structure of their elements, rather than just the p^n formula. This information was used later in the course to construct finite vector spaces using modular arithmetic.

Part (a) offers a chance to perform Actions to construct n-tuples from set V and calculate its cardinality: the student can construct a list of the elements for $V = \left(\mathbb{Z}_5\right)^2$ by pairing the elements of $\mathbb{Z}_5$ and use this list to provide the requested examples. Reflecting on these Actions can help the student interiorize them into a set Process. Parts (b) and (c) offer a chance to reflect on the differences and similarities in the procedure used to generate the elements of $\left(\mathbb{Z}_p\right)^n$ sets—more than just selecting a specific element or listing them. This reflection implies recognizing each set as a whole, which can be compared based on their definitions. The reflection can promote the encapsulation of the set Process into an Object.

The following task was included in activity set three to work on binary operations and equality concepts. This task introduces vector space axioms as properties that binary operations may satisfy.

Task 2

Consider $W = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R}^+ \bigcup 0 \right\}$, that is, W is the set of matrices with real entries such that the component in the second row, first column, is zero, and the remaining components are positive real numbers or zero. We define two operations on W: an operation "oplus" of two elements in W, given by $A \oplus B = AB$ (the usual matrix product of A and B); and an "otimes" operation of a scalar from $\mathbb{R}$ by a matrix in W, given by: $t \otimes \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a^t & b^t \\ 0 & c^t \end{pmatrix}$.

If A and B are matrices and t is a real number, is it true that $t \otimes (A \oplus B) = (t \otimes A) \oplus (t \otimes B)$?

A priori analysis

In this task, students can find a counterexample to probe that the property is false. A general proof of this can be based on the fact that $t \otimes (A \oplus B) = \begin{pmatrix} (ad)^t & (ae + bf)^t \\ 0 & (cf)^t \end{pmatrix}$ and $(t \otimes A) \oplus (t \otimes B) = \begin{pmatrix} (ad)^t & (ae)^t + (bf)^t \\ 0 & (cf)^t \end{pmatrix}$.

To verify the properties in this task, it is necessary to demonstrate equality by examining how binary operations apply to all elements within the given sets. Students can perform Actions to evaluate the operations on both sides of each equality using specific values and then repeat them with different elements to check whether the property holds in each case. Reflecting on how these equalities work in specific cases helps understanding how the property applies to all elements of the corresponding sets. This reflection fosters the interiorization of a new Process, allowing students to validate or invalidate an axiom with a universal quantifier by using generic elements to argue about it.

The concept of vector space was introduced in the next task, part of the fourth set of activities.

Task 3

Verify if the set with the given operations is a vector space over $\mathbb{R}$.

The set $V = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : x, y, z \in \mathbb{R}^+ \right\}$ with the operations defined as follows:

- Vector addition: $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \oplus \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & be \\ 0 & cf \end{pmatrix}$.
- Scalar multiplication: $t \otimes \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a^t & b^t \\ 0 & c^t \end{pmatrix}$.

A priori analysis

Defining $x, y, z \in \mathbb{R}^+$ implies students cannot use negative numbers or zero as arguments for the existence of zero vectors or additive inverses. This constraint requires students to reflect on the elements of set V and the defined vector addition to prove these properties. Other vector space properties can be proved using real number products and power laws.

The task helps to evidence constructions related to vector space. Students can perform Actions to confirm the closure of binary operations by evaluating specific elements. Interiorizing this Action into the Process of set with binary operations allows for the description of how these operations act on all the set elements, using generic elements.

To confirm the validity of the vector space axioms involving the equality sign and universal quantifier, the student could evaluate both sides of the equality with specific elements. Repeating this Action for several elements of the set can promote interiorizing those Actions into a new Process, which can be coordinated with the Set Process to construct a Process for verifying a set with binary operations satisfying an axiom with a universal quantifier. This would enable the student to argue about the validity of these axioms for all elements of V.

When working with the existence of the additive identity and the additive inverses, students can select elements from V and evaluate them in the given operations to verify these properties. Repeating these Actions in the search for specific elements satisfying the properties can promote their interiorization in the corresponding Process and its coordination with the Set Process into a new one, involving verifying a property with an existential quantifier for binary operations. This would enable the student to argue about the validity of these axioms on V.

