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Hermite-Hadamard type inequalities for n-polynomial generalized convex functions of Raina type and some related inequalities

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Abstract: In this paper, we introduce the concept of a new family of convex functions namely n–polynomial generalized convex functions of Raina type. We investigate the algebraic properties of a newly introduced idea and discuss their connections with convex functions. Furthermore, we establish the new version of Hermite–Hadamard and some refinements of Hermite–Hadamard type inequalities this class of functions. Finally, we investigate some applications to special means of real numbers. Results obtained in this paper can be viewed as a significant improvement of previously known results and also may stimulate and energize for further activities in this research area field.

Keywords: Hermite–Hadamard inequality; Hölder's inequality; Hölder-İscan inequality; Improved power-mean integral inequality; Generalized convexity; n–polynomial generalized convexity of Raina type.

1. Introduction

The term "convexity" is the most important, interesting, natural, and fundamental notations in mathematics and was used for the first time widely in the classical book by Hardy, Little, and Polya (see [1]). In recent times, the theory of convexity has played a very fascinating and amazing role in the world of science, of course, no one can refuse its significance and importance. Many researchers always try to do and use new ideas for the enjoyment and beautification of convex analysis. This theory provides us with interesting and powerful numerical tools, on the basis of these tools, we solve a lot of problems that appear in mathematics. During the last century, many researchers have contributed to the theory of convexity. The theory of convexity and its generalizations also play a magnificent role in the analysis of extremum problems. For the applications and interesting literature about convex analysis, readers refer to [2].

The theory of convexity also played an important and central tool in the development of the theory of inequalities. The subject of "Inequalities" is a very attractive and captivating field of research. The theory of inequalities is a subject of many mathematician's work in the last century. Many mathematicians always keep and continually try to do their work in the field of inequalities with hardworking and to collaborate different ideas and concepts. Thus the theory of inequalities and algorithms may be regarded as an independent area of mathematics. In recent years, due to its diverse and widespread applications, the ideas about the topics of convexity and integral inequalities have been extended, improved, and generalized in many different ways and the researchers are always inspired by the relationship of these two fields and consequently, many new inequalities have been obtained via the convexity property of the functions. For the importance of inequalities, interested readers are refer to [3].

The aim of this paper is to introduce a new family of convex functions namely n–polynomial generalized convex functions of Raina type. Interesting algebraic properties, several new integral inequalities, example with logic, and applications to means via the newly introduced class of functions are provided.

2. Preliminaries

In this section we recall some known concepts.

Definition 1. [4] A function $\Psi : I \to \mathbb{R}$ is said to be convex, if

$$\Psi\left(\kappa \varrho_1 + (1-\kappa)\varrho_2\right) \leq \kappa \Psi\left(\varrho_1\right) + (1-\kappa)\Psi\left(\varrho_2\right), \tag{1}$$

holds $\forall \varrho_1, \varrho_2 \in I$ and $\kappa \in [0,1]$.

The most important inequality concerning convex function is Hermite–Hadamard inequality given as:

Theorem 1. *If $\Psi : [\varrho_1, \varrho_2] \to \mathbb{R}$ is a convex function, then*

$$\Psi\left(\frac{\varrho_1 + \varrho_2}{2}\right) \leq \frac{1}{\varrho_2 - \varrho_1}\int_{\varrho_1}^{\varrho_2} \Psi(x)dx \leq \frac{\Psi(\varrho_1) + \Psi(\varrho_2)}{2}. \tag{2}$$

The double inequality (2) is in reverse order if Ψ is a concave function. The researchers have shown keep interest in above inequality and as a result various generalizations and improvements have been appeared in the literature. Due to widespread views and applications, this inequality has a lot of importance in the field of analysis.

In 2005, Raina [5] introduced a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(z) = \mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(z) = \sum_{k=0}^{+\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} z^k, \tag{3}$$

where $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ and $\rho, \lambda > 0, |z| < R$. The above class of function is the generalization of classical Mittag–Leffler function and the Kummer function, if $\rho = 1, \lambda = 0$ and $\sigma(k) = \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}$ for $k = 0, 1, 2, \ldots$.

