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SOME RESULTS ON THE JOIN GRAPH OF FINITE GROUPS

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ABSTRACT. Let G be a finite group which is not cyclic of prime power order. The join graph $\Delta(G)$ of G is a graph whose vertex set is the set of all proper subgroups of G, which are not contained in the Frattini subgroup G and two distinct vertices H and K are adjacent if and only if $G = \langle H, K\rangle$. Among other results, we show that if G is a finite cyclic group and H is a finite group such that $\Delta(G) \cong \Delta(H)$, then H is cyclic. Also we prove that $\Delta(G) \cong \Delta(A_5)$ if and only if $G \cong A_5$.

1. **Introduction and results**

There are many ways to associate a graph to a ring or a group, see for example [5, 6, 9, 10, 12, 14, 20]. In this context, it is natural to ask for the relation between the structure of the group, given in group theoretical terms, and the structure of the graph, given in graph theoretical terms. Sharma and Bhatwadekar [19] introduced the co-maximal graph associated with algebraic structures. The generating graph $\Gamma(G)$ of a finite group G is the graph whose vertex set is set of the elements of G and two distinct vertices a and b are connected by an edge if and only if $G = \langle a, b\rangle$. A series of papers investigate some questions concerning generating graph, see for example [7, 8, 13, 14, 15, 16].

For a finite group G different from a cyclic group of prime power order, Ahmadi and the second author [1] defined an undirected simple graph $\Delta(G)$ whose vertices are the proper subgroups of G, which are not contained in $\Phi(G)$, the Frattini subgroup of G, and two vertices H and K are adjacent if and only if $G = \langle H, K\rangle$. Note that by the condition on vertices, $\Delta(G)$ has no isolated vertices.

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The contraction of a pair of vertices u and v of a graph produces a graph such that u and v are replaced with a single vertex w such that w is adjacent to the union of the vertices to which u and v were originally adjacent. In vertex contraction, it doesn't matter if u and v are adjacent if they are adjacent, the edge is simply removed upon contraction (see [17, page 231]).

Let G be a finite non-cyclic group. Note that an element of $\Phi(G)$ is not adjacent to any element of G. Thus all elements of $\Phi(G)$ are isolated vertices in the generating graph $\Gamma(G)$ of G, and for every $a \in G \setminus \Phi(G)$, the subgroup $\langle a \rangle$ is a vertex in $\Delta(G)$. By contracting some suitable vertices of $\Gamma(G)$ we obtain a subgraph of $\Delta(G)$: in $\Gamma(G)$, we contract all elements of $\Phi(G)$ with any element of G, this means that we remove all isolated vertices. Then we contract all generators of $\langle a \rangle$, for all $a \in G \setminus \Phi(G)$; this means all nonincident vertices of the form a^j, where $\gcd(j, |a|) = 1$, are identified. In this way we obtain a subgraph of $\Delta(G)$. For example, if G is isomorphic to one of the groups $\mathbb{Z}_p \rtimes \mathbb{Z}_q$ or $\mathbb{Z}_p \times \mathbb{Z}_p$, ($\mathbb{Z}_n$ is the cyclic group of order n), where p and q are distinct primes, then the vertex contraction of generating graph of G is just the join graph of G. As an another example, by vertex contraction of $\Gamma(S_3)$, (S_n is the symmetric group on n letters), we obtain $\Delta(S_3)$ (see Figure 1).

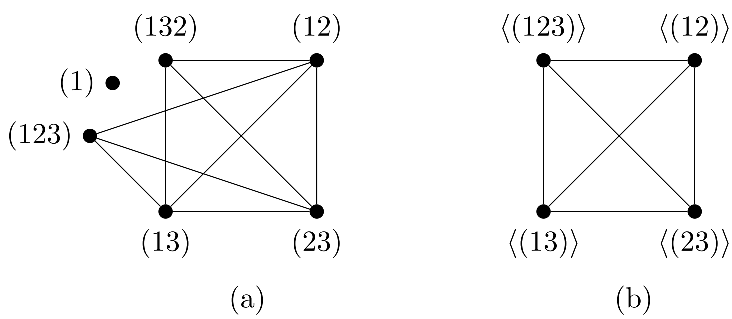

FIGURE 1. (a) $\Gamma(S_3)$ and (b) $\Delta(S_3)$

Let Δ be a graph with vertex set $V(\Delta)$. The degree of a vertex $v \in V(\Delta)$, denoted by $\deg(v)$, is the number of edges incident to v. If the vertices v and u are adjacent (non-adjacent), then we write $v \sim u$ ($v \nsim u$). The maximum degree, the minimum degree and the diameter of Δ are denoted by $\theta(\Delta)$, $\delta(\Delta)$ and $\mathrm{diam}(\Delta)$, respectively. If Δ_1 and Δ_2 are isomorphic graphs we write $\Delta_1 \cong \Delta_2$.