Task 4 was used at the course's end to interview selected students and explore their understanding of the vector space concept.

Task 4

Let S be a vector space over the field $\mathbb{R}$ and suppose that $U, V \subseteq S$ are two vector spaces over the field $\mathbb{R}$ with the same addition and scalar multiplication operations defined on S. Let W be the intersection (suppose it is non-empty) of U and V ($W = U \bigcap V$), this is, W is the set formed by all vectors that are in both U and V. If we know that the additive identity of U belongs to W, Prove that W is a vector space.

A priori analysis

This task addresses a well-known result about vector subspaces, but it was assigned deliberately because the students had not yet worked with them when they completed the activity. This task helps us to gather evidence about the possible coordination of the Processes needed to understand axioms with quantifiers (universal and existential) and each of the specific Processes corresponding to the axioms defining vector spaces. The activity seeks to obtain evidence of constructing a single Process for verifying if sets with binary operations satisfy the axioms defining vector spaces, as proposed in the GD.

4.5 Data collection and analysis

All student discussions were recorded during the six online sessions. Each intervention was assigned an identifier to organize the transcripts of the online class sessions. Additionally, the names of the students in the dialogues were anonymized. All collected data have been translated from Spanish.

Using APOS theory, each researcher independently analyzed the data to identify the procedures and constructions described in the GD in students' statements. These instances were coded in terms of the Actions, Processes, or Objects proposed in the GDs and the tasks' analysis. For example, if a student showed memorized arguments or dependence on specific vectors to verify vector space properties in task 3, it was interpreted as evidence of the use of Actions in that instance. If the student responded to most problems and the questions posed during the interview using Actions, we considered he evidenced an Action conception of the involved concept. When a student was able to generalize operations, describe a set of vectors, or argue about the validity of a property in general over a given set, we refer to it as a Process. When a student was able to consider a Process as a totality and was able to do Actions on a Process, it was considered that the student constructed the concept as an Object.

After individually analyzing each session's data using the criteria mentioned above, researchers compared and negotiated their results to reach a consensus on the mental constructions demonstrated by the students. This analysis was repeated weekly on all teams' data to determine how the construction of prerequisite concepts supported the development of the vector space concept and which of the constructions from the GDs were manifested by the students.

When the six online sessions ended, four students were selected based on their responses during online sessions: one demonstrating a solid vector space construction, two showing progress, and two evidencing difficulties throughout the sessions. Each participated in an individual interview where questions about vector space and task four were posed. These interviews aimed to contrast the structures evidenced by students during the lessons with new ones shown in the interviews to discern whether the constructions proposed by GDs adequately explained performance variations or if additional constructions not addressed by the GDs were shown (Arnon et al., 2014). This allowed us to determine whether these students had constructed the constructions outlined in the GDs at the end of the study.

5 Results

This section presents the main results of the online sessions and individual interviews. We highlight the GD constructions that students evidenced while solving the assigned tasks. This section also presents empirical data that reflect the general trends, typical achievements, and common difficulties encountered in the group.

5.1 How did the construction of prerequisite concepts support the construction of the concept of vector space in this course?

We followed the recommendation from the literature about the need to introduce prerequisites. A GD for each was used, and activities were designed to foster the construction described according to APOS theory and the ACE cycle. When working on the first task, some students showed difficulties with the letters. They tested the proposed identities by substituting letters with numbers. When asked to perform non-standard operations with elements of a defined set, students would use a standard operation and, sometimes, they disregarded the set elements. Only two students clearly understood equivalence properties and sets as Objects, ten as Process, and eight as Action when introducing prerequisite concepts.

All students had difficulties to understand the purpose of using different binary operations when working with vectors, and other sets. For example, when working with task 1, defined above, asking to give examples of $\left(\mathbb{Z}_p\right)^n$ sets, student D proposed:

> D: I chose numbers for p and n, $\left(\mathbb{Z}_7\right)^3$, then I have to use numbers from 0 to 6, the set elements would be (0,2)(0,6) and (0,0).

