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INDEPENDENT ROMAN {3}-DOMINATION

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Abstract. Let G be a simple, undirected graph. In this paper, we initiate the study of independent Roman {3}-domination. A function $g : V(G) \rightarrow \{0, 1, 2, 3\}$ having the property that $\sum_{v \in N_G(u)} g(v) \geq 3$, if $g(u) = 0$, and $\sum_{v \in N_G(u)} g(v) \geq 2$, if $g(u) = 1$ for any vertex $u \in V(G)$, where $N_G(u)$ is the set of vertices adjacent to u in G, and no two vertices assigned positive values are adjacent is called an *independent Roman {3}-dominating function* (IR3DF) of G. The weight of an IR3DF g is the sum $g(V) = \sum_{v \in V} g(v)$. Given a graph G and a positive integer k, the independent Roman {3}-domination problem (IR3DP) is to check whether G has an IR3DF of weight at most k. We investigate the complexity of IR3DP in bipartite and chordal graphs. The minimum independent Roman {3}-domination problem (MIR3DP) is to find an IR3DF of minimum weight in the input graph. We show that MIR3DP is linear time solvable for bounded tree-width graphs, chain graphs and threshold graphs. We also show that the domination problem and IR3DP are not equivalent in computational complexity aspects. Finally, we present an integer linear programming formulation for MIR3DP.

1. Introduction

Consider $G = (V, E)$ be a simple, undirected and connected graph with no isolated vertices. For a vertex $v \in V$, the *open neighborhood* of v in G is $N_G(v) = \{u \in V \mid (u, v) \in E\}$ and the *closed neighborhood* of v is defined as $N_G[v] = N_G(v) \cup \{v\}$. The *degree* $deg(v)$ of a vertex v is $|N_G(v)|$. Δ and δ denote, respectively the maximum degree and minimum degree of G. An *induced subgraph* is a graph formed from a subset D of vertices of G and all of the edges in G connecting pairs of vertices in that subset, denoted by $\langle D \rangle$. A *clique* is a subset of vertices of G such that every two distinct

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vertices in the subset are adjacent. An *independent set* is a set of vertices in which no two vertices are adjacent. A vertex v of G is called *universal vertex* if $deg(v)$ is equal to Δ. A vertex u is *simplicial* if its neighborhood $N_G(u)$ induces a complete subgraph of G. An ordering of vertices $\sigma = \{u_1, u_2, \ldots, u_n\}$ is called *Perfect Elimination Ordering* (PEO), if each u_i is simplicial in the subgraph induced by the vertices u_i to u_n. A graph G is *chordal graph* if and only if G admits a PEO. For undefined terminology and notations refer to [9].

A vertex v in G dominates the vertices of its closed neighborhood. A set of vertices $S \subseteq V$ is a *dominating set* (DS) in G if for every vertex $u \in V \setminus S$, there exists at least one vertex $v \in S$ such that $(u, v) \in E$, i.e., $N_G[S] = V$. If S is independent set then S is called an *independent dominating set* (IDS) of G. The (*independent*) *domination number* is the minimum cardinality of a (independent) dominating set in G and is denoted by $\gamma(G)$ ($i(G)$). The MINIMUM INDEPENDENT DOMINATING SET problem is to find an IDS of minimum cardinality [24].

Roman domination was introduced in 2004 by Cockayne et al. in [10]. A function $f : V \to \{0, 1, 2\}$ is a *Roman Dominating Function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The literature on Roman domination in graphs has been surveyed in [6, 19, 21].

Independent Roman domination was introduced in 2004 by Cockayne et al. in [10]. An *Independent Roman Dominating Function* (IRDF) is a RDF f with the additional property that the subgraph induced by the set of vertices with positive weight contains only isolated vertices. The concept of independent Roman domination has been studied in [5, 12, 13].

Roman $\{2\}$-domination was introduced in 2016 by Chellali et al. in [16]. A *Roman* $\{2\}$*-Dominating Function* (R2DF) $f : V \to \{0, 1, 2\}$ has the property that for every vertex $v \in V$ with $f(v) = 0$, either there exists a vertex $u \in N_G(v)$, with $f(u) = 2$, or at least two vertices $x, y \in N_G(v)$ with $f(x) = f(y) = 1$. The literature on Roman $\{2\}$-domination in graphs has been surveyed in [7, 11].

