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A short and sharpened way to approach fixed point results involving fuzzy $\mathcal{H}$-contractive mappings

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Abstract

In the present paper, we adopt a short and sharpened approach to prove fixed point results involving fuzzy $\mathcal{H}$-contractive mappings utilized in (Wardowski, Fuzzy Sets Syst. 125:245–252, 2013) and other related articles. In this process, we are able to relax some conditions utilized by earlier authors which in turn yields affirmative answers to some open questions raised by earlier authors.

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1 Introduction

The existing literature on fuzzy sets and systems contains several definitions of fuzzy metric spaces [2–4]. The most popular definition of fuzzy metric space is essentially due to Kramosil and Michalek [5]. With a view to having a Hausdorff topology, George and Veeramani [6] modified the concept of fuzzy metric spaces initiated by Kramosil and Michalek [5]. Like other areas in mathematics, fuzzy metric fixed point theory is also flourishing and by now there exists a considerable body of literature on fuzzy metric fixed point theory. Gregori and Sapena [1] introduced the idea of fuzzy contractive mappings and proved a fuzzy version of the Banach contraction principle for such mappings in fuzzy metric spaces. Motivated by Samet et al. [7], Salimi et al. [8] introduced some classes of fuzzy contractive mappings and gave fixed point results which generalize and extend some comparable results in the existing literature.

In 2013, Wardowski [9] generalized the concept of fuzzy contractive mapping by introducing the concept of fuzzy $\mathcal{H}$-contractive mapping and proved a fixed point result in M-complete fuzzy metric space. Thereafter, Shukla [10] defined fuzzy $\mathcal{H}$-weak contractive mapping and utilized the same to extend the fixed point results due to Wardowski [9]. Later, Beg et al. [11] defined the notion of α-fuzzy-$\mathcal{H}$-contractive mapping and established some existence and uniqueness of fixed point results in fuzzy M-complete metric spaces. For more results in this direction, we refer the reader to [12–24].

Inspired by Wardowski [9], Shukla [10], and Beg et al. [11], in this paper, we extend and improve some existing results involving fuzzy $\mathcal{H}$-contractive, fuzzy $\mathcal{H}$-weak contractive, and α-fuzzy-$\mathcal{H}$-contractive mappings besides answering two open questions raised by Wardowski [9] and Beg et al. [11].

2 Preliminaries

In this section, we recall some known definitions, properties, and results about fuzzy metric spaces.

Definition 2.1 ([25]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous triangular norm (or continuous t-norm) if it satisfies the following conditions:

($T1$) $*$ is associative and commutative;

($T2$) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$;

($T3$) $a * 1 = a$ for all $a \in [0,1]$;

($T4$) $*$ is continuous.

For all $a, b \in [0,1]$, the most commonly used t-norms are:

- $a *_p b = a \cdot b$ (product t-norm),
- $a *_m b = \min\{a, b\}$ (minimum t-norm),
- $a *_L b = \max\{a + b - 1, 0\}$ (Lukasiewicz t-norm).

A positive t-norm $*$ is a t-norm satisfying $a * b > 0$ for $a, b \in (0,1]$. If $*$ is a t-norm and $a * b$ is continuous such that for any $a \in (0,1)$ there exists $n \in \mathbb{N}$ with $\prod_{i=1}^{n} a_i = 0$, then $*$ is called a nilpotent t-norm.

The concept of fuzzy metric space in the sense of George and Veeramani [6] is defined as follows.

Definition 2.2 ([6]) An ordered triple $(X, M, *)$ is said to be a fuzzy metric space if X is a nonempty set, $*$ is a continuous t-norm, and $M : X^2 \times (0, \infty) \to [0,1]$ is a fuzzy set satisfying the following conditions:

($G1$) $M(x, y, t) > 0$,

($G2$) $M(x, y, t) = 1$ if and only if $x = y$,

($G3$) $M(x, y, t) = M(y, x, t)$,

($G4$) $M(x, y, t + s) \geq M(x, z, t) * M(z, y, s)$,

($G5$) $M(x, y, \cdot) : (0, \infty) \to (0,1]$ is continuous

for all $x, y, z \in X$ and $t, s > 0$.

A fuzzy metric space $(X, M, *)$ is called strong if condition ($G4$) in Definition 2.2 is replaced by the following condition:

($G4$)′ $M(x, y, t) \geq M(x, z, t) * M(z, y, t)$ for all $x, y, z \in X$ and $t > 0$.

Definition 2.3 ([6]) Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $\mathfrak{B}(x, r, t)$ with a center $x \in X$ and radius $r \in (0,1)$ is defined by

$$\mathfrak{B}(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if, for each $x \in A$, there exist $t > 0$ and $r \in (0,1)$ such that $\mathfrak{B}(x, r, t) \subset A$. The family of all open subsets of X is a topology on X, called the topology induced by the fuzzy metric M.

Example 2.1 ([6]) Let (X,d) be a metric space. Define $M : X \times X \times (0,\infty) \to [0,1]$ as follows:

$$M(x,y,t) = \frac{t}{t + d(x,y)}, \quad \forall x,y \in X \text{ and } t > 0.$$

Then $(X,M,*)$ is a fuzzy metric space with respect to the product t-norm (or minimum t-norm) for all $x,y \in [0,1]$. M is known as the standard fuzzy metric.