The above parameters α, β and γ can take arbitrary real or complex values (provided that $\gamma \neq 0, -1, -2, \ldots$), and the symbol α_k represents the quantity

$$(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} = \alpha(\alpha+1)\ldots(\alpha+k-1), \quad k = 0, 1, 2, \ldots,$$

and restrict its domain to $|z| \leq 1$ (with $z \in \mathbb{C}$), then we have the classical hypergeometric function, that is

$$\mathcal{F}(\alpha, \beta; \gamma; z) = \sum_{k=0}^{+\infty} \frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k} z^k.$$

Also, if $\sigma = (1, 1, \ldots)$ with $\rho = \alpha$, $(Re(\alpha) > 0), \lambda = 1$, then

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(1 + \alpha k)}.$$

The above function is called a classical Mittag–Leffler function.

Cortez established the new class of set and function involving Raina's function in [6,7], which is said to be generalized convex set and convex function.

Definition 2. [8] Let $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ and $\rho, \lambda > 0$. A set $X \neq \emptyset$ is said to be generalized convex, if

$$\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) \in X, \tag{4}$$

for all $\varrho_1, \varrho_2 \in X$ and $\kappa \in [0,1]$.

Definition 3. [8] Let σ denote a bounded sequence then $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ and $\rho, \lambda > 0$. If $\Psi : X \to \mathbb{R}$ satisfies the following inequality

$$\Psi\Big(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)\Big) \leq \kappa \Psi(\varrho_1) + (1-\kappa)\Psi(\varrho_2), \tag{5}$$

for all $\varrho_1, \varrho_2 \in X$, where $\varrho_1 < \varrho_2$ and $\kappa \in [0,1]$, then Ψ is called generalized convex function.

Remark 1. We have $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) = \varrho_1 - \varrho_2 > 0$, and so we obtain Definition 1.

Condition 1. Let $X \subseteq \mathbb{R}$ be an open generalized convex subset with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}(\cdot)$. For any $\varrho_1, \varrho_2 \in X$ and $\kappa \in [0,1]$,

$$\mathcal{F}^{\sigma}_{\rho,\lambda}\Big(\varrho_2 - (\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))\Big) = -\kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2),$$
$$\mathcal{F}^{\sigma}_{\rho,\lambda}\Big(\varrho_1 - \Big(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)\Big)\Big) = (1-\kappa)\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2).$$

Note that, for every $\varrho_1,\ \varrho_2 \in X$ and for all $\kappa_1,\ \kappa_2 \in [0,1]$ from Condition 1, we have

$$\mathcal{F}^{\sigma}_{\rho,\lambda}\Big(\varrho_2 + \kappa_2\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) - (\varrho_2 + \kappa_1\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))\Big) = (\kappa_2 - \kappa_1)\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2). \tag{6}$$

Definition 4. [9] Let $\forall \varrho_1, \varrho_2 \in I$, $n \in \mathbb{N}$ and $\kappa \in [0,1]$, then an inequality of the form

$$\Psi\left(\kappa\varrho_1 + (1-\kappa)\,\varrho_2\right) \le \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-\kappa)^s\right]\Psi\left(\varrho_1\right) + \frac{1}{n}\sum_{s=1}^{n}\left[1-\kappa^s\right]\Psi\left(\varrho_2\right). \tag{7}$$

is called an n–polynomial convex function.

Definition 5. [10] Two functions Ψ and Φ are said to be similarly ordered, if

$$(\Psi(\varrho_1) - \Psi(\varrho_2))(\Phi(\varrho_1) - \Phi(\varrho_2)) \ge 0, \quad \forall \varrho_1, \varrho_2 \in \mathbb{R}. \tag{8}$$

Theorem 2. [11] *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\Psi, \Phi \in \mathbb{R}$ defined on $[\varrho_1, \varrho_2]$ and if $|\Psi|^p, |\Phi|^q$ are $L[\varrho_1, \varrho_2]$, then*

$$\int_{\varrho_1}^{\varrho_2} |\Psi(x)\Phi(x)|dx \le \frac{1}{\varrho_2 - \varrho_1}\left\{\left(\int_{\varrho_1}^{\varrho_2}(\varrho_2 - x)|\Psi(x)|^p dx\right)^{\frac{1}{p}}\left(\int_{\varrho_1}^{\varrho_2}(\varrho_2 - x)|\Phi(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_{\varrho_1}^{\varrho_2}(x - \varrho_1)|\Psi(x)|^p dx\right)^{\frac{1}{p}}\left(\int_{\varrho_1}^{\varrho_2}(x - \varrho_1)|\Phi(x)|^q dx\right)^{\frac{1}{q}}\right\}.$$