Independent Roman $\{2\}$-domination was introduced in 2018 by Rahmouni et al. in [1]. An *Independent Roman* $\{2\}$*-Dominating Function* (IR2DF) is a R2DF f with the additional property that the subgraph induced by the set of vertices with positive weight contains only isolated vertices. The literature on independent Roman $\{2\}$-domination in graphs has been surveyed in [5, 22].

Recently, Mojdeh et al. in [8] initiated the study of Roman $\{3\}$-domination. A function $g : V \to \{0, 1, 2, 3\}$ having the property that $\sum_{v \in N_G(u)} g(v) \geq 3$, if $g(u) = 0$, and $\sum_{v \in N_G(u)} g(v) \geq 2$, if $g(u) = 1$ for any vertex $u \in G$ is called a *Roman* $\{3\}$*-Dominating Function* (R3DF) of G.

In this paper, we initiate the study of independent Roman $\{3\}$-domination. An *Independent Roman* $\{3\}$*-Dominating Function* (IR3DF) is a R3DF g with the additional property that the subgraph of G induced by the set $\{v \in V : g(v) \geq 1\}$ contains only isolated vertices. The weight of a IR3DF g is the value $g(V) = \sum_{v \in V} g(v)$. The *independent Roman* $\{3\}$*-domination number* equals the minimum weight of an IR3DF on G, denoted by $i_{\{R3\}}(G)$. The minimum independent Roman $\{3\}$-domination problem (MIR3DP) is to find an IR3DF of minimum weight in the input graph.
The following are the results without proofs.

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Proposition 1.1. *Let P_n be a path graph with n vertices. Then*

$$i_{\{R3\}}(P_n) = \begin{cases} n, & \text{if } n\%3 = 0 \\ n+1, & \text{otherwise} \end{cases}$$

Proposition 1.2. *Let C_n be a cycle graph with n vertices. Then*

$$i_{\{R3\}}(C_n) = \begin{cases} n, & \text{if } n\%3 = 0 \\ n+1, & \text{otherwise} \end{cases}$$

Proposition 1.3. *Let $K_{p,q}$, where $p, q \geq 1$, be a complete bipartite graph. Then*

$$i_{\{R3\}}(K_{p,q}) = \begin{cases} 3, & \text{if } G \text{ is a star graph} \\ 4, & \text{if } p = 2 \text{ or } q = 2 \\ 2\,min\{p,\, q\}, & \text{otherwise} \end{cases}$$

A decision version of independent Roman $\{3\}$-domination problem is defined as below.

INDEPENDENT ROMAN $\{3\}$-DOMINATION PROBLEM(IR3DP)

INSTANCE : Graph $G = (V, E)$ and a positive integer k.
QUESTION : Does G have an IR3DF of weight at most k

In this paper, we show that IR3DP is NP-complete for chordal and bipartite graphs. ReVelle and Rosing [4] and Ivanović [15] have proposed integer linear programming (ILP) formulations for the Roman domination problem. Motivated by this, we propose an ILP formulation for the MIR3DP.

2. Complexity Results

In this section, we show that IR3DP is NP-complete for bipartite graphs and chordal graphs, by giving a polynomial time reduction from a well-known NP-complete problem, Exact-3-Cover ($X3C$)[17], which is defined as follows.

EXACT-3-COVER (X3C)

INSTANCE : A finite set X with $\mid X \mid = 3q$ and a collection C of 3-element subsets of X.
QUESTION : Is there a subcollection C' of C such that every element of X appears in exactly one member of C'

A variant of $X3C$ in which each element appears in at least two subsets has also been proved as NP-complete [25]. Through out this subsection, we use this variant of $X3C$ problem.

Theorem 2.1. *IR3DP is NP-complete for bipartite graphs.*

Proof. Given a graph G and a function f, whether f is an IR3DF of size at most k can be checked in polynomial time. Hence IR3DP is a member of NP. Now we show that IR3DP is NP-hard by transforming an instance $\langle X, C\rangle$ of X3C, where $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$, to an instance $\langle G, k\rangle$ of IR3DP as follows.