Definition 2.4 ([6]) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a fuzzy metric space $(X,M,*)$ is called convergent and converges to $x \in X$ if $\lim_{n\to\infty} M(x_n,x,t) = 1$ for all $t > 0$, that is, for each $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n,x,t) > 1 - r$ for all $n \geq n_0$.

Definition 2.5 ([6]) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a fuzzy metric space $(X,M,*)$ is called M-Cauchy if, for each $\epsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_m,x_n,t) > 1 - \epsilon$ for all $m,n \geq n_0$.

The fuzzy metric space $(X,M,*)$ is called M-complete if every M-Cauchy sequence in X converges to a point of X.

Remark 2.1 ([26, 27])

(a) The limit of a convergent sequence in the setting of fuzzy metric spaces is unique.

(b) The mapping $M(x,y,\cdot)$ is nondecreasing on $(0,\infty)$ for all $x,y \in X$.

(c) In a fuzzy metric space $(X,M,*)$, the mapping M is continuous on $X \times X \times (0,\infty)$.

Definition 2.6 ([1]) Let $(X,M,*)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be fuzzy contractive if there exists $k \in (0,1)$ such that

$$\frac{1}{M(Tx,Ty,t)} - 1 \leq k\left(\frac{1}{M(x,y,t)} - 1\right)$$

for all $x,y \in X$ and $t > 0$.

Let $\mathcal{H}$ be the class of all functions $\eta : (0,1] \to [0,\infty)$ which satisfy the following:

($\mathcal{H}1$) η transforms $(0,1]$ onto $[0,\infty)$;

($\mathcal{H}2$) η is strictly decreasing, that is, for all $a,b \in (0,1], a < b \Longrightarrow \eta(a) > \eta(b)$.

Definition 2.7 ([9]) Let $(X,M,*)$ be a fuzzy metric space. A mapping $T : X \to X$ is called fuzzy $\mathcal{H}$-contractive w.r.t. some $\eta \in \mathcal{H}$ if there exists $\lambda \in (0,1)$ such that

$$\eta\bigl(M(Tx,Ty,t)\bigr) \leq k\eta\bigl(M(x,y,t)\bigr), \quad \forall x,y \in X \text{ and } t > 0. \tag{2.1}$$

In [9], Wardowski proved the following fixed point theorem in M-complete fuzzy metric spaces.

Theorem 2.1 ([9]) *Let $(X,M,*)$ be an M-complete fuzzy metric space. Assume that $T : X \to X$ is a fuzzy $\mathcal{H}$-contractive mapping w.r.t some $\eta \in \mathcal{H}$ such that*

(a) $\prod_{i=1}^{k} M(x,Tx,t_i) \neq 0, \forall x \in X, k \in \mathbb{N}$ *and* $(t_i) \subset (0,\infty), t_i \searrow 0$;

(b) $r * s > 0 \Rightarrow \eta(r * s) < \eta(r) * \eta(s), \forall r,s \in \{M(x,Tx,t) : x \in X, t > 0\}$;

(c) $\{\eta(M(x,Tx,t_i)) : i \in \mathbb{N}\}$ *is bounded* $\forall x \in X$ *and* $(t_i) \subset (0,\infty)$, $t_i \searrow 0$.
Then T *possesses a unique fixed point.*

Definition 2.8 ([10]) Let $(X,M,*)$ be a fuzzy metric space. A mapping $T : X \to X$ is called fuzzy $\mathcal{H}$-weak contractive w.r.t. some $\eta \in \mathcal{H}$ if there exists $\lambda \in (0,1)$ such that

$$\eta\big(M(Tx,Ty,t)\big) \le \lambda \mathcal{N}(x,y,t), \tag{2.2}$$

where $\mathcal{N}(x,y,t) = \max\{\eta(M(x,y,t)), \eta(M(x,Tx,t)), \eta(M(y,Ty,t))\}$ for all $x,y \in X$ and $t > 0$.

The following theorem by Shukla comes as a generalization of Theorem 2.1.

Theorem 2.2 ([10]) *Let* $(X,M,*)$ *be an* M*-complete fuzzy metric space. Assume that* $T : X \to X$ *is a fuzzy* $\mathcal{H}$*-weak contractive mapping w.r.t. some* $\eta \in \mathcal{H}$ *such that*
(a) $\prod_{i=1}^{k} M(x,Tx,t_i) \ne 0$, $\forall x \in X$, $k \in \mathbb{N}$ *and* $(t_i) \subset (0,\infty)$, $t_i \searrow 0$;
(b) $r * s > 0 \Rightarrow \eta(r*s) < \eta(r) * \eta(s)$, $\forall r,s \in \{M(x,Tx,t) : x \in X, t > 0\}$;
(c) $\{\eta(M(x,Tx,t)) : i \in \mathbb{N}\}$ *is bounded* $\forall x \in X$ *and any sequence* $(t_i) \subset (0,\infty)$, $t_i \searrow 0$.
Then T *possesses a unique fixed point.*

Definition 2.9 ([11]) Let $(X,M,*)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be α-fuzzy $\mathcal{H}$-contractive w.r.t. some $\eta \in \mathcal{H}$ if there exist $\lambda \in (0,1)$ and $\alpha : X \times X \times (0,\infty) \to [0,\infty)$ such that

$$\alpha(x,y,t)\eta\big(M(Tx,Ty,t)\big) \le \lambda\eta\big(M(x,y,t)\big), \quad \forall x,y \in X \text{ and } t > 0. \tag{2.3}$$

Definition 2.10 ([28]) Let $(X,M,*)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be α-admissible if there exists a function $\alpha : X \times X \times (0,\infty) \to [0,\infty)$ such that

$$\alpha(x,y,t) \ge 1 \quad \Longrightarrow \quad \alpha(Tx,Ty,t) \ge 1 \tag{2.4}$$

for all $x,y \in X$ and $t > 0$.

