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Strong convergence of an inertial algorithm for maximal monotone inclusions with applications

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Abstract

An inertial iterative algorithm is proposed for approximating a solution of a maximal monotone inclusion in a uniformly convex and uniformly smooth real Banach space. The sequence generated by the algorithm is proved to converge *strongly* to a solution of the inclusion. Moreover, the theorem proved is applied to approximate a solution of a convex optimization problem and a solution of a Hammerstein equation. Furthermore, numerical experiments are given to compare, in terms of CPU time and number of iterations, the performance of the sequence generated by our algorithm with the performance of the sequences generated by three recent inertial type algorithms for approximating zeros of maximal monotone operators. In addition, the performance of the sequence generated by our algorithm is compared with the performance of a sequence generated by another recent algorithm for approximating a solution of a Hammerstein equation. Finally, a numerical example is given to illustrate the implementability of our algorithm for approximating a solution of a convex optimization problem.

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1 Introduction

Let H be a real Hilbert space. A set-valued map $A : H \rightrightarrows H$ is called *monotone* if for each $u, v \in H$, $\eta_u \in Au$, $\gamma_v \in Av$, the following inequality holds:

$$\langle \eta_u - \gamma_v, u - v \rangle \geq 0. \tag{1.1}$$

Monotone maps in Hilbert spaces were first introduced by Minty [34] to aid the abstract study of electrical networks and later studied by Browder [6] and his school in the setting of partial differential equations. The map A is called *maximal monotone* if it is monotone, and in addition, its graph is not included in the graph of any other monotone map. The extension of the monotonicity definition from a Hilbert space to itself

... to operators from a Banach space into its dual has been the starting point for the development of non-linear functional analysis ... The monotone mappings appear in a rather wide variety of contexts, since they can be found in many functional equations. Many of them appear also in calculus of variations, as sub-differentials of convex functions (Pascali and Sburian [36], p. 101).

For example, consider the following: Let E be a real Banach space with dual space E^* and let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous (lsc) and convex function. The *subdifferential* of f, $\partial f : E \rightrightarrows E^*$ is defined by

$$\partial f(u) := \{u^* \in E^* : f(v) - f(u) \geq \langle v - u, u^* \rangle, \forall v \in E\}, \quad u \in E. \tag{1.2}$$

It is well known that ∂f is a monotone operator and that $0 \in \partial f(u^*)$ *if and only if* u^* *is a minimizer of* f. Setting $\partial f \equiv A$, it follows that solving the inclusion $0 \in Au$, in this case, is equivalent to solving for a minimizer of f. It is well known that any maximal monotone map $A : \mathbb{R} \rightrightarrows \mathbb{R}$ is the subdifferential of a proper, convex, and lsc function (see, e.g., Cioranescu, [24], Corollary 4.5, p. 170).

In general, a fundamental problem in the study of monotone maps in Banach spaces is the following:

$$\text{Find } u \in E \text{ such that } 0 \in Au. \tag{1.3}$$

This problem has been investigated in Hilbert spaces by numerous researchers. The *proximal point algorithm* (PPA) introduced by Martinet [33] and studied extensively by Rockafellar [41] and numerous other authors is concerned with an iterative method for approximating a solution of the inclusion $0 \in Au$, where A is a maximal monotone map. Specifically, given $x_n \in H$, the proximal point algorithm generates the next iterate x_{n+1} by solving the following equation:

$$x_{n+1} = \left(I + \frac{1}{\lambda_n} A\right)^{-1} x_n + e_n, \tag{1.4}$$

where $\lambda_n > 0$ is a regularizing parameter. Rockafellar [41] proved that if the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is bounded from above, then the resulting sequence $\{x_n\}_{n=1}^{\infty}$ of proximal point iterates converges *weakly* to a solution of (1.3), when $E = H$, provided that a solution exists. Several alternatives and modifications of the PPA have been proposed to obtain strong convergence under suitable conditions. For a brief review of these alternatives and modifications in Banach spaces more general than Hilbert spaces, interested readers may see, e.g., [14, 30, 31, 39, 43, 46] and the references therein.

Chidume et al. [21] recently proved the following strong convergence theorem.

Theorem 1.1 (Chidume et al. [21]) *Let E be a uniformly convex and uniformly smooth real Banach space and let E^* be its dual. Let $A : E \rightrightarrows E^*$ be a maximal monotone and bounded mapping with $A^{-1}(0) \neq \emptyset$. For arbitrary $u_1 \in E$, define a sequence $\{u_n\}$ iteratively by*

$$u_{n+1} = J^{-1}\big(Ju_n - \lambda_n \eta_n - \lambda_n \theta_n (Ju_n - Ju_1)\big), \quad \eta_n \in Au_n, n \geq 1, \tag{1.5}$$

where $\{\lambda_n\}$ *and* $\{\theta_n\}$ *are sequences in* $(0,1)$ *satisfying certain conditions and* J *is the normalized duality map on* E. *Then, the sequence* $\{u_n\}$ *converges strongly to a solution of* $0 \in Au$.

It is well known that the convergence of iterative algorithms for approximating zeros of monotone maps are generally slow. This is expected since monotone maps are generally not differentiable. Thus, fast converging algorithms such as the *Newton–Kantorovich* algorithm cannot be used. Consequently, a lot of effort is now being put into iterative algorithms for approximating zeros of maximal monotone maps that improve the speed of convergence of known algorithms. One method that is now studied is to incorporate the *inertial extrapolation term* in algorithms.

In a recent paper, Alvarez [3] studied the asymptotic *weak convergence* of three *inertial* implicit iterative methods for solving the inclusion $0 \in Au$, when A is a maximal monotone operator on a real Hilbert space, which generalizes the classical PPA. The motivation for the first of these three methods, called *Inertial Proximal Point Algorithm* (IPPA), stems from a discretization of the equation for an oscillator with damping and conservative restoring force: $x''(t) + \gamma x'(t) + \nabla f(x(t)) = 0$, where $\gamma > 0$ and $f : H \to \mathbb{R}$ is differentiable. In the context of optimization problems, this dynamical system which is called *Heavy Ball with Friction* (HBF) was first considered by Polyak [37]. It has been known that the inertial nature of the HBF could be exploited in numerical computations to accelerate the trajectories and speed up convergence (see, e.g., [17, 18]). Concerning asymptotic convergence, Alvarez [2] showed that if f is differentiable, i.e., if ∇f is monotone and $(\nabla f)^{-1}(0) \neq \emptyset$, then, every trajectory of HBF converges *weakly* to some $x^* \in H$ with $(\nabla f)(x^*) = 0$. Considering the implicit discretization of the HBF, the following recursion formula, in terms of resolvents, has been obtained (see, e.g., Alvarez [3], p. 774):

$$x_{k+1} = J_{\lambda}^{\nabla f}\bigl(x_k + \alpha(x_k - x_{k-1})\bigr), \quad k = 1, 2, \ldots, \tag{1.6}$$

where λ is a regularizing parameter that combines the damping factor of γ and the actual step size $h > 0$. Replacing ∇f with a maximal monotone operator A, and considering variable parameters $\lambda_k > 0$ and $\alpha_k \in [0, 1)$, the discussion above motivated the introduction of the *inertial-type* iteration:

$$x_{k+1} = J_{\lambda_k}^{A}\bigl(x_k + \alpha_k(x_k - x_{k-1})\bigr), \quad k = 1, 2, \ldots, \qquad \text{(IPPA)} \tag{1.7}$$

where the extrapolation term $\alpha_k(x_k - x_{k-1})$ is intended to speed up convergence. The *Inertial Proximal Point Algorithm* (IPPA) was first considered in [2] for nonsmooth conservative operator $A = \partial f$, the subdifferential of a closed, proper, and convex function $f : H \to \mathbb{R} \cup \{\infty\}$. Alvarez [2, Theorem 3.1] proved, under suitable conditions, that $\{x_k\}$ converges *weakly* to a minimizer of f. For the nonconservative case, a partial positive result for cocoercive operators was obtained in [29], where comparisons with first-order-in-time methods are also given through numerical facts, showing improvements in the speed of convergence.

The case of arbitrary maximal monotone operators is treated in [4] under the following conditions:

(i) $\lambda = \inf_{k\geq 0} \lambda_k > 0$,

(ii) $\forall k \in \mathbb{N}$, $\alpha_k \in [0, 1)$, $\alpha := \sup_{k\geq 0} \alpha_k < 1$,

(iii) $\sum \alpha_k \|x_k - x_{k-1}\|^2 < \infty$.