For a group G, we denote by $\pi(G)$ and $\mathscr{M}(G)$ the set of all prime divisors of $|G|$ and the set of all maximal subgroups of G, respectively. The set of all Sylow p-subgroups of G, where $p \in \pi(G)$ is denoted by $\mathrm{Syl}_p(G)$. For a subgroup H of G, we denote the number of subgroups of H, which are not contained in the Frattini subgroup $\Phi(G)$, by $\overline{n}(H)$. In particular if $\Phi(G) = 1$, then $\overline{n}(H)$ is the number of non-trivial subgroups of H. The core of H in G is denoted by $\mathrm{core}_G(H)$.

Ahmadi and the second author [1], studied some properties of the join graph $\Delta(G)$; in particular, they showed that $\Delta(G)$ is connected and its clique number (the size of the largest of complete subgraph) is equal to $|\mathscr{M}(G)|$. It follows that if $\Delta(G) \cong \Delta(H)$, then $|\mathscr{M}(G)| = |\mathscr{M}(H)|$. They classified finite groups with planner join graph [2]. Also they classified finite groups with regular join graph [3].

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In [4] we classified finite groups whose join graphs have domination number ≤ 2 and independence number ≤ 3, and showed that $\Delta(G) \cong \Delta(A_4)$ if and only if $G \cong A_4$. Also we showed that if the independence number of $\Delta(G)$ is less than 15, then G is solvable; moreover if the equality holds and G is non-solvable, then $G/\Phi(G) \cong A_5$.

In this paper, we continue the study of $\Delta(G)$. One may pose the following question is: If G and H are finite groups such that $\Delta(G) \cong \Delta(H)$, then is it true that $G \cong H$? The weaker version of the question is: if G and H are finite groups such that $\Delta(G) \cong \Delta(H)$ and G belongs to a class $\mathfrak{X}$, then is it true that $H \in \mathfrak{X}$? In section 2, we show that if G is abelian (resp. nilpotent) and H is a group such that $\Delta(G) \cong \Delta(H)$, then H may be non-abelian (resp. non-nilpotent). Thus in general the answer to both question is not yes. Therefore we may investigate some particular classes of groups or particular groups. To start we consider the class of cyclic groups and prove that

Theorem A. *Let H be a finite cyclic group and G be a finite group such that $\Delta(H) \cong \Delta(G)$. Then G is cyclic.*

We show that if G is cyclic of order $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where p_i's are distinct primes, then we can determine $n_1, \ldots, n_k$, from the join graph of G. Furthermore given the join graph of any finite group G, we can determine whether G is cyclic or not.

In section 4 we consider the alternating group A_5 and obtain a characterization of A_5 in term of its join graph:

Theorem B. *Let G be a finite group. Then $\Delta(G) \cong \Delta(A_5)$ if and only if $G \cong A_5$.*

Based on the works of this paper we pose the following questions.

Question 1. *Let G be a finite group. If $\Delta(G) \cong \Delta(A_n)$, $n \geq 6$, then $G \cong A_n$?*

Question 2. *Let G be a finite simple group and H is a finite group such that $\Delta(G) \cong \Delta(H)$, then H is a simple group?*

Question 3. *Let G and H are finite simple groups such that $\Delta(G) \cong \Delta(H)$, then $G \cong H$?*

2. Some non-isomorphic groups with the same join graph

In this section we show that if G is abelian (resp. nilpotent) and H is a group such that $\Delta(G) \cong \Delta(H)$, then H may be non-abelian (resp. non-nilpotent). We show there exist non-isomorphic groups with the same join graphs.

We start with the following easy Lemma.

Lemma 2.1. *If G is a finite group, then $\theta(\Delta(G)) = \max\{\deg(M) \mid M \in \mathscr{M}(G)\}$.*

Proof. For every vertex H in $\Delta(G)$, there exists $M \in \mathscr{M}(G)$ such that $H \leq M$. If $H \sim K$, then it is clear that $M \sim K$, and so $\deg(H) \leq \deg(M)$. This completes the proof. $\square$

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Note that for $M \in \mathscr{M}(G)$, $\deg(M)$ is equal to the number of proper subgroups of G which are not contained in M. The following Lemmas 2.2 and 2.3 below play a fundamental rule in the study of isomorphism of join graphs.

Lemma 2.2. *Let G and H be finite groups. Let $\varphi : \Delta(G) \longrightarrow \Delta(H)$ be an isomorphism and $M \in \mathscr{M}(G)$. Then there exists a unique maximal subgroup N of H such that $\varphi(M) \leq N$ and $N = \varphi(K)$, where $K \leq M$. In particular $\deg(M) = \deg(N)$ and $\overline{n}(M) = \overline{n}(N)$.*

Proof. Suppose, for a contradiction, that there exist maximal subgroups N_1 and N_2 of H such that $\varphi(M) \leq N_1$ and $\varphi(M) \leq N_2$. Since φ is bijective, there exist subgroups K_1 and K_2 of G such that $\varphi(K_1) = N_1$ and $\varphi(K_2) = N_2$. Since $\varphi(M)$ is not adjacent to N_1 and N_2, it follows that K_1 and K_2 are not adjacent to M. Hence $K_1, K_2 \leq M$. Therefore $K_1 \nsim K_2$, and so $\varphi(K_1) \nsim \varphi(K_2)$, which implies that $N_1 \nsim N_2$, a contradiction. Thus there exists a unique maximal subgroup N of H such that $\varphi(M) \leq N$. Now if K is a vertex in $\Delta(G)$ such that $\varphi(K) = N$, then $\varphi(M) \nsim \varphi(K)$, which implies that $M \nsim K$. Thus $K \leq M$.