Based on the above definitions, Beg et al. [11] proved the following fixed point theorem.

Theorem 2.3 ([11]) *Let* $(X,M,*_L)$ *be an* M*-complete strong fuzzy metric space such that* $*_L$ *is nilpotent. Assume that* $T : X \to X$ *is* α*-fuzzy* $\mathcal{H}$*-contractive w.r.t. some* $\eta \in \mathcal{H}$ *such that*
(i) *there exists* $x_0 \in X$ *with* $\alpha(x_0,Tx_0,t) \ge 1$, $t > 0$;
(ii) T *is* α*-admissible*;
(iii) $\eta(r*s) < \eta(r) * \eta(s)$ *for all* $r,s \in \{M(x,Tx,t) : x \in X, t > 0\}$;
(iv) *each subsequence* $\{x_{n_k}\}$ *of a sequence* $\{x_n = T^n x_0\}$ *has the following property*: $\alpha(x_{n_k},x_{n_l},t) \ge 1$, $k,l \in \mathbb{N}$, $k > l$, $t > 0$;
(v) *if* x_n *is a sequence in* X *such that* $\alpha(x_n,x_{n+1},t) \ge 1$, $n \in \mathbb{N}$, $t > 0$, *and* $\lim_{n\to\infty} x_n = x$, *then* $\alpha(x_n,x,t) \ge 1$, $n \in \mathbb{N}$, $t > 0$.
Then T *admits a fixed point.*

3 Main results

We begin this section with some observations which play a significant role in proving our results which led to withdrawal of certain conditions utilized by earlier authors in their corresponding results. In this course, we are also able to obtain affirmative answers to certain questions raised by the authors of the corresponding results.

The following remark is clear.

Remark 3.1 For any $\eta \in \mathcal{H}$, we have the following (in view of $\mathcal{H}_1$ and $\mathcal{H}_2$):

(a) η is continuous and bijective;

(b) $\eta(t) = 0$ if and only if $t = 1$.

Now, we have the following proposition.

Proposition 3.1 *If the mapping T is fuzzy $\mathcal{H}$-contractive, then it is continuous.*

To accomplish this, let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \textit{and} \quad M(Tx_n, Tx, t) < 1, \quad \forall n \in \mathbb{N},$$

for some $x \in X$. Using (2.1), we obtain

$$\eta\big(M(Tx_n, Tx, t)\big) \le \lambda\eta\big(M(x_n, x, t)\big) < \eta\big(M(x_n, x, t)\big), \quad \textit{as } \lambda \in (0, 1).$$

Since η is strictly decreasing, we have

$$M(Tx_n, Tx, t) > M(x_n, x, t),$$

so that

$$\lim_{n\to\infty} M(Tx_n, Tx, t) = 1,$$

which shows that T is continuous.

Our next result offers an affirmative answer to the open question posed by Wardowski [9], which runs as follows:

"*Can condition (a) in Theorem 2.1 be omitted for nilpotent t-norms?*" Moreover, it can be pointed out that even conditions (b) and (c) of Theorem 2.1 (due to Wardowski [9]) can also be omitted. In fact, we prove the following.

Theorem 3.1 *Let $(X, M, *)$ be an M-complete fuzzy metric space. Assume that $T : X \to X$ is a fuzzy $\mathcal{H}$-contractive mapping w.r.t. some $\eta \in \mathcal{H}$. Then T possesses a unique fixed point.*

Proof Let $x_0 \in X$ and define $\{x_n\}$ by

$$x_n = Tx_{n-1}, \quad \forall n \in \mathbb{N}.$$

Using condition (2.1), we obtain

$$\begin{aligned} 0 &< \eta\bigl(M(Tx_{n-1}, Tx_n, t)\bigr) \\ &\leq \lambda\eta\bigl(M(x_{n-1}, x_n, t)\bigr) \\ &= \lambda\eta\bigl(M(Tx_{n-2}, Tx_{n-1}, t)\bigr) \\ &\leq \lambda^2\eta\bigl(M(x_{n-2}, x_{n-1}, t)\bigr) \\ &\vdots \\ &\leq \lambda^n\eta\bigl(M(x_0, x_1, t)\bigr), \end{aligned} \tag{3.1}$$

which on making $n \to \infty$ in (3.1) gives rise to

$$\lim_{n\to\infty} \eta\bigl(M(x_n, x_{n+1}, t)\bigr) = 0 \quad \Rightarrow \quad \eta\Bigl(\lim_{n\to\infty} M(x_n, x_{n+1}, t)\Bigr) = 0 \quad (\text{as } \eta \text{ is continuous}).$$

