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Branched covers induced by semisymmetric quasigroup homomorphisms

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Abstract. Finite semisymmetric quasigroups are in bijection with certain mappings between abstract polyhedra and directed graphs, termed alignments. We demonstrate the polyhedra of any given alignment can always be realized as compact, orientable surfaces. For any $n \in \mathbb{N}$, the class of quasigroups having associated surfaces with sum genus $\leqslant n$ is closed under subobjects and homomorphic images. Further, we demonstrate semisymmetric quasigroup homomorphisms may be translated into branched covers between their respective surfaces.

1. Introduction

Semisymmetric quasigroups are among the more well-studied classes of quasigroup, in part because of their parastrophic symmetry and significance in regard to quasigroup homotopisms [17] [28], as well as their connection to combinatorial design theory [7] [16] and discrete geometry [23] [29]. In the author's previous work [19], it was established that finite semisymmetric quasigroups are in bijection with objects we refer to as *alignments on polyhedra* or simply *alignments*, which represent mappings between abstract polytopes (a combinatorial generalization of the more familiar geometric polytopes) and directed graphs.

The motivation for this paper stems from the observation that the abstract polyhedra associated with semisymmetric quasigroups exhibit certain well-behaved properties when considered as topological surfaces. More specifically, we demonstrate that each polyhedron of any given alignment can, in a straightforward way, be realized as a compact, orientable 2-manifold. The particular set of surfaces obtained via this method is shown

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to be an isomorphism invariant, and as such it is possible to classify semisymmetric quasigroups according to total genus. This topological information, in turn, places restrictions on the possible algebraic relations between quasigroups – in particular, for any $n \in \mathbb{N}$, the class of quasigroups of genus $\leqslant n$ is closed under subobjects and homomorphic images.

One might note that the above result effectively amounts to a version of the Riemann-Hurwitz formula as applied to semisymmetric quasigroups; indeed, from any semisymmetric quasigroup homomorphism, one obtains a set of continuous mappings on their respective surfaces. Furthermore, we demonstrate these induced mappings are, in fact, branched covers.

2. Preliminaries

A *partial quasigroup* $(Q, \cdot)$ is a set Q with a binary operation $(\cdot)$ such that for some $a, b \in Q$ there exist (at most) unique elements $x, y \in Q$ such that $a \cdot x = b, y \cdot a = b$; if this relation is satisfied for all $a, b \in Q$, then it is *complete* or simply a *quasigroup* [5] [8]. For brevity, we may denote $x \cdot y$ by juxtaposition xy. A partial quasigroup is *semisymmetric* if it satisfies the identity $x(yx) = y$, or equivalently $(xy)x = y$ [28]. A function $h : Q_1 \to Q_2$ between quasigroups Q_1, Q_2 is a *homomorphism* if $h(x) \cdot h(y) = h(xy)$ for all $x, y \in Q_1$; if h is bijective, then it is an *isomorphism*.

A *multiset* is a generalization of a set which allows for multiple instances of each element. We will consider a *cyclic order* on a multiset of 3 elements $\{x, y, z\}$ to be a ternary relation θ such that $\theta(x, y, z) \Leftrightarrow \theta(z, x, y)$ and if $x \neq y \neq z$ then $\theta(x, y, z) \Leftrightarrow \neg\theta(x, z, y)$ [12]. We call a pair of cyclic orders of the form $\theta_1(x, y, a), \theta_2(y, x, b)$ *partial opposites*; that is, to say, they share ≥ 2 common elements which are in reversed order in regards to each other. If partial opposites share all 3 elements, then they are simply *opposites* – note that any cyclic order of the form $\theta(x, x, y)$ or $\theta(x, x, x)$ is opposite to itself.

Define a *Mendelsohn triple* (x, y, z) to be a 3 element multiset $\{x, y, z\}$ with a cyclic order $\theta(x, y, z)$; a triple (x, y, z) will be said to *contain* the ordered pairs $(x, y), (y, z), (z, x)$ and no others. A *type n* triple contains n distinct elements. A *partial extended Mendelsohn triple system* is a pair (W, B) where W is a set and B is a set of Mendelsohn triples composed of elements of W such that for any $x, y \in W$, the ordered pair (x, y) is contained in at most 1 triple of B [7] [16]. If every possible pair (x, y) of W is contained in some triple of B, then the system is *complete* and simply an *extended Mendelsohn system.* (Partial) extended Mendelsohn triple systems are the only kind of combinatorial block design appearing in this paper, so we may abbreviate to *(partial) Mendelsohn systems* or *triple systems.*

There exists a well-known bijection between semisymmetric quasigroups and extended Mendelsohn systems [8]; for some partial semisymmetric quasigroup Q, let $M : Q \to M(Q)$ send it to the partial Mendelsohn system $M(Q)$ on the same underlying set such that $(x, y, z) \in M(Q)$ if and only if $xy = z, yz = x, zx = y$ in Q.

Suppose some graded partially ordered set $(P, \leqslant)$ with strictly monotone rank function $\rho : P \to \{-1, 0, 1, 2, \ldots, n\}$ sending elements $f_i \in P$, called *faces*, to integer values. Faces of rank n are *n-faces*; if there is no ambiguity, we may refer to 0-faces as *vertices* and 1-faces as *edges*. Faces f_1, f_2 are *incident* if $f_1 \leqslant f_2$ or $f_2 \leqslant f_1$. Any maximal totally ordered subset of $F_i \subset P$ is called a *flag*, and any 2 flags are *adjacent* if they differ by exactly 1 face. If for any 2 flags $F_x, F_y \subset P$, there exists some sequence of flags $(F_0, F_1, ..., F_n)$ where $F_0 = F_x$ and $F_n = F_y$ such that any 2 successive flags F_i, F_{i+1} are adjacent and $F_x \cap F_y \subseteq F_i$ for all i, then P is *strongly flag-connected.* P is said to satisfy the *diamond condition* if any pair of incident faces that differ in rank by 2 have exactly 2 incident faces strictly between them.

A graded poset $(P, \leqslant)$ is an *abstract n-polytope* if it is strongly flag-connected, satisfies the diamond condition, contains a unique least face of rank -1 and a unique greatest face of rank n, and all flags of P contain exactly $n + 2$ faces [9] [26]. The least and greatest faces are referred to as *improper faces* and all others are referred to as *proper faces* of P. An abstract 3-polytope is an *abstract polyhedron.* We will call a polyhedron *cubic* if each of its vertices is incident to exactly 3 edges. From hereon, all polytopes are assumed to be abstract and all quasigroups are assumed to be finite.

A CW complex is *regular* if the characteristic map of each closed cell is a homeomorphism onto its image [20] [21]. We will refer to a compact 2-manifold as a *closed surface.* A closed surface is *orientable* iff its second homology group H_2 is isomorphic to $\mathbb{Z}$ [13] [18]. For a topological space T_1 and subset $S \subseteq T_1$, let $\mathrm{int}_{T_1}(S)$ be the interior of S with respect to T_1. If γ is a continuous function from T_1 to space T_2, let $\gamma|_S$ be the restriction of γ mapping S to $\gamma(S)$, with both S and $\gamma(S)$ equipped with their respective subspace topologies.