From a different point of view, the following *Relaxed Proximal Point Algorithm* (RPPA) was proposed in [28] to accelerate the standard PPA:

$$x_{k+1} = \big[(1-\rho_k)I + \rho_k J_{\lambda_k}^A\big](x_k), \qquad \text{(RPPA)} \tag{1.8}$$

where $\{\rho_k\} \subset (0,2)$ is a *relaxing factor* which is assumed to satisfy the following conditions: $\inf_{k\geq 0} \rho_k > 0$ and $\sup_{k\geq 0} \rho_k < 2$.

Alvarez [3] recently coupled the IPPA and RPPA, two acceleration strategies, to propose the following iterative method:

$$x_{k+1} = \big[(1-\rho_k)I + \rho_k J_{\lambda_k}^A\big]\big(x_k + \alpha_k(x_k - x_{k-1})\big). \qquad \text{(RIPPA)} \tag{1.9}$$

He proved *weak convergence* of the sequence $\{x_k\}$ to some $x^* \in A^{-1}(0)$.

We remark that each of the algorithms, IPPA, RPPA, and RIPPA, involves the resolvent operator, J_λ^A.

In this paper, an *inertial iterative algorithm* is proposed for approximating a solution of a maximal monotone inclusion in a uniformly convex and uniformly smooth real Banach space. The sequence generated by the algorithm is proved to converge *strongly* to a solution of the inclusion. Moreover, the theorem proved is applied to approximate a solution of a convex optimization problem, and a solution of a Hammerstein equation. Furthermore, numerical experiments are given to compare, in terms of CPU time and number of iterations, the performance of the sequence generated by our algorithm with the sequences generated by IPPA, RPPA, and RIPPA, respectively, for approximating a solution of a maximal monotone inclusion in Hilbert spaces. Finally, numerical examples are given to illustrate the implementability of our algorithm for approximating a solution of a convex optimization problem and for approximating a solution of a Hammerstein equation.

2 Preliminaries

Let E be a real normed space with dual space, E^*. A map $J : E \rightrightarrows E^*$ defined by

$$J(v) := \big\{v^* \in E^* : \langle v, v^*\rangle = \|v\|\,\|v^*\|, \|v\| = \|v^*\|\big\},$$

is called the *normalized duality map*.

A normed space E is called *uniformly convex*, if for all $\epsilon \in (0,2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that, if $u, v \in E$, with $\|u\| \leq 1$, $\|v\| \leq 1$, and $\|u - v\| \geq \epsilon$, then $\|\frac{1}{2}(u+v)\| \leq 1 - \delta$.

A normed space E is called *strictly convex*, if for all $u, v \in E$, with $u \neq v$, $\|u\| = \|v\| = 1$, we have that $\|\lambda u + (1-\lambda)v\| < 1$, for all $\lambda \in (0,1)$.

A normed space E is called *uniformly smooth*, if given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, for all $u, v \in E$, with $\|u\| = 1$, $\|v\| \leq \delta$, one has

$$\|u + v\| + \|u - v\| < 2 + \epsilon\|v\|.$$

A normed space E is called *smooth*, if for every $u \in E$, $\|u\| = 1$, there exists a unique u^* in E^* such that $\|u^*\| = 1$ and $\langle u, u^*\rangle = \|u\|$.

Remark 1 It is well known that if E is a smooth, strictly convex, and reflexive Banach space, the normalized duality map, J, is single-valued, one-to-one and onto, respectively. Also, if E is uniformly smooth, then J is uniformly continuous on bounded subsets of E. For more properties of the normalized duality map, see e.g., Alber and Ryazantseva [1], Lindenstrauss and Tzafriri [32], Chidume [9], and Cioranescu [24].

Let E be a real normed space with $\dim E \geq 2$. The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \left\{1 - \left\|\frac{u+v}{2}\right\| : \|u\| = \|v\| = 1; \epsilon = \|u-v\|\right\}.$$

The following properties of the modulus of convexity will be needed in the sequel (see, e.g., Chidume [9], page 9):

(a) $\frac{\delta_E(\epsilon)}{\epsilon}$ is a decreasing function on $(0,2]$;

(b) $\delta_E : (0,2] \to [0,1]$ is a convex and continuous function;

(c) $\delta_E : (0,2] \to [0,1]$ is a strictly increasing function.

Let E be a smooth real normed space and let $\phi : E \times E \to \mathbb{R}^+$ be a map defined by

$$\phi(u,v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2, \quad \text{for all } u, v \in E.$$

This map was introduced by Alber [1] and has been extensively studied by Alber [1] and a host of other authors (see, e.g., [18, 20, 30]). It is obvious from the definition of the map ϕ that, for any $u, v \in E$, we have:

$$\big(\|u\| - \|v\|\big)^2 \leq \phi(u,v) \leq \big(\|u\| + \|v\|\big)^2, \tag{2.1}$$

$$\phi(v,u) = \phi(u,v) + 2\langle u, Jv\rangle - 2\langle v, Ju\rangle. \tag{2.2}$$

Define a map $V : E \times E^* \to \mathbb{R}$ by

$$V\big(u,u^*\big) = \|u\|^2 - 2\big\langle u,u^*\big\rangle + \big\|u^*\big\|^2, \quad \text{for } u \in E, u^* \in E^*. \tag{2.3}$$

Then, it is easy to see that

$$V\big(u,u^*\big) = \phi\big(u, J^{-1}\big(u^*\big)\big), \quad \forall u \in E, u^* \in E^*. \tag{2.4}$$

We shall use the following lemmas in the sequel, where $\text{Int}(D(A))$ denotes the interior of the domain of A.

Lemma 2.1 (Alber and Ryazantseva [1]) *Let E be a reflexive, strictly convex and smooth Banach space with E^* as its dual. Then,*

$$V\big(u,u^*\big) + 2\big\langle J^{-1}u^* - u, v^*\big\rangle \leq V\big(u, u^* + v^*\big), \tag{2.5}$$

for all $u \in E$ and $u^*, v^* \in E^*$.

Lemma 2.2 (Pascali and Sburian [36], Lemma 3.6, Chap. III) *Let E be a real normed space and $A : E \rightrightarrows E^*$ be a monotone map with $0 \in \operatorname{Int}(D(A))$. Then, A is quasi-bounded, i.e., for any $M > 0$, there exists $C > 0$ such that*:

(i) $(y, v) \in G(A)$;
(ii) $\langle v, y\rangle \leq M\|y\|$; *and*
(iii) $\|y\| \leq M$, *imply* $\|v\| \leq C$.

Lemma 2.3 (Kamimura and Takahashi [30]) *Let E be a uniformly convex and uniformly smooth real Banach space and $\{x_n\}$, $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.4 (Alber and Ryazantseva [1], p. 50) *Let E be a reflexive, strictly convex and smooth Banach space with E^* as its dual. Let $W : E \times E \to \mathbb{R}^1$ be defined by $W(x,y) = \frac{1}{2}\phi(y,x)$. Then,*

$$W(x,y) - W(z,y) \geq \langle Jx - Jz, z - y\rangle,$$

i.e.,

$$\phi(y,x) - \phi(y,z) \geq 2\langle Jx - Jz, z - y\rangle,$$

and also

$$W(x,y) \leq \langle Jx - Jy, x - y\rangle,$$

for all $x, y, z \in E$.

Lemma 2.5 (Alber and Ryazantseva [1], p. 45) *Let E be a uniformly convex Banach space. Then, for any $R > 0$ and any $x, y \in E$ such that $\|x\| \leq R$, $\|y\| \leq R$, the following inequality holds*:

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1}\delta_X\big(c_2^{-1}\|x - y\|\big),$$

where $c_2 = 2\max\{1, R\}$, $1 < L < 1.7$.