Therefore, in view of Remark 3.1, we conclude that

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1, \quad \forall n \in \mathbb{N}, t > 0. \tag{3.2}$$

Now, we show that $\{x_n\}$ is an M-Cauchy sequence. On the contrary, let us assume that the sequence $\{x_n\}$ is not Cauchy. Then there are $\epsilon \in (0,1)$, $t_0 > 0$ and two subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ of $\{x_n\}$ such that $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ and

$$M(x_{m_k}, x_{n_k}, t_0) \leq 1 - \epsilon. \tag{3.3}$$

In view of Remark 2.1(b), we infer

$$M\left(x_{m_k}, x_{n_k}, \frac{t_0}{2}\right) \leq 1 - \epsilon. \tag{3.4}$$

Suppose that n_k is the least integer exceeding m_k satisfying inequality (3.4). Then we have

$$M\left(x_{m_k-1}, x_{n_k}, \frac{t_0}{2}\right) > 1 - \epsilon. \tag{3.5}$$

Using the contractive inequality (2.1) with $x = x_{m_k-1}$, $y = x_{n_k-1}$, and $t = t_0$, we get

$$\begin{aligned} \eta\bigl(M(x_{m_k}, x_{n_k}, t_0)\bigr) &\leq \lambda\eta\bigl(M(x_{m_k-1}, x_{n_k-1}, t_0)\bigr) \\ &< \eta\bigl(M(x_{m_k-1}, x_{n_k-1}, t_0)\bigr). \end{aligned} \tag{3.6}$$

Since η is strictly decreasing, therefore we have

$$M(x_{m_k}, x_{n_k}, t_0) > M(x_{m_k-1}, x_{n_k-1}, t_0). \tag{3.7}$$

Making use of (3.3) and (3.5) in (3.7) and using ($G4$), we get

$$\begin{aligned}1-\epsilon &\geq M(x_{m_k},x_{n_k},t_0)\\ &> M(x_{m_k-1},x_{n_k-1},t_0)\\ &\geq M\left(x_{m_k-1},x_{n_k},\frac{t_0}{2}\right)*M\left(x_{n_k},x_{n_k-1},\frac{t_0}{2}\right)\\ &> (1-\epsilon)*M\left(x_{n_k},x_{n_k-1},\frac{t_0}{2}\right),\end{aligned}$$

which on letting $k\to\infty$ and making use of (3.2) and ($T3$) yields

$$\lim_{n\to\infty} M(x_{m_k},x_{n_k},t_0)=1-\epsilon \tag{3.8}$$

and

$$\lim_{n\to\infty} M(x_{m_k-1},x_{n_k-1},t_0)=1-\epsilon. \tag{3.9}$$

Taking the limit $k\to\infty$ over both sides of inequality (3.6) and taking into account the continuity of η, we have

$$\eta\Bigl(\lim_{k\to\infty} M(x_{m_k},x_{n_k},t_0)\Bigr)\leq\lambda\eta\Bigl(\lim_{k\to\infty} M(x_{m_k-1},x_{n_k-1},t_0)\Bigr).$$

Making use of (3.8) and (3.9) in the above inequality, we obtain

$$\eta(1-\epsilon)\leq\lambda\eta(1-\epsilon)<\eta(1-\epsilon),$$

which is a contradiction. Hence, $\{x_n\}$ is an M-Cauchy sequence in X.

The completeness of $(X,M,*)$ ensures the existence of $x\in X$ such that

$$\lim_{n\to\infty} M(x_n,x,t)=1,\quad t>0.$$

As T is continuous (see Proposition 3.1), therefore $\lim_{n\to\infty} M(x_{n+1},Tx,t)=1$. Owing to the uniqueness of the limit, we get $Tx=x$.

Now, we show that the fixed point of T is unique. Suppose that x_1 and x_2 are two different fixed points of T, that is, $Tx_1=x_1\neq x_2=Tx_2$. Then, using (2.1), we get

$$\eta\bigl(M(x_1,x_2,t_0)\bigr)\leq\lambda\eta\bigl(M(x_1,x_2,t_0)\bigr)<\eta\bigl(M(x_1,x_2,t_0)\bigr),\quad \text{where } \lambda\in(0,1),$$

which is a contradiction. Thus, T has a unique fixed point which concludes the proof. □

Remark 3.2 Notice that Wardowski [9] used conditions (a), (b), and (c) in Theorem 2.1 to prove the Cauchyness of the sequence $\{x_n\}$, but we prove the same in a different way wherein such conditions are useless.

To establish the genuineness of Theorem 3.1 over Theorem 2.1, we adopt the following example wherein conditions (b) and (c) of Theorem 2.1 are not satisfied but the conclusion of Theorem 2.1 continues to hold.

Example 3.1 Consider X to be the set of real numbers. Define a fuzzy set $M : X^2 \times (0,\infty) \to [0,1]$ by $M(x,y,t) = e^{-\frac{|x-y|}{t}}$ for all $x,y \in X$ and $t > 0$. Then $(X,M,*)$ is an M-complete fuzzy metric space where $*$ is a t-norm given by $a * b = a \cdot b$ for all $a,b \in [0,1]$.