This student chose numbers included in $\mathbb{Z}_7$ followed by Actions to choose some set members. He incorrectly associated the role of 3 in the expression with the number of elements in the set. After discussing with his team, he corrected:

> D: I see now, I don't need 3 elements in each set member, it is, for example, (1,3, 5) would be an element in the set and others with three components until completing all the set's triads.

Students did not initially grasp the need to prove statements and often overlooked valuable information about the sets involved in the task. A common procedure is illustrated by the work of a team discussing a task similar to task 3, asking to verify the commutative property of the defined sum for a set of matrices. Students used the standard definition of matrix addition instead of the one proposed in the task.

These examples show that students did indeed need the introduction of prerequisites. It took them time to understand the purpose of introducing different sets with a diversity of binary operations. They performed Actions on specific numbers. Working on the proposed activities in teams and discussing them with the teacher fostered their reflection and promoted some students' interiorization of Actions into Processes.

As students progressed through the lessons, they became more fluent. Eleven showed the construction of equality and set as Processes and three as Objects. They showed they had interiorized or encapsulated equivalence properties and decisions on the truth of propositions. Most of them understood the need to validate them.

For example, when asked to generalize the product by a scalar for elements in $\left(\mathbb{Z}_p\right)^n$ eleven students showed the interiorization of Actions into a Process when giving a similar response to R's:

> R: for all vectors resulting from $\left(\mathbb{Z}_p\right)^n$, those vectors would need to be multiplied by the scalar, but then… it is necessary to find the reminder of dividing them by p… it would be from 0 to $p-1$ in order for the operation to be valid, that is, all the coordinates should result in a number belonging to the set….

These students showed the construction of the Process of set while being able to work on given sets and verify their defined properties. They showed to be able to consider equality, sets and binary operations as Processes as they did not need to use specific examples and were aware of the need to validate statements and developed general coherent arguments.

During this part of the course, students had many opportunities to reflect on and develop the prerequisite concepts. When finalizing this part of the experience, all students could operate on equality and sets: three as Object, eleven as Process, and six as Action. Students who, at this point, constructed prerequisites as Actions had other opportunities to reflect on them through their work on vector space.

> L: To me they are equal (Fig. 3), since this side has the same operations, and when doing those two sums, here (referring to both sides of the equality) … they are the same.

Although this argument is not a formal proof, it shows the student's need to make sure the property was valid.

Most students found binary operations complex. At the end of the prerequisites' introduction, eight students showed their construction as Processes and two as Objects. They could recognize the generality of the operations needed and argue about their application and properties.

How did the construction of prerequisite concepts support the construction of the concept of vector space in this course?

Regarding the first research question, this course's work on prerequisite concepts provided students with the conceptual foundations necessary to begin constructing vector space. Students' work supported their introduction to vector space by stimulating the construction of Processes and

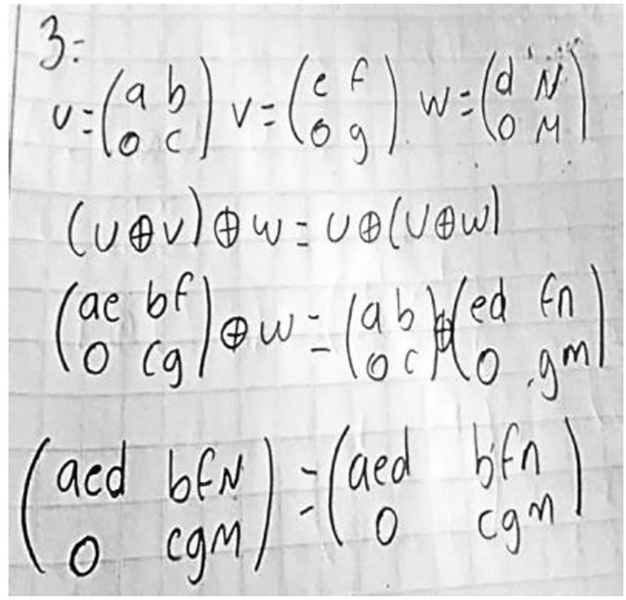

Fig. 3 Student's work on a task designed to evidence a Process construction. The student did not develop the procedure expected for a proof, but was able to validate the proposition

Objects related to the meaning of equality identification of elements in various sets with binary operations and making them all aware of the need to validate propositions through argumentation or proof. These constructions supported the construction of vector space by making students aware of its definition's main components and helping them distinguish different kinds of statements and how to validate them. These have been found in the literature and are behind students' difficulties when facing vector space.