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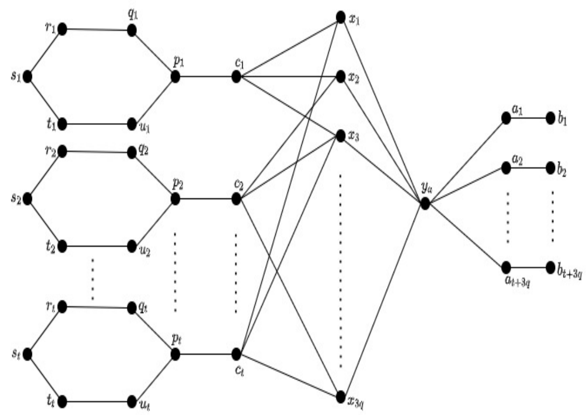

FIGURE 1. An illustration to the construction of bipartite graph from an instance of X3C

Create vertices x_i for each $x_i \in X$, $p_i, q_i, r_i, s_i, t_i, u_i, c_i$ for each $C_i \in C$ and y_a. Also create vertices a_i, b_i, where $1 \le i \le t + 3q$. Add edges $(p_i, q_i), (q_i, r_i), (r_i, s_i), (s_i, t_i), (t_i, u_i), (u_i, p_i), (p_i, c_i)$ for each c_i, (y_a, x_i) for each x_i and (c_j, x_i) if $x_i \in C_j$. Also, add edges $(y_a, a_i), (a_i, b_i)$, where $1 \le i \le t + 3q$. Clearly, G is a bipartite graph and can be constructed from the given instance $\langle X, C\rangle$ of X3C in polynomial time. Next we show that, $X3C$ has a solution if and only if G has an IR3DF with weight at most $8t + 8q + 2$.

Suppose C' is a solution for $X3C$ with $|C'| = q$. We define a function $f : V \to \{0, 1, 2, 3\}$ as follows.

$$f(v) = \begin{cases} 3, & \text{if } v \in \{r_i, u_i : C_i \in C'\} \cup \{p_i, s_i : C_i \notin C'\} \\ 2, & \text{if } v \in C' \cup \{y_a\} \cup \{b_i : 1 \le i \le t + 3q\} \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

It can be easily verified that f is an IR3DF of G and $f(V) = 8t + 8q + 2$.

Conversely, suppose that G has an IR3DF g with weight $8t + 8q + 2$. Clearly, $\forall i, 1 \le i \le t$, $g(q_i) + g(r_i) + g(s_i) + g(t_i) + g(u_i) \ge 6$ irrespective of label of c_i and $\forall j, 1 \le j \le t+3q, g(a_i) + g(b_i) \ge 2$.

Claim 2.2. *If $x_i \in X$ then $g(x_i) = 0$.*

Proof. (Proof by contradiction) Assume there exista a vertex x_a, where $1 \le a \le 3q$ such that $g(x_a) \ne 0$. Clearly, if $g(x_a) = 1$ then a neighbor of x_a would have been assigned a label greater than or equal to one and hence g is not an IR3DF. Therefore $g(x_a) \ge 2$, $g(y_a) = 0$ and $\forall c_j \in N_G(x_a), g(c_j) = 0$. Then it follows that $g(a_i) + g(b_i) \ge 3$, where $1 \le i \le t + 3q$. Also $\forall i, g(q_i) + g(r_i) + g(s_i) + g(t_i) + g(u_i) \ge 6$. Hence $g(V) \ge 3(t + 3q) + 6t + g(x_a) > 8t + 8q + 2$, a contradiction. Therefore for each $x_i \in X$, $g(x_i) = 0$. □

Since each c_i has exactly three neighbors in X, clearly, there exist q number of c_i's with weight 2 such that $\left(\bigcup_{g(c_i)=2} N_G(c_i)\right) \cap X = X$. Consequently, $C' = \{C_i : g(c_i) = 2\}$ is an exact cover for C. □

Theorem 2.3. *IR3DP is NP-complete for chordal graphs.*

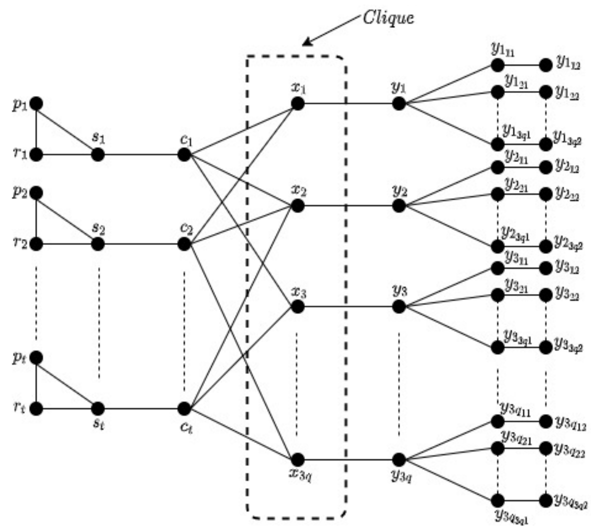

FIGURE 2. An illustration to the construction of chordal graph from an instance of X3C

Proof. Clearly, IR3DP for chordal graphs is a member of *NP*. We transform an instance $\langle X, C\rangle$ of $X3C$, where $X = \{x_1, x_2, . . ., x_{3q}\}$ and $C = \{C_1, C_2, . . ., C_t\}$, to an instance $\langle G, k\rangle$ of IR3DP as follows.