Define a mapping $T : X \to X$ as follows:

$$T(x) = \frac{x}{3}, \quad \forall x \in X,$$

and let $\eta(\alpha) = \ln(\frac{1}{\alpha})$ for all $\alpha \in (0,1]$. Then, for all $x,y \in X$, $t > 0$, and $k = \frac{1}{3}$, we have

$$\begin{aligned}\eta\bigl(M(Tx,Ty,t)\bigr) &= \ln\bigl(e^{\frac{|Tx-Ty|}{t}}\bigr) = \frac{1}{3}\frac{|x-y|}{t} \\ &= \frac{1}{3}\ln\bigl(e^{\frac{|x-y|}{t}}\bigr) \\ &= k\eta\bigl(M(x,y,t)\bigr),\end{aligned}$$

which shows that T is $\mathcal{H}$-contractive. Hence, by Theorem 3.1, T has a unique fixed point (namely $x = 0$). Observe that condition (b) of Theorem 2.1 does not hold (e.g. choose $r = 0.3$ and $s = 0.5$). Moreover, for any $x \in X$, we have

$$\eta\bigl(M(x,Tx,t)\bigr) = \frac{|x|}{2t} \to \infty, \quad \text{as } t \to 0^+,$$

which confirms the failure of condition (c) of Theorem 2.1.

In a similar way, we refine and improve Theorem 2.2 due to Shukla [10] by relaxing conditions (a), (b), and (c).

Theorem 3.2 *Let $(X,M,*)$ be an M-complete fuzzy metric space. Assume that $T : X \to X$ is a fuzzy $\mathcal{H}$-weak contractive mapping w.r.t. some $\eta \in \mathcal{H}$. Then T admits a unique fixed point.*

Proof Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ as follows:

$$x_n = Tx_{n-1} \quad \text{for all } n \in \mathbb{N}.$$

For all $n \in \mathbb{N}$ and $t > 0$, using (2.2) we have

$$\begin{aligned}\eta\bigl(M(x_n,x_{n+1},t)\bigr) &= \eta\bigl(M(Tx_{n-1},Tx_n,t)\bigr) \\ &\le \lambda\mathcal{N}(x_{n-1},x_n,t),\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}\mathcal{N}(x_{n-1},x_n,t) &= \max\bigl\{\eta\bigl(M(x_{n-1},x_n,t)\bigr),\eta\bigl(M(x_{n-1},Tx_{n-1},t)\bigr),\eta\bigl(M(x_n,Tx_n,t)\bigr)\bigr\} \\ &= \max\bigl\{\eta\bigl(M(x_{n-1},x_n,t)\bigr),\eta\bigl(M(x_n,x_{n+1},t)\bigr)\bigr\}.\end{aligned}$$

If for some $n \in \mathbb{N}$, $\mathcal{N}(x_{n-1},x_n,t) = \eta(M(x_n,x_{n+1},t))$, then (3.10) turns into

$$\eta\bigl(M(x_n,x_{n+1},t)\bigr) \le \lambda\eta\bigl(M(x_n,x_{n+1},t)\bigr) < \eta\bigl(M(x_n,x_{n+1},t)\bigr),$$

a contradiction. Hence, we must have $\mathcal{N}(x_{n-1},x_n,t) = \eta(M(x_{n-1},x_n,t))$ for all $n \in \mathbb{N}$, and therefore (3.10) becomes

$$\eta\bigl(M(x_n,x_{n+1},t)\bigr) \leq \lambda\eta\bigl(M(x_{n-1},x_n,t)\bigr), \quad \forall n \in \mathbb{N}. \tag{3.11}$$

Inductively, from (3.11), we find that

$$0 < \eta\bigl(M(x_n,x_{n+1},t)\bigr) \leq \lambda^n\eta\bigl(M(x_0,x_1,t)\bigr), \quad \forall n \in \mathbb{N}. \tag{3.12}$$

By taking $n \to \infty$ in (3.12), we get

$$\lim_{n\to\infty} \eta\bigl(M(x_n,x_{n+1},t)\bigr) = 0,$$

and hence, it follows from Remark 2.1 that

$$\lim_{n\to\infty} M(x_n,x_{n+1},t) = 1. \tag{3.13}$$

Now, we show the Cauchyness of the sequence $\{x_n\}$ by contradiction. We assume that the sequence $\{x_n\}$ is not M-Cauchy. Then there are $\epsilon \in (0,1)$, $t_0 > 0$ and two subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ of $\{x_n\}$ such that $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ and

$$M(x_{m_k},x_{n_k},t_0) \leq 1 - \epsilon. \tag{3.14}$$

By Remark 2.1(b), we get

$$M\left(x_{m_k},x_{n_k},\frac{t_0}{2}\right) \leq 1 - \epsilon. \tag{3.15}$$

Suppose that n_k is the least integer exceeding m_k satisfying inequality (3.15). Then we have

$$M\left(x_{m_k-1},x_{n_k},\frac{t_0}{2}\right) > 1 - \epsilon. \tag{3.16}$$

Applying inequality (2.2), we get

$$\begin{aligned}\eta\bigl(M(x_{m_k},x_{n_k},t_0)\bigr) &= \eta\bigl(M(Tx_{m_k-1},Tx_{n_k-1},t_0)\bigr)\\ &\leq \lambda\mathcal{N}(x_{m_k-1},x_{n_k-1},t_0),\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}\mathcal{N}(x_{m_k-1},x_{n_k-1},t_0) = \max\bigl\{&\eta\bigl(M(x_{m_k-1},x_{n_k-1},t_0)\bigr),\\ &\eta\bigl(M(x_{m_k-1},x_{m_k},t_0)\bigr),\eta\bigl(M(x_{n_k-1},x_{n_k},t_0)\bigr)\bigr\}.\end{aligned}$$