Define

$$K := 4RL\sup\big\{\|Jx - Jy\| : \|x\| \leq R, \|y\| \leq R\big\} + 1. \tag{2.6}$$

Lemma 2.6 (Alber and Ryazantseva [1], p. 46) *Let E be a uniformly smooth and strictly convex Banach space. Then for any $R > 0$ and any $x, y \in E$ such that $\|x\| \leq R$, $\|y\| \leq R$, the following inequality holds*:

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1}\delta_{E^*}\big(c_2^{-1}\|Jx - Jy\|\big),$$

where $c_2 = 2\max\{1, R\}$, $1 < L < 1.7$, and δ_E is the modulus of convexity *of E.*

Lemma 2.7 (Reich [38]) *Let E^* be a strictly convex dual Banach space with a Fréchet differentiable norm, and let $A : E \rightrightarrows E^*$ be a maximal monotone map with a zero and $z \in E^*$. For each $\lambda > 0$, there exists a unique $x_\lambda \in E$ such that $z \in Jx_\lambda + \lambda Ax_\lambda$. Furthermore, x_λ converges strongly to a unique zero of A.*

Lemma 2.8 *From Lemma 2.7, setting $\lambda_n := \frac{1}{\theta_n}$, where $\theta_n \to 0$, as $n \to \infty$, $\theta_n \le \theta_{n-1}$, $\forall n \ge 1$, $\frac{1}{2}(\frac{\theta_{n-1}-\theta_n}{\theta_n}K) \le 1$, $z = Jh$, for some $h \in E$, $v_n \in Ay_n$ and $y_n := (J + \frac{1}{\theta_n}A)^{-1}z$, we have:*

$$v_n - \theta_n(Jh - Jy_n) = 0 \quad \text{and} \quad y_n \to y^* \in A^{-1}(0), \tag{2.7}$$

where $A : E \rightrightarrows E^*$ is maximal monotone.

Lemma 2.9 (Xu [45]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - \sigma_n)a_n + \sigma_n b_n + c_n, \quad n \ge 1,$$

where $\{\sigma_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy the conditions:

(i) $\{\sigma_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} \sigma_n = \infty$;
(ii) $\limsup_{n\to\infty} b_n \le 0$;
(iii) $c_n \ge 0$, $\sum_{n=1}^{\infty} c_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3 Main results

The following conditions are required in the combined proofs of Lemma 3.1 and Theorem 3.2 below, where $\{\lambda_n\}$, $\{\beta_n\}$, and $\{\theta_n\}$ are sequences in $(0, 1)$:

(i) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$,
(ii) $\delta_E^{-1}(\lambda_n K) \le \theta_n^2\gamma_0$,
(iii) $\delta_{E^*}^{-1}(\lambda_n K) \le \theta_n^2\gamma_0$,
(iv) $\omega_J(\beta_n K) \le \lambda_n^4\theta_n\gamma_0$,
(v) $\delta_E^{-1}(\eta_n) \to 0$,
(vi) $\delta_{E^*}^{-1}(\eta_n) \to 0$,
(vii) $\frac{\delta_E^{-1}(\eta_n)}{\lambda_n\theta_n} \to 0$,
(viii) $\frac{\delta_{E^*}^{-1}(\eta_n)}{\lambda_n\theta_n} \to 0$,
(ix) $\lambda_n \le \theta_n\gamma_0$,

where $\eta_n = (\frac{\theta_{n-1}}{\theta_n} - 1)K$, for some constants $\gamma_0 > 0$, $K > 0$; and δ_E is the modulus of convexity of E, ω_J is the modulus of continuity of J.

Estimates for the moduli of convexity of $E = L_p$, $1 < p < \infty$.

The following estimates have been obtained for δ_E in L_p spaces, $1 < p < \infty$,

$$\delta_E(\epsilon) \ge \begin{cases} \frac{p-1}{8}\epsilon^2, & \text{if } 1 < p < 2; \\ \frac{1}{p}(\frac{\epsilon}{2})^p, & \text{if } p \ge 2; \end{cases}$$

where $\epsilon \in (0, 2]$ (see e.g., Lindenstrauss and Tzafriri [32], see also, Chidume [9], p. 44).

Also, in L_p spaces, J is Lipschitz if $2 \le p < \infty$ and it satisfies the following inequality:

$$\|Jx - Jy\| \le H\|x - y\|^{p-1},$$

if $1 < p < 2$. Consequently, we have the following estimates:

$$\omega_J(\epsilon) \leq \begin{cases} H\epsilon^{(p-1)}, & \text{if } 1 < p < 2; \\ M\epsilon, & \text{if } p \geq 2; \end{cases}$$

where $\epsilon > 0$, H and M are positive constants, and J is the normalized duality map (see, e.g., Lindenstrauss and Tzafriri [32], see also Chidume [9]).

Prototypes of the parameters for Lemma 3.1 and Theorem 3.2 below in the case that $E = L_p$, $1 < p < \infty$ are:

For L_p spaces, $2 \leq p < \infty$,

$$\lambda_n = (n+1)^{-\frac{1}{2}}, \qquad \theta_n = (n+1)^{-\frac{1}{4p}} \quad \text{and} \quad \beta_n = (n+1)^{-(2+\frac{1}{4p})}, \quad n \geq 1.$$

For L_p spaces, $1 < p < 2$,

$$\lambda_n = (n+1)^{-\frac{1}{4}}, \qquad \theta_n = (n+1)^{-\frac{1}{16}} \quad \text{and} \quad \beta_n = (n+1)^{-\frac{17}{16(p-1)}}, \quad n \geq 1.$$

With these choices, conditions (i)–(ix) given in Lemma 3.1 and Theorem 3.2 are easily satisfied.

Furthermore, we have the following formulae, for J and J^{-1} in L_p and l_p, $1 < p < \infty$, $p^{-1} + q^{-1} = 1$ (see, e.g., Alber and Ryazantseva [1], p. 36):

$$Ju = \|u\|_{l_p}^{2-p} v \in l_q, \quad v = \{|u_1|^{p-2}u_1, |u_2|^{p-2}u_2, \dots\}, u = \{u_1, u_2, \dots\},$$

$$J^{-1}u = \|u\|_{l_q}^{2-q} v \in l_p, \quad v = \{|u_1|^{q-2}u_1, |u_2|^{q-2}u_2, \dots\}, u = \{u_1, u_2, \dots\},$$

$$Ju = \|u\|_{L_p}^{2-p} |u(s)|^{p-2} u(s) \in L_q(G), \quad s \in G,$$

$$J^{-1}u = \|u\|_{L_q}^{2-q} |u(s)|^{q-2} u(s) \in L_p(G), \quad s \in G.$$

We now prove the following lemma.

Lemma 3.1 *Let E be a uniformly smooth and uniformly convex real Banach space and $A : E \rightrightarrows E^*$ be a maximal monotone operator with $D(A) = E$ such that the inclusion $0 \in Az$ has a solution. For arbitrary $z_0, z_1 \in E$, define a sequence $\{z_n\}$ by*

$$\begin{cases} w_n = z_n + \beta_n(z_n - z_{n-1}), \\ z_{n+1} = J^{-1}(Jw_n - \lambda_n\mu_n - \lambda_n\theta_n Jw_n), \quad \mu_n \in Aw_n, n \geq 1. \end{cases} \tag{3.1}$$

Then, the sequence $\{z_n\}$ is bounded.

Proof We show that the sequence $\{z_n\}$ is bounded.

Let z^* be a solution of $0 \in Az$, i.e., $0 \in Az^*$. Then, there exists $r > 0$ (sufficiently large) such that

$$r > \max\{4\|v^*\|^2, \phi(z^*, z_1)\}. \tag{3.2}$$

Define $B := \{z \in E : \phi(z^*, z) < r\}$, with $0 \in B$. Clearly, $B \subset \text{Int}(D(A))$. It suffices to show that $\{\phi(z^*, z_n)\}$ is bounded. We proceed by induction. For $n = 1$, by construction, we have that $\phi(z^*, z_1) < r$. Assume that $\phi(z^*, z_n) < r$, for some $n \geq 1$. Using inequality (2.1), we have that $\|z_n\| < \|z^*\| + \sqrt{r}$. Now, we show that $\phi(z^*, z_{n+1}) < r$. Suppose for contradiction that $\phi(z^*, z_{n+1}) < r$ does not hold. Then, $\phi(z^*, z_{n+1}) \geq r$.