5.2 Which of the constructions from the GD do students manifest through a didactic approach designed with APOS theory?

We begin by presenting a dialogue among a team of students as they tried to prove propositions on task 3 in the fourth session:

> J: … We have to verify equality. First, we need to perform the operations both on the left and the right, and then verify whether it is true… As I understood, we have to assign values to the matrices and then do the operations to see if we get the same result.
> E: Well, we have different properties. I remember that one (describes the distributive property), I mean, the properties, that no matter how you do it, the results will be the same.

J's proposal shows he needs to perform Actions to verify the distributive property. He needs to use specific numbers to determine its truth. E's response suggests an awareness of the need to test propositions in general, which is related to constructing a Process. She did not ask the group to change their approach, so they continued using numbers to validate the involved propositions, disregarding the definition of the operations. They evidence the construction of operations and the need to validate equality propositions but not the application of the corresponding binary operations.

This example illustrates the work of other teams where students verified vector space properties by choosing specific vectors from the sets and that of other students who used letters all representing specific numbers in the proof. They demonstrated the needed to perform Actions to validate properties defined for different sets with binary operations. They did not consider that verifying all the elements in the set should satisfy each property. They also illustrate how students who rely on Actions had not yet constructed the need for generalization to determine the truth of propositions including equality. This finding revealed that despite introducing prerequisite concepts, two students in the group had not interiorized the role of binary operations as a Process to verify general propositions including an equality. This result supports previous findings indicating that the concept of vector space is challenging for many students.

Returning to the second research question about the constructions from the GD manifested by students who could only use numbers in the verification, they have only constructed the part of the GD associated with the construction of Actions (Fig. 4). We can also observe that students who have constructed the prerequisites as Actions continue performing Actions regarding vector space, which means that they can test the properties defining it in terms of specific examples satisfying the properties.

Eight students showed evidence throughout their work of interiorizing the Actions into Processes needed to verify the properties of the operation in given sets to determine if they were or not vector spaces. They could all argue their response and operate without following memorized procedures. For example, when working on task 2, F can explain the validity of the distributive property:

> F: First, we need to add both matrices, $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ because the task tells you that oplus doesn't mean adding matrices. It means matrix multiplication. You are not going to do an element-by-element addition; you are going to do a multiplication, in which you multiply both the rows of one and the columns of the other. It is matrix multiplication. What he did was an addition, and in the one where he did scalar multiplication by a matrix, it tells us that the otimes that appear there is not a multiplication. It is like raising the elements of the matrix to a power.

F described, in general, how he would prove the distributive property. He showed he had interiorized Actions into a Process. He underlined the fact that the distributive property is valid for any number included in the set he was working with and using the defined binary operations, thus demonstrating that he also considered the property valid for all the vectors, in this case, matrices, in the set.

Seven other students demonstrated the construction of Processes regarding other vector space properties with universal quantifiers. When asked about the validity of the addition closure in task 3, F replied:

> F: In this set all matrices have the form with a, b, c in the reals and the zero in the second row first column. This is one matrix, other in the same space is $\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$. Then, we add them following the given definition for the sum and it results in the matrix $\begin{pmatrix} ad & be \\ 0 & cf \end{pmatrix}$. You can notice that it has the same structure, because ad, be and cf are real numbers, we could have changed them by $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ so the result of the sum has the same form as the others, it pertains to the same set, which is a vector space… so the addition is closed in that set.