Create vertices x_i, y_i for each $x_i \in X$, p_i, r_i, s_i, c_i for each $C_i \in C$ and $y_{i_{kl}}$, where $1 \leq i, k \leq 3q$ and $1 \leq l \leq 2$. Add edges $(c_i, s_i), (s_i, p_i), (s_i, r_i), (p_i, r_i)$ for each c_i and (c_j, x_i) if $x_i \in C_j$. Next add edges $(x_i, y_i), (y_i, y_{i_{kl}})$, where $1 \leq i, k \leq 3q$ and $1 \leq l \leq 2$. Also add edges (x_i, x_j), $\forall x_i, x_j \in X$, where $i \neq j$. The graph constructed is shown in the Figure 2. Since G admits a PEO $\{y_{i_{k2}} : 1 \leq i, k \leq 3q\} \cup \{y_{i_{k1}} : 1 \leq i, k \leq 3q\} \cup \{y_i : 1 \leq i \leq 3q\} \cup \{p_i : 1 \leq i \leq t\} \cup \{r_i : 1 \leq i \leq t\} \cup \{s_i : 1 \leq i \leq t\} \cup \{c_i : 1 \leq i \leq t\} \cup \{x_i : 1 \leq i \leq 3q\}$, it is a chordal graph and the construction of G can be accomplished in polynomial time. Next we show that, $X3C$ has a solution if and only if G has an IR3DF with weight at most $3t + 18q^2 + 8q$.

Suppose C' is a solution for $X3C$ with $|C'| = q$. We define a function $f : V \to \{0, 1, 2, 3\}$ as follows.

$$f(v) = \begin{cases} 3, & \text{if } v \in \{s_i : C_i \notin C'\} \cup \{r_i : C_i \in C'\} \\ 2, & \text{if } v \in C' \cup \{y_i, y_{i_{k2}} : 1 \leq i, k \leq 3q\} \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

It can be easily verified that f is an IR3DF of G and $f(V) = 3t + 18q^2 + 8q$.

Conversely, suppose that G has an IR3DF g with weight $3t + 18q^2 + 8q$. The following claim holds.

Claim 2.4. *For each $x_i \in X$, $g(x_i) = 0$.*

Proof. Assume there exista a vertex x_a, where $1 \leq a \leq 3q$ such that $g(x_a) \neq 0$. Clearly, if $g(x_a) = 1$ then a neighbor of x_a would have been assigned a label ≥ 1 and g is not an IR3DF. Therefore $g(x_a) \geq 2$, $g(y_a) = 0$, $\forall x_i$, where $i \neq a, g(x_i) = 0$ and $\forall c_j \in N_G(x_a), g(c_j) = 0$. Then, each $y_{a_{k1}} - y_{a_{k2}}$, where

$1 \leq k \leq 3q$, path requires a weight of at least 3 and each $\langle\{y_i, y_{i_{k1}}, y_{i_{k2}}\}\rangle$, where $1 \leq i, k \leq 3q$ and $y_i \neq y_a$, requires a weight of at least $6q+2$. And also each $\langle\{p_i, r_i, s_i\}\rangle$, where $1 \leq i \leq t$, requires a weight of at least 3. Hence $g(V) \geq 3(3q)+(3q-1)(6q+2)+3t+g(x_a) > 3t+18q^2+8q$, a contradiction. Therefore for each $x_i \in X$, $g(x_i) = 0$. □

Clearly, $g(y_i) + g(y_{i_{k1}}) + g(y_{i_{k2}}) \geq 6q + 2$, where $1 \leq i, k \leq 3q$ and $g(p_i) + g(r_i) + g(s_i) \geq 3$, where $1 \leq i \leq t$. Since each c_i has exactly three neighbors in X, clearly, there exist q number of c_i's with weight 2 such that $\left(\bigcup_{g(c_i)=2} N_G(c_i)\right) \cap X = X$. Consequently, $C' = \{C_i : g(c_i) = 2\}$ is an exact cover for C. □

3. Threshold graphs

In this section, we determine the independent Roman $\{3\}$-domination number of threshold graph.