Now suppose that $\varphi(M) \leq N$ and $N = \varphi(K)$, where $K \leq M$. Since $\varphi(M) \leq N$ and $\deg(M) = \deg(\varphi(M))$, by Lemma 2.1, we have $\deg(M) \leq \deg(N)$. On the other hand $\deg(K) = \deg(N)$ and by Lemma 2.1, $\deg(N) \leq \deg(M)$. Hence $\deg(N) = \deg(M)$. Clearly for every maximal subgroup M of G, all non-adjacent vertices to M in $\Delta(G)$ are subgroups of M, which are not contained in $\Phi(G)$. I follows easily from $\deg(N) = \deg(M)$, that $\overline{n}(M) = \overline{n}(N)$. □

Lemma 2.3. *Let G and H be finite groups and $\varphi : \Delta(G) \longrightarrow \Delta(H)$ be an isomorphism. If $\Phi(G) = \bigcap_{M \in \mathscr{A}} M$, where $\mathscr{A} \subseteq \mathscr{M}(G)$, then $\Phi(H) = \bigcap_{N \in \mathscr{B}} N$, where $\mathscr{B} = \{N \in \mathscr{M}(H) \mid \varphi(M) \leq N, M \in \mathcal{A}\}$.*

Proof. Assume, for a contradiction, that $\Phi(H) \lneqq \bigcap_{N \in \mathscr{B}} N$. Thus $\bigcap_{N \in \mathscr{B}} N$ is a vertex of $\Delta(H)$, and so there exists a vertex K of $\Delta(G)$ such that $\varphi(K) = \bigcap_{N \in \mathscr{B}} N$. Hence for every $N \in \mathscr{B}$, $\varphi(K) \leq N$, which implies that $\varphi(K) \nsim N$. Thus $\varphi(K) \nsim \varphi(M)$, for all $M \in \mathscr{A}$. Therefore for every $M \in \mathscr{A}$, $K \nsim M$, which implies that $K \leq M$. It follows that $K \leq \Phi(G)$, which is a contradiction. □

In order to obtain examples of groups G and H with isomorphic join graphs such that G is abelian but H is non-abelian, we prove the following Proposition.

Proposition 2.4. *Let G be a p-group of order p^n with $|\Phi(G)| = p^{n-2}$. If $\Delta(G) \cong \Delta(H)$, then H is a p-group or $H \cong \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_{q^m}$, where q is a prime number and $\Phi(H) \cong \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{q^{m-1}}$.*

Proof. Since G is nilpotent, $G = M_iM_j$, for every distinct $M_i, M_j \in \mathscr{M}(G)$. Hence $|M_i \cap M_j| = |M_i||M_j|/|M_iM_j| = p^{n-2}$, and so $\Phi(G) = M_i \cap M_j$, for every distinct $M_i, M_j \in \mathscr{M}(G)$. Therefore Lemma 2.3 implies that $\Phi(H) = N_i \cap N_j$, for every $N_i, N_j \in \mathscr{M}(H)$.