This paper heavily references the author's previous work in [19], the relevant details of which will be summarized here in brief:

Given a partial semisymmetric quasigroup Q, define a function $D : Q \to D(Q)$ sending it to the directed graph such that there exists exactly 1 vertex in $D(Q)$ for every type 1 and type 2 triple of $M(Q)$, and for any vertices v_1, v_2 corresponding to triples t_1, t_2, then v_1 directly succeeds v_2 if and only if there exists some element x in both t_1, t_2 such that more instances of x are contained within t_1 than within t_2. Define function $G : Q \to G(Q)$ sending Q to the undirected multigraph such that there exists a vertex in $G(Q)$ for every type 3 triple of $M(Q)$, and for any triples t_1, t_2 mapping to vertices v_1, v_2, there is exactly 1 edge linking v_1 and v_2 for every pair of elements t_1 and t_2 have in common. We refer to the partial quasigroups corresponding to the maximally connected components of $G(Q)$ as the *components* of Q. Note that for any component q, its graph $G(q)$ is always 3-regular, and if q does not correspond to a *commutative pair* of triples of the form $\{(x, y, z), (z, y, x)\}$, then $G(q)$ is a simple graph. Define a *free component* to be a partial semisymmetric quasigroup q such that $G(q)$ is connected and 3-regular – then every component of any semisymmetric quasigroup is isomorphic to some free component. In general, if there is little chance for confusion we will use the same terminology between Q, $M(Q)$, and $G(Q)$, *e.g.* we may refer to a triples in $M(Q)$ corresponding to adjacent vertices in $G(Q)$ as "adjacent triples," or a vertex corresponding to a triple containing an element x as a "vertex containing x" etc.

Given a free component q and an element $x \in q$, we call a cycle in $G(q)$ an *element-cycle* for x iff for every vertex in the cycle, its corresponding triple in $M(q)$ contains x. Each vertex of $G(q)$ is contained in exactly 3 element-cycles, and each edge in exactly 2 element-cycles. Let $P : q \to P(q)$ send q to the abstract polyhedron $P(q)$ such that for each vertex, edge, and element-cycle of $G(q)$ there exists a unique vertex, edge, or 2-face of $P(q)$, and such that the incidence structure is preserved. Explicitly: a 0-face f_w is incident to a 1-face f_x of $P(q)$ iff the vertex corresponding to f_w is incident to the edge represented by f_x in $G(q)$, and a 0- or 1-face f_y is incident to a 2-face f_z iff the vertex or edge corresponding to f_y is contained within the element-cycle represented by f_z. Let $P_M : M(q) \to P(q)$ be the function constructed in the same manner as P, but with $M(q)$ as its domain.

Define an *oriented vertex* as a pair $\hat{v} = (v, \theta)$ where v is a vertex of some polyhedron and θ is a cyclic order on the 2-faces incident to v. Define an *oriented polyhedron* as a pair $\hat{p} = (p, \Theta)$ where p is some cubic polyhedron and $\Theta : V \to \Theta(V)$ a function on the vertices $V \subset p$ sending each vertex

$v_i \mapsto \hat{v}_i$ to an oriented vertex such that the orientation on any $\hat{v}_1$ is partial opposite to that of any adjacent $\hat{v}_2$; we call Θ an *orientation* on p. Then let $\hat{P} : q \to \hat{P}(q)$ send free component q to the oriented polyhedron $\hat{P}(q) = (P(q), \Theta_q)$ such that Θ_q sends each vertex $v_i \in P$ to an oriented vertex $\hat{v}_i$ with a cyclic order matching that of its corresponding triple in $M(q)$. Likewise, let $\hat{P}_M : M(q) \to \hat{P}(q)$ be constructed in the same manner, except with $M(q)$ as its domain.

Let an *alignment* be an ordered triple (d, O, Ψ) such that d is a directed graph without 2-cycles where each vertex has outdegree $\leqslant 1$, $O = \{\hat{p}_1, \hat{p}_2, ..., \hat{p}_n\}$ some set of oriented polyhedra, and $\Psi = \{\psi_1, \psi_2, ..., \psi_n\}$ a set of functions $\psi_i : \hat{p}_i \to d$ taking each 2-face of its respective $\hat{p}_i \in O$ to some vertex in d such that the following requirements are satisfied: letting U be the set of all unordered pairs of 2-faces of any $\hat{p}_i$ sharing some incident edge, for each $\{f_x, f_y\} \in U$ the pair $\{\psi_i(f_x), \psi_i(f_y)\}$ is unique. Further, there is no v_1 directly succeeded by v_2 such that some face $f_1 \in \Psi^{-1}(v_1)$ shares an incident edge with some $f_2 \in \Psi^{-1}(v_2)$, where $\Psi^{-1}(v_i) = \{f_x | \psi_x(f_x) = v_i\}$, that is to say Ψ^{-1} is the preimage of $v_i \in d$ across all $\psi_x \in \Psi$. Finally, for each $v_x \in d$, the sum of the outdegree of v_x + the number of other vertices in d it directly succeeds + the total number of edges of each 2-face mapped to v_x across all $\psi_x \in \Psi$ + 1 is exactly equal to the order of d.

Then alignments and semisymmetric quasigroups are in bijection up to isomorphism. Given any semisymmetric quasigroup Q, let $\alpha : Q \to \alpha(Q)$ take Q to its corresponding alignment. One can construct $\alpha(Q)$ by taking $d = D(Q)$, $O = \{\hat{P}(q_1), \hat{P}(q_2), ...\hat{P}(q_n)\}$ for each component q_i of Q, and letting each ψ_i send each 2-face of $\hat{P}(q_i)$ representing an element cycle for some $x \in Q$ to the vertex in $D(Q)$ corresponding to the triple (x, x, x) or $(x, x, -)$ in $M(Q)$. Conversely, given an alignment one can recover the full structure of its associated quasigroup up to isomorphism by arbitrarily assigning a unique label to every vertex of d, then deriving the Mendelsohn triples for the vertices of d and O accordingly.

3. Maps on polyhedra

Suppose semisymmetric quasigroups Q_1, Q_2 and a homomorphism $h : Q_1 \to Q_2$; this can straightforwardly be extended to a map $h_M : M(Q_1) \to M(Q_2)$ sending $(x, y, z) \mapsto (h(x), h(y), h(z))$. Then for a given component q of Q_1, let $h_M|_q : M(q) \to h_M(M(q))$ denote the restriction of h_M to the image of $M(q)$.

Lemma 3.1. *Given semisymmetric quasigroups Q_1, Q_2, a homomorphism $h : Q_1 \to Q_2$, and a component q of Q_1, if any triple $t_1 \in M(q)$ maps to a type 1 or type 2 triple $t_2 \in h_M(M(q))$, then all triples in $M(q)$ map to t_2.*

Proof. If $t_1 = (x, y, z)$ maps to a type 2 triple $t_2 = (a, a, b)$ such that $h(x) = h(y) = a$ and $h(z) = b$, then any triple t_3 adjacent to t_1 by definition must have at least 2 elements in common with t_1, and thus $h_M(t_3)$ must be of the form $(a, b, -)$, $(b, a, -)$, or $(a, a, -)$. Since we already know $(a, a, b) \in h_M(M(q))$, these can only be completed to $(a, a, b) = t_2$. Likewise, any triple adjacent to t_3 must also map to t_2; then because $G(q)$ is connected, all triples in $M(q)$ must map to t_2.The same logic applies if t_1 maps to a type 1 triple. □

Lemma 3.2. *Given semisymmetric quasigroups Q_1, Q_2, a homomorphism $h : Q_1 \to Q_2$, and components q_1 of Q_1 and q_2 of Q_2, if any triple $t_1 \in M(q_1)$ maps to a triple $t_2 \in h_M(M(q_1))$ which lies within $M(q_2)$, then all triples in $M(q_1)$ map to triples in $M(q_2)$. Further, $h_M|_{q_1}$ is surjective onto $M(q_2)$.*

Proof. Any triple $t_3 \in M(q_1)$ adjacent to t_1 must share at least 2 elements in common with t_1, and thus $h_M(t_3)$ must share at least two elements in common with $h_M(t_1) = t_2$. Then by definition, $h_M(t_3)$ is adjacent to t_2 and is contained in q_2. Likewise, any triple t_4 adjacent to t_3 must map to some $h_M(t_4) \in M(q_2)$ having at least 2 elements in common with $h_M(t_3)$, therefore $h_M(t_3)$ and $h_M(t_4)$ are adjacent. Then because $G(q_2)$ is connected, all triples in $h_M(M(q_1))$ must be within $M(q_2)$.

