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Letter to the Editor

Remarks on "On a Converse of Jensen's Discrete Inequality" of S. Simić

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We show that the main results by S. Simić are special cases of results published many years earlier by J. E. Pečarić et al. (1992).

Let I be an interval in $\mathbb{R}$ and $\phi : I \to \mathbb{R}$ a convex function on I. If $\mathbf{x} = (x_1, \ldots, x_n)$ is any n-tuple in I^n, and $\mathbf{p} = (p_1, \ldots, p_n)$ a positive n-tuple such that $\sum_{i=1}^n p_i = 1$, then the well known Jensen's inequality

$$\phi\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i \phi(x_i) \tag{1}$$

holds (see, e.g., [1, page 43]). If ϕ is strictly convex, then (1) is strict unless $x_i = c$ for all $i \in \{j : p_j > 0\}$.

The following results are given in [2].

Theorem 1. *Let $I = [a, b]$, where $a < b$, $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and $\mathbf{p} = (p_1, \ldots, p_n)$, $\sum_{i=1}^n p_i = 1$, be a sequence of positive weights associated with $\mathbf{x}$. Let ϕ be a (strictly) positive, twice continuously differentiable function on I and $0 \leq p, q \leq 1$, $p + q = 1$. One has that*

(i) *if ϕ is a (strictly) convex function on I, then*

$$1 \leq \frac{\sum_{i=1}^n p_i \phi(x_i)}{\phi\left(\sum_{i=1}^n p_i x_i\right)} \leq \max_p \left[\frac{p\phi(a) + q\phi(b)}{\phi(pa + qb)}\right] := S_\phi(a, b), \tag{2}$$

(ii) *if ϕ is a (strictly) concave function on I, then*

$$1 \leq \frac{\phi(\sum_{i=1}^{n} p_i x_i)}{\sum_{i=1}^{n} p_i \phi(x_i)} \leq \max_{p} \left[\frac{\phi(pa+qb)}{p\phi(a)+q\phi(b)} \right] := S'_{\phi}(a,b). \tag{3}$$

Both estimates are independent of $\mathbf{p}$.

Theorem 2. *There is unique $p_0 \in (0,1)$ such that*

$$S_{\phi}(a,b) = \frac{p_0\phi(a) + (1-p_0)\phi(b)}{\phi(p_0 a + (1-p_0)b)}. \tag{4}$$

The main aim of our paper is to show that the main results presented in [2] are simple consequences of more general results published in [3]. For this purpose, we will first introduce the concept of positive linear functionals defined on a linear class of real-valued functions.

Let E be a nonempty set, and let L be a linear class of functions $f : E \to \mathbb{R}$ having the following properties:

(L1) if $f, g \in L$, then $(af + bg) \in L$ for all $a, b \in \mathbb{R}$,

(L2) $1 \in L$, that is, $f(t) = 1$ for all $t \in E$, then $f \in L$.

We consider positive linear functionals $A : L \to \mathbb{R}$; that is, we assume the following

(A1) $A(af + bg) = aA(f) + bA(g)$ for all $f, g \in L$, $a, b \in \mathbb{R}$ (linearity),

(A2) if $f \in L$, $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$ (positivity).

If in addition $A(1) = 1$ is satisfied, then we say that A is a positive normalized linear functional.

Pečarić and Beesack [3] proved the next result which presents generalization of Knopp's inequality for convex functions (see also [4], [1, pages 101–103]).

Theorem 3 (see [3, Theorem 1]). *Let L satisfy properties (L1), (L2), and let A be a positive normalized linear functional on L. Let ϕ be a convex function on an interval $I = [m, M] \subset \mathbb{R}$ $(-\infty < m < M < \infty)$, and let J be an interval in $\mathbb{R}$ such that $\phi(I) \subset J$. If $F : J \times J \to \mathbb{R}$ is an increasing function in the first variable, then, for all $g \in L$ such that $g(E) \subset I$ and $\phi(g) \in L$, one has*

$$\begin{aligned} F(A(\phi(g)), \phi(A(g))) &\leq \max_{x \in [m,M]} F\left(\frac{M-x}{M-m}\phi(m) + \frac{x-m}{M-m}\phi(M), \phi(x) \right) \\ &= \max_{\theta \in [0,1]} F(\theta\phi(m) + (1-\theta)\phi(M), \phi(\theta m + (1-\theta)M)). \end{aligned} \tag{5}$$

Furthermore, the right-hand side in (5) *is an increasing function of M and a decreasing function of m.*

Remark 4. Analogous discrete version of Theorem 3 can be found in [5, Theorem 8, pages 9-10].