Let $y \in B$ be arbitrary and $(y, v) \in G(A)$, $u \in Ax$. Since A is locally bounded at 0, there exist $h_0 > 0$, $m_0 > 0$ such that

$$\|u\| \leq m_0, \quad \forall x \in B_{h_0}(0) \subset B.$$

By the monotonicity of A, we have that:

$$\langle v, y\rangle \geq \langle u, y - x\rangle + \langle v, x\rangle, \quad \forall x \in B_{h_0}(0), v \in Ay,$$
$$\langle v, -y\rangle \leq \langle u, x - y\rangle + \langle v, -x\rangle.$$

Setting $s = -y$, we have that:

$$\begin{aligned}\langle v, s\rangle &\leq \langle u, x + s\rangle + \langle v, -x\rangle \\ &\leq \|u\|\big(\|x\| + \|s\|\big) + \|v\|\|x\|, \\ \sup_{\|s\| \leq (\|z^*\| + \sqrt{r})} \big|\langle v, s\rangle\big| &\leq m_0\big(h_0 + \big\|z^*\big\| + \sqrt{r}\big) + \|v\| h_0,\end{aligned}$$

so that

$$\|v\| \leq \frac{m_0(h_0 + \|z^*\| + \sqrt{r})}{\|z^*\| + \sqrt{r} - h_0} := M_0, \quad \forall y \in B.$$

Define $M := \max\{M_0, \|z^*\| + \sqrt{r}\}$. Then, $\langle v, y\rangle \leq M\|y\|$ and $\|y\| \leq M$. By Lemma 2.2, there exists $C > 0$ such that $\|v\| \leq C$, $\forall y \in B$. Define

$$M_1 = \sup\big\{\|\mu + \theta J w\|, w \in B, \mu \in Aw, \theta \in (0,1)\big\} + 1; \tag{3.3}$$
$$M_2 = \sup\big\{\big\|J^{-1}(Jw - \lambda\mu - \lambda\theta Jw)\big\|, w \in B, \mu \in Aw, \theta \in (0,1)\big\} + 1. \tag{3.4}$$

From the recursion formula, Lemma 2.5, and the fact that J and J^{-1} are uniformly continuous on bounded sets, we have that

$$\|Jz_{n+1} - Jw_n\| \leq \lambda_n M_1 \quad \text{and} \quad \|z_{n+1} - w_n\| \leq c_2 \delta_E^{-1}\big(\lambda_n M^*\big), \quad \text{for some } M^* > 0. \tag{3.5}$$

Define

$$\gamma_0 := \min\left\{1, \frac{r}{32K^*}\right\}, \tag{3.6}$$

where $K^* = \max\{M^*, M_1, M_2, M_1M_2, M, c_2M_1\}$. Using Lemma 2.1 and denoting $0 \in Az^*$ by 0^*, we compute:

$$\begin{aligned}
\phi(z^*, z_{n+1}) &= V(z^*, Jw_n - \lambda_n\mu_n - \lambda_n\theta_n w_n) \\
&\leq V(z^*, Jw_n) - 2\lambda_n\langle z_{n+1} - z^*, \mu_n + \theta_n Jw_n\rangle \\
&= \phi(z^*, w_n) - 2\lambda_n\langle z_{n+1} - w_n, \mu_n + \theta_n Jw_n\rangle - 2\lambda_n\langle w_n - z^*, \mu_n + \theta_n Jw_n\rangle \\
&\leq \phi(z^*, w_n) + 2c_2\lambda_n\delta_E^{-1}(\lambda_n M^*)M_1 - 2\lambda_n\langle w_n - z^*, \mu_n - 0^*\rangle \\
&\quad - 2\theta_n\lambda_n\langle w_n - z^*, Jw_n\rangle \\
&\leq \phi(z^*, w_n) + 2c_2\lambda_n\delta_E^{-1}(\lambda_n M^*)M_1 - 2\theta_n\lambda_n\langle w_n - z^*, Jw_n - Jz_{n+1}\rangle \\
&\quad - 2\theta_n\lambda_n\langle w_n - z^*, Jz_{n+1}\rangle.
\end{aligned} \tag{3.7}$$

By Lemma 2.4, we have that

$$\begin{aligned}
&-2\lambda_n\theta_n\langle w_n - z^*, Jz_{n+1}\rangle \leq \lambda_n\theta_n\|z^*\|^2 + 2M\lambda_n\theta_n\|w_n - z_{n+1}\| - \lambda_n\theta_n\phi(z^*, z_{n+1}), \\
&\phi(z^*, w_n) \leq \phi(z^*, z_n) + 2M_2\omega_J(\beta_n M).
\end{aligned} \tag{3.8}$$

It follows from inequality (3.7) that

$$\begin{aligned}
r &\leq \phi(z^*, z_{n+1}) \\
&\leq \phi(z^*, z_n) + 2M_2\omega_J(\beta_n M) + 2c_2\lambda_n\delta_E^{-1}(\lambda_n M^*)M_1 + \lambda_n\theta_n\|z^*\|^2 \\
&\quad + 2\lambda_n\theta_n M\|Jw_n - Jz_{n+1}\| + 2\lambda_n\theta_n M\|w_n - z_{n+1}\| - \lambda_n\theta_n\phi(z^*, z_{n+1}) \\
&\leq \phi(z^*, z_n) + 2M_2\omega_J(\beta_n M) + 2c_2\lambda_n\delta_E^{-1}(\lambda_n M^*)M_1 + \lambda_n\theta_n\|z^*\|^2 \\
&\quad + 2\lambda_n^4\theta_n MM_1 + 2c_2\lambda_n\theta_n\delta_E^{-1}(\lambda_n M^*)M - \lambda_n\theta_n\phi(z^*, z_{n+1}) \\
&< r + 2M_2\lambda_n\theta_n\gamma_0 + 2c_2\lambda_n\theta_n M_1\gamma_0 + \lambda_n\theta_n\frac{r}{4} + 2\lambda_n\theta_n MM_1\gamma_0 \\
&\quad + 2\lambda_n\theta_n\gamma_0 M - \lambda_n\theta_n r \\
&< r + \lambda_n\theta_n\frac{r}{2} - \lambda_n\theta_n r < r.
\end{aligned}$$

This is a contradiction. Hence, $\phi(z^*, z_{n+1}) < r$. Therefore, $\phi(z^*, z_n) < r$, for all $n \geq 1$. □

Theorem 3.2 *Let E be a uniformly smooth and uniformly convex real Banach space. Let $A : E \rightrightarrows E^*$ be a maximal monotone operator with $D(A) = E$ such that the inclusion $0 \in Az$ has a solution. For arbitrary $z_0, z_1 \in E$, define a sequence $\{z_n\}$ by algorithm (3.1). Then, the sequence $\{z_n\}$ converges strongly to a zero of A (see Remark 2 below).*

Proof Using Lemma 2.1 and equation (2.2), we have

$$\begin{aligned}
\phi(y_n, z_{n+1}) &= V(y_n, Jw_n - \lambda_n\mu_n - \lambda_n\theta_n Jw_n) \\
&\leq V(y_n, Jw_n) - 2\lambda_n\langle z_{n+1} - y_n, \mu_n + \theta_n Jw_n\rangle \\
&= \phi(w_n, y_n) + 2\langle w_n, Jy_n\rangle - 2\langle y_n, Jw_n\rangle - 2\lambda_n\langle z_{n+1} - y_n, \mu_n + \theta_n Jw_n\rangle.
\end{aligned} \tag{3.9}$$

Observe that

$$\begin{aligned}\phi(w_n,y_n) &= V(w_n,Jy_n) = V(w_n,Jy_{n-1}+Jy_n-Jy_{n-1})\\&\le V(w_n,Jy_{n-1})-2\langle y_n-w_n,Jy_{n-1}-Jy_n\rangle.\end{aligned}\tag{3.10}$$

Thus, from inequalities (3.9), (3.10), and the fact that $v_n\in Ay_n$, we obtain

$$\begin{aligned}\phi(y_n,z_{n+1}) &\le V(w_n,Jy_{n-1})-2\langle y_n-w_n,Jy_{n-1}-Jy_n\rangle+2\langle w_n,Jy_n\rangle-2\langle y_n,Jw_n\rangle\\&\quad-2\lambda_n\langle z_{n+1}-y_n,\mu_n+\theta_nJw_n\rangle\\&=\phi(y_{n-1},w_n)+2\langle y_{n-1},Jw_n\rangle-2\langle w_n,Jy_{n-1}\rangle-2\langle y_n-w_n,Jy_{n-1}-Jy_n\rangle\\&\quad+2\langle w_n,Jy_n\rangle-2\langle y_n,Jw_n\rangle-2\lambda_n\langle z_{n+1}-y_n,\mu_n+\theta_nJw_n\rangle\\&=\phi(y_{n-1},w_n)+2\langle y_{n-1}-y_n,Jw_n\rangle+2\langle w_n,Jy_n-Jy_{n-1}\rangle\\&\quad-2\langle y_n-w_n,Jy_{n-1}-Jy_n\rangle-2\lambda_n\langle z_{n+1}-y_n,\mu_n+\theta_nJw_n\rangle\\&\le\phi(y_{n-1},w_n)+2\|y_{n-1}-y_n\|\|w_n\|+2\|w_n\|\|Jy_n-Jy_{n-1}\|\\&\quad+2\|y_n-w_n\|\|Jy_{n-1}-Jy_n\|+2\lambda_n\|z_{n+1}-w_n\|M_1\\&\quad-2\lambda_n\langle w_n-y_n,\mu_n-v_n\rangle-\underline{2\lambda_n\langle w_n-y_n,v_n\rangle}-\underline{2\lambda_n\theta_n\langle w_n-y_n,Jw_n\rangle}.\end{aligned}\tag{3.11}$$