First suppose that H is nilpotent. Then $H \cong P_1 \times \cdots \times P_m$, where $P_i \in \mathrm{Syl}_{p_i}(H)$, $1 \leq i \leq m$. Since $p+1 = |\mathscr{M}(G)| = |\mathscr{M}(H)| = \sum_{i=1}^{m} |\mathscr{M}(P_i)|$. So $m = 1$ and $p_1 = p$. It follows that H is a p-group and $H/\Phi(H) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Now suppose that H is not nilpotent. Then H contains a non-normal maximal subgroup N. Since $N \cap N^h = \Phi(H)$, for all $h \notin N$, we see that $\overline{H} = H/\Phi(H)$, where $-: H \to H/\Phi(H)$ is the natural epimorphism, is a Frobenius group with the Frobenius complement $\overline{N}$. Suppose that $\overline{K}$ is the Frobenius kernel of $\overline{H}$. Then $\overline{K}$ is unique normal maximal subgroup of $\overline{H}$. For if $\overline{K} \leq \overline{M}$, where M is a maximal subgroup of H, then $M = M \cap H = M \cap KN = K(M \cap N) = K\Phi(H)$, which implies that $\overline{K} = \overline{M}$. Thus $|\overline{N}| = |\overline{H} : \overline{K}| = q$, where q is a prime number. Since N is a maximal subgroup of H, it follows that $\overline{K}$ is a normal minimal subgroup of $\overline{H}$. Therefore $p + 1 = |\mathscr{M}(H)| = 1 + |\overline{K}|$, and so $|\overline{K}| = p$. Hence $\overline{H} \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$, where $q \mid p - 1$. Now suppose that $P \in \mathrm{Syl}_p(H)$ and $Q \in \mathrm{Syl}_q(H)$. Since $P\Phi(H)/\Phi(H) \lhd H/\Phi(H)$, we have $P\Phi(H) \lhd H$. Hence $H = P\Phi(H)N_H(P) = N_H(P)$, and so $P \lhd H$. Also $\Phi(P) \leq P \cap \Phi(H)$. If $\Phi(P) \lneq P \cap \Phi(H)$, then there exists maximal subgroup M of P such that $P = M(P \cap \Phi(H))$. So $H = PQ = M(P \cap \Phi(H))Q = MQ$, which is a contradiction. Therefore $\Phi(P) = P \cap \Phi(H)$, and so $P/\Phi(P) \cong \mathbb{Z}_p$, which implies that P is cyclic. Since for every maximal subgroup M of Q, PM is a maximal subgroup of H, we have $Q \cap \Phi(H) \leq Q \cap PM = (Q \cap P)M = M$, and so $Q \cap \Phi(H) \leq \Phi(Q)$. Thus $\Phi(Q) = Q \cap \Phi(H)$, and hence $Q/\Phi(Q) \cong \mathbb{Z}_q$. Therefore $H \cong \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_{q^m}$ and $\Phi(H) = \Phi(P) \times \Phi(Q) \cong \mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_{q^{m-1}}$. □

By above Proposition if G is a 2-group of order 2^n with $|\Phi(G)| = 2^{n-2}$ and $\Delta(G) \cong \Delta(H)$, then it is clear that H is a 2-group of order 2^m with $|\Phi(H)| = 2^{m-2}$. For example, we have $\Delta(Q_8) \cong \Delta(\mathbb{Z}_2 \times \mathbb{Z}_2)$. From $\Delta(\mathbb{Z}_3 \times \mathbb{Z}_3) \cong \Delta(S_3)$ and $S_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ we see that, if G is abelian or nilpotent and H is a group such that $\Delta(G) \cong \Delta(H)$, then H may be non-abelian or non-nilpotent respectively.

3. **Join graph of cyclic groups**

In this section we proceed to prove the Theorem A and obtain some results about the join graph of a cyclic group. We show that if G is cyclic of order $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where p_i's are distinct primes, then we can determine $n_1, \ldots, n_k$, by the join graph of G. Furthermore we see that by the join graph of any finite group G, we can determine whether G is cyclic or not. We find non-identity automorphism of the join graph of a finite cyclic group.

In what follows we assume (unless otherwise stated) that G is a cyclic group of order $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where p_i's are distinct primes, $n_1 \leq n_2 \leq \cdots \leq n_k$ and $k \neq 1$. We say that $(n_1, n_2, \ldots, n_k)$ is the type of G. Also for every subgroup H of G of order $p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, we say that $(r_1, r_2, \ldots, r_k)$ is the type of H, where $0 \leq r_i \leq n_i$ and $1 \leq i \leq k$. For every $1 \leq i \leq k$, let $M_i \in \mathscr{M}(G)$ be the unique maximal subgroup of G, which is of order $|G|/p_i$.

Lemma 3.1. *By above assumption if $H \in V(\Delta(G))$, then* $\deg(H) = \prod_{j \in J}(n_j + 1) - 1$, *where* $J = \{j \mid p_j^{n_j} \mid |H|\}$.

Proof. Put $J' = \{1, 2, \ldots, k\} \setminus J$. It is clear that

$$\deg(H) = |\{K \in V(\Delta(G)) \mid p_i^{n_i} \text{ divides } |K|, \text{ for all } i \in J'\}|.$$

Hence $|K| = \prod_{i\in J'} p_i^{n_i} \prod_{j\in J} p_j^{r_j}$, where $0 \leq r_j \leq n_j$. Therefore the number of subgroups K such that $\langle H, K\rangle = G$ is equal to $\prod_{j\in J}(n_j + 1)$. Since G is not a vertex, the number of adjacent vertices to H is equal to $\prod_{j\in J}(n_j + 1) - 1$. $\square$

Corollary 3.2. *Let G be a cyclic group. Then the degrees of $P_i \in \mathrm{Syl}_{p_i}(G)$ and M_i are equal to n_i and $\prod_{i\neq j=1}^{k}(n_j + 1) - 1$, respectively.*

Now we consider the following condition which is important to study the join graph of a finite cyclic group.

Condition Φ: We say that finite group G satisfies the condition Φ, if $\Phi(G) \lneq \bigcap_{M\in\mathscr{C}} M$, where $\mathscr{C} \subsetneq \mathscr{M}(G)$.