Definition 1. A graph $G = (V, E)$ is called a *threshold graph* if there is a real number T and a real number $w(v)$ for every $v \in V$ such that a set $S \subseteq V$ is independent if and only if $\sum_{v \in S} w(S) \leq T$.

Although several characterizations are defined for threshold graphs, we use the following characterization of threshold graphs given in [20] to prove that independent Roman $\{3\}$-domination number can be computed in linear time for threshold graphs.

A graph G is a threshold graph if and only if it is a split graph and, for split partition (C, I) of V where C is a clique and I is an independent set, there is an ordering $\{x_1, x_2, \ldots, x_p\}$ of vertices of C such that $N_G[x_1] \subseteq N_G[x_2] \subseteq N_G[x_3] \subseteq \ldots \subseteq N_G[x_p]$, and there is an ordering $\{y_1, y_2, \ldots, y_q\}$ of the vertices of I such that $N_G(y_1) \supseteq N_G(y_2) \supseteq N_G(y_3) \supseteq \ldots \supseteq N_G(y_q)$.

Theorem 3.1. *Let $G(V, E)$, where $|V| = n$, be a threshold graph with split partition (C, I). Then*

$$i_{\{R3\}}(G) = \begin{cases} 2n, & \text{if } G \cong \overline{K_n}, \\ 2k+1, & \text{otherwise} \end{cases} \tag{3.1}$$

where k is the number of connected components of G.

Proof. Let $G(V, E)$ be a threshold graph with p clique vertices such that $N_G[x_1] \subseteq N_G[x_2] \subseteq N_G[x_3] \subseteq \ldots \subseteq N_G[x_p]$. Now, define a function $f : V \to \{0, 1, 2, 3\}$ as follows. If G is complement of K_n, then $\forall v \in V, f(v) = 2$. Clearly, $f(V) = 2n$. Otherwise,

$$f(v) = \begin{cases} 2, & \text{if } deg(v) = 0 \\ 3, & \text{if } v = x_p \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

Clearly, f is an IR3DF $i_{\{R3\}}(G) \leq 2k + 1$. From the definition of IR3DF, it follows that $i_{\{R3\}}(G) \geq 2k + 1$. Therefore $i_{\{R3\}}(G) = 2k + 1$. □

Since the ordering of the vertices of the clique and the number of connected components in a threshold graph can be determined in linear time [3, 20], the following result is immediate from Theorem 3.1.

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Theorem 3.2. *MIR3DP can be solvable in linear time for threshold graphs.*

4. Chain graphs

In this section, we propose a method to compute the independent Roman $\{3\}$-domination number of a chain graph in linear time. A bipartite graph $G = (X, Y, E)$ is called a *chain graph* if the neighborhoods of the vertices of X form a *chain*, that is, the vertices of X can be linearly ordered, say $x_1, x_2, \ldots, x_p$, such that $N_G(x_1) \subseteq N_G(x_2) \subseteq \cdots \subseteq N_G(x_p)$. If $G = (X, Y, E)$ is a chain graph, then the neighborhoods of the vertices of Y also form a chain. An ordering $\alpha = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$ of $X \cup Y$ is called a *chain ordering* if $N_G(x_1) \subseteq N_G(x_2) \subseteq \cdots \subseteq N_G(x_p)$ and $N_G(y_1) \supseteq N_G(y_2) \supseteq \cdots \supseteq N_G(y_q)$. Every chain graph admits a chain ordering [18]. If G is a complete bipartite graph, then $i_{\{R3\}}(G)$ is obtained directly from Proposition 1.3. Otherwise, the following theorem holds.

Theorem 4.1. *Let $G(X, Y, E)$ be a chain graph. Then,*

$$i_{\{R3\}}(G) = \begin{cases} 1 + 2|X|, & \textit{if } |X| \leq |Y| \\ 1 + 2|Y|, & \textit{otherwise} \end{cases} \tag{4.1}$$

Proof. If $G \cong K_1$ then $i_{\{R3\}}(G) = 2$. Otherwise, let $G(X, Y, E)$ be a chain graph with $|X| = p$ and $|Y| = q$, where $p, q \geq 1$. Now, define a function $f : V \to \{0, 1, 2, 3\}$ as follows.

$$\textit{Case } 1: \text{If } |X| \leq |Y|, \text{then } f(v) = \begin{cases} 3, & \text{if } v = x_p \\ 2, & \text{if } v \in \{x_i : 1 \leq i < p\} \\ 0, & \text{otherwise} \end{cases}$$