Now suppose some triples $t_x, t_y \in M(q_2)$ such that $t_x \in h_M(M(q_1))$ and t_x is adjacent to t_y. Then there exists some triple of the form $(a, b, c) \in M(q_1)$ where $t_x = (h(a), h(b), h(c))$ and $t_y = (h(b), h(a), w)$. Because Q_1 is semisymmetric and $G(q_1)$ is maximally connected, there must also be a triple $(b, a, d) \in M(q_1)$. Then because h is a homomorphism, $h(d) = h(b \cdot a) = h(b) \cdot h(a) = w$. Therefore if a triple in $M(q_2)$ has at least 1 preimage in $M(q_1)$ under h, then any adjacent triple also must have at least 1 premimage in $M(q_1)$ under h, and so because $G(q_2)$ is connected, $h_M|_{q_1}$ surjects onto $M(q_2)$. □

Corollary 3.3. *Given semisymmetric quasigroups Q_1, Q_2, a homomorphism $h : Q_1 \to Q_2$, and components q_1 of Q_1 and q_2 of Q_2 such that $h_M(M(q_1)) = M(q_2)$, then the preimages of $h_M|_{q_1}$ for any 2 triples in $M(q_2)$ are of the same cardinality.*

Proof. Suppose triples $(a, b, c) \in M(q_1)$ and $(x, y, z), (x, z, w) \in M(q_2)$ such that $h(a) = x, h(b) = y, h(c) = z$. By Lemma 3.2 there must be some triple $(a, c, d) \in M(q_1)$ where $h(d) = w$. Now suppose some additional triple $t_1 \in M(q_1)$ such that $h_M(t_1) = (x, y, z)$ and $t_1 \neq (a, b, c)$. Again, by Lemma 3.2 there must be some triple $t_2 \in M(q_1)$ where $h_M(t_2) = (x, z, w)$, however, it remains to be proven that $t_2 \neq (a, c, d)$.

If $t_2 = (a, c, d)$, then because t_1 is adjacent, it must be of the form $(c, a, -)$, $(a, d, -)$, or $(d, c, -)$. But it cannot be $(c, a, -)$ because the ordered pair (c, a) already occurs in (a, b, c), and it cannot be $(a, d, -)$ or $(d, c, -)$ because, as $h_M(t_1) = (h(a), h(b), h(c)) = (x, y, z)$, either of these possibilities would imply $h(d) = y$, when we already know $h(d) = w$. Thus $t_2 \neq (a, c, d)$, which means that every triple in $M(q_2)$ must have the same number of preimages as any adjacent triple, and by extension any triple in $M(q_2)$. □

Lemma 3.4. *Given semisymmetric quasigroups Q_1, Q_2, a surjective homomorphism $h : Q_1 \to Q_2$, and a component q_2 of Q_2, there exists at least 1 component q_1 of Q_1 such that $h_M(M(q_1)) = M(q_2)$.*

Proof. Because h is surjective, every element $x_2 \in Q_2$ must have at least 1 element $x_1 \in Q_1$ where $h(x_1) = x_2$. Suppose some triple $t_2 = (x_2, y_2, z_2) \in M(Q_2)$, and select elements $x_1, y_1 \in Q_1$ such that $h(x_1) = x_2$ and $h(y_1) = y_2$. Then there must exist some triple $t_1 = (x_1, y_1, x_1 \cdot y_1) \in M(Q_1)$, and because h is a homomorphism $h(x_1 \cdot y_1) = h(x_1) \cdot h(y_1) = x_2 \cdot y_2 = z_2$, and so $h_M(t_1) = t_2$. Then by Lemma 3.1, t_1 must be a type 3 triple, so there is some component q_1 of Q_1 such that $t_1 \in M(q_1)$, and by Lemma 3.2 $h_M(M(q_1)) = M(q_2)$. □

Given oriented polyhedra $\hat{p}_1, \hat{p}_2$, let a monotone surjection $\beta : \hat{p}_1 \to \hat{p}_2$ be called *orientation preserving* iff every oriented vertex $\hat{v}_i \in \hat{p}_1$ with cyclic order on incident 2-faces $\theta_i = (f_x, f_y, f_z)$ is sent to some oriented vertex $\beta(\hat{v}_i) = \hat{v}_j \in \hat{p}_2$ with cyclic order θ_j such that $\theta_j = (\beta(f_x), \beta(f_y), \beta(f_z))$. Then given semisymmetric quasigroups Q_1, Q_2, a homomorphism $h : Q_1 \to Q_2$, and components q_1 of Q_1 and q_2 of Q_2 such that $M(h(q_1)) = M(q_2)$, let the *induced map on polyhedra* be the orientation preserving map $h_{\hat{P}} : \hat{P}(q_1) \to \hat{P}(q_2)$ such that the following diagram commutes:

$$\begin{array}{ccc} M(q_1) & \xrightarrow{h_M} & M(q_2) \\ {\scriptstyle \hat{P}_M}\downarrow & & \downarrow{\scriptstyle \hat{P}_M} \\ \hat{P}(q_1) & \xrightarrow[h_{\hat{P}}]{} & \hat{P}(q_2) \end{array}$$

Figure 1: Diagram for induced map $h_{\hat{P}}$

Proposition 3.5. *Given semisymmetric quasigroups Q_1, Q_2, a homomorphism $h : Q_1 \to Q_2$, and components q_1 of Q_1 and q_2 of Q_2 such that $h_M(M(q_1)) = M(q_2)$, there exists a unique induced map $h_{\hat{P}} : \hat{P}(q_1) \to \hat{P}(q_2)$.*

Proof. By definition, $\hat{P}_M$ is bijective between Mendelsohn triples of $M(q_x)$ and vertices of $\hat{P}(q_x)$, and by Lemma 3.2 h_M is surjective from $M(q_1)$ onto $M(q_2)$; therefore, for any vertex $\hat{v}_1 \in \hat{P}(q_1)$, there exists exactly 1 vertex $h_{\hat{P}}(\hat{v}_1) \in \hat{P}(q_2)$ such that figure 1 commutes, given by $h_{\hat{P}}(\hat{v}_1) = \hat{P}_M(h_M(\hat{P}_M^{-1}(\hat{v}_1)))$, the image of its $\hat{P}_M$ preimage under the composition of h_M and $\hat{P}_M$, and further, $\hat{P}_M$ surjects from the vertices of $\hat{P}(q_1)$ onto the vertices of $\hat{P}(q_2)$.

There exists an edge $e_x \in \hat{P}_M(M(q_x))$ linking vertices v_i, v_j for every pair of elements the triples $\hat{P}_M^{-1}(v_i)$ and $\hat{P}_M^{-1}(v_j)$ have in common, so because h is a homomorphism, for any edge $e_1 \in \hat{P}(q_1)$ linking vertices v_1, v_2, then for their triples $\hat{P}_M^{-1}(v_1) = t_1, \hat{P}_M^{-1}(v_2) = t_2 \in M(q_1)$, there is a corresponding pair of elements which $h_M(t_1)$ and $h_M(t_2)$ have in common, and thus there is a single corresponding edge in $\hat{P}(q_2)$ such that $h_{\hat{P}}$ is monotone.

By definition, $\hat{P}_M$ is bijective between the element-cycles of $G(q_x)$ and the 2-faces of $\hat{P}(q_x)$. Suppose some cycle o_a of $G(q_1)$ such that o_a is an element cycle for an element $a \in q_1$. Then the vertices of o_a correspond to some sequence of elements $(b_1, b_2, ...b_n)$ where $a \cdot b_1 = b_2, a \cdot b_2 = b_3, ...a \cdot b_n = b_1$. Therefore, because h is a homomorphism, there must be some sequence of elements $(h(b_1), h(b_2), ...h(b_n))$ in c_2 where $h(a) \cdot h(b_1) = h(b_2), h(a) \cdot h(b_2) = h(b_3), ...h(a) \cdot h(b_n) = h(b_1)$. Then there are corresponding triples $(h(a), h(b_1), h(b_2)), (h(a), h(b_2), h(b_3))$ etc. where each triple is adjacent to the next, and all triples contain $h(a)$ – thus, the associated vertices in $G(c_2)$ constitute an element-cycle for $h(a)$. Then $h_{\hat{P}}$ sends any given 2-face of $\hat{P}(q_1)$ representing an element cycle for a to a 2-face in $\hat{P}(q_2)$ representing an element-cycle for $h(a)$, and any other function on the 2-faces of $\hat{P}(q_1)$

would fail to be monotone.