When questioned about the existence of additive inverse vectors, L said:

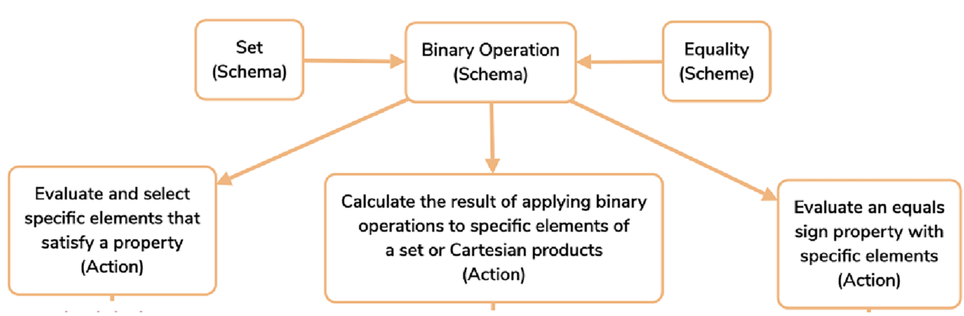

Fig. 4 Part of the GD constructed by students demonstrating an Action conception of vector space. These students work with vector spaces mostly through concrete examples

L:You can see we are dividing a by a, well if you have… let us say that a is 3, 3 divided by 3 is 1. If a over a is 1 and, the others are all 1 as this… (Fig. 5).

F and L show the construction of a Process for using and proving properties. They evidence of general thinking about the matrix's components and operate fluidly on them. This evidences their Process construction in finding a neutral element. L says:

L: We took what we already had for the fourth [property] as a reference, since it indicates that, if $-u$ exists, it should lead us to the value of the additive neutral element… Well, I thought about possible numbers, but then I changed to $(1/a)$, since when multiplying, I mean, when using addition, it would be a over a, (a/a), and as a result, we get the additive neutral element that we had obtained before.

L proposes a general structure for the additive inverse vector based on a generic vector u of the given set and applies the binary operation defined on this set. L describes how the general structure he proposes can prove the existence of the additive inverse elements for a given value, thus showing a Process construction for a general proposition with an existential quantifier.

These interventions evidence the construction of at least the Processes required to prove properties in general, thus demonstrating F and L's construction of Processes for verifying axioms for binary operations in general for the given set.

When looking for a neutral additive element in task 3 for the set. J and R discussed:

J: As L said, we need unknowns to make it easier.... We used zero as the unknown for the neutral value to obtain the same result…

J followed L's argument and tried to explain it, but she was unclear. J shows she can perform Actions on variables. Her use of Actions can be seen when she refers to the use of "unknowns" when looking for an inverse element.

R: Since it is a product, the additive identity for product uses 1, and I thought we could use this (in all the elements of the matrix), when we multiply it by 1... that is, it will give us again... nothing changes.

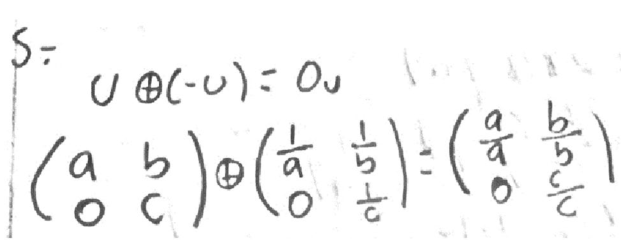

Fig. 5 Vector proposed by L to verify the existence of additive inverses

R also tries to understand J's argument but is confused by his teammates' use of zero. His proposal shows that he has interiorized those Actions into a Process.

J: No, I mean… it is the zero matrix. When we perform an addition, it gives us the same value as u…Well, here we have the result of a plus 0, b plus 0, 0 plus 0, and c plus 0, which, when replaced by any positive real value, gives us the same result as matrix u.

J shows she has not interiorized her Actions into a Process since the neutral additive she uses does not follow the given definition.

A: Look, here it says vector addition. You say it is an addition, but it really is a product, so the additive identity would not be zeros but ones as R said.

A shows she understands that, in this case, addition has been defined as a product. This makes J reconsider her argument and support R.

R: Professor, can two values be an additive identity, or is there only one for all?
P: That is interesting. When looking at the axiom, it does not say that there is only one. It just says it exists.
R: I see, it is just that with L's example, it generally works for all, but I imagine that using numbers like one would also work… For example, if we have a matrix, the additive identity could only have ones in both rows. If we apply oplus, we would get the same result, it would be the same, as in L's example… and it's even more general.