If $\mathcal{N}(x_{m_k-1},x_{n_k-1},t_0) = \eta(M(x_{m_k-1},x_{m_k},t_0))$, then (3.17) becomes

$$\eta\bigl(M(x_{m_k},x_{n_k},t_0)\bigr) \leq \lambda\eta\bigl(M(x_{m_k-1},x_{m_k},t_0)\bigr).$$

Letting $k \to \infty$ in the above inequality and making use of (3.13), Remark 2.1 and taking into account the continuity of η yields

$$\eta\Big(\lim_{k\to\infty} M(x_{m_k}, x_{n_k}, t_0)\Big) = 0,$$

and hence $\lim_{k\to\infty} M(x_{m_k}, x_{n_k}, t_0) = 1$, which contradicts Equation (3.14).

Similarly, if we consider $\mathcal{N}(x_{m_k-1}, x_{n_k-1}, t_0) = \eta(M(x_{n_k-1}, x_{n_k}, t_0))$, then again we arrive at a contradiction. Therefore, we must have $\mathcal{N}(x_{m_k-1}, x_{n_k-1}, t_0) = \eta(M(x_{m_k-1}, x_{n_{k-1}}, t_0))$ and hence (3.17) gives rise to

$$\eta\big(M(x_{m_k}, x_{n_k}, t_0)\big) \le \lambda\eta\big(M(x_{m_k-1}, x_{n_k-1}, t_0)\big) < \eta\big(M(x_{m_k-1}, x_{n_k-1}, t_0)\big). \tag{3.18}$$

Since η is strictly decreasing, then we have

$$M(x_{m_k}, x_{n_k}, t_0) > M(x_{m_k-1}, x_{n_k-1}, t_0). \tag{3.19}$$

Making use of (3.19), (3.14), and (3.16), we have

$$\begin{aligned}
1-\epsilon &\ge M(x_{m_k}, x_{n_k}, t_0) \\
&> M(x_{m_k-1}, x_{n_k-1}, t_0) \\
&\ge M\left(x_{m_k-1}, x_{n_k}, \frac{t_0}{2}\right) * M\left(x_{n_k}, x_{n_k-1}, \frac{t_0}{2}\right) \\
&> (1-\epsilon) * M\left(x_{n_k}, x_{n_k-1}, \frac{t_0}{2}\right),
\end{aligned}$$

which on letting $k \to \infty$ and using (3.13) along with $(T3)$ yields

$$\lim_{k\to\infty} M(x_{m_k-1}, x_{n_k-1}, t_0) = 1-\epsilon = \lim_{k\to\infty} M(x_{m_k}, x_{n_k}, t_0). \tag{3.20}$$

Now, using (3.18), (3.20), and the continuity of η, we obtain

$$\eta(1-\epsilon) \le \lambda\eta(1-\epsilon) < \eta(1-\epsilon),$$

which is a contradiction. Hence, $\{x_n\}$ is an M-Cauchy sequence in X. Due to the M-completeness of $(X, M, *)$, there exists $x \in X$ such that

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad t > 0. \tag{3.21}$$

Next, we show that x is a fixed point of T. Suppose that there exists $t_1 > 0$ such that $M(x, Tx, t_1) < 1$, then $\eta(M(x, Tx, t_1)) > 0$. Also, as T is a fuzzy $\mathcal{H}$-weak contractive mapping, we have

$$\begin{aligned}
\eta\big(M(x_{n+1}, Tx, t_1)\big) &= \eta\big(M(Tx_n, Tx, t_1)\big) \\
&\le \lambda\mathcal{N}(x_n, x, t_1),
\end{aligned} \tag{3.22}$$

where $\mathcal{N}(x_n, x, t_1) = \max\{\eta(M(x_n, x, t_1)), \eta(M(x_n, x_{n+1}, t_1)), \eta(M(x, Tx, t_1))\}$. As η is continuous, we have $\lim_{n\to\infty} \eta(M(x_n, x, t_1)) = \lim_{n\to\infty} \eta(M(x_n, x_{n+1}, t_1)) = 0$ for all $t > 0$, and hence

$$\begin{aligned}\lim_{n\to\infty} \mathcal{N}(x_n, x, t_1) &= \max\bigl\{0, 0, \eta\bigl(M(x, Tx, t_1)\bigr)\bigr\} \\ &= \eta(M(x, Tx, t_1)).\end{aligned}$$

Now, making $n \to \infty$ in (3.22), we get

$$\eta\bigl(M(x, Tx, t_1)\bigr) \le \lambda\eta\bigl(M(x, Tx, t_1)\bigr) < \eta\bigl(M(x, Tx, t_1)\bigr), \quad \text{as } \lambda \in (0, 1),$$

which is a contradiction. Therefore, we must have $M(x, Tx, t) = 1$, $t > 0$, which shows that x remains fixed under T.