Clearly, the only 3-face and only -1-face of $\hat{P}(q_1)$ must be sent to the only 3-face and -1 face of $\hat{P}(q_2)$, respectively.

Suppose an oriented vertex $\hat{v}_1 \in \hat{P}_M(M(q_1))$ where $\theta_1 = (f_x, f_y, f_z)$ in regard to its incident 2-faces; then there is a corresponding triple $(x, y, z) \in M(q_1)$. So because $h_M(x, y, z) = (h(x), h(y), h(z))$, we have orientation $\theta_2 = (h_{\hat{P}}(f_x), h_{\hat{P}}(f_y), h_{\hat{P}}(f_z))$ on $h_{\hat{P}}(v_1)$. Thus, there exists a unique orientation preserving map $h_{\hat{P}}$ induced on $\hat{P}(q_1)$ by h. □

For example, consider Q_4, the Mendelsohn quasigroup of order 4, and the natural projection $h : Q_4 \times \mathbb{Z}_2 \to Q_4$ from its direct product with $\mathbb{Z}_2$. Let q_1 and q_2 be the components of $Q_4 \times \mathbb{Z}_2$ and Q_4 of greatest cardinality. $P(q_1)$ is equivalent to the face lattice of a truncated tetrahedron, and $P(q_2)$ to that of a tetrahedron; h induces an orientation preserving map $h_{\hat{P}} : \hat{P}(q_1) \to \hat{P}(q_2)$ identifying 2-faces on opposite sides of the polyhedron, and identifying edges and vertices on opposite sides of their respective "hexagons."

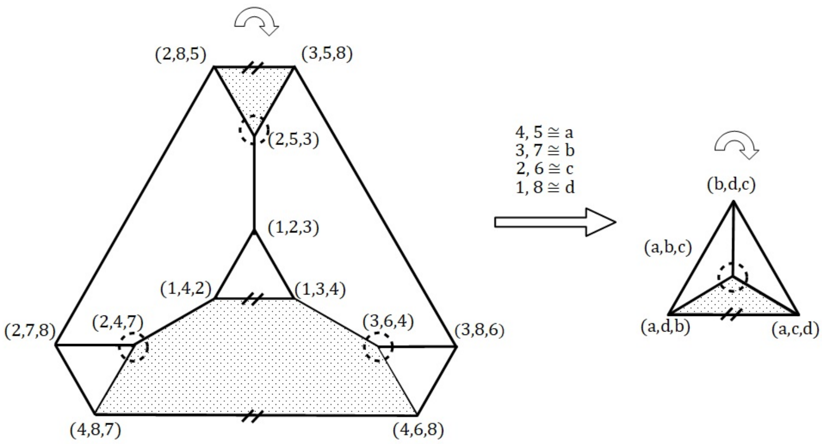

Figure 2: Illustration of $h_{\hat{P}}$ with selected vertices, edges, and 2-faces marked; vertices are labeled with their corresponding triples, curved arrows indicate orientation

Proposition 3.6. *Given semisymmetric quasigroups Q_1, Q_2, a homomorphism $h : Q_1 \to Q_2$, components q_1, q_2 such that $h_{\hat{P}}(\hat{P}(q_1)) = \hat{P}(q_2)$, and a pair of 2-faces $f_1, f_2 \in \hat{P}(q_1)$ where $f_1 \neq f_2$ and there exists some vertex v such that $v \leqslant f_1, f_2$, then $h_{\hat{P}}(f_1) \neq h_{\hat{P}}(f_2)$.*

Proof. The vertex v represents some triple $\hat{P}_M(v)^{-1} = (x, y, z) \in M(q_1)$, and since f_1, f_2 are incident to v, they must correspond to distinct element-cycles for some $x, y \in Q_1$ which contain (x, y, z). If $h_{\hat{P}}(f_1) = h_{\hat{P}}(f_2)$, then necessarily $h(x) = h(y)$, meaning $h_M((x, y, z))$ is a type 1 or type 2 triple of $M(q_2)$ – but q_2 is a component, which by definition implies $M(q_2)$ contains only type 3 triples. So then $h_{\hat{P}}(f_1) \neq h_{\hat{P}}(f_2)$, that is to say, no pair of distinct 2-faces of $\hat{P}(q_1)$ incident to a common vertex can map to the same 2-face of $\hat{P}(q_2)$. □

Corollary 3.7. *Given semisymmetric quasigroups Q_1, Q_2, a homomorphism $h : Q_1 \to Q_2$, components q_1, q_2 such that $h_{\hat{P}}(\hat{P}(q_1)) = \hat{P}(q_2)$, and a pair of edges $e_1, e_2 \in \hat{P}(q_1)$ where $e_1 \neq e_2$ and there exists some vertex v such that $v \leqslant e_1, e_2$, then $h_{\hat{P}}(e_1) \neq h_{\hat{P}}(e_2)$.*

Proof. As $\hat{P}(q_1)$ is a cubic polyhedron, there must exist distinct 2-faces f_1, f_2, f_3 where $e_1 \leqslant f_1$, $e_2 \leqslant f_2$, $e_1, e_2 \leqslant f_3$, and $v \leqslant f_1, f_2, f_3$. Now suppose $h_{\hat{P}}(e_1) = h_{\hat{P}}(e_2)$; because $h_{\hat{P}}$ is monotone, we have $h_{\hat{P}}(e_1) \leqslant h_{\hat{P}}(f_1), h_{\hat{P}}(f_2), h_{\hat{P}}(f_3)$. However, the diamond condition implies there must be exactly 2 incident 2-faces for any given edge of an abstract polyhedron, thus $h_{\hat{P}}$ must map at least 2 of f_1, f_2, f_3 to the same 2-face of $\hat{P}(q_2)$ – but this is impossible by proposition 3.6. Therefore $h_{\hat{P}}(e_1) \neq h_{\hat{P}}(e_2)$, that is to say, no pair of distinct edges of f $\hat{P}(q_1)$ incident to a common vertex can map to the same edge of $\hat{P}(q_2)$. □

4. Constructing surfaces from quasigroups

The *Euler characteristic* χ of a finite CW complex C of dimension n is a topological invariant defined as [18] [21]:

$$\chi(C) = \sum_{k=0}^{n} (-1)^k n_k \tag{1}$$

where n_k is the number of k-cells of C.

Note that χ can be calculated via purely combinatorial means, which allows us to extend this formula to abstract polytopes as well. Define the *Euler characteristic* of a free component q to be:

$$\chi(q) = \sum_{k=0}^{n-1} (-1)^k n_k \tag{2}$$

where n is the dimension and n_k the number of proper k-faces of $P(q)$.

As $P(q)$ is always a cubic polyhedron, this simplifies to $\chi(q) = |F| - |V|/2$ where $F, V \subset P(q)$ are the sets of 2- and 0-faces. It is not difficult to see that χ is an isomorphism invariant.