Theorem 3. [12] *Let $q \ge 1$. If $\Psi, \Phi \in \mathbb{R}$ defined on $[\varrho_1, \varrho_2]$ and if $|\Psi|, |\Psi||\Phi|^q$ are $L[\varrho_1, \varrho_2]$, then*

$$\int_{\varrho_1}^{b} |\Psi(x)\Phi(x)|dx \le \frac{1}{\varrho_2 - \varrho_1}\left\{\left(\int_{\varrho_1}^{\varrho_2}(\varrho_2 - x)|\Psi(x)|dx\right)^{1-\frac{1}{q}}\left(\int_{\varrho_1}^{\varrho_2}(\varrho_2 - x)|\Psi(x)||\Phi(x)|^q dx\right)^{\frac{1}{q}} + \left(\int_{\varrho_1}^{\varrho_2}(x - \varrho_1)|\Psi(x)|dx\right)^{1-\frac{1}{q}}\left(\int_{\varrho_1}^{\varrho_2}(x - \varrho_1)|\Psi(x)||\Phi(x)|^q dx\right)^{\frac{1}{q}}\right\}.$$

Owing to the aforementioned trend and energized by the in-progress research activities in this captivating field, we organize the paper in the following pattern. In Section 3, we give the idea and algebraic properties of n–polynomial generalized convex functions of Raina type. In Section 4 and 5, using the newly introduced idea, we attain the new version of Hermite–Hadamard inequality and some of its refinements. In Section 6, we will add some applications and conclusion.

3. n–polynomial generalized convex functions of Raina type and its Properties

In this section, we introduce a new family of convex functions namely n–polynomial generalized convex functions of Raina type and also will study some of its properties. One important thing to keep in mind throughout the paper, n–poly represents n–polynomial.

Definition 6. Let $\mathbb{X} \in \mathbb{R}$ be a nonempty generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$. Then a nonnegative function $\Psi : \mathbb{X} \to \mathbb{R}$ is called n–poly generalized convex function of Raina type, if

$$\Psi(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)) \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-\kappa)^s\right]\Psi(\varrho_1) + \frac{1}{n}\sum_{s=1}^{n}\left[1-\kappa^s\right]\Psi(\varrho_2), \tag{9}$$

holds for every $\varrho_1, \varrho_2 \in \mathbb{X}$, $n \in \mathbb{N}$, $\kappa \in [0,1]$, $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$, $\rho, \lambda > 0$ and $|\varrho_1 - \varrho_2| < R$.

Remark 2. It is easy to show that, if nonnegative function Ψ is generalized convex function of Raina type then nonnegative function Ψ is n–poly generalized convex function of Raina type. Indeed for all $\kappa \in [0,1]$, then we attain the following inequalities

$$\kappa \leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s] \quad \text{and} \quad 1-\kappa \leq \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s].$$

This mean that, the new family of n–poly generalized convex functions of Raina type is very larger with respect the known class of functions, like n–poly convex functions. This is an advantage of the proposed new Definition 6.

Example 1. Since $\Psi(x) = e^x$ is nonnegative convex function, so obviously it is a generalized convex function of Raina type. So according to Remark 2, it is n–poly generalized convex function of Raina type.

Theorem 4. *Let $\Psi, \Phi : [\varrho_1, \varrho_2] \to \mathbb{R}$. If Ψ and Φ are two n–poly generalized convex functions of Raina type, then*

(i) $\Psi + \Phi$ is an n–poly generalized convex function of Raina type.
(ii) For nonnegative real number c, $c\Psi$ is an n–poly generalized convex function of Raina type.

Proof. (i) Since Ψ and Φ be two n–poly generalized convex functions of Raina type, then

$$\begin{aligned}(\Psi+\Phi)(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)) &= \Psi(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)) + \Phi(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)) \\ &\leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s]\Psi(\varrho_1) + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s]\Psi(\varrho_2) \\ &\quad + \frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s]\Phi(\varrho_1) + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s]\Phi(\varrho_2). \\ &= \frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s][\Psi(\varrho_1)+\Phi(\varrho_1)] + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s][\Psi(\varrho_2)+\Phi(\varrho_2)] \\ &= \frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s](\Psi+\Phi)(\varrho_1) + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s](\Psi+\Phi)(\varrho_2).\end{aligned}$$

