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A NOTE ON GROUPS WITH A FINITE NUMBER OF PAIRWISE PERMUTABLE SEMINORMAL SUBGROUPS

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Abstract. A subgroup A of a group G is called *seminormal* in G, if there exists a subgroup B such that $G = AB$ and AX is a subgroup of G for every subgroup X of B. The group $G = G_1G_2 \cdots G_n$ with pairwise permutable subgroups $G_1, \ldots, G_n$ such that G_i and G_j are seminormal in G_iG_j for any $i, j \in \{1, \ldots, n\}$, $i \neq j$, is studied. In particular, we prove that if $G_i \in \mathfrak{F}$ for all i, then $G^{\mathfrak{F}} \leq (G')^{\mathfrak{N}}$, where $\mathfrak{F}$ is a saturated formation and $\mathfrak{U} \subseteq \mathfrak{F}$. Here $\mathfrak{N}$ and $\mathfrak{U}$ are the formations of all nilpotent and supersoluble groups respectively, the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G is the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{F}$.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notations and terminology of [1, 2]. The monograph [3] contains the necessary information of the theory of formations.

It is well known that AB is a subgroup of G if and only if $AB = BA$, that is, if the subgroups A and B permute. Should it happen that AB coincides with the group G, then G is said to be *factorized* by its subgroups A and B.

We say that two subgroups A and B of G are *mutually permutable* if A permutes with every subgroup of B and B permutes with every subgroup of A. If $G = AB$ and A and B are mutually

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permutable, then G is called the *mutually permutable product* of A and B, see [4]. The monograph [5] contains detailed information on the structure of groups, which are mutually permutable products of two subgroups. More generally, a group $G = G_1G_2\cdots G_n$ is said to be the *product of the pairwise mutually permutable subgroups* $G_1, \dots, G_n$ if G_i and G_j are mutually permutable subgroups of G for all $i, j \in \{1, \dots, n\}$, $i \neq j$. The structure of groups factorized by finitely many pairwise mutually permutable subgroups is obtained in [6, 7, 8] and also in [5, Chapters 4–5].

A subgroup A of G is called *seminormal* in G, if there exists a subgroup B such that $G = AB$ and AX is a subgroup of G for every subgroup X of B, see [9]. There are many papers devoted to studying groups with seminormal subgroups, see, for example, [9]–[16]. It's obvious that if $G = AB$ is the mutually permutable product of A and B, then A and B are seminormal in G. The converse is not true. Let Z_n be a cyclic group of order n. A group

$$G = Z_7 \rtimes \mathrm{Aut}\, Z_7 = Z_7 \rtimes (Z_2 \times Z_3)$$

is the product of seminormal in G subgroups $A \simeq Z_2 \times Z_3$ and $B \simeq Z_7 \rtimes Z_2$. But A and B are not mutually permutable, since A is not permute with some subgroups of order 2 of B.

Recall that a formation $\mathfrak{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathfrak{F}$ implies $G \in \mathfrak{F}$. In present paper, we develop the result [5, Theorem 5.2.21]: *if $G = G_1G_2\cdots G_n$ is the product of the pairwise mutually permutable subgroups $G_1, \dots, G_n$ such that G_i belongs to $\mathfrak{F}$ for all $i \in \{1, \dots, n\}$ and the derived subgroup G' is nilpotent, then G belongs to $\mathfrak{F}$. Here $\mathfrak{F}$ be a saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F}$.*

Now we state our main result in the following.

Theorem 1.1. *Let $G = G_1G_2\cdots G_n$ be the product of the pairwise permutable subgroup $G_1 \dots, G_n$ such that G_i and G_j are seminormal in G_iG_j for any $i, j \in \{1, \dots, n\}$, $i \neq j$. Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F}$. Suppose that G_i belongs to $\mathfrak{F}$ for all $i \in \{1 \dots, n\}$. Then $G^{\mathfrak{F}} \leq (G')^{\mathfrak{N}}$.*

Corollary 1.2. *Let $G = G_1G_2\cdots G_n$ be the product of the pairwise mutually permutable subgroups $G_1, \dots, G_n$. Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F}$. Then:*

1) *if G_i belongs to $\mathfrak{F}$ for all $i \in \{1 \dots, n\}$ and G' is nilpotent, then G belongs to $\mathfrak{F}$, see* [5, Theorem 5.2.21]*;*

2) *if G_i belongs to $\mathfrak{U}$ for all $i \in \{1 \dots, n\}$ and G' is nilpotent, then G belongs to $\mathfrak{U}$, see* [5, Corollary 4.1.39].