Clearly, f is an IR3DF and $i_{\{R3\}}(G) \leq 1 + 2|X|$. From the definition of independent Roman $\{3\}$-domination, it follows that $i_{\{R3\}}(G) \geq 1 + 2|X|$. Therefore $i_{\{R3\}}(G) = 1 + 2|X|$.

$$\textit{Case } 2: \text{Otherwise, } f(v) = \begin{cases} 3, & \text{if } v = y_1 \\ 2, & \text{if } v \in \{y_i : 2 \leq i \leq q\} \\ 0, & \text{otherwise} \end{cases}$$

Clearly, f is an IR3DF and $i_{\{R3\}}(G) \leq 1 + 2|Y|$. From the definition of independent Roman $\{3\}$-domination, it follows that $i_{\{R3\}}(G) \geq 1 + 2|Y|$. Therefore $i_{\{R3\}}(G) = 1 + 2|Y|$. □

If the chain graph G is disconnected with k connected components $G_1, G_2, \ldots, G_k$ then it is easy to verify that $i_{\{R3\}}(G) = \sum_{i=1}^{k} i_{\{R3\}}(G_i)$. Since the chain ordering can be computed in linear time [23], the following result is immediate from Theorem 4.1.

Theorem 4.2. *MIR3DP can be solvable in linear time for chain graphs.*

5. Bounded tree-width graphs

Let G be a graph, T be a tree and v be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices t of T. The pair (T, v) is called a tree-decomposition of G if it satisfies the following three conditions:

(i) $V(G) = \bigcup_{t \in V(T)} V_t$, (ii) for every edge $e \in E(G)$ there exists a $t \in V(T)$ such that both ends of e lie in V_t, (iii) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_1, t_2, $t_3 \in V(T)$ and t_2 is on the path in T from t_1 to t_3. The *width* of $(T, v\ )$ is the number $max\{|V_t| - 1 : t \in T\}$, and the *tree-width* $tw(G)$ of G is the minimum width of any tree-decomposition of G. By Courcelle's Thoerem, it is well known that every graph problem that can be described by counting monadic second-order logic (CMSOL) can be solved in linear-time in graphs of bounded tree-width, given a tree decomposition as input [2]. We show that IR3DP can be expressed in CMSOL.

Theorem 5.1 (*Courcelle's Theorem*)**.** [2] *Let P be a graph property expressible in CMSOL and k be a constant. Then, for any graph G of tree-width at most k, it can be checked in linear-time whether G has property P.*

The following notations are used in the rest of this section.

(1) *adj*(p, q) is the binary adjacency relation which holds if and only if, p, q are two adjacent vertices of G.
(2) *inc*(v, e) is the binary incidence relation which holds if and only if edge e is incident to vertex v in G.
(3) A set X is independent if and only if there does not exist a partition of X into two sets X_A and X_B such that there is an edge between a vertex in X_A and a vertex in X_B. A CMSOL formula to express that the set X is independent is given below.

$$Independent(X) = \neg(\exists X_A, X_A \subseteq X, (\exists e \in E, \exists u \in X_A, \exists v \in X \setminus X_A, (inc(u, e) \wedge inc(v, e)))).$$

Theorem 5.2. *Given a graph G and a positive integer k, IR3DP can be expressed in CMSOL.*

Proof. Let $g : V \to \{0, 1, 2, 3\}$ be a function on a graph G, where $V_i = \{v|g(v) = i\}$ for $i \in \{0, 1, 2, 3\}$. A CMSOL formula for the R3DF is expressed as follows.

$Rom\_3\_Dom(V) = (g(V) \leq k) \wedge \exists V_0, V_1, V_2, V_3, \forall p((p \in V_0 \wedge ((\exists q, r, s \in V_1 \wedge adj(p, q) \wedge adj(p, r) \wedge adj(p, s)) \vee ((\exists t \in V_1 \wedge \exists u \in V_2 \wedge adj(p, t) \wedge adj(p, u)) \vee (\exists q, r \in V_2 \wedge adj(p, q) \wedge adj(p, r)) \vee (\exists v \in V_3 \wedge adj(p, v))))) \vee (p \in V_1 \wedge (\exists w, x \in V_1 \wedge adj(p, w) \wedge adj(p, x)) \vee (\exists y \in (V_2 \cup V_3) \wedge adj(p, y))) \vee (p \in V_2) \vee (p \in V_3))$.