Observe that

$$\begin{aligned}-\underline{2\lambda_n\theta_n\langle w_n-y_n,Jw_n\rangle} &= 2\lambda_n\theta_n\langle w_n-y_{n-1},Jy_{n-1}-Jw_n\rangle-2\lambda_n\theta_n\langle y_{n-1}-y_n,Jw_n-Jy_{n-1}\rangle\\&\quad-2\lambda_n\theta_n\langle w_n-y_n,Jy_n\rangle-2\lambda_n\theta_n\langle w_n-y_n,Jy_{n-1}-Jy_n\rangle\\&\le-\lambda_n\theta_n\phi(y_{n-1},w_n)+2\lambda_n\theta_n\|y_{n-1}-y_n\|M\\&\quad-\underline{2\lambda_n\theta_n\langle w_n-y_n,Jy_n\rangle}+2\lambda_n\theta_n\|Jy_{n-1}-Jy_n\|M.\end{aligned}\tag{3.12}$$

Also, from Lemma 2.8, we obtain that

$$-\underline{2\lambda_n\langle w_n-y_n,v_n\rangle}-\underline{2\lambda_n\theta_n\langle w_n-y_n,Jy_n\rangle}=-2\lambda_n\langle w_n-y_n,v_n+\theta_nJy_n\rangle=0.\tag{3.13}$$

Hence, substituting inequality (3.12) and equation (3.13) into inequality (3.11), we have that

$$\begin{aligned}\phi(y_n,z_{n+1}) &\le(1-\lambda_n\theta_n)\phi(y_{n-1},w_n)+2\|y_{n-1}-y_n\|M+4\|Jy_{n-1}-Jy_n\|M\\&\quad+2\lambda_n\|z_{n+1}-w_n\|M+2\lambda_n\theta_n\|y_{n-1}-y_n\|M+2\lambda_n\theta_n\|Jy_{n-1}-Jy_n\|M\\&\le(1-\lambda_n\theta_n)\phi(y_{n-1},w_n)+2M\big(\delta_E^{-1}(\eta_n)+\delta_{E^*}^{-1}(\eta_n)\big)+2\lambda_n\delta_E^{-1}(\lambda_nM)M\\&\quad+2\lambda_n\theta_nM\big(\delta_E^{-1}(\eta_n)+\delta_{E^*}^{-1}(\eta_n)\big)\\&\le(1-\lambda_n\theta_n)\phi(y_{n-1},z_n)+2M_2\omega_J(\beta_nM)+2M\lambda_n\theta_n^2\gamma_0\\&\quad+2\lambda_n\theta_n\left(\frac{\delta_E^{-1}(\eta_n)}{\lambda_n\theta_n}+\frac{\delta_{E^*}^{-1}(\eta_n)}{\lambda_n\theta_n}+\delta_E^{-1}(\eta_n)+\delta_{E^*}^{-1}(\eta_n)\right)M.\end{aligned}\tag{3.14}$$

Applying inequalities (3.8) and (3.14), we obtain

$$\begin{aligned}\phi(y_n, z_{n+1}) &\leq (1-\lambda_n\theta_n)\phi(y_{n-1}, z_n) + 2M\lambda_n^4\theta_n\gamma_0 \\ &\quad + 2\lambda_n\theta_n\biggl(\frac{\delta_E^{-1}(\eta_n)}{\lambda_n\theta_n} + \frac{\delta_{E^*}^{-1}(\eta_n)}{\lambda_n\theta_n} + \theta_n\gamma_0 + \delta_E^{-1}(\eta_n) + \delta_{E^*}^{-1}(\eta_n)\biggr)M. \end{aligned} \tag{3.15}$$

Set $a_n := \phi(y_{n-1}, z_n)$, $\sigma_n := \lambda_n\theta_n$, $c_n := \lambda_n^4\theta_n$, and

$$b_n := M\biggl(\frac{\delta_E^{-1}(\eta_n)}{\lambda_n\theta_n} + \frac{\delta_{E^*}^{-1}(\eta_n)}{\lambda_n\theta_n} + \delta_E^{-1}(\eta_n) + \delta_{E^*}^{-1}(\eta_n) + \theta_n\gamma_0\biggr).$$

Hence, inequality (3.15) becomes $a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n b_n + c_n$, $n \geq 1$. It follows from Lemma 2.9 that $\lim_{n\to\infty}\phi(y_{n-1}, z_n) = 0$. By Lemma 2.3, we have $\lim_{n\to\infty}\|z_n - y_{n-1}\| = 0$. Since $\lim_{n\to\infty} y_n = y^* \in A^{-1}0$, we have that $\{z_n\}$ converges to $y^* \in A^{-1}0$. This completes the proof. □

Remark 2 This zero of A may be a minimum norm zero of A, for example, if A is the subdifferential, ∂f, of a proper lower semicontinuous and convex function f.

4 Applications

4.1 Application to a convex optimization problem

The following lemma will be crucial in what follows.

Lemma 4.1 (Rockafellar [40]) *Let E be a Banach space and let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper, convex and lower semicontinuous function. Then, the subdifferential of f, ∂f, is maximal monotone. Furthermore, $0 \in \partial f(u^*)$ if and only if u^* is a minimizer of f.*

We now have the following theorem.

Theorem 4.2 *Let E be a uniformly convex and uniformly smooth real Banach space with dual E^*. Let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper, lower semicontinuous, and convex function such that $(\partial f)^{-1}0 \neq \emptyset$. For given $z_0, z_1 \in E$, let $\{z_n\}$ be generated by the algorithm*

$$\begin{cases} w_n = z_n + \beta_n(z_n - z_{n-1}), \\ z_{n+1} = J^{-1}(Jw_n - \lambda_n\partial f(w_n) - \lambda_n\theta_n Jw_n), \quad n \geq 1. \end{cases} \tag{4.1}$$

Then, the sequence $\{z_n\}$ converges strongly to a minimizer of f.

Proof By Lemma 4.1, ∂f is maximal monotone. The conclusion follows from Theorem 3.2. □

4.2 Applications to Hammerstein integral equations

Definition 4.3 Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation (generally nonlinear) of Hammerstein-type has the form

$$u(x) + \int_\Omega k(x,y)f\bigl(y, u(y)\bigr)\,dy = w(x), \tag{4.2}$$

where the unknown function u and inhomogeneous function w lie in a Banach space E of measurable real-valued functions.

If we define an operator K by $K(v) := \int_{\Omega} \kappa(x,y)v(y)\,dy$; $x \in \Omega$, and the so-called *superposition* or *Nemytskii* operator by $Fu(y) := f(y,u)$, then equation (4.2) can be put in the form

$$u + KFu = 0. \tag{4.3}$$

Without loss of generality, we have taken $w \equiv 0$.

Interest in Hammerstein integral equations stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear part possesses Green's function can, as a rule, be transformed into the form of equation (4.2) (see, e.g., Pascali and Sburian [36], Chap. IV). Consider, for example, the following pendulum problem:

$$\begin{cases} \frac{d^2 v(t)}{dt^2} - a^2 \sin v(t) = z(t), & t \in [0,1], \\ v(0) = v(1) = 0, \end{cases} \tag{4.4}$$

where the driving the force z is periodical and odd. The constant $a \neq 0$ depends on the length of the pendulum and on gravity. Since the Green's function of the problem

$$v''(t) = 0, \qquad v(0) = v(1) = 0$$

is the function defined by

$$k(t,x) = \begin{cases} t(1-x), & 0 \le t \le x, \\ t(1-x), & x \le x \le 1, \end{cases}$$

problem (4.4) is equivalent to the nonlinear integral equation

$$v(t) = -\int_0^1 k(t,x)\bigl[z(x) - a^2 \sin v(x)\bigr]\,dx. \tag{4.5}$$

If $\int_0^1 k(t,x)z(x)\,dx = g(t)$ and $v(t) + g(t) = u(t)$, then (4.5) can be written as the Hammerstein integral equation

$$u(t) + \int_0^1 k(t,x) f\bigl(x, u(x)\bigr)\,dx = 0,$$

where $f(x,u(x)) = a^2 \sin[u(x) - g(x)]$.