Since $\mathfrak{U} \subseteq \mathfrak{N}\mathfrak{A}$, it follows that $G^{(\mathfrak{N}\mathfrak{A})} = (G^{\mathfrak{A}})^{\mathfrak{N}} = (G')^{\mathfrak{N}} \leq G^{\mathfrak{U}}$ by [17, Proposition 2.2.8, Proposition 2.2.11]. Therefore for $\mathfrak{F} = \mathfrak{U}$, we obtain the following corollary.

Corollary 1.3. *Let A and B be supersoluble subgroups of G and $G = AB$. Then:*

1) *if A and B are seminormal in G, then $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$, see* [16, Theorem 2.3]*;*

2) *if A and B are seminormal in G and G' is nilpotent, then G is supersoluble, see* [16, Theorem 2.2]*;*

3) *if A and B are mutually permutable and G' is nilpotent, then G is supersoluble, see* [18, Theorem 3.8]*;*

4) *if A and B are mutually permutable, then $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$, see* [19, Theorem 2.1].

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2. Preliminaries

In this section, we give some definitions and basic results which are essential in the sequel.

Recall that a *p-closed* group is a group with a normal Sylow p-subgroup.

Denote by $Z(G)$, $F(G)$ and $\Phi(G)$ the centre, Fitting and Frattini subgroups of G respectively, and by $\mathrm{O}_p(G)$ the greatest normal p-subgroup of G. Denote by $\pi(G)$ the set of all prime divisors of order of G. The semidirect product of a normal subgroup A and a subgroup B is written as follows: $A \rtimes B$. If H is a subgroup of G, then $H_G = \bigcap_{x\in G} H^x$ is called *the core* of H in G.

Let $\mathbb{P}$ be the set of all prime numbers. A *formation function* is a function f defined on $\mathbb{P}$ such that $f(p)$ is a (possibly empty) formation. A formation $\mathfrak{F}$ is said to be *local* if there exists a formation function f such that $G \in \mathfrak{F}$ if and only if for any chief factor H/K of G and any $p \in \pi(H/K)$, one has $G/C_G(H/K) \in f(p)$. We write $\mathfrak{F} = LF(f)$ and f is a local definition of $\mathfrak{F}$. By [3, Theorem IV.3.7], among all possible local definitions of a local formation $\mathfrak{F}$ there exists a unique f such that f is *integrated* (i.e., $f(p) \subseteq \mathfrak{F}$ for all $p \in \mathbb{P}$) and *full* (i.e., $f(p) = \mathfrak{N}_p f(p)$ for all $p \in \mathbb{P}$). Here $\mathfrak{N}_p$ is the formation of all p-groups. Such local definition f is said to be *canonical local definition* of $\mathfrak{F}$. By [3, Theorem IV.4.6], a formation is saturated if and only if it is local.

If G contains a maximal subgroup M with a trivial core, then G is said to be *primitive* and M is its *primitivator*.

Lemma 2.1. (1) *If H is seminormal in G and $H \leq X \leq G$, then H is seminormal in X.*

(2) *If H is seminormal in G and N is normal in G, then HN is seminormal in G and HN/N is seminormal in G/N.*

(3) *Let $H \leq K \leq G$ and N be a normal subgroup in G. If H is seminormal in K, then HN/N is seminormal in KN/N.*

Proof. The first two properties easily follow from the definition of a seminormal subgroup. We check the third property. Since H is seminormal in K, it follows that there exists a subgroup Y such that $K = HY$ and HX is a subgroup for every subgroup X of Y. Hence $KN/N = (HN/N)(YN/N)$. Let B/N be an arbitrary subgroup of YN/N. Since $N \leq B \leq YN$, we have by Dedekind's identity, $B = B \cap YN = (B \cap Y)N$. Hence

$$(HN/N)(B/N) = H(B \cap Y)N/N = (B \cap Y)HN/N = (B/N)(HN/N),$$

because $B \cap Y \leq Y$. Therefore HN/N is seminormal in KN/N. □

The following Lemma is easily verified.