$ROM\_3\_Dom(V)$ ensures that for every vertex $p \in V$, either (i) $p \in V_2$ or (ii) $p \in V_3$, or (iii) if $p \in V_0$ then either there exist three vertices $q, r, s \in V_1$ such that p is adjacent to q, r and s, or there exists two vertives $t \in V_1$, $u \in V_2$ such that p is adjacent to both t and u, or there exist two vertices $q, r \in V_2$ such that p is adjacent to both q and r, or there exist a vertex $v \in V_3$ such that p is adjacent to v (iv) if $p \in V_1$ then either there exists two vertices $w, x \in V_1$ such that p is adjacent to both w and x or there exists a vertex $y \in V_2 \cup V_3$ such that p is adjacent to y.
Now, we can express IR3DP in CMSOL as follows.

$$Independent\_Rom\_3\_Dom(V) = (g(V) \leq k) \wedge Rom\_3\_Dom(V) \wedge Independent(V_1 \cup V_2 \cup V_3). \qquad \square$$

Now, the following result is immediate from Theorems 5.1 and 5.2.

Theorem 5.3. *IR3DP can be solvable in linear time for bounded tree-width graphs.*

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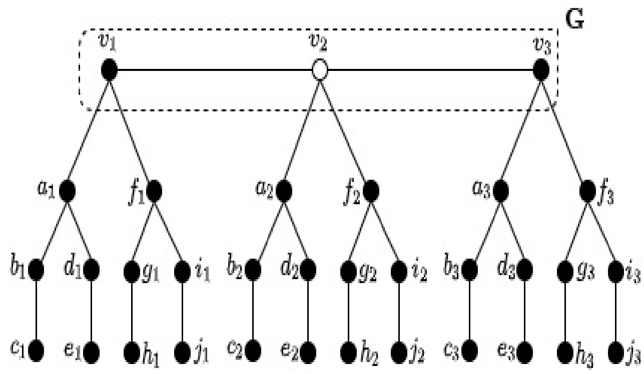

FIGURE 3. An illustration to the construction of GP graph from G

6. Complexity difference in domination and independent Roman $\{3\}$-domination

We show the complexity difference in domination and independent Roman $\{3\}$-domination by constructing a new class of graphs in which the MIR3DP can be solved trivially, whereas the DOMINATION DECISION problem is NP-complete.

Definition 3. (GP graph). A graph is *GP graph* if it can be constructed from a connected graph $G = (V, E)$ where $V = \{v_1, v_2, \dots, v_n\}$, in the following way :

(1) Create four copies of P_2 graphs such as $b_i - c_i, d_i - e_i, g_i - h_i$ and $i_i - j_i$, for each i.
(2) Consider $2n$ additional vertices $\{a_1, a_2, \dots, a_n, f_1, f_2, \dots, f_n\}$.
(3) Add edges $\{(v_i, a_i), (a_i, b_i), (a_i, d_i), (v_i, f_i), (f_i, g_i), (f_i, i_i) : 1 \le i \le n\}$.

General GP graph construction is shown in Figure 3.

Theorem 6.1. *If G' is a GP graph obtained from a graph $G = (V, E)$ ($|V| = n$), then $i_{\{R3\}}(G') = 12n$.*

Proof. Let $G' = (V', E')$ is a GP graph constructed from G. Let $f : V' \to \{0, 1, 2, 3\}$ be a function on graph G', which is defined as below

$$f(v) = \begin{cases} 2, & \text{if } v \in \{a_i, c_i, e_i, f_i, h_i, j_i : 1 \le i \le n\} \\ 0, & \text{otherwise} \end{cases} \tag{6.1}$$

Clearly, f is an IR3DF and $i_{\{R3\}}(G') \le 12n$.

Next, we show that $i_{\{R3\}}(G') \ge 12n$. Let g be an IR3DF on graph G'. Then following claim holds.

Claim 6.2. *If $g(V) = 12n$ then for each $v_i \in V$, $g(v_i) = 0$.*

Proof. (Proof by contradiction) Assume $g(V) = 12n$ and there exist some v_i's such that $g(v_i) \ne 0$. Let v_p be a vertex such that $g(v_p) \ge 1$. Clearly, $g(v_p) = 2$, $g(a_p) = 0$, $g(f_p) = 0$ and $\forall v_j \in N_G(v_p), g(v_j) = 0$. Then each of the four p_2's $b_p - c_p, d_p - e_p, g_p - h_p, i_p - j_p$ requires a weight of at least 3 and each $\langle\{a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i, i_i, j_i : 1 \le i \le n, i \ne p\}\rangle$, requires a weight of at least 12. Hence $g(V) \ge 12n + g(v_p) > 12n$, a contradiction. Therefore for each $v_i \in V$, $g(v_i) = 0$. □