Equations of Hammerstein-type also play a special role in the theory of optimal control systems and in automation and network theory (see, e.g., Dolezale [27]).

In the case when K and F are *maximal monotone*, several existence and uniqueness theorems have been proved for equations of Hammerstein type (see, e.g., [5, 7, 8, 25]).

Iterative methods for approximating solutions of problem (4.3) have been studied (see e.g., [10, 12, 13, 15, 19, 22, 23, 26, 35, 42] and the references therein).

In this section, we shall apply Theorem 3.1, for the case where the map A is single-valued, to approximate a solution of equation (4.3). First we state the following important lemmas.

Lemma 4.4 (Chidume and Idu [16]) *Let X be a uniformly convex and uniformly smooth real Banach space with dual space X^* and $E = X \times X^*$. Let $F : X \to X^*$ and $K : X^* \to X$ be monotone maps with $R(F) = D(K)$, where $R(F)$ is the range of F and $D(K)$ is the domain of K. Let $A : E \to E^*$ be defined by $A[u,v] = [Fu - v, Kv + u]$. Then, A is maximal monotone.*

Let $\{\lambda_n\}$, $\{\beta_n\}$, and $\{\theta_n\}$ be sequences in $(0,1)$ and satisfy the conditions as given in Theorem 3.2.

Theorem 4.5 *Let E be a uniformly convex and uniformly smooth real Banach space with dual space E^*. Let $F : E \to E^*$, $K : E^* \to E$ be maximal monotone maps. Let $X := E \times E^*$ and $A : X \to X^*$ be defined by $A[u,v] := [Fu - v, Kv + u]$. For arbitrary $z_0, z_1 \in X$, define the sequence $\{z_n\}$ in X by*

$$\begin{cases} w_n = z_n + \beta_n(z_n - z_{n-1}), \\ z_{n+1} = J^{-1}(Jw_n - \lambda_n Aw_n - \lambda_n\theta_n w_n), \quad n \geq 1. \end{cases} \tag{4.6}$$

Assume that the equation $u + KFu = 0$ has a solution, then the sequence $\{z_n\}_{n=1}^{\infty}$ converges strongly to a solution of $u + KFu = 0$.

Proof By a result of Chidume [9], E is uniformly smooth and uniformly convex, also, by Lemma 4.4, A maximal monotone. Therefore, the conclusion follows from Theorem 3.2. □

Theorem 4.5 can also be stated as follows.

Theorem 4.6 *Let E be a uniformly smooth and uniformly convex real Banach space with dual space E^*. Let $F : E \to E^*$, $K : E^* \to E$ be maximal monotone maps with $R(F) = D(K)$, where $R(F)$ is the range of F and $D(K)$ is the domain of K.*

For arbitrary $(u_0, v_0), (u_1, v_1) \in E \times E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in $E \times E^*$ by

$$\begin{cases} c_n = u_n + \beta_n(u_n - u_{n-1}), \qquad d_n = v_n + \beta_n(v_n - v_{n-1}), \\ u_{n+1} = J^{-1}(Jc_n - \lambda_n(Fc_n - d_n) - \lambda_n\theta_n Jc_n), \quad n \geq 1, \\ v_{n+1} = J(J^{-1}d_n - \lambda_n(Kd_n + c_n) - \lambda_n\theta_n J^{-1}d_n), \quad n \geq 1. \end{cases} \tag{4.7}$$

Assume that the equation $u + KFu = 0$ has a solution, then the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge strongly to u^* and v^*, respectively, where u^* is a solution of $u + KFu = 0$, with $v^* = Fu^*$.

Remark 3 Algorithm (4.7) (Inertial Algorithm 2) will be compared with Algorithm (4.8) of Uba et al. [44] and Algorithm (4.9) of Chidume et al. [11] below. We state the theorems for completeness.

Theorem 4.7 (Uba et al. [44]) *Let E be a uniformly convex and uniformly smooth real Banach space and and $F : E \to E^*$, $K : E^* \to E$ be maximal monotone and bounded maps.*

For $u_1 \in E$ and $v_1 \in E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively, by

$$\begin{cases} u_{n+1} = J^{-1}(Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)), & n \geq 1, \\ v_{n+1} = J(J^{-1}v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(J^{-1}v_n - J^{-1}v_1)), & n \geq 1, \end{cases} \tag{4.8}$$

where λ_n and θ_n are sequences in $(0,1)$ satisfying appropriate conditions. Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is the solution of $u + KFu = 0$ with $v^* = Fu^*$.

Theorem 4.8 (Chidume et al. [11]) *Let E be a uniformly convex and uniformly smooth real Banach space and $F : E \to E^*$, $K : E^* \to E$ be maximal monotone maps. For $u_1 \in E$ and $v_1 \in E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in E and E^*, respectively, by*

$$\begin{cases} u_{n+1} = J^{-1}(Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n Ju_n), & n \geq 1, \\ v_{n+1} = J(J^{-1}v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n J^{-1}v_n), & n \geq 1, \end{cases} \tag{4.9}$$

where λ_n and θ_n are sequences in $(0,1)$ satisfying appropriate conditions. Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}$ and $\{v_n\}$ converge strongly to u^* and v^*, respectively, where u^* is the solution of $u + KFu = 0$ with $v^* = Fu^*$.

5 Numerical illustration

In this section, we present numerical examples to compare the convergence of the sequence of our inertial algorithms and some recent important algorithms. First, we compare the convergence of the sequence of Inertial Algorithm (3.1) (Inertial Algorithm 1) with IPPA (1.5), (1.7) (IPPA), (1.8) (RPPA), and (1.9) (RIPPA), respectively. Also, we present numerical examples to compare the convergence of the sequence of Algorithm (4.7) (Inertial Algorithm 2) with Algorithms (4.8) and (4.9), respectively. Finally, we present a numerical example to illustrate the implementability of Algorithm (4.1) whose sequence approximates a solution of a convex optimization problem.

Example 1 (Zeros of a maximal monotone map in a real Hilbert space) In Theorem 1.1, IPPA, RPPA, RIPPA, and Theorem 3.2 set $E = L_2([0,1])$. Consider the map $A : E \to E$ defined by

$$(Au)(t) := (t+1)u(t).$$

Then, it is easy to see that A is maximal monotone. Furthermore, the function $u(t) = 0$, $\forall t \in [0,1]$ is the solution of the equation $Au(t) = 0$. In Theorem 1.1, we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$; in Algorithm (1.7) (IPPA), take $\lambda_k = \frac{k}{k+1}$, $\alpha_k = \frac{1}{(k+1)^2}$; in Algorithm (1.8) (RPPA), take $\lambda_k = \frac{k}{k+1} = \rho_k$; in Algorithm (1.9) (RPPA), take $\lambda_k = \frac{k}{k+1} = \rho_k$, $\alpha_k = \frac{1}{(k+1)^2}$; and in Theorem 3.2, we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $\beta_n = \frac{1}{(n+1)^2}$, $n = 1, 2, \ldots$, as our parameters. Clearly, these parameters satisfy the hypothesis of the respective theorems. In all the tables below, we use the following notions:

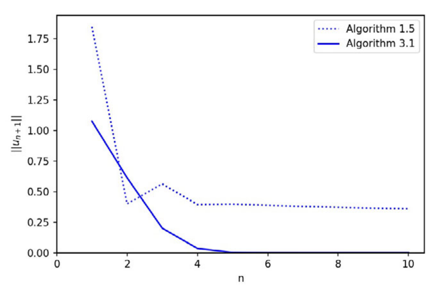
- IP—initial point,
- n—number of iterations,

- $\|u_{n+1}\|$—norm of the approximate solution at the $(n+1)$th iteration,
- $T(s)$—time in seconds.

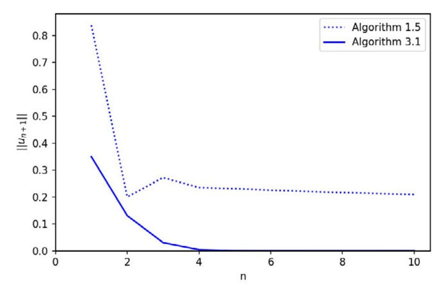
Setting a tolerance of 10^{-6} and maximum number of iterations $n = 10$, we obtain the iterates which are shown in Tables 1 and 2.