Lemma 3.3. *Let G be a finite cyclic group and H be a finite group. Then the following hold*

(1) *If $\Delta(G) \cong \Delta(H)$, then H satisfies condition Φ.*

(2) *If H is cyclic, then $\Delta(G) \cong \Delta(H)$ if and only if G and H have the same type.*

Proof. (1) It is clear that every cyclic group satisfies the condition Φ. So by Lemma 2.3, H satisfies the condition Φ.

(2) We use the notation preceding Lemma 3.1. Since G satisfies the condition Φ, it follows that there exists at least one vertex $T_i \leq \bigcap\{M_j \mid 1 \leq j \leq k,\ j \neq i\}$ of $\Delta(G)$. Clearly T_i is adjacent to M_i and is non-adjacent to M_j, where $j \neq i$. On the other hand it is easy to see that $\deg(T_i) = n_i$. Therefore from the join graph of a cyclic group we can determine the type of the group. Suppose that H is cyclic and let $\varphi : \Delta(G) \longrightarrow \Delta(H)$ be an isomorphism. By Lemma 2.2, there exist $N_j \in \mathscr{M}(H)$, $1 \leq j \leq k$, such that $\varphi(M_j) \leq N_j$. Clearly $\varphi(T_i) \sim N_i$. Now if there exists an index j distinct from i such that $\varphi(T_i) \sim N_j$, then by Lemma 2.2, $N_j = \varphi(K)$, where $K \leq M_j$. Therefore $T_i \sim K$, and so $T_i \sim M_j$, which is a contradiction. Thus $\varphi(T_i)$ is adjacent to exactly one maximal subgroup of H and is not adjacent to other maximal subgroups of H. On the other hand $\deg(T_i) = \deg(\varphi(T_i)) = n_i$. Hence above argument implies that H is of type $(n_1, n_2, \ldots, n_k)$.

Now we prove the converse. Suppose that the type of H is $(n_1, n_2, \ldots, n_k)$ and it is equal to the type of G. Let $|H| = q_1^{n_1} q_2^{n_2} \cdots q_k^{n_k}$, where q_i is a prime. Define $\varphi : \Delta(G) \longrightarrow \Delta(H)$ such that T and $\varphi(T)$ have the same type for every vertex T of $\Delta(G)$. Two distinct vertices T_1 and T_2 are adjacent if and only if $p_i^{n_i}$ divides $|T_1|$ or $|T_2|$ for every $1 \leq i \leq k$. Since T_j and $\varphi(T_j)$ have the same type, where $j \in \{1, 2\}$, it follows that for every $1 \leq i \leq k$, $q_i^{n_i}$ divides $|\varphi(T_1)|$ or $|\varphi(T_2)|$. Thus $\varphi(T_1)$ and $\varphi(T_2)$ are adjacent, and so $\Delta(G) \cong \Delta(H)$. $\square$

Using the notation preceding Lemma 3.1, for every M_i, $1 \leq i \leq k$, there exists at least one vertex H such that $H \leq \bigcap\{M_j \mid 1 \leq j \leq k,\ j \neq i\}$. Therefore $|H| = p_1^{r_1} p_2^{r_2} \cdots p_i^{n_i} \cdots p_k^{r_k}$, with $0 \leq r_j < n_j$ and $j \neq i$. By Lemma 3.1, $\deg(H) = n_i$, and so n_i is the degree of vertices which are adjacent to M_i and not adjacent to other maximal subgroups. Thus we can find the type of a cyclic group by its join graph.

Now we prove the Theorem A.

Proof of Theorem A. Let $\mathscr{M}(G) = \{M_1, \ldots, M_n\}$. Since by Lemma 3.3, G satisfies the condition Φ, we have $\bigcap_{i\neq j=1}^{n} M_j \nsubseteq M_i$, for all $1 \leq i \leq n$. Now we choose $1 \neq a_i \in \bigcap_{i\neq j=1}^{n} M_j \setminus M_i$, for all $1 \leq i \leq n$. Hence $a_i \notin M_i$ and $a_i \in M_j$, for all $j \neq i$. Put $a = a_1a_2\cdots a_n$. If $G \neq \langle a\rangle$, then there exists maximal subgroup M_l such that $a \in M_l$. We can write $a = a_1\cdots a_{l-1}a_la_{l+1}\cdots a_n$, clearly $a_1\cdots a_{l-1}, a_{l+1}\cdots a_n \in M_l$, so $a_l \in M_l$, which is a contradiction. Thus $G = \langle a\rangle$. □

Notice that by Lemma 3.3 and proof of Theorem A we have the following result.