To prove the uniqueness of the fixed point of T, let x_1, x_2 be two fixed points of T. Then using (2.2) we have

$$\begin{aligned}\eta\bigl(M(x_1, x_2, t)\bigr) &= \eta\bigl(M(Tx_1, Tx_2, t)\bigr) \\ &\le \lambda \max\bigl\{\eta\bigl(M(x_1, x_2, t)\bigr), \eta\bigl(M(x_1, Tx_1, t)\bigr), \eta\bigl(M(x_2, Tx_2, t)\bigr)\bigr\} \\ &= \lambda \max\bigl\{\eta\bigl(M(x_1, x_2, t)\bigr), 0, 0\bigr\}. \end{aligned} \tag{3.23}$$

If $M(x_1, x_2, t) < 1$, then $\eta(M(x_1, x_2, t)) > 0$ and hence (3.23) becomes

$$\eta\bigl(M(x_1, x_2, t)\bigr) \le \lambda\eta\bigl(M(x_1, x_2, t)\bigr) < \eta\bigl(M(x_1, x_2, t)\bigr),$$

which is a contradiction. Therefore, $M(x_1, x_2, t) = 1$, yielding thereby $x_1 = x_2$, this concludes the proof. □

In what follows, we answer the open question raised by Beg et al. [11]:

Can the assumption of strong fuzzy metric in Theorem 2.3 be omitted/further relaxed?

The answer to this question is in the affirmative. To substantiate this claim, we prove the following theorem in which we have also withdrawn condition (iii) besides relaxing the requirement of nilpotent t-norm from Theorem 2.3.

Theorem 3.3 *Let $(X, M, *)$ be an M-complete fuzzy metric space. Assume that $T : X \to X$ is an α-fuzzy $\mathcal{H}$-contractive mapping w.r.t some $\eta \in \mathcal{H}$ such that*

(a) *there exists $x_0 \in X$ with $\alpha(x_0, Tx_0, t) \ge 1$ and $t > 0$;*

(b) *T is α-admissible;*

(c) *each subsequence $\{x_{n_k}\} \subset \{x_n\} = \{T^n x_0\}$ has the following property:*

$$\alpha(x_{n_k}, x_{n_l}, t) \ge 1, \quad \text{where } k, l \in \mathbb{N}, k > l \text{ and } t > 0;$$

(d) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \ge 1$, $n \in \mathbb{N}$, $t > 0$, and $\lim_{n\to\infty} x_n = x$, then $\alpha(x_n, x, t) \ge 1$, $n \in \mathbb{N}$, $t > 0$.*

Then T has a fixed point.

Proof In view of condition (a), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$, $t > 0$. Define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$, $\forall n \in \mathbb{N}$.

Since T is α-admissible, then we have

$$\alpha(x_0, x_1, t) = \alpha(x_0, Tx_0, t) \geq 1, \quad t > 0 \quad \Longrightarrow \quad \alpha(Tx_0, Tx_1, t) = \alpha(x_1, x_2, t) \geq 1, \quad t > 0.$$

Continuing this process, we get

$$\alpha(x_n, x_{n+1}, t) \geq 1, \quad \forall n \in \mathbb{N}, t > 0. \tag{3.24}$$

Since T is α-fuzzy-$\mathcal{H}$-contractive mapping and due to (3.24), we obtain

$$\begin{aligned} 0 &\leq \eta\big(M(x_n, x_{n+1}, t)\big) \\ &= \eta\big(M(Tx_{n-1}, Tx_n, t)\big) \\ &\leq \alpha(x_{n-1}, x_n, t)\eta\big(M(Tx_{n-1}, Tx_n, t)\big) \\ &\leq \lambda\eta\big(M(x_{n-1}, x_n, t)\big) \\ &\vdots \\ &\leq \lambda^n\eta\big(M(x_0, x_1, t)\big). \end{aligned} \tag{3.25}$$

By taking $n \to \infty$ in (3.25), we get

$$\lim_{n\to\infty} \eta\big(M(x_n, x_{n+1}, t)\big) = 0,$$

and hence $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ for all $t > 0$.

Now, we show the Cauchyness of the sequence $\{x_n\}$. On the contrary, we assume that the sequence $\{x_n\}$ is not M-Cauchy. Then there are $\epsilon \in (0, 1)$, $t_0 > 0$ and two subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ of $\{x_n\}$ such that

$$m_k > n_k \geq k \quad \text{and} \quad M(x_{m_k}, x_{n_k}, t_0) \leq 1 - \epsilon \quad \text{for all } k \in \mathbb{N}.$$