In this part of the interview, it is observed that R has interiorized the additive identity into a Process. He is still confused by L's response (Fig. 5) and does not consider it equal to his proposal, which indicates he has not constructed it as an Object since he considers both responses different.

L: …instead of using 1, I used letters so… it is a division of a by a (Fig. 5)… again, let's say a is 3, 3 divided by 3 will give you 1, so, if you notice, that a divided by a is one. So, the others are also ones, all of them. What I did is different because I used variables.

Here, L shows that he has constructed the additive identity as an Object. He can do the Action needed to compare two representations of the same result. He considers its different representations equivalent and defends his procedure in terms its generality.

This discussion shows how J, R, and L make different decisions regarding the same property. They show differences in their construction of this concept, which may be behind their difficulties in agreeing on a specific response. The discussion continues:

> R: That is what I meant by whether there can be more than one additive identity. That made me wonder about if there could be more than one.

Comparing different representations for the same element involves constructing the additive identity as an Object. R demonstrates that he has not encapsulated his constructed Process, while L has.

We also find evidence of the coordination of two Processes: Applying a binary operation to all the elements of a set and evaluating axioms with existence and uniqueness quantifiers. These students show they can use the meaning of the neutral element and the additive inverses to justify their decisions. This coordination involves reflection on the characteristics of a set's elements and how binary operations act on all these elements. It led J, R, and L to reflect on the uniqueness and universality of those properties. This excerpt clearly evidences the construction of vector space as a Process.

Other students demonstrated the ability to consider general elements and operations in most tasks to validate properties of different sets. They were also able to explain their reasoning and coordinate two Processes into a new Process as outlined in the GD. They showed evidence of having constructed vector space as a Process. Figure 6 shows the part of vector space GD constructed by these students.

Going back to the constructions these students showed, there is evidence that L and R demonstrated the construction of vector space as a Process. According to APOS theory, they have constructed all the needed constructions as Processes. They could prove if different sets were or not vector spaces. However, they may find it challenging to apply Actions, for example, to prove statements about vector spaces in general. At this point, R, L, and W may have developed the concept of vector space as an Object, but we do not have sufficient evidence from the data collected to confirm this.

Interviews' results

At the end of the experience, the teacher selected four students, each from a different team and representing students with different levels of understanding, to be interviewed. The interview was planned as another opportunity for these students to show what they had learned during the lessons and for the researchers to verify student's constructions through their responses to the interviewer.

The analysis of two interviewed students confirmed our previous appreciation of their vector space construction. L and W were able to verify vector space axioms for new sets with binary operations given. To confirm, P asked L whether the set of second-degree polynomials with integer coefficients forms a vector space. L tested the addition