(ii) Since Ψ be an n–poly generalized convex function of Raina type, then

$$\begin{aligned}(c\Psi)(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)) &\leq c\left[\frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s]\Psi(\varrho_1) + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s]\Psi(\varrho_2)\right] \\ &= \frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s]c\Psi(\varrho_1) + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s]c\Psi(\varrho_2) \\ &= \frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s](c\Psi)(\varrho_1) + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s](c\Psi)(\varrho_2),\end{aligned}$$

which completes the proof.
□

Theorem 5. *Suppose $\Psi : I \to J$ be n–poly generalized convex function of Raina type and $\Phi : J \to \mathbb{R}$ is non-decreasing function. Then the composition of two functions i.e., $\Phi \circ \Psi : I \to \mathbb{R}$ is an n–poly generalized convex function of Raina type.*

Proof. Let $\kappa \in [0,1]$ and for all $\varrho_1, \varrho_2 \in I$, we have

$$\begin{aligned}(\Phi \circ \Psi)\left(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)\right) &= \Phi(\Psi(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))) \\ &\leq \Phi\left[\frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s]\,\Psi(\varrho_1) + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s]\,\Psi(\varrho_2)\right] \\ &\leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s]\,\Phi(\Psi(\varrho_1)) + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s]\,\Phi(\Psi(\varrho_2)) \\ &= \frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s]\,(\Phi\circ\Psi)(\varrho_1) + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s]\,(\Phi\circ\Psi)(\varrho_2).\end{aligned}$$

This completes the proof. □

Remark 3. (i) Consider $n = 1$ in above Theorem 5, then we attain the following inequality

$$(\Phi \circ \Psi)\left(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)\right) \leq \kappa(\Phi\circ\Psi)(\varrho_1) + (1-\kappa)(\Phi\circ\Psi)(\varrho_2).$$

(ii) If we put $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) = \varrho_1 - \varrho_2$ in above Theorem 5, then we attain the following inequality

$$(\Phi \circ \Psi)(\kappa\varrho_1 + (1-\kappa)\varrho_2) \leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s]\,(\Phi\circ\Psi)(\varrho_1) + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s]\,(\Phi\circ\Psi)(\varrho_2).$$

(iii) If we put $n = 1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) = \varrho_1 - \varrho_2$ in above Theorem 5, then we attain the following inequality

$$(\Phi \circ \Psi)(\kappa\varrho_1 + (1-\kappa)\varrho_2) \leq \kappa(\Phi\circ\Psi)(\varrho_1) + (1-\kappa)(\Phi\circ\Psi)(\varrho_2).$$

Theorem 6. *Let $0 < \varrho_1 < \varrho_2$, $\Psi_j : [\varrho_1, \varrho_2] \to [0, +\infty)$ be a family of n–poly generalized convex functions of Raina type and $\Psi(u) = \sup_j \Psi_j(u)$. Then Ψ is an n–poly generalized convex function of Raina type and $U = \{u \in [\varrho_1, \varrho_2] : \Psi(u) < +\infty\}$ is an interval.*

Proof. Let $\varrho_1, \varrho_2 \in U$ and $\kappa \in [0,1]$, then

$$\begin{aligned}\Psi(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)) &= \sup_j \Psi_j(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)) \\ &\leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s]\sup_j \Psi_j(\varrho_1) + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s]\sup_j \Psi_j(\varrho_2) \\ &= \frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s]\,\Psi(\varrho_1) + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s]\,\Psi(\varrho_2) < +\infty.\end{aligned}$$

This completes the proof. □

Remark 4. Taking $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) = \varrho_1 - \varrho_2$ in above Theorem 6, then we attain Theorem 3 in [9].

Theorem 7. *Suppose $\Psi, \Phi : [\varrho_1, \varrho_2] \to \mathbb{R}$. If Ψ and Φ are two n–poly generalized convex functions of Raina type. If the above defined functions Ψ and Φ are similarly ordered and $\frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s] + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s] \leq 1$, then the product $\Psi\Phi$ is also an n–poly generalized convex function of Raina type.*