Lemma 2.2. *Let $\mathfrak{F}$ be a saturated formation. Assume that $G \notin \mathfrak{F}$, but $G/N \in \mathfrak{F}$ for all non-trivial normal subgroups N of G. Then G is a primitive group.*

Lemma 2.3. [13, Lemma 2.3] *Let G be a soluble primitive group and M is a primitivator of G. Then the following statements hold:*

(1) $\Phi(G) = 1$*;*

(2) $F(G) = C_G(F(G)) = O_p(G)$ and $F(G)$ is an elementary abelian subgroup of order p^n for some prime p and some positive integer n;

(3) G contains a unique minimal normal subgroup N and moreover, $N = F(G)$;

(4) $G = F(G) \rtimes M$ and $O_p(M) = 1$.

Lemma 2.4. *Let $G = G_1G_2 \cdots G_n$ be a group with pairwise permutable subgroups $G_1, \ldots, G_n$ such that G_i and G_j are seminormal in G_iG_j for any $i, j \in \{1, \ldots, n\}$, $i \neq j$. If a Sylow p-subgroup P of G is normal in G and is abelian, then $P \cap G_i$ is normal in G for every i.*

Proof. We consider the following representation

$$G = (G_1G_i) \cdots G_i \cdots (G_nG_i).$$

Since G_i is seminormal in G_iG_j, there exists a subgroup H_j such that $G_iH_j = G_iG_j$. Let $(H_j)_{p'}$ be a Hall p'-subgroup of H_j and $(H_j)_p$ is a Sylow p-subgroup of H_j. Then $G_i(H_j)_{p'}$ is a subgroup of G. It's obvious that $P \cap G_i$ is a Sylow p-subgroup of G_i. Denote by $(G_i)_p = P \cap G_i$. Hence $P \cap G_i(H_j)_{p'} = (G_i)_p$ and $(G_i)_p$ is normal in $G_i(H_j)_{p'}$. Therefore $(H_j)_{p'} \leq N_G((G_i)_p)$. Since $(H_j)_p$ and $(G_i)_p$ are contained in abelian subgroup P, we have $(H_j)_p \leq C_G((G_i)_p)$. So $(G_i)_p$ is normal in

$$G = (G_1G_i) \cdots G_i \cdots (G_nG_i) = (H_1G_i) \cdots G_i \cdots (H_nG_i) =$$

$$= ((H_1)_p(H_1)_{p'}G_i) \cdots G_i \cdots ((H_n)_p(H_n)_{p'}G_i).$$

□

3. Proof of Theorem

We consider the case when the derived subgroup G' is nilpotent. Then G is soluble. Assume that $G \notin \mathfrak{F}$. Let N be a non-trivial normal subgroup of G. The quotients

$$G/N = \prod_i (G_iN/N),\ G_iN/N \simeq G_i/G_i \cap N.$$

Hence the subgroups $G_iN/N \in \mathfrak{F}$ for all i and by Lemma 2.1 (3), G_iN/N and G_jN/N are seminormal in G_iG_jN/N for any $i \neq j$.

Since

$$(G/N)' = G'N/N \simeq G'/G' \cap N,$$

it follows that the derived subgroup $(G/N)'$ is nilpotent. Thus the hypotheses of the theorem hold for G/N. By induction, $G/N \in \mathfrak{F}$. Since $\mathfrak{F}$ is saturated, we have that G is primitive by Lemma 2.2. Hence $\Phi(G) = 1$, $G = N \rtimes M$ and $N = C_G(N) = F(G) = \mathrm{O}_p(G)$ is a unique minimal normal subgroup of G by Lemma 2.3. Because G' is nilpotent, we have $N = G'$ and G/N is abelian. Let P be a Sylow p-subgroup of G. Then $PN/N = P/N$ is a Sylow p-subgroup of G/N. Hence P is normal in G, $N = P$ and $M = G_{p'}$ is abelian.