Remark 5. The results of this type are considered in [6], where generalizations for positive linear operators are obtained. Further generalizations for positive operators are given in [7]. Recently, Ivelić and Pečarić [8] obtained generalizations of Theorem 3 for convex functions defined on convex hulls.

Remark 6. The general results for concave functions can be proved in an analogous way, that is, for example, in case of positive linear operators given in [6, page 37]. Therefore, we will look back only on case (i) of Theorem 1.

By applying Theorem 3 to the function $F(x, y) = x/y$, we obtain the following result.

Theorem 7. *Suppose that all the conditions of Theorem 3 are satisfied. Then one has*

$$\begin{aligned} \frac{A(\phi(g))}{\phi(A(g))} &\leq \max_{x\in[m,M]}\left[\frac{(M-x)/(M-m)\phi(m)+(x-m)/(M-m)\phi(M)}{\phi(x)}\right] \\ &= \max_{\theta\in[0,1]}\left[\frac{\theta\phi(m)+(1-\theta)\phi(M)}{\phi(\theta m+(1-\theta)M)}\right]. \end{aligned} \tag{6}$$

Furthermore, the right-hand side in (6) *is an increasing function of M and a decreasing function of m.*

Theorem 8. *Let L, A, and I be as in Theorem 3. Let ϕ be a positive convex function on I such that $\phi''(x) \geq 0$ with equation for at most isolated points of I (so that ϕ is strictly convex on I), $g \in L$ such that $g(E) \subset I$ and $\phi(g) \in L$. Then,*

(i)

$$\frac{A(\phi(g))}{\phi(A(g))} = \frac{(M-\overline{x})/(M-m)\phi(m)+(\overline{x}-m)/(M-m)\phi(M)}{\phi(\overline{x})}, \tag{7}$$

where $\overline{x} \in (m, M)$ is uniquely determinated,

(ii)

$$\frac{A(\phi(g))}{\phi(A(g))} = \frac{\overline{\theta}\phi(m)+\left(1-\overline{\theta}\right)\phi(M)}{\phi\left(\overline{\theta}m+\left(1-\overline{\theta}\right)M\right)}, \tag{8}$$

where $\overline{\theta} \in (0, 1)$ is uniquely determinated.

Proof. (i) Proof is given in [3, Corollary 1, Remark 2] (see also [1, Remark 3.43 pages 102-103]).

(ii) This case follows immediately from (i) by changing of variable

$$\theta = \frac{M-x}{M-m}, \tag{9}$$

so that

$$x = \theta m + (1 - \theta) M \tag{10}$$

with $0 \leq \theta \leq 1$. □

Remark 9. In the case of a discrete positive functional $A(f) = \sum_{i=1}^{n} p_i f(x_i)$, $\sum_{i=1}^{n} p_i = 1$, $p_i > 0$, we can get a discrete version of Theorem 8. It is obvious that the main results presented in [2] are special cases of results given in [3, Theorem 1, Corollary 1, Remark 2].

Note that there is a difference in formulation between Theorems 2 and 8; that is, in Theorem 2, the differentiability of a function ϕ is not emphasized which is used in the proof. Also, the proof of Theorem 2 is completely analogous to the proof of [3, Corollary 1, Remark 2] with the above substitution $\theta = (M - x)/(M - m)$.

References

[1] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, vol. 187 of *Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1992.

[2] S. Simić, "On a converse of Jensen's discrete inequality," *Journal of Inequalities and Applications*, vol. 2009, Article ID 153080, 6 pages, 2009.

[3] J. E. Pečarić and P. R. Beesack, "On Knopp's inequality for convex functions," *Canadian Mathematical Bulletin*, vol. 30, no. 3, pp. 267–272, 1987.

[4] K. Knopp, "Über die maximalen Abstände und Verhältnisse verschiedener Mittelwerte," *Mathematische Zeitschrift*, vol. 39, no. 1, pp. 768–776, 1935.

[5] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, vol. 61 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[6] T. Furuta, J. Mićić Hot, J. Pečarić, and Y. Seo, *Mond-Pečarić Method in Operator Inequalities: Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, Croatia, 2005.

[7] J. Mićić, J. Pečarić, and Y. Seo, "Converses of Jensen's operator inequality," *Operators and Matrices*, vol. 4, no. 3, pp. 385–403, 2010.

[8] S. Ivelić and J. Pečarić, "Generalizations of converse Jensen's inequality and related results," *Journal of Mathematical Inequalities*, vol. 5, no. 1, pp. 43–60, 2011.