Table 1 Numerical results for Example 1

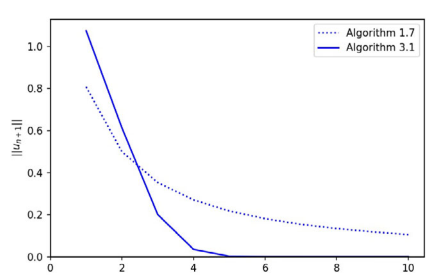
Algorithm (1.5)				Algorithm (1.7) (Inertial PPA)				Algorithm (3.1) (Inertial Algorithm 1)			
IP	n	$\|u_{n+1}\|$	T (s)	IP	n	$\|u_{n+1}\|$	T (s)	IP	n	$\|u_{n+1}\|$	T (s)
$u_1(t) = t^2 + 1$	10	0.3587	0.032	$u_1(t) = t^2 + 1$	10	0.0762	0.081	$u_0(t) = 2t$ $u_1(t) = t^2 + 1$	10	1.999E–6	15.69
$u_1(t) = \frac{1}{t+1}$	10	0.2093	0.058	$u_1(t) = \frac{1}{t+1}$	10	0.1056	0.082	$u_0(t) = 2t$ $u_1(t) = \frac{1}{t+1}$	10	1.87E–6	17.65
$u_1(t) = te^t$	10	0.2984	0.056	$u_1(t) = te^t$	10	0.0552	0.095	$u_0(t) = 2t$ $u_1(t) = te^t$	8	1.89E–6	92.44



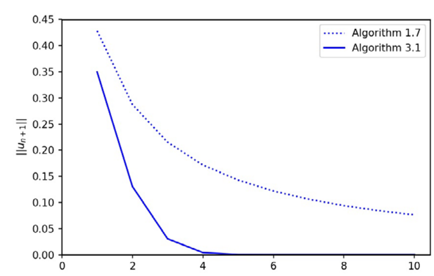
Graph of the first 10 iterates of Algorithms (1.5) and (3.1) with $u_1(t) = t^2 + 1$.



Graph of the first 10 iterates of Algorithms (1.5) and (3.1) with $u_1(t) = \frac{1}{t+1}$.



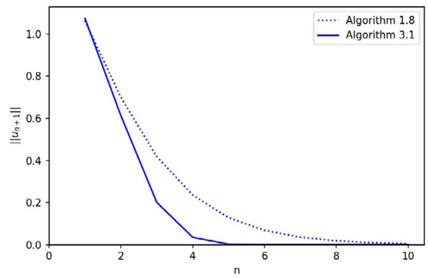
Graph of the first 10 iterates of Algorithms (1.7) and (3.1) with $u_1(t) = t^2 + 1$.



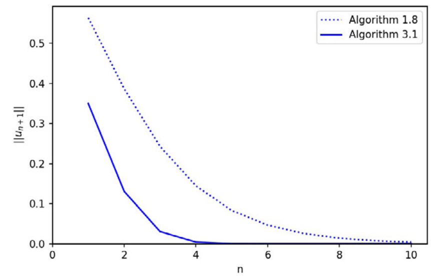
Graph of the first 10 iterates of Algorithms (1.7) and (3.1) with $u_1(t) = \frac{1}{t+1}$.

Table 2 Numerical results for Example 1

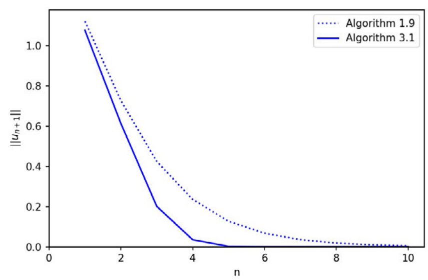
Algorithm (1.8) (RPPA)				Algorithm (1.9) (RIPPA)				Algorithm (3.1) (Inertial Algorithm 1)			
IP	n	$\|u_{n+1}\|$	T (s)	IP	n	$\|u_{n+1}\|$	T (s)	IP	n	$\|u_{n+1}\|$	T (s)
$u_1(t) = t^2 + 1$	10	0.005	0.025	$u_1(t) = t^2 + 1$	10	0.0051	16.68	$u_0(t) = 2t$ $u_1(t) = t^2 + 1$	10	1.999E–6	15.69
$u_1(t) = \frac{1}{t+1}$	10	0.0041	0.0381	$u_1(t) = \frac{1}{t+1}$	10	0.0042	17.95	$u_0(t) = 2t$ $u_1(t) = \frac{1}{t+1}$	10	1.87E–6	17.65
$u_1(t) = te^t$	10	0.0021	0.0392	$u_1(t) = te^t$	10	0.0017	21.13	$u_0(t) = 2t$ $u_1(t) = te^t$	8	1.89E–6	92.44



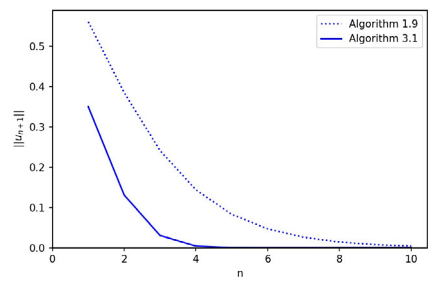
Graph of the first 10 iterates of Algorithms (1.8) and (3.1) with $u_1(t) = t^2 + 1$.



Graph of the first 10 iterates of Algorithms (1.8) and (3.1) with $u_1(t) = \frac{1}{t+1}$.



Graph of the first 10 iterates of Algorithms (1.9) and (3.1) with $u_1(t) = t^2 + 1$.



Graph of the first 10 iterates of Algorithms (1.9) and (3.1) with $u_1(t) = \frac{1}{t+1}$.

Example 2 (Numerical example for solutions of Hammerstein equation) In Theorems 4.7, 4.8, and 4.6 (Inertial algorithm 2), respectively, set $E = L_5([0,1])$, then, $E^* = L_{\frac{5}{4}}([0,1])$ and $F : L_5([0,1]) \to L_{\frac{5}{4}}([0,1])$ is defined by

$$(Fu)(t) = Ju(t).$$

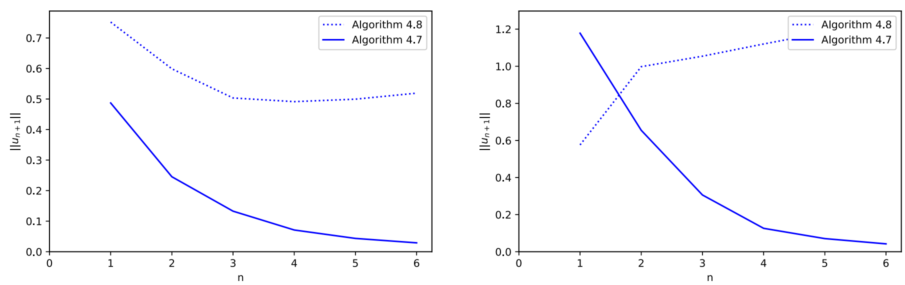
Then, it is easy to see that F is maximal monotone. Let $K : L_{\frac{5}{4}}([0,1]) \to L_5([0,1])$ be defined by

$$(Kv)(t) = tv(t).$$

Observe that K is linear. Furthermore, it is easy to see that K maximal monotone and the function $u^*(t) = 0$, $\forall t \in [0,1]$ is the only solution of the equation $u + KFu = 0$. In the algorithm of Theorem 3.1 in [44], we take $\lambda_n = \theta_n = \frac{1}{(n+1)^{\frac{1}{2}}}$; in the algorithm of Theorem 3.4 in Theorem [11], $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\beta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $n = 1, 2, \ldots$; and in the algorithm of Theorem 3.1, we take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $\beta_n = \frac{1}{(n+1)^2}$, $n = 1, 2, \ldots$, as our parameters and fixed $u_0(t) = t$ and $v_0(t) = t + 1$. Clearly, these parameters satisfy the hypotheses of the respective theorems. Setting a tolerance of 10^{-6} and maximum number of iterations $n = 6$, we obtain the iterates which are shown in Table 3.