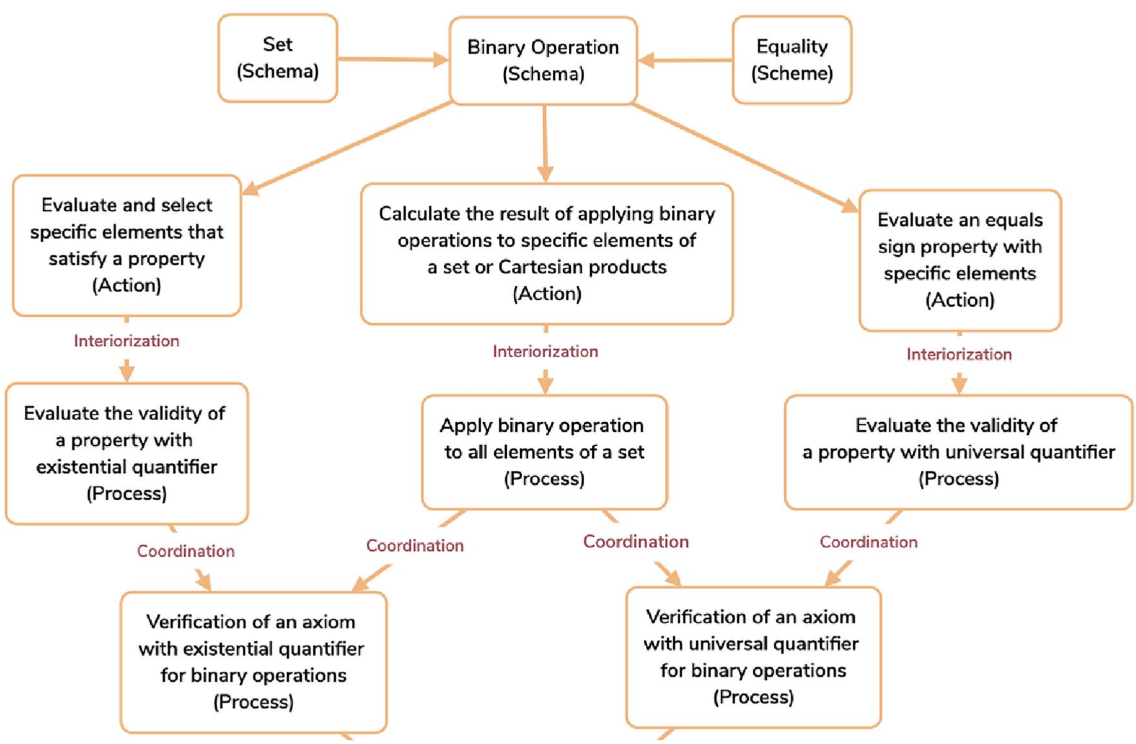

Fig. 6 Construction of vector space as a Process

properties and said, “It seems to be. I would check the scalar multiplication”. When he finished these operations, he confirmed that the set was a vector space. P asked him if he was sure or if something else was needed. L responded that he was sure, as it held true for all real numbers. L did not realize that coefficients needed to be in Z. He did not consider that he was including, for example, rational numbers, thus demonstrating he had not constructed vector space as an Object.

Student W was confident in his knowledge. His arguments were always clear. He gave evidence through the lessons and the interview of consistently constructing the last Process coordination (Fig. 2). The interviewer decided to test his ability to perform Actions on general vector spaces. What follows is his discussion with interviewer P when asked if the intersection of two vector spaces was a vector space (task 4).

This question might seem basic to those who have studied advanced introductory linear algebra or have been introduced to vector subspaces. However, none of these conditions apply to this course, as students had not previously worked on proving theorems or on subspaces. W response follows:

> W: If I have two vector spaces, the intersection means that there exists another, let us say, a W, which is not yet defined as a vector space, but which has elements from both V and U, and the intersection consists of those elements that coincide and are enclosed in another space, which we will call W. Now, here we want to verify that W is indeed a vector space.
> P: Do you need to prove that the addition is closed in W?
> W: Here addition is not defined, so... would I need to create an addition operation here?
> P: No, we know there is an addition operation defined in U and V. We do not know it yet. We will use the same addition operation for W since it is defined as their intersection.
> W:…if the intersection of U and V is W, then the addition operations in U and V are already defined and they are the same, …if the elements that coincide for W can be added, then it would be as adding those elements of W… I could apply the addition of elements from U and V, but it would be the same because the intersections that coincide in U and V and adding them would be the same as adding the elements that belong to W… I get it now! …If these elements have already been added in U and V and always resulted in closed sets in U and V, then the addition of elements in W would also be closed in W.
> P: What about scalar multiplication? Would it be closed?
> W: If for U and V, the multiplication by a scalar is already defined in both and was closed for U and V so it would be the same in W.
> P: And what about additive inverse? Does W have an additive inverse? …Okay, let me repeat, suppose w_1 is in W, what do you know about w_1 then?
> W: …Yes, I got it! If there is an additive inverse both in U and V for w_1, the additive inverse is… it repeats in both, which means that in W, it has to exist too because W is the intersection of those elements that are repeated in both, and if it is for both, it has to be for W too.

W had shown before the construction of a Process conception of vector space. His arguments show his ability to compare sets and discuss the properties in their intersection without referring to their elements. When asked to describe the addition and scalar multiplication defined in U, he showed evidence of encapsulation of the verification of properties related to the universal quantifier while describing addition and scalar multiplication as those defined in U and V to argue that this operation was also closed in W. Finally, W discussed the validity of properties related to vector space axioms. He compared spaces U, V, and W, thus evidencing the construction of vector space as an Object. W was not asked about the additive inverses, but his ability to act on vector space is enough to consider his construction of vector space as an Object.