Corollary 3.4. *Let G be a finite group. Then G is cyclic if and only if G satisfies condition Φ.*

Proposition 3.5. *Let G be a finite group. Then G is a cyclic group if and only if for every vertex $M_i \in \mathscr{M}(G)$, $i \in I = \{1, 2, \ldots, k\}$, there exists vertex H_i such that $H_i \sim M_i$ and $H_i \nsim M_j$, for $j \in I$ and $j \neq i$.*

Proof. Suppose that G is cyclic. Since G satisfies the condition Φ, for all $i \in I$, $\bigcap\{M_j \mid j \in I, j \neq i\}$ is a vertex which is adjacent to M_i and is non-adjacent to all M_j's, $i \neq j \in I$. Now we prove the converse. Suppose that for $i \in I$ there exists vertex H_i such that $H_i \sim M_i$ and $H_i \nsim M_j$, $i \neq j \in I$. Therefore $H_i \nleq M_i$ and $H_i \leq \bigcap\{M_j \mid j \in I, j \neq i\}$. Since $H_i \nleq \Phi(G)$, it follows that $\bigcap\{M_j \mid j \in I, j \neq i\} \nleq \Phi(G)$. Hence for all $i \in I$, $\Phi(G) \lneq \bigcap\{M_j \mid j \in I, j \neq i\}$. Clearly G satisfies the condition Φ, and so by Corollary 3.4, G is cyclic. □

By Proposition 3.5, from the join graph of any finite group G, we can determine whether G is cyclic or not. For example, let G be a finite group whose join graph shown in Figure 2. The clique number of $\Delta(G)$ is 3 and so $|\mathscr{M}(G)| = 3$. Since each maximal subgroup of G is adjacent to a vertex H such that H is non-adjacent to other maximal subgroups, by Proposition 3.5, G is cyclic. We note that G is of type (n_1, n_2, n_3). For every vertex H which is adjacent to exactly one maximal subgroup M_i, we have $\deg(H) = n_i$. Thus it is clear that the type of G is $(1, 1, 2)$ and so $G \cong \mathbb{Z}_{p_1p_2p_3^2}$.

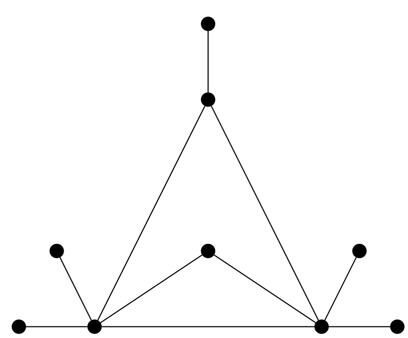

FIGURE 2.

Recall that a connected graph is Eulerian if and only if all vertices have an even degree (see [21, Theorem 1.2.26.]).

Proposition 3.6. *Let G be a cyclic group of type $(n_1, \dots, n_k)$, where $k > 1$, then,*

(1) $\delta(\Delta(G)) = n_1$ *and* $\theta(\Delta(G)) = \prod_{i=2}^{k}(n_i + 1) - 1$.
(2) $\mathrm{diam}(\Delta(G)) \leq 2$ *if and only if* $|\pi(G)| = 2$.
(3) $\Delta(G)$ *is Eulerian if and only if all n_i are even numbers.*

Proof. (1) It is clear, by Lemma 3.1.

(2) First suppose that $\mathrm{diam}(\Delta(G)) \leq 2$ and $|\pi(G)| \geq 3$. Let P_i and P_j be two Sylow subgroups. It is clear that P_i and P_j are not adjacent. Since $\mathrm{diam}(\Delta(G)) = 2$, so there exists a vertex H such that P_i and P_j are adjacent to it. Therefore $|H| = |G|$, a contradiction. Thus $|\pi(G)| = 2$.

Now to prove the converse suppose that $|\pi(G)| = 2$ and $|G| = p^n q^m$. We have $\mathrm{daim}(\Delta(G)) = 1$ if and only if $\Delta(G)$ is complete if and only if $n = m = 1$ (see [1, Theorem 2.5]). Let $\mathrm{daim}(\Delta(G)) \neq 1$. Hence there exist non-adjacent vertices in $\Delta(G)$. Suppose that H and K are non-adjacent. Since for every vertex T, $p^n \mid |T|$ or $q^m \mid |T|$, we conclude that p^n divides $|H|$ and $|K|$ (or q^m divides $|H|$ and $|K|$). Therefore H and K are adjacent to $Q \in \mathrm{Syl}_q(G)$ (or $P \in \mathrm{Syl}_p(G)$). Thus $\mathrm{diam}(\Delta(G)) = 2$.

(3) First suppose that n_i's are even numbers. By Lemma 3.1, for every $H \in V(\Delta(G))$, $\deg(H) = \prod_{j \in J}(n_j + 1) - 1$, and so $\deg(H)$ is even. Thus $\Delta(G)$ is Eulerian.

Now to prove the converse suppose that $\Delta(G)$ is Eulerian. For every $1 \leq i \leq k$, by Corollary 3.2, there exists a vertex of degree n_i, which implies that n_i's are even. □

We conclude this section by proving that the automorphism of the join graph of cyclic group is non-trivial.