By Remark 2.1(b), we get

$$M\left(x_{m_k}, x_{n_k}, \frac{t_0}{2}\right) \leq 1 - \epsilon.$$

Suppose that n_k is the least integer exceeding m_k satisfying the above inequality. Then we have

$$M\left(x_{m_k-1}, x_{n_k}, \frac{t_0}{2}\right) > 1 - \epsilon.$$

Applying inequality (2.3) and using condition (c), we get

$$\begin{aligned} \eta\big(M(x_{m_k}, x_{n_k}, t_0)\big) &\leq \alpha(x_{m_k-1}, x_{n_k-1}, t_0)\eta\big(M(x_{m_k}, x_{n_k}, t_0)\big) \\ &\leq \lambda\eta\big(M(x_{m_k-1}, x_{n_k-1}, t_0)\big) \\ &< \eta\big(M(x_{m_k-1}, x_{n_k-1}, t_0)\big). \end{aligned}$$

The rest of the proof of Cauchyness can be shown as in Theorem 3.1. Now, since $(X, M, *)$ is M-complete, then there exists $x \in X$ such that

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad t > 0. \tag{3.26}$$

Using condition (d) and (2.3), we obtain

$$0 < \eta\bigl(M(Tx_n, Tx, t)\bigr) \le \alpha(x_n, x, t)\eta\bigl(M(Tx_n, Tx, t)\bigr) \le \lambda\eta\bigl(M(x_n, x, t)\bigr).$$

Taking the limit as $n \to \infty$ in the above inequality, using (3.26) and the continuity of η, we deduce

$$\lim_{n\to\infty} \eta\bigl(M(x_{n+1}, Tx, t)\bigr) = 0,$$

which implies that $\lim_{n\to\infty} M(x_{n+1}, Tx, t) = 1$. By the uniqueness of the limit, we conclude that $Tx = x$, that is, x is a fixed point of T. □

Finally, we conclude this section by proving a result which ensures the existence of fixed point besides unifying Theorems 3.1, 3.2, and 3.3. In doing so, first we introduce the notion of an α-fuzzy-$\mathcal{H}$-weak contractive mapping as follows.

Definition 3.1 Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is called α-fuzzy $\mathcal{H}$-weak contractive w.r.t. some $\eta \in \mathcal{H}$ if there exist $\lambda \in (0, 1)$ and $\alpha : X \times X \times (0, \infty) \to [0, \infty)$ such that

$$\alpha(x, y, t)\eta\bigl(M(Tx, Ty, t)\bigr) \le \lambda\mathcal{N}(x, y, t), \tag{3.27}$$

where $\mathcal{N}(x, y, t) = \max\{\eta(M(x, y, t)), \eta(M(x, Tx, t)), \eta(M(y, Ty, t))\}$ for all $x, y \in X$ and $t > 0$.

Theorem 3.4 *Let $(X, M, *)$ be an M-complete fuzzy metric space. Assume that $T : X \to X$ is an α-fuzzy-$\mathcal{H}$- weak contractive mapping w.r.t. some $\eta \in \mathcal{H}$ such that*

(a) *there exists $x_0 \in X$ with $\alpha(x_0, Tx_0, t) \ge 1$, $t > 0$;*
(b) *T is α-admissible;*
(c) *each subsequence $\{x_{n_k}\} \subset \{x_n = T^n x_0\}$ has the following property:*

$$\alpha(x_{n_k}, x_{n_l}, t) \ge 1, \quad \text{where } k, l \in \mathbb{N}, k > l \text{ and } t > 0;$$

(d) *if $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}, t) \ge 1$ and $\lim_{n\to\infty} x_n = x$, then $\alpha(x_n, x, t) \ge 1$, $\forall n \in \mathbb{N}$, $t > 0$.*

Then T admits a fixed point.

Proof Condition (a) ensures the existence of a point $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \ge 1$, $t > 0$. Define a sequence $\{x_n\}$ in X by

$$x_n = Tx_{n-1}, \quad \forall n \in \mathbb{N}.$$

Since T is α-admissible, then we have

$$\alpha(x_0, x_1, t) = \alpha(x_0, Tx_0, t) \geq 1, \quad t > 0 \quad \Longrightarrow \quad \alpha(Tx_0, Tx_1, t) = \alpha(x_1, x_2, t) \geq 1, \quad t > 0.$$

Continuing this process, we get

$$\alpha(x_{n-1}, x_n, t) \geq 1 \quad \text{for all } n \in \mathbb{N}, t > 0. \tag{3.28}$$

Applying (3.27) and using (3.28), we obtain

$$\begin{aligned} \eta\big(M(x_n, x_{n+1}, t)\big) &= \eta\big(M(Tx_{n-1}, Tx_n, t)\big) \\ &\leq \alpha(x_{n-1}, x_n, t)\eta\big(M(Tx_{n-1}, Tx_n, t)\big) \\ &\leq \lambda\mathcal{N}(x_{n-1}, x_n, t), \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} \mathcal{N}(x_{n-1}, x_n, t) &= \max\big\{\eta\big(M(x_{n-1}, x_n, t)\big), \eta\big(M(x_{n-1}, Tx_{n-1}, t)\big), \eta\big(M(x_n, Tx_n, t)\big)\big\} \\ &= \lambda \max\big\{\eta\big(M(x_{n-1}, x_n, t)\big), \eta\big(M(x_n, x_{n+1}, t)\big)\big\}. \end{aligned}$$

The rest of the proof can be completed in line with the proof of Theorem 3.2, wherein conditions (c) and (d) are also exploited, and hence the details of the proof are omitted. □

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