Proposition 4.1. *Given semisymmetric quasigroups Q_1, Q_2, an isomorphism $h : Q_1 \to Q_2$, and components q_1 of Q_1 and q_2 of Q_2 such that $h_{\hat{P}}(\hat{P}(q_1)) = \hat{P}(q_2)$, then $\chi(q_1) = \chi(q_2)$.*

Proof. If h is an isomorphism, then its inverse h^{-1} is also an isomorphism, so $h_{\hat{P}}$ is a monotone bijection with a monotone inverse $h_{\hat{P}}^{-1}$ and is thus an order isomorphism. Clearly then, $\hat{P}(q_1)$ and $\hat{P}(q_2)$ have the same number of vertices, edges, and 2-faces. □

Of course, abstract polytopes are merely partially ordered sets, and thus taking the Euler characteristic of an abstract polyhedron does not come equipped with a direct topological interpretation. Ideally, we would want there to exist some reasonably natural method of translating $\hat{P}(q)$ of any free component q into a surface C such that $\chi(q) = \chi(C)$. To this end, let the *surface realization* $\sigma : q \to \sigma(q)$ send q to the regular CW complex $\sigma(q)$ such that for each proper n-face of $\hat{P}(q)$ there exists an n-cell of $\sigma(q)$, and such that the poset formed by the closed cells of $\sigma(q)$ under inclusion is isomorphic to the poset formed by the proper faces of $\hat{P}(q)$ under its incidence relation $\leq$.

Lemma 4.2. *Given a free component q, up to homeomorphism there exists a unique $\sigma(q)$.*

Proof. A closed 1-cell has 2 end points, which must be distinct from each other if the boundary map for said 1-cell is to be injective onto its image. Hence, there exists a 1 dimensional regular CW complex having the same order with respect to inclusion as that of the 0- and 1-faces of $\hat{P}(q)$ with respect to incidence if and only if for each 1-face, there exist exactly 2 incident 0-faces v_1, v_2 where $v_1 \neq v_2$ - this is given by the diamond condition, as there must be a pair of 0-faces between any 1-face and the unique -1-face of the polyhedron. Likewise, the boundary of a closed 2-cell is a circle, so there exists a regular CW complex $\sigma(q)$ with the proper incidence structure iff for every 2-face of $\hat{P}(q)$, its incident 1-faces correspond to cycles in the 1-skeleton of $\sigma(q)$, which holds true as the 2-faces of $\hat{P}(q)$ correspond to element-cycles by definition. Then because $\sigma(q)$ is regular and its incidence structure has been specified, there exists only 1 choice of boundary map for any cell and therefore any 2 CW complexes satisfying the conditions of $\sigma(q)$ are homeomorphic, thus $\sigma(q)$ is unique up to homeomorphism [4] [14]. □

Lemma 4.3. *Given any free component q, $\sigma(q)$ is a closed surface.*

Proof. $\sigma(q)$ is a finite CW complex, and therefore a compact, second-countable Hausdorff space [18]. Any point $z \in \sigma(q)$ is within some 0-cell, or the interior of some 1- or 2-cell. If z is in the interior of a 2-cell, then clearly it has an open neighborhood homeomorphic to Euclidean space $\mathbb{R}^2$. Suppose z is on the interior of a 1-cell; because $\hat{P}(q)$ is a cubic polyhedron, each 1-cell of $\sigma(q)$ is is within the boundary of exactly 2 closed 2-cells. Then, where $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 | y \geq 0\}$ is the closed upper half-plane, z must have an open neighborhood homeomorphic to $\mathbb{H}^2_1 \coprod \mathbb{H}^2_2 / \sim_2$, the disjoint union of 2 copies of $\mathbb{H}^2$ under the equivalence relation identifying points $(x, 0) \in \mathbb{H}^2_1 \sim_2 (x, 0) \in \mathbb{H}^2_2$; that is to say, 2 half-planes with their boundaries identified, which is homeomorphic to $\mathbb{R}^2$. Similarly, supposing instead z is on a 0-cell, then because $\hat{P}(q)$ is cubic, z must have an open neighborhood homeomorphic to $(\mathbb{H}^2_1 \coprod \mathbb{H}^2_2 \coprod \mathbb{H}^2_3) / \sim_3$, the disjoint union of 3 copies of $\mathbb{H}^2$ under the relation identifying $(x \geq 0, 0) \in \mathbb{H}^2_1 \sim_3 (x \leqslant 0, 0) \in \mathbb{H}^2_2$ and $(x \geq 0, 0) \in \mathbb{H}^2_2 \sim_3 (x \leqslant 0, 0) \in \mathbb{H}^2_3$ and $(x \geq 0, 0) \in \mathbb{H}^2_3 \sim_3 (x \leqslant 0, 0) \in \mathbb{H}^2_1$; that is, 3 half-planes with their adjacent boundaries identified, which is likewise homeomorphic to $\mathbb{R}^2$. □

Proposition 4.4. *Given any free component q, $\sigma(q)$ is orientable.*

Proof. By Lemma 4.3, $\sigma(q)$ is a closed surface. An orientation on the 2-cells of $\sigma(q)$ can be constructed from the orientations on the vertices of $\hat{P}(q)$ in the following manner: given a closed 2-cell $a_x \subset \sigma(q)$, for any 2 closed 1-cells $e_1, e_2 \subset a_x$, let e_1 directly precede e_2 with respect to a_x if and only if both are incident to some point corresponding to an oriented vertex $\hat{v}_x \in \hat{P}(q)$ with orientation $\theta_x = (f_1, f_2, f_3)$ on its incident 2-faces such that f_1 corresponds to a 2-cell in $\sigma(q)$ containing e_1, f_2 corresponds to a 2-cell containing e_2, and f_3 corresponds to a_x. Then because adjacent vertices in $\hat{P}(q)$ must be partial opposites, adjacent 2-cells in $\sigma(q)$ will always have opposite orientations in regard to their shared edge.

Now consider the cellular chain complex generated by the n-cells of $\sigma(q)$ with coefficients in $\mathbb{Z}$:

$$\dots \xrightarrow{d_4} 0 \xrightarrow{d_3} \mathbb{Z}^f \xrightarrow{d_2} \mathbb{Z}^e \xrightarrow{d_1} \mathbb{Z}^v \xrightarrow{d_0} 0$$

Figure 3: Cellular chain complex of $\sigma(q)$

There are no 3-cells, so the image of the boundary operator d_3 must be 0. Then consider the element c of the 2nd chain group $\mathbb{Z}^f$ representing 1 copy of each 2-cell. Every 1-cell $e_x \subset \sigma(q)$ has exactly 2 incident 2-cells, which must have opposite orientations in regard to e_x, and the attaching maps of $\sigma(q)$ are by definition injective hence degree 1, therefore [10] [21]:

$$d_2(c) = \sum_{x=1}^{f} e_x - e_x = 0 \tag{3}$$

and thus the kernel of d_2 is nontrivial. Then $H_2 = \ker(d_2)/\operatorname{im}(d_3)$ must be nontrivial, and H_2 of a closed surface is nontrivial if and only if $H_2 = \mathbb{Z}$[13][18]. □

In light of this, one can obtain a version of the Riemann-Hurwitz formula as applied to the surfaces associated with semisymmetric quasigroups:

Corollary 4.5. *Given semisymmetric quasigroups Q_1, Q_2, a homomorphism $h : Q_1 \to Q_2$, and components q_1, q_2 such that $h_{\hat{P}}(\hat{P}(q_1)) = \hat{P}(q_2)$, then $\chi(q_1) \leqslant \chi(q_2)$.*

Proof. Let V and F be the numbers of vertices and 2-faces of $\hat{P}(q_1)$; by Corollary 3.3 and Proposition 3.5, $\hat{P}(q_2)$ must have V/d vertices for some positive integer d. Because $h_{\hat{P}}$ surjects onto $\hat{P}(q_2)$, the number of 2-faces of $\hat{P}(q_2)$ must be $\leqslant F$. Further, the 2-faces of $\hat{P}(q_2)$ must be $\geq F/d$, else there would necessarily be some 2-face $f_i \in \hat{P}(q_1)$ and vertex $v_i \leqslant f_i$ such that $h_{\hat{P}}(v_i) \nleq h_{\hat{P}}(f_i)$, which would violate the monotonicity of $h_{\hat{P}}$. Then there exists some real number $1 \leqslant r \leqslant d$ such that there are F/r 2-faces of $\hat{P}(q_2)$, and so $\chi(q_1) = F - V/2$ and $\chi(q_2) = F/r - V/(2d)$. Clearly, if all other variables remain fixed, increasing the value of r will decrease the value of $\chi(q_2)$.