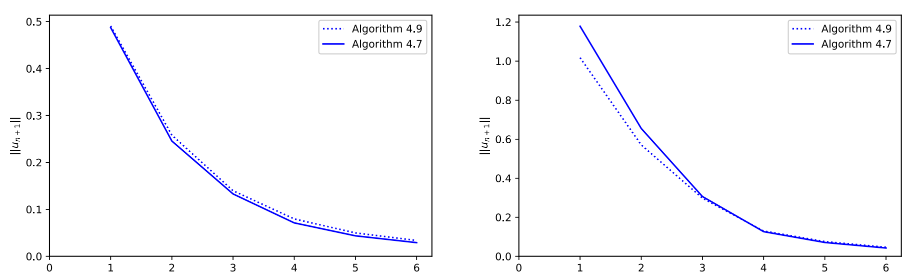
Table 3 Numerical results for Example 2

Algorithm (4.8)				Algorithm (4.9)				Algorithm (4.7) (Inertial Algorithm 2)			
IP	n	$\Vert u_{n+1}\Vert$	T (s)	IP	n	$\Vert u_{n+1}\Vert$	T (s)	IP	n	$\Vert u_{n+1}\Vert$	T (s)
$u_1(t) = \sin t$ $v_1(t) = \cos t$	6	0.5193	41.56	$u_1(t) = \sin t$ $v_1(t) = \cos t$	6	0.0337	92.78	$u_1(t) = \sin t$ $v_1(t) = \cos t$	6	0.0291	4129.97
$u_1(t) = t^2 - 2$ $v_1 = e^t - 1$	6	1.2381	244.31	$u_1 = t^2 - 2$ $v_1 = e^t - 1$	6	0.0463	28.55	$u_1 = t^2 - 2$ $v_1 = e^t - 1$	6	0.0424	884.05
$u_1(t) = 2t^3 - 2$ $v_1 = te^t + 2t$	6	1.4154	647.69	$u_1 = 2t^3 - 2$ $v_1 = te^t + 2t$	6	0.0720	57.03	$u_1 = 2t^3 - 2$ $v_1 = te^t + 2t$	6	0.0519	2268.58



Graph of the first 6 iterates of Algorithms (4.8) and (4.7) with $u_1(t) = \sin t$ and $v_1(t) = \cos t$.

Graph of the first 6 iterates of Algorithms (4.8) and (4.7) with $u_1(t) = t^2 - 2$ and $v_1(t) = e^t - 1$.



Graph of the first 6 iterates of Algorithms (4.9) and (4.7) with $u_1(t) = \sin t$ and $v_1(t) = \cos t$.

Graph of the first 6 iterates of Algorithms (4.9) and (4.7) with $u_1(t) = t^2 - 2$ and $v_1(t) = e^t - 1$.

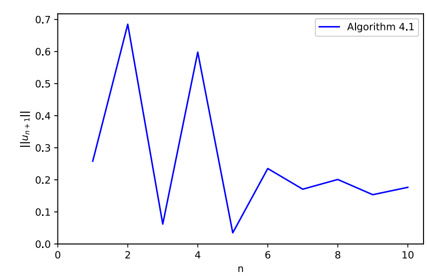
Example 3 (Numerical example for solutions of convex optimization problem) In Theorem 4.2, set set $E = L_2([0,1])$. Let $f : E \to \mathbb{R} \cup \{\infty\}$ be defined by

$$f(z) = \|z\|, \quad \text{then } \partial f(z) = \begin{cases} \frac{z(t)}{\|z\|}, & z(t_0) \neq 0, t_0 \in [0,1]; \\ B(0,1), & z(t) = 0, \forall t \in [0,1]. \end{cases} \tag{5.1}$$

Then it is easy to see that ∂f is maximal monotone. Furthermore, the function $z(t) = 0$, $\forall t \in [0,1]$ is the solution of the equation $\partial f z(t) = 0$. We take $\alpha_n = \frac{1}{(n+1)^{\frac{1}{2}}}$, $\theta_n = \frac{1}{(n+1)^{\frac{1}{4}}}$, $\beta_n = \frac{1}{(n+1)^2}$, $n = 1, 2, \ldots$, as our parameters. Clearly, these parameters satisfy the hypotheses of Theorem 4.2. Setting a tolerance of 10^{-6} and maximum number of iterations $n = 10$, we obtain the iterates which are shown in Table 4.

Table 4 Numerical results for Example 3

n	Algorithm (4.1)
	$\|z_{n+1}\|$
1	0.2581
2	0.685
3	0.6209
4	0.5979
5	0.0347
6	0.2352
7	0.1707
8	0.2011
9	0.1536
10	0.1766



Graph of the first 10 iterates of Algorithm (4.1) with $z_1(t) = t + \sin t$.

Observations

1. In Example 1, we presented a numerical experiment for zeros of a maximal monotone map A on E, where $E = L_2([0,1])$. With a tolerance of 10^{-6}, setting the maximum number of iterations to $n = 10$ and considering $u_1(t) = t^2 + 1$, the sequence generated by Algorithm (1.5) and the sequence generated by the IPPA (1.7) are yet to converge to zero, whereas the sequence generated by our algorithm, Algorithm (3.1), converges to zero in less than 6 iterations already with the 8th iterate as 1.89E–6, a very good approximation to a zero.

   Furthermore, the sequence generated by the RPPA converges to zero with the 10th iterate as 0.005, and the sequence generated by the RIPPA converges to zero with the 10th iterate as 0.0051 in 16.68 seconds, whereas the sequence generated by our algorithm, Algorithm (3.1), converges to zero as in the example. The convergence of the sequence generated by our algorithm is better than the convergence of the sequence generated by either the RPPA and RIPPA. A similar trend is observed when the initial vector is changed to $u_1(t) = \frac{1}{t+1}$.
2. In Example 2, we presented a numerical experiment for solutions of a Hammerstein integral equation, where $E = L_5([0,1])$, $F : E \to E^*$ and $K : E^* \to E$ are maximal monotone. With a tolerance of 10^{-6}, setting the maximum number of iterations to $n = 6$, and taking $u_1(t) = \sin t$ and $v_1(t) = \cos t$, the sequence generated by Algorithm (4.8), after 6 iterations in 41.56 seconds is yet to converge to any zero of A, whereas Algorithms (4.7) and (4.9) after 6 iterations, in 4129.97 and 92.78 seconds, respectively, converged to a zero of A. Furthermore, Algorithm (4.9) and our

Algorithm (4.7), with these initial vectors, both converge to zero almost jointly with the 6th iterate as 0.0337 in 92.78 and 0.0291 in 4129.97 seconds, respectively. Similar trends are observed when the initial vectors are changed to $u_1(t) = t^2 - 2$, $v_1(t) = e^t - 1$, and $u_1(t) = 2t^3 - 2$, $u_1(t) = te^t$.

3. Example 3, where $E = L_2([0,1])$, $f : E \to \mathbb{R} \cup \{\infty\}$ and $\partial f : E \rightrightarrows E^*$ are maps defined in equation (5.1), demonstrates the implementability of the sequence generated by Algorithm (4.1) with a tolerance of 10^{-6}, maximum number of iterations $n = 10$, and $z_1(t) = t + \sin t$.

Remark 4 Theorems involving IPPA, RPPA, and RIPPA as cited above are proved in real Hilbert spaces, whereas our theorems in this paper are proved in much more general, uniformly smooth and uniformly convex real Banach spaces. Moreover, a *strong convergence* theorem is proved in Theorem 3.2, whereas a *weak convergence* theorem is proved for IPPA, RPPA and RIPPA, respectively.

6 Conclusions

An inertial iterative algorithm which does not involve the *resolvent operator* is proposed for approximating a solution of a maximal monotone inclusion in uniformly convex and uniformly smooth real Banach spaces. The sequence generated by the algorithm is proved to converge *strongly* to a solution of the inclusion. Furthermore, the theorem proved is applied to approximate a solution of a convex optimization problem, and a solution of a Hammerstein integral equation. In addition, numerical experiments are given to compare, in terms of CPU time and number of iterations, the performance of the sequence generated by our algorithm with the performance of the sequences generated by IPPA, RPPA and RIPPA, respectively. In these examples, the performance of the sequence generated by our algorithm is much better than the performance of the sequence generated by any of IPPA, RPPA, and RIPPA. A numerical example is also given to illustrate the implementability of our algorithm for approximating a solution of a convex optimization problem and for approximating a solution of a Hammerstein integral equation. Finally, it is clear that our algorithm is a welcome addition to the inertial proximal point type algorithms for approximating solutions of maximal monotone inclusions and their applications.

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