Proof. Since Ψ and Φ are two n–poly generalized convex functions of Raina type, then

$$\begin{aligned}
&\Psi(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))\Phi(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))\\
&\le\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-(1-\kappa)^s\right]\Psi\left(\varrho_1\right)+\frac{1}{n}\sum_{s=1}^{n}\left[1-\kappa^s\right]\Psi\left(\varrho_2\right)\right]\\
&\quad\times\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-(1-\kappa)^s\right]\Phi\left(\varrho_1\right)+\frac{1}{n}\sum_{s=1}^{n}\left[1-\kappa^s\right]\Phi\left(\varrho_2\right).\right]\\
&\le\frac{1}{n^2}\sum_{s=1}^{n}\left[1-(1-\kappa)^s\right]^2\Psi(\varrho_1)\Phi(\varrho_1)+\frac{1}{n^2}\sum_{s=1}^{n}\left[1-\kappa^s\right]^2\Psi(\varrho_2)\Phi(\varrho_2)\\
&\quad+\frac{1}{n^2}\sum_{s=1}^{n}\left[1-(1-\kappa)^s\right]\left[1-\kappa^s\right]\left[\Psi(\varrho_1)\Phi(\varrho_2)+\Psi(\varrho_2)\Phi(\varrho_1)\right]\\
&\le\frac{1}{n^2}\sum_{s=1}^{n}\left[1-(1-\kappa)^s\right]^2\Psi(\varrho_1)\Phi(\varrho_1)+\frac{1}{n^2}\sum_{s=1}^{n}\left[1-\kappa^s\right]^2\Psi(\varrho_2)\Phi(\varrho_2)\\
&\quad+\frac{1}{n^2}\sum_{s=1}^{n}\left[1-(1-\kappa)^s\right]\left[1-\kappa^s\right]\left[\Psi(\varrho_1)\Phi(\varrho_1)+\Psi(\varrho_2)\Phi(\varrho_2)\right]\\
&=\left[\frac{1}{n}\sum_{s=1}^{n}\left[1-(1-\kappa)^s\right]\Psi(\varrho_1)\Phi(\varrho_1)+\frac{1}{n}\sum_{s=1}^{n}\left[1-\kappa^s\right]\Psi(\varrho_2)\Phi(\varrho_2)\right]\\
&\quad\times\left(\frac{1}{n}\sum_{s=1}^{n}\left[1-(1-\kappa)^s\right]+\frac{1}{n}\sum_{s=1}^{n}\left[1-\kappa^s\right]\right)\\
&\le\frac{1}{n}\sum_{s=1}^{n}\left[1-(1-\kappa)^s\right]\Psi(\varrho_1)\Phi(\varrho_1)+\frac{1}{n}\sum_{s=1}^{n}\left[1-\kappa^s\right]\Psi(\varrho_2)\Phi(\varrho_2).
\end{aligned}$$

This shows that the product of two n–poly generalized convex functions of Raina type is again an n–poly generalized convex function of Raina type. $\square$

Remark 5. (i) Taking $n=1$ in above Theorem 7, then we obtain

$$\Psi(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))\Phi(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))\le\kappa\Psi\left(\varrho_1\right)\Phi(\varrho_1)+(1-\kappa)\Psi\left(\varrho_2\right)\Phi(\varrho_2).$$

(ii) Taking $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)=\varrho_1-\varrho_2$ in above Theorem 7, then we obtain

$$\Psi(\kappa\varrho_1+(1-\kappa)\varrho_2)\Phi(\kappa\varrho_1+(1-\kappa)\varrho_2)\le\frac{1}{n}\sum_{s=1}^{n}\left[1-(1-\kappa)^s\right]\Psi\left(\varrho_1\right)+\frac{1}{n}\sum_{s=1}^{n}\left[1-\kappa^s\right]\Psi\left(\varrho_2\right).$$

(iii) Taking $n=1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)=\varrho_1-\varrho_2$ in above Theorem 7, then we obtain

$$\Psi(\kappa\varrho_1+(1-\kappa)\varrho_2)\Phi(\kappa\varrho_1+(1-\kappa)\varrho_2)\le\kappa\Psi\left(\varrho_1\right)\Phi(\varrho_1)+(1-\kappa)\Psi\left(\varrho_2\right)\Phi(\varrho_2).$$

4. Hermite–Hadamard type inequality for n–polynomial generalized convex functions of Raina type

The focus of this section is to establish Hermite–Hadamard inequality for n–poly generalized convex functions of Raina type.