Let us assume that $r = d$, the maximum possible value: then $\chi(q_2) = F/d - V/(2d) = (F - V/2)/d = \chi(q_1)/d$. Suppose $\chi(q_1) > \chi(q_2)$; this implies $\chi(q_1) > \chi(q_1)/d$, which can only be true if $\chi(q_1)$ is positive. Now, given any free component q_x, by Lemma 4.3 and Proposition 4.4 $\sigma(q_x)$ is a closed, orientable surface, and by definition $\chi(\sigma(q_x)) = \chi(q_x)$, therefore $\chi(q_x)$ is an even integer $\leqslant 2$ [18] [25]. Thus, in this case, the only possible value for $\chi(q_1)$ would be 2 – but then $\chi(q_2) = \chi(q_1)/d = 2/d$, and so $0 < \chi(q_2) < 2$, which is impossible. So even supposing the minimum possible value for $\chi(q_2)$, still necessarily $\chi(q_1) \leqslant \chi(q_2)$. □

Given semisymmetric quasigroup Q with a set of components $\{q_1, q_2, ..., q_n\}$, define the *genus* of Q to be:

$$g(Q) = \sum_{x=1}^{n} 1 - \chi(q_n)/2 \tag{4}$$

$g(Q)$ represents the total number of tori under connected sums and disjoint unions which would be required to construct a set of surfaces homeomorphic to $\{\sigma(q_1), \sigma(q_2), ..., \sigma(q_n)\}$, with the sphere considered to require 0 tori [18] [25].

Theorem 4.6. *For any $n \in \mathbb{N}$, the class of quasigroups of genus $\leqslant n$ is closed under subquasigroups and homomorphic images.*

Proof. Given a semisymmetric quasigroup Q_1 and component q_i of Q, then $\chi(q_i) \leqslant 2$, therefore $1 - \chi(q_i)/2 \geq 0$. Semisymmetric quasigroups form a variety in the sense of universal algebra [28], and are thus closed under subobjects and homomorphic images [3]; therefore, any subquasigroup $Q_2 \subseteq Q_1$ is likewise semisymmetric, and so the set of polyhedra of its alignment $\alpha(Q_2)$ must form a subset of the polyhedra of $\alpha(Q_1)$ [19]. Clearly then, if S_1 is a set of natural numbers and $S_2 \subseteq S_1$, the sum of all elements of S_2 cannot be greater than the sum of all elements of S_1, hence $g(Q_1) \geq g(Q_2)$.

Now suppose quasigroup Q_3 and homomorphism $h : Q_1 \to Q_3$, and let $Q_4 \subseteq Q_3$ be the image of Q_1 in Q_3. Because Q_1 surjects onto Q_4, by Lemma 3.2, Corollary 3.3, and Proposition 3.5, every polyhedron of $\alpha(Q_4)$ has at least 1 $h_{\hat{P}}$ preimage in the polyhedra of $\alpha(Q_1)$. So any component q_4 of Q_4 must have be at least 1 preimage q_1 of Q_1, and moreover by Corollary 4.5, we have $1 - \chi(q_1)/2 \geq 1 - \chi(q_2)/2$. Therefore, the sum $g(Q_1)$ must be greater than or equal to the sum $g(Q_4)$. □

However, semisymmetric quasigroups of genus $\leqslant n$ do not form an equational variety, as they do not appear to be particularly well-behaved under direct products, the minimal example of this being $Q_3 \times Q_4$, letting Q_3 be the Steiner quasigroup of order 3 and Q_4 be the Mendelsohn quasigroup of order 4: then $g(Q_3) = 0$ and $g(Q_4) = 0$ but $g(Q_3 \times Q_4) = 1$.

5. Maps on surfaces

In the context of topology, the Riemann-Hurwitz formula describes relations between piecewise linear manifolds under branched covers, a specific kind of continuous mapping [2] [27]. Given that we have established that the surfaces associated with semisymmetric quasigroups obey similar relations under quasigroup homomorphisms, it seems reasonable to ask if said homomorphisms can be somehow translated into continuous mappings between those surfaces.

For the next few proofs, it will be convenient to use a slightly finer subdivision of $\sigma(q)$. Each open 2-cell $f \subset \sigma(q)$ is bounded by a cycle composed of k 0-cells and k 1-cells of $\bar{f}$ for some integer $k > 1$, where $\bar{f}$ denotes the closure of f. One may obtain a further cellular decomposition of f by including an additional 0-cell v_0 at the origin, and k additional 1-cells $e_x \subset f$ such that for every $v_x \in \bar{f}$ there is some e_x where $v_x \in \bar{e}_x$, and for any e_x, e_y then $\bar{e}_x \cap \bar{e}_y = v_0$. Then the remainder of f has been partitioned into k open 2-cells, each bounded by cycles of length 3 consisting of a 1-cell in the boundary of f and the pair of 1-cells linking its end points to v_0 [6]. Define $\Delta : q \to \Delta(q)$ to be the function sending a given free component q to the CW complex $\Delta(q)$ consisting of the 1-skeleton of $\sigma(q)$ along with the subdivision of each 2-cell of $\sigma(q)$ constructed via the above method.

Likewise, let $\Delta_{\hat{P}} : \hat{P}(q) \to \Delta(q)$ be the same function, except with $\hat{P}(q)$ as its domain. We will refer to the 0- and 1- cells inherited from $\sigma(q)$ as *polyhedral* and the 0- and 1-cells present in $\Delta(q)$ but not $\sigma(q)$ as *central*.

Remark 5.1. $\Delta(q)$ is combinatorially similar to a triangulation of $\sigma(q)$, although $\Delta(q)$ does not necessarily have a geometric realization as a simplicial complex with the same incidence structure – take, for instance, when q is a commutative pair.

Suppose semisymmetric quasigroups Q_1, Q_2, a homomorphism $h : Q_1 \to Q_2$, and components q_1, q_2 such that $h_{\hat{P}}(\hat{P}(q_1)) = \hat{P}(q_2)$. Then define the *induced map on surfaces* $h_\Delta : \Delta(q_1) \to \Delta(q_2)$ to be the function such that for any closed 2-cell c_i of $\Delta(q_1)$, the restriction $h_\Delta|_{c_i}$ is a homeomorphism onto its image in $\Delta(q_2)$, and such that the following diagram commutes:

$$\begin{array}{ccc} \hat{P}(q_1) & \xrightarrow{h_{\hat{P}}} & \hat{P}(q_2) \\ {\scriptstyle \Delta_{\hat{P}}}\downarrow & & \downarrow{\scriptstyle \Delta_{\hat{P}}} \\ \Delta(q_1) & \xrightarrow{h_{\Delta}} & \Delta(q_2) \end{array}$$

Figure 4: Diagram for induced map h_Δ

Proposition 5.2. *Given semisymmetric quasigroups Q_1, Q_2, a homomorphism $h : Q_1 \to Q_2$, and components q_1, q_2 such that $h_{\hat{P}}(\hat{P}(q_1)) = \hat{P}(q_2)$, there exists some induced map $h_\Delta : \Delta(q_1) \to \Delta(q_2)$; further, h_Δ is continuous and unique up to homotopy class.*

Proof. By definition, $\Delta_{\hat{P}}$ is bijective between the 0-, 1-, and 2-faces of $\hat{P}(q_x)$ and the polyhedral 0-cells, polyhedral 1-cells, and central 0-cells of $\Delta(q_x)$, respectively. Further, any central 1-cell must be bounded by a unique pair of 0-cells $(\Delta_{\hat{P}}(f_a), \Delta_{\hat{P}}(f_b))$, where f_a is 0-face and f_b a 2-face of $\hat{P}(q_x)$ such that $f_a \leqslant f_b$. Similarly, because each 1-face of $\hat{P}(q_x)$ is incident to a unique pair of 2-faces, and given every 2-cell of $\Delta(q_x)$ must contain exactly 1 polyhedral 1-cell in its boundary, it follows that each closed 2-cell contains a unique polyhedral 1-cell and central 0-cell pair $(\Delta_{\hat{P}}(f_c), \Delta_{\hat{p}}(f_d))$, where f_c is a 1-face and f_d a 2-face of $\hat{P}(q_x)$ such that $f_c \leqslant f_d$. So if figure 4 is to commute, then the cell of $\Delta(q_2)$ which any cell of $\Delta(q_1)$ is sent to under h_Δ is fully determined by the face of $\hat{P}(q_2)$ which its $\Delta_{\hat{P}}$ preimage in $\hat{P}(q_1)$ is sent to under $h_{\hat{P}}$.