When considering the development of students’ conceptions at the end of the experience, we agreed that two students showed an Action conception of vector space, fifteen students showed a Process conception, and three students showed an Object conception. Given the conditions in which this experience took place, compared to previous literature results, the findings obtained in this study may inform future development of strategies to teach vector space.

6 Discussion and conclusion

This study aimed to design a didactic strategy based on APOS theory for teaching vector space to university students and testing it with engineering students in a first-year Linear Algebra course during the second semester of their program. The course was designed according to a GD, including the prerequisite concepts of equality, sets, and binary operations. These prerequisites were introduced through collaborative activities, helping students to gradually construct their understanding of vector space.

6.1 Integration of prior research

Previous research has pointed to several obstacles that students face in learning vector space, mainly due to the formalism obstacle (Dorier & Sierpinska, 2001). The challenge lies in coordinating different semiotic representations and applying vector space properties beyond traditional vector spaces, which often require more profound levels of abstraction (Gueudet-Chartier, 2004; Kú et al., 2008). Our study builds on this literature by addressing the need to formalize prerequisite concepts such as equality, sets and binary operations (Parraguez & Oktaç, 2010), which had not been explored as part of a didactic strategy for introducing vector spaces. This contribution enriches the existing body of research by incorporating these prerequisites into vector space instruction and demonstrating, through empirical evidence, how they positively influence students' understanding and ability to engage with abstract concepts.

6.2 Contribution to research on linear algebra instruction

One of the main contributions of this study is the development and implementation of a GD tailored explicitly for the concepts of equality, sets, and binary operations, which are critical for understanding vector space. While previous studies (e.g., Arnon et al., 2014) have developed similar strategies, particularly with mathematics students and the use of programming, this is the first time such an approach has been applied to engineering students, incorporating the prerequisite concepts as foundational elements. Moreover, the activities were designed to introduce students to these concepts and to guide them through verifying whether given sets of vectors with various binary operations satisfied vector space axioms. This successful teaching approach is a significant addition to the literature on linear algebra teaching methodologies.

6.3 Response to research questions

In addressing the first research question—"How did the construction of prerequisite concepts support the construction of the concept of vector space in this course?"—we found that introducing prerequisite concepts was essential for all students because they provided the necessary groundwork to engage with a more abstract concept like vector space. As students entered the vector space topic, these prerequisites served as a foundation for validating whether different sets, defined by various binary operations, satisfied each of the axioms defining vector spaces. This process supported students' gradual transition from dealing with isolated operations to recognizing the necessity of proving axiomatic properties and the distinction of the role of existential and universal quantifiers, as evidenced by their improved ability to argue about these properties in class discussions and assignments.

For the second research question—"Which of the constructions from the GD do students manifest through a didactic approach designed with APOS theory?"—our results indicate that students manifested all the predicted constructions from the GD during classroom activities, except for the Object construction of vector space. However, these constructions appeared with varying frequency among students, with many still relying heavily on Actions rather than Processes, highlighting the challenges of reaching a more advanced understanding. Nevertheless, in the interviews, two students demonstrated the construction of vector space as an Object: one in a specific set and another in a general sense, although still limited by prior knowledge regarding proof techniques.

6.4 Challenges encountered

This study faced some challenges. First, while it was initially planned that students would engage in programming activities to deepen their understanding, this was only partially implemented due to the COVID-19 pandemic and technological limitations (e.g., some students lacked access to computers and relied on mobile phones with limited data). Despite these limitations, students adapted well to the situation. Second, some students encountered difficulties with the abstract nature of linear algebra, as this was their first exposure to the subject. While this presented a significant challenge, many students were still able to engage in the proposed activities using Actions and demonstrated an understanding of validating propositions regarding vector spaces. Although not all students could construct general proofs, others successfully constructed nearly all the structures outlined in the GD. By the end of the course, these students could discuss and use logical propositions to prove statements related to vector space, indicating a deeper understanding of the subject.

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