Proposition 3.7. *Let G be a cyclic group, then* $\mathrm{Aut}(\Delta(G)) \neq \{\mathrm{id}\}$.

Proof. We consider two cases:

Case (i). There exists i such that $n_i \geq 2$. Then there exists vertices M and H of orders $|G|/p_i$ and $|G|/p_i^{n_i}$, respectively. Put $S = \{M, H\}$. Define $\varphi : \Delta(G) \longrightarrow \Delta(G)$ as:

$$\varphi(K) = \begin{cases} M & \text{if} \quad K = H \\ H & \text{if} \quad K = M \\ K & \text{if} \quad K \notin S \end{cases}$$

Let $T_1, T_2 \in V(\Delta(G))$. We show that $T_1 \sim T_2$ if and only if $\varphi(T_1) \sim \varphi(T_2)$. Hence we consider three cases:

(1) First suppose that $T_1, T_2 \in S$. We have $\varphi(T_1), \varphi(T_2) \in S$ and it is clear that $T_1 \nsim T_2$ if and only if $\varphi(T_1) \nsim \varphi(T_2)$.

(2) Suppose that $T_1, T_2 \notin S$ so $\varphi(T_1) = T_1$ and $\varphi(T_2) = T_2$. Thus the results is clear.

(3) Finally if $T_1 \in S$ and $T_2 \notin S$. Let $T_1 = M$. So $\varphi(T_1) = H$ and $\varphi(T_2) = T_2$. Therefore $M \sim T_2$ if and only if $p_i^{n_i}$ divides $|T_2|$ if and only if $H \sim T_2$. Similarly for $T_1 = H$ we have $H \sim T_2$ if and only if $M \sim T_2$.

Case (ii). $n_i = 1$ for every $1 \leq i \leq k$. Then $|G| = p_1 p_2 \cdots p_k$. Consider maximal subgroups M_1 and M_2 (recall that the order of M_i is $|G|/p_i$). Put

$$S_i = \{H \in V(\Delta(G)) \mid H \leq M_i, H \nleq M_j\}$$

where $i, j \in \{1, 2\}$. Let $H \in V(\Delta(G))$ and $|H| = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, where $r_i \in \{0, 1\}$. Hence H is of type $(r_1, r_2, \ldots, r_k)$. If H is in S_1 or S_2, then H is of type $(0, 1, r_3, \ldots, r_k)$ or $(1, 0, r_3, \ldots, r_k)$, respectively. If $H \notin S_1 \cup S_2$, then H is of type $(1, 1, r_3, \ldots, r_k)$ or $(0, 0, r_3, \ldots, r_k)$. Note that $S_1 \cap S_2 = 1$. We define $\varphi : \Delta(G) \longrightarrow \Delta(G)$ by

$$\varphi(H) = \begin{cases} H & \text{if} \quad H \notin S_1 \cup S_2 \\ H_1 & \text{if} \quad H \in S_1 \\ H_2 & \text{if} \quad H \in S_2 \end{cases}$$

Such that if H is of type $(0, 1, r_3, \ldots, r_k)$, then H_1 is of type $(1, 0, r_3, \ldots, r_k)$. Similarly if H is of type $(1, 0, r_3, \ldots, r_k)$, then H_2 is of type $(0, 1, r_3, \ldots, r_k)$. It is easy to see that φ is an isomorphism. This completes the proof. $\square$

4. Characterization of the alternating group A_5 by its join graph

Recall that the number of maximal subgroups of an elementary abelian p-group of rank n is equal to $(p^n - 1)/(p - 1)$. Now we ready to prove the Theorem B.

Proof of Theorem B. If $G \cong A_5$, then it is clear that $\Delta(G) \cong \Delta(A_5)$.

Now to prove the converse, let $\Delta(G) \cong \Delta(A_5)$ and $\varphi : \Delta(A_5) \longrightarrow \Delta(G)$ be an isomorphism. We know that $\Phi(A_5) = 1$ and $|\mathscr{M}(A_5)| = 21$. Also A_5 has two maximal subgroups with trivial intersection, and so by Lemma 2.3, there exist two maximal subgroups M_1 and M_2 of G such that $M_1 \cap M_2 = \Phi(G)$. We claim that G is non-nilpotent. Assume, for a contradiction, that G is nilpotent. Since $\overline{G} = \overline{M_1} \times \overline{M_2}$ and $|G : M_i| = p_i$, where p_i is a prime, $i = 1, 2$, we see that $\overline{G} \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2}$. If $p_1 = p_2 = p$, then $|\overline{G}| = p^2$. Since $|\mathscr{M}(G)| = p + 1 = 21$, $p = 20$, a contradiction. Thus p_1 and p_2 are distinct and $\overline{G} \cong \mathbb{Z}_{p_1 p_2}$. Hence $|\mathscr{M}(G)| = 2$, which is a contradiction.