For any closed 2-cell $c_i \subset \Delta(q_1)$, denote $h_\Delta|_{c_i}$ by ϕ_i. The domains of any set of ϕ_i are disjoint on the interiors of each closed 2-cell, overlapping only on the intersections between cells of $\Delta(q_1)$. Then there exists some h_Δ satisfying the above definition if and only if for any 2-cells $c_a, c_b \subset \Delta(q_1)$ with restriction maps ϕ_a, ϕ_b, then $\phi_a(c_a \cap c_b) = \phi_b(c_a \cap c_b)$; that is to say, if each ϕ_x agrees on the overlaps between their domains. This condition is trivially met if $c_a \cap c_b = \emptyset$. If $c_a \cap c_b \neq \emptyset$, then $c_a \cap c_b$ must be equal to some 0- or 1-cell of $\Delta(q_1)$, so ϕ_a, ϕ_b agree iff $\phi_a(c_a \cap c_b) \subset \phi_b(c_b)$ and $\phi_b(c_a \cap c_b) \subset \phi_a(c_a)$. By definition $\Delta_{\hat{P}}^{-1}(c_a \cap c_b) \leqslant \Delta_{\hat{P}}^{-1}(c_a), \Delta_{\hat{P}}^{-1}(c_b)$, thus the monotonicity of $h_{\hat{P}}$ implies $h_{\hat{P}}(\Delta_{\hat{P}}^{-1}(c_a \cap c_b)) \leqslant h_{\hat{P}}(\Delta_{\hat{P}}^{-1}(c_a)), h_{\hat{P}}(\Delta_{\hat{P}}^{-1}(c_b))$. Therefore, $\Delta_{\hat{P}}(h_{\hat{P}}(\Delta_{\hat{P}}^{-1}(c_a \cap c_b))) \subset \Delta_{\hat{P}}(h_{\hat{P}}(\Delta_{\hat{P}}^{-1}(c_a))), \Delta_{\hat{P}}(h_{\hat{P}}(\Delta_{\hat{P}}^{-1}(c_b)))$, meaning h_Δ always sends intersecting 2-cells of $\Delta(q_1)$ to intersecting 2-cells of $\Delta(q_2)$, hence there always exists some h_Δ. Because each ϕ_i is continuous, then $h_\Delta = \cup\phi_i$, the union of said mappings across all 2-cells of $\Delta(q_1)$, must likewise be continuous [22].

Now suppose δ_1, δ_2 are both maps from $\Delta(q_1)$ to $\Delta(q_2)$ induced by h. If c_i is some closed 2-cell of $\Delta(q_1)$, the restriction $\delta_x|_{c_i}$ is a homeomorphism onto its image in $\Delta(q_2)$, and furthermore must be orientation preserving i.e. of positive degree, else the 0- and 1-cells in the boundary of c_i would fail to commute with figure 4. There exists only 1 isotopy class of orientation preserving homeomorphism $D^2 \to D^2$ [1] [11], therefore $\delta_1|_{c_i}$ is isotopic to $\delta_2|_{c_i}$. Then we can construct a homotopy between δ_1 and δ_2 by selecting an appropriate isotopy for each 2-cell, and thus h_Δ is unique up to homotopy class. □

Lemma 5.3. *Given semisymmetric quasigroups Q_1, Q_2, a homomorphism $h : Q_1 \to Q_2$, and components q_1, q_2 such that $h_\Delta(\Delta(q_1)) = \Delta(q_2)$, there exists some positive integer d such that for any point $x \in \Delta(q_2)$ not on a central 0-cell, $|h_\Delta^{-1}(x)| = d$.*

Proof. Each n-cell of $\Delta(q_1)$ is mapped homeomorphically by h_Δ onto its image in $\Delta(q_2)$, therefore for any point $x \in \Delta(q_2)$, the cardinality of its preimage $|h_\Delta^{-1}(x)|$ is equal to the number of open n-cells mapped onto the open n-cell containing x. The polyhedral 0-cells of $\Delta(q_x)$ are in bijection with the 0-faces of $\hat{P}(q_x)$ and thus the triples of $M(q_x)$, so by Corollary 3.3, h_Δ^{-1} has equal cardinality d on all polyhedral 0-cells. Likewise, the polyhedral 1-cells are in bijection with the 1-faces of $\Delta(q_x)$, and there must be a single 1-face between each pair of adjacent 0-faces. If h_Δ^{-1} does not also have cardinality d on all polyhedral 1-cells, then there must exist some 1-face f_a and 0-faces $f_b, f_c \in \hat{P}(q_1)$ such that $f_b, f_c \leqslant f_a$ but $h_{\hat{P}}(f_b), h_{\hat{P}}(f_c) \nleqslant h_{\hat{P}}(f_a)$, which is impossible because $h_{\hat{P}}$ is monotone.

Central 0-cells of $\Delta(q_x)$ are in bijection with 2-faces of $\hat{P}(q_x)$, and each central 1-cell e_c is bounded by a polyhedral 0-cell v_p and a central 0-cell v_c such that $\Delta_{\hat{P}}^{-1}(v_p) \leqslant \Delta_{\hat{P}}^{-1}(v_c)$. So if e_c, v_p, v_c are within $\Delta(q_2)$, then for every $v_i \in h_\Delta^{-1}(v_p)$, we have a unique central 1-cell $e_i \in h_\Delta^{-1}(e_c)$ linking v_i to $h_\Delta^{-1}(v_c)$. In a similar manner, each 2-cell f_a of $\Delta(q_x)$ contains exactly 1 polyhedral 1-cell e_p and 1 central 0-cell v_c within its boundary such that $\Delta_{\hat{P}}^{-1}(e_p) \leqslant \Delta_{\hat{P}}^{-1}(v_c)$. If f_a, e_p, v_c are in $\Delta(q_2)$, then for every $e_i \in h_\Delta^{-1}(e_p)$ we obtain a unique 2-cell $f_i \in h_\Delta^{-1}(f_a)$ with e_i and $h_\Delta^{-1}(v_c)$ within its boundary. Therefore, $|h_\Delta^{-1}(x)| = d$ for any $x \in \Delta(q_2)$ not on a central 0-cell. □

Lemma 5.4. *Given semisymmetric quasigroups Q_1, Q_2, a homomorphism*

$h : Q_1 \to Q_2$, components q_1, q_2 such that $h_\Delta(\Delta(q_1)) = \Delta(q_2)$, and open cells c_i, c_j of $\Delta(q_1)$ such that $c_i \neq c_j$ and $\bar{c}_i \cap \bar{c}_j \setminus C_1 \neq \emptyset$, where C_1 is the set of central 0-cells of $\Delta(q_1)$, then $h_\Delta(c_i) \cap h_\Delta(c_j) = \emptyset$.

Proof. As demonstrated in Proposition 5.2, h_Δ maps n-cells to n-cells, therefore $h_\Delta(c_i) \cap h_\Delta(c_j) = \emptyset \Leftrightarrow h_\Delta(c_i) \neq h_\Delta(c_j)$; furthermore, h_Δ maps central cells to central cells and likewise for polyhedral cells.