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Clearly, $g(a_i)+g(b_i)+g(c_i)+g(d_i)+g(e_i) \geq 6$, $g(f_i)+g(g_i)+g(h_i)+g(i_i)+g(j_i) \geq 6$, where $1 \leq i \leq n$. Hence $g(V) \geq 12n$. Therefore $g(V) = 12n$. □

Lemma 6.3. *Let G' be a GP graph constructed from a graph $G = (V, E)$. Then G has a dominating set of size at most k if and only if G' has a dominating set of size at most $k + 4n$.*

Proof. Suppose D be dominating set of G of size at most k, then it is clear that $D \cup \{b_i, d_i, g_i, i_i : 1 \leq i \leq n\}$ is a dominating set of G' of size at most $k + 4n$.

Conversely, suppose D' is a dominating set of G' of size at most $k + 4n$. Then at least one vertex from each piar of the vertices $\{b_i, c_i\}$, $\{d_i, e_i\}$, $\{g_i, h_i\}$, $\{i_i, j_i\}$ must be included in D'. Let D'' be the set formed by replacing all a_i's (or f_i's) in D' by the corresponding v_i's. Clearly, $D'' \cap V$ is a dominating set of G of size at most k. Hence the lemma. □

The following result is well known for the DOMINATION DECISION problem.

Theorem 6.4. [17] *The DOMINATION DECISION problem is NP-complete for general graphs.*

From Theorem 6.4 and Lemma 6.3, the following result is immediate.

Theorem 6.5. *The DOMINATION DECISION problem is NP-complete for GP graphs.*

7. Integer linear programming formulation

Let $G = (V, E)$ be an undirected graph, with $|V| = n$, $|E| = m$ and $f : V \to \{0, 1, 2, 3\}$ be a IR3DF on G. The MIR3DP can now be modeled as Integer Linear Program (ILP).

Here we present an ILP model for MIR3DP. This model uses four sets of binary variables. Specifically, for each vertex $v \in V$, we define

$$a_v = \begin{cases} 1, & f(v) = 0 \\ 0, & \text{otherwise} \end{cases} \qquad b_v = \begin{cases} 1, & f(v) = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$c_v = \begin{cases} 1, & f(v) = 2 \\ 0, & \text{otherwise} \end{cases} \qquad d_v = \begin{cases} 1, & f(v) = 3 \\ 0, & \text{otherwise} \end{cases}$$

The ILP model of the MIR3DP can now be formulated as

Determine : $min(\sum_{v \in V}(b_v + 2c_v + 3d_v))$ (8.1)

subject to

$$3(1 - a_v) + \textstyle\sum_{u \in N(v)}(b_u + 2c_u + 3d_v) \geq 3,\ v \in V \tag{8.2}$$

$$2(1 - b_v) + \textstyle\sum_{u \in N(v)}(b_u + 2c_u + 3d_v) \geq 2,\ v \in V \tag{8.3}$$

$$b_u + c_u + d_u + b_v + c_v + d_v \leq 1, (u, v) \in E \tag{8.4}$$

$$a_v + b_v + c_v + d_v = 1,\ v \in V \tag{8.5}$$

$$a_v, b_v, c_v, d_v \in \{0, 1\},\ v \in V \tag{8.6}$$

The objective function (8.1) minimizes the weight of a IR3DF. The condition in (8.2), guarantees that for every vertex labeled zero, the sum of labels in its open neighborhood is three or more. The condition

in (8.3), guarantees that for every vertex labeled one, the sum of labels in its open neighborhood is at least two. The condition in (8.4), ensures that no two vertices with label greater than zero are adjacent. The condition in (8.5), guarantees that exactly one label is assigned to every vertex and the condition in (8.6) ensures that the variables are binary in nature.
The number of variables is $4n$ and the number of constraints is $3n + m$.

8. Conclusion

In this paper, we have shown that IR3DP is NP-complete for bipartite graphs and chordal graphs. Next, we have shown that MIR3DP is linear time solvable for threshold graphs, chain graphs and bounded tree-width graphs. We remark, however, that the two problems, domination and independent Roman {3}-domination are not equivalent in computational complexity aspects. Finally, we have proposed an ILP formulation for the MIR3DP. Designing better ILP formulation methods for the MIR3DP is an interesting direction for future work.

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