Therefore G is non-nilpotent. We know that A_5 has 5 maximal subgroups $N_i \cong A_4$, $1 \leq i \leq 5$. If there exist i, j such that $N_i \cap N_j = 1$, then $60 = |A_5| = |A_5 : N_i \cap N_j| \leq |A_5 : N_i||A_5 : N_j| = 25$, which is a contradiction. Hence for distinct $1 \leq i, j \leq 5$, $N_i \cap N_j \neq 1$. By GAP [11], one can check that for any pairwise distinct $1 \leq i, j, k \leq 5$, $N_i \cap N_j \cap N_k = 1$. Now by Lemma 2.2, for $1 \leq i \leq 5$, there exist $M_i \in \mathscr{M}(G)$ such that $\varphi(N_i) \leq M_i$, and $\overline{n}(M_i) = 9$. Also Lemma 2.3 implies that for any pairwise distinct $1 \leq i, j, k \leq 5$, $M_i \cap M_j \neq \Phi(G)$ and $M_i \cap M_j \cap M_k = \Phi(G)$. First suppose that M_i is normal, for all $i = 1, 2, \ldots, 5$. It is clear that for $1 \leq i \leq 5$, $|G : M_i|$ is a prime number. Since $G = M_i M_j$, for all distinct $1 \leq i, j \leq 5$, we have $|G : M_i| = |M_i M_j : M_i| = |M_i M_j|/|M_i|$. Suppose that for $i \neq j$,

$|G:M_i|=p, |G:M_j|=q$, where p and q are prime numbers, then

$$\begin{aligned} p &= |M_iM_j:M_i| \\ &= \frac{|M_iM_j|}{|M_i|} \\ &= \frac{|M_iM_j||M_j|}{|M_i||M_j|} \\ &= q|M_j|/|M_i|, \end{aligned}$$

and hence $|M_j|=|M_i|$ and $p=q$. Thus $|G:M_i|=p$ for all $1\leq i\leq 5$. Note that since $G=M_iM_j$,

$$\begin{aligned} |G:M_i\cap M_j| &= \frac{|M_iM_j|}{|M_i\cap M_j|} \\ &= \frac{|M_i||M_j|}{|M_i\cap M_j||M_i\cap M_j|} \\ &= \frac{|M_iM_j|/|M_i|}{|M_iM_j|/|M_j|} \\ &= |G:M_i||G:M_j|. \end{aligned}$$

On the other hand $G=M_k(M_i\cap M_j)$, and so

$$\begin{aligned} |G:M_k\cap(M_i\cap M_j)| &= \frac{|M_k(M_i\cap M_j)|}{|M_k\cap(M_i\cap M_j)|} \\ &= \frac{|M_k(M_i\cap M_j)|}{|M_k\cap(M_i\cap M_j)|}\,\frac{|M_k|}{|M_k|} \\ &= |G:M_i\cap M_j||G:M_k| \\ &= |G:M_i||G:M_j||G:M_k|. \end{aligned}$$

Hence $|G:\Phi(G)|=|G:M_i\cap M_j\cap M_k|=p^3$, which implies that G is nilpotent, which is a contradiction. Therefore there exists i such that $M_i\ntrianglelefteq G$, hence $d=|G:M_i|\geq 3$. Since $\Phi(G)=\mathrm{core}_G(M_i)$, the core of M_i in G, it follows that $G/\Phi(G)$ is isomorphic to a subgroup of S_d. Suppose, if possible, that $|G:M_i|\geq 6$. Then M_i is conjugate to a maximal subgroup T of G, where $\overline{n}(T)=5$ or 7, a contradiction. So $|G:M_j|\leq 5$. Now if $|G:M_j|=3$ or 4, then $G/\Phi(G)$ is isomorphic to a subgroup of S_3 or S_4, respectively. Since S_3 and S_4 have no subgroup H with $|\mathscr{M}(H)|=21$, we have $|\mathscr{M}(G/\Phi(G))|\neq 21$, a contradiction.

Therefore $|G:M_i|=5$, which implies that $G/\Phi(G)$ is isomorphic to a subgroup of S_5. Since A_5 is the only subgroup of S_5 with 21 maximal subgroups, $G/\Phi(G)\cong A_5$. It follows that $\Delta(G/\Phi(G))\cong\Delta(A_5)$. On the other hand we have $\Delta(A_5)\cong\Delta(G)$, which implies that $\Delta(G)\cong\Delta(G/\Phi(G))$. Therefore $|V(\Delta(G/\Phi(G)))|=|V(\Delta(G))|$, and so for every $H\in V(\Delta(G))$, $\Phi(G)\leq H$. For every $P_j\in\mathrm{Syl}_{p_j}(G)$, where $p_j\in\pi(G)$, P_j is a vertex in $\Delta(G)$ and $\Phi(G)\leq P_j$. Since G is non-nilpotent, $|\pi(G)|\neq 1$. Hence for $j\neq k$, $\Phi(G)\leq P_j\cap P_k=1$, and so $\Phi(G)=1$. Thus $G\cong A_5$. This completes the proof. $\square$

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