The statement holds vacuously true if $n = 0$, as there would be no case where $\bar{c}_i \cap \bar{c}_j \neq \emptyset$. If $n = 1$, then $\bar{c}_i \cap \bar{c}_j$ is some polyhedral 0-cell c_k. If c_i, c_j are both polyhedral, then $h_\Delta(c_i) \neq h_\Delta(c_j)$ follows straightforwardly from Crollary 3.7. Now suppose c_i, c_j are central; then they correspond to pairs $(\Delta^{-1}_{\hat{P}}(c_k), f_i), (\Delta^{-1}_{\hat{P}}(c_k), f_j)$ where f_i, f_j are 2-faces of $\hat{P}(q_1)$ sharing some incident 0-face $\Delta^{-1}_{\hat{P}}(c_k)$. Thus, $h_\Delta(c_i) = h_\Delta(c_j)$ would imply $h_{\hat{P}}(f_i) = h_{\hat{P}}(f_j)$, which is impossible by Proposition 3.6.

In the final case $n = 2$, if $\bar{c}_i \cap \bar{c}_j \setminus C_1 \neq \emptyset$ then there exists some polyhedral 0-cell $c_x \in \bar{c}_i \cap \bar{c}_j \setminus C_1$. Given any pair of central 1-cells c_a, c_b within the boundary of any 2-cell, $\bar{c}_a \cap \bar{c}_b$ must be a central (not polyhedral) 0-cell; therefore, there must exist polyhedral 1-cells $c_c \subset \bar{c}_i, c_d \subset \bar{c}_j$ where $c_x \in \bar{c}_c \cap \bar{c}_d$. As the boundary of any 2-cell contains exactly 1 polyhedral 1-cell, then if $h_\Delta(c_i) = h_\Delta(c_j)$, necessarily $h_\Delta(c_c) = h_\Delta(c_d)$, which we just established cannot be the case as $c_x \in \bar{c}_c \cap \bar{c}_d \setminus C_1$. □

Suppose topological spaces T_1, T_2, a nowhere dense subset $t_1 \subset T_1$, and a continuous surjection $B : T_1 \to T_2$. Then B is a *branched cover* if for every point $x \in T_2$ where $x \notin B(t_1)$, there is some open set $X \subseteq T_2$ containing x such that $B^{-1}(X)$ is the union of disjoint open sets, each mapped homeomorphically onto X by B [24] [30]. The subset t_1 is referred to as the *singular set* of B, and its image $B(t_1)$ is called the *branch set*. Essentially, a branched cover is simply a covering map which fails to be a local homeomorphism on its singular set. For components of semisymmetric quasigroups, the reason we have defined the induced map on surfaces in relation to the function Δ rather than the ostensibly simpler σ is because this allows us to place all points in the singular and branch sets on central vertices.

Theorem 5.5. *Given semisymmetric quasigroups Q_1, Q_2, a homomorphism $h : Q_1 \to Q_2$, and components q_1, q_2 such that $h_{\hat{P}}(\hat{P}(q_1)) = \hat{P}(q_2)$, the induced map h_Δ is a branched cover.*

Proof. Every open cell $c_a \subset \Delta(q_1)$ is contained within the closure of some 2-cell c_b, and by definition $h_\Delta|_{\bar{c}_b}$ is a homeomorphism onto its image in $\Delta(q_2)$, thus the further restriction $h_\Delta|_{c_a}$ is also a homeomorphism onto its image, which as shown in Proposition 5.2, is an open cell of $\Delta(q_2)$. So letting C_1, C_2 denote the set of central 0-cells of $\Delta(q_1), \Delta(q_2)$, by Lemma 5.3 the h_Δ preimage of any cell c_c of $\Delta(q_2)$ where $c_c \notin C_2$ is a set of k disjoint cells of $\Delta(q_1)$ for some positive integer k, each mapped homeomorphically onto c_c. Then consider some point $x \in \Delta(q_2)$ such that $x \notin C_2$. If x is within some open 2-cell, then we already have disjoint open sets and homeomorphisms given by the 2-cells of $\Delta(q_1)$ containing each point of $h_\Delta^{-1}(x)$ and the 2-cell of $\Delta(q_2)$ containing x. However, if x is within a 0- or 1-cell, the proof will be somewhat more involved.

By Lemma 5.3, the preimage $h_\Delta^{-1}(x)$ is a set of k points $p_i \in \Delta(q_1)$. Because by Lemma 4.3 $\Delta(q_1)$ is a compact 2-manifold and thus an Urysohn space [15] [22], it is always possible to select a closed 2-disk $d_i \subset \Delta(q_1)$ for each p_i where $p_i \in \mathrm{int}_{\Delta(q_1)}(d_i)$ such that $d_i \cap C_1 = \emptyset$ and for any $d_i \neq d_j$ then $d_i \cap d_j = \emptyset$. Then as $\Delta(q_1)$ has a finite number of cells, one can always select a set E of k closed 2-disks $e_i \subseteq d_i$ such that for every $p_i \in h_\Delta^{-1}$ there is an e_i where $p_i \in \mathrm{int}_{\Delta(q_1)}(e_i)$, and such for any given e_i, the number of cells c_x where $c_x \cap e_i \neq \emptyset$ is minimal. So any $e_i \in E$ is a closed subset of a compact space and therefore compact when granted the subspace topology[22], and by Proposition 5.2 the restriction $h_\Delta|_{e_i}$ is a continuous surjection onto its image, which is a subset of the surface $\Delta(q_2)$ and thus Hausdorff. As h_Δ maps n-cells homeomorphically to n-cells, then $h_\Delta|_{e_i}$ is also injective and hence a homemorphism if and only if for any open cells $c_i, c_j \subset \Delta(q_2)$ such that $c_i \cap h_\Delta|_{e_i}(e_i) \neq \emptyset, c_j \cap h_\Delta|_{e_i}(e_i) \neq \emptyset$ we have $h_\Delta|_{e_i}(c_i) \cap h_\Delta|_{e_i}(c_j) = \emptyset$ [22].

Because the number of cells intersecting any e_i must be minimal, if x is on a 1-cell then e_i will have nonempty intersection with the 1-cell containing $h_\Delta^{-1}(x)$ and both its adjacent 2-cells. If x is on a (necessarily polyhedral) 0-cell, then given $\hat{P}(q_1)$ is cubic, e_i will have nonempty intersection with the 0-cell containing $h_\Delta^{-1}(x)$ and its 3 adjacent polyhedral 1-cells, 3 adjacent central 1-cells, and 6 adjacent 2-cells. In either case, it is obvious that any pair of closed n-cells c_i, c_j chosen from e_i will have at minimum 1 point of intersection outside C_1, thus by Lemma 5.4, $h_\Delta|_{e_i}(c_i) \cap h_\Delta|_{e_i}(c_j) = \emptyset$. Therefore, $h_\Delta|_{e_i}$ is a homeomorphism for any given e_i, and an embedding when considered as a map into $\Delta(q_2)$, implying interiors are preserved and so $x \in \mathrm{int}_{\Delta(q_2)}(h_\Delta|_{e_i}(e_i))$. Define o_x to be the intersection $\cap_{i=1}^{n} \mathrm{int}_{\Delta(q_2)}(h_\Delta|_{e_i}(e_i))$ across all $e_1, e_2, ..., e_n \in E$; then o_x is the intersection of finitely many open sets containing x and hence itself an open set containing x.

For any $h_\Delta|_{e_i}$, let $\mu_i : h_\Delta|_{e_i}(e_i) \to e_i$ be its inverse function – then $\mu_i|_{o_x}$ is likewise a homeomorphism onto its image, and because h_Δ is continuous, the h_Δ^{-1} image of o_x must be open. Thus, for any point $x \in \Delta(q_2) \setminus C_2$, we have an open set o_x containing x, and k disjoint open sets $\mu_i|_{o_x}(o_x) \subset \Delta(q_1) \setminus C_1$ mapped homeomorphically onto o_x by h_Δ. The set of central 0-cells $C_1 \subset \Delta(q_1)$ is a finite set of points within a 2-manifold, therefore the singular set is nowhere dense in $\Delta(q_1)$, and h_Δ is a branched cover. □

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