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Without loss of generality, we assume that p divides the order of G_1. By Lemma 2.4, $N \le G_1$. Since G_1 is normal in G, it follows that $\Phi(G_1) = 1$ and $F(G_1) = N$. Hence $N = Y_1 \times Y_2 \times \dots \times Y_k$, where Y_s is a minimal normal subgroup of G_1 for every $s \in \{1\dots, k\}$. Furthermore,

$$C_{G_1}(N) = G_1 \cap C_G(N) = N.$$

By [2, Appendix C, Theorem 2.5], we have

$$N = C_{G_1}(N) = F(G_1) = \bigcap_s C_{G_1}(Y_s).$$

Since $\mathfrak{F}$ is saturated, there exists the canonical local definition f. Hence $\mathfrak{F} = LF(f)$, $f(p) \subseteq \mathfrak{F}$ and $f(p) = \mathfrak{N}_p f(p)$. By hypothesis, $G_1 \in \mathfrak{F}$. Then by definition of formation function, $G_1/C_{G_1}(Y_s) \in f(p)$ for any s. Because $f(p)$ is a formation, it follows that $G_1/N \in f(p)$.

Let $K = G_2G_3 \cdots G_n$ and p divides the order of K. Hence p divides the order at least one of the subgroups G_i, $i \in \{2\dots, n\}$. By Lemma 2.4, $N \le K$. By induction, $K \in \mathfrak{F}$. Then, proving as above, $K/N \in f(p)$.

Suppose p does not divide the order of K. Since for every $i \in \{2\dots, n\}$ G_i is seminormal in G_iG_1, it follows that there exists a subgroup T such that $G_iT = G_iG_1$ and G_iX is a subgroup of G for every subgroup X of T. Because $N \le T$ and G is p-closed, we have $G_i \le N_G(U)$ for every subgroup U of N. Since N is an elementary abelian p-subgroup, it follows that $N = N_1 \times N_2 \times \dots \times N_t$, where $|N_r| = p$ for every $r \in \{1\dots, t\}$. Because $K \le N_G(N_r)$, we have $KN/C_{KN}(N_r) = N_{KN}(N_r)/C_{KN}(N_r)$ is a cyclic group of order dividing $p-1$. It is clear that

$$N = C_{KN}(N) = \bigcap_r C_{KN}(N_r).$$

Hence $K \simeq KN/N = KN/C_{KN}(N)$ is abelian of exponent dividing $p-1$. Then $K \in g(p)$, where g is the canonical local definition of $\mathfrak{U}$. Since $\mathfrak{U} \subseteq \mathfrak{F}$, we have by [3, Proposition IV.3.11], $g(p) \subseteq f(p)$, where f is the canonical local definition of $\mathfrak{F}$. Hence $K \in f(p)$.

Thus G/N is the product of normal subgroups G_1/N and KN/N such that each of them belongs to $f(p)$. We consider the direct product $G_1/N \times KN/N = \{(aN, bN), a \in G_1, b \in K\}$. Let $\varphi : G_1/N \times KN/N \to G/N = (G_1/N)(KN/N)$ be a function from $G_1/N \times KN/N$ to G/N and $\varphi(aN, bN) = (ab)N$. Since G/N is abelian, then $G_1/N \le C_{G/N}(KN/N)$. It is clear that φ is an epimorphism. Then by fundamental homomorphism theorem

$$(G_1/N \times KN/N)/\mathrm{Ker}\ \varphi \simeq \mathrm{Im}\ \varphi = G/N.$$

Since $f(p)$ is a formation, it follows that $G/N \in f(p)$. Because $N \in \mathfrak{N}_p$, we have $G \in \mathfrak{N}_p f(p) = f(p) \subseteq \mathfrak{F}$. Hence the assumption is wrong.

Let $(G')^{\mathfrak{N}} \neq 1$. We show that the quotient $G/(G')^{\mathfrak{N}}$ belongs to $\mathfrak{F}$. Since

$$(G/(G')^{\mathfrak{N}})' = G'(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} = G'/(G')^{\mathfrak{N}},$$

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we have the $(G/(G')^{\mathfrak{N}})'$ is nilpotent. The quotients

$$G/(G')^{\mathfrak{N}} = \prod_i (G_i(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}),$$

$$G_i(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} \simeq G_i/G_i \cap (G')^{\mathfrak{N}},$$

hence the subgroups $G_i(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}$ for any $i \in \{1\dots,n\}$ belong to $\mathfrak{F}$ and by Lemma 2.1 (3), $G_i(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}$ and $G_j(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}$ are seminormal in $G_iG_j(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}$. Arguing as above, we see that $G/(G')^{\mathfrak{N}}$ belongs to $\mathfrak{F}$. The theorem is proved.

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