Theorem 8. *Let $\Psi:[\varrho_1,\varrho_2]\in\mathbb{R}$ be an n–poly generalized convex function of Raina type, if $\varrho_1<\varrho_2$ and $\Psi\in L[\varrho_1,\varrho_2]$, then*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\Psi(\varrho_2+\frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))\le\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\Psi(x)dx\le\frac{\Psi(\varrho_1)+\Psi(\varrho_2)}{n}\sum_{s=1}^{n}\frac{s}{s+1}.$$

Proof. From the property of an n–poly generalized convex function of Raina type Ψ, we have that

$$\Psi(\varrho_2+\kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)) \leq \frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s]\Psi(\varrho_1)+\frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s]\Psi(\varrho_2)$$

$$\int_0^1 \Psi(\varrho_2+\kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))d\kappa \leq \frac{\Psi(\varrho_1)}{n}\sum_{s=1}^{n}\int_0^1[1-(1-\kappa)^s]d\kappa+\frac{\Psi(\varrho_2)}{n}\sum_{s=1}^{n}\int_0^1[1-\kappa^s]d\kappa$$

but

$$\int_0^1 \Psi(\varrho_2+\kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))d\kappa = \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\Psi(x)dx$$

so

$$\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\Psi(x)dx \leq \frac{\Psi(\varrho_1)+\Psi(\varrho_2)}{n}\sum_{s=1}^{n}\frac{s}{s+1}.$$

This completes the proof of right side of above inequality. For the left side using the property of an n–poly generalized convex function of Raina type and Condition A for $\mathcal{F}^{\sigma}_{\rho,\lambda}$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&\Psi\Big(\varrho_2+\frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)\Big)\\
&\quad=\Psi(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)+\frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_2+(1-\kappa)\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)-(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)))\\
&\quad\leq\frac{1}{n}\sum_{s=1}^{n}\left[1-\left(\frac{1}{2}\right)^s\right]\left[\int_0^1\Psi(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))d\kappa+\int_0^1\Psi(\varrho_2+(1-\kappa)\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))d\kappa\right]\\
&\quad\leq\frac{1}{n}\sum_{s=1}^{n}\left[1-\left(\frac{1}{2}\right)^s\right]\frac{2}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\Psi(x)dx\\
&\quad\leq\left[\frac{n+2^{-n}-1}{n}\right]\frac{2}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\Psi(x)dx.
\end{aligned}$$

This completes the proof. □

Corollary 1. *If we put $n=1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)=\varrho_1-\varrho_2$ in Theorem 8, then we get Hermite-Hadamard inequality in [13].*

Corollary 2. *If we put $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)=\varrho_1-\varrho_2$ in Theorem 8, then we get inequality (3.1) in [9].*

Remark 6. Under the assumption of Theorem 8, if we take $\sigma=(1,1,...)$ with $\rho=\alpha$, $\lambda=1$, we get the following inequality involving classical Mittag–Leffler function

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\Psi(\varrho_2+\frac{1}{2}E_\alpha(\varrho_1-\varrho_2)) \leq \frac{1}{E_\alpha(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+E_\alpha(\varrho_1-\varrho_2)}\Psi(x)dx \leq \frac{\Psi(\varrho_1)+\Psi(\varrho_2)}{n}\sum_{s=1}^{n}\frac{s}{s+1}.$$

5. New inequalities for n–polynomial generalized convex functions of Raina type

The intention of this section is to derived the refinements of Harmite–Hadamard type inequalities for n–poly generalized convex functions of Raina type.

Lemma 1. *Let $X\subseteq\mathbb{R}$ be an generalized convex subset with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}: X\times X\to\mathbb{R}$ and $\varrho_1,\varrho_2\in X$ with $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)\neq 0$. Suppose that $\Psi: X\to\mathbb{R}$ is a differentiable function. If Ψ is integrable on the $\mathcal{F}^{\sigma}_{\rho,\lambda}$, then the following equality holds*

$$\begin{aligned}
&-\frac{\Psi(\varrho_2)+\Psi(\varrho_2+\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))}{2}+\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\Psi(x)dx\\
&\quad=\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\int_0^1(1-2\kappa)\Psi'(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))d\kappa.
\end{aligned}$$

Proof. Suppose that $\varrho_1, \varrho_2 \in X$. Since X is generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, for every $\kappa \in [0,1]$, we have $\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) \in X$. Integrating by parts implies that

$$\begin{aligned}
&\int_0^1 (1-2\kappa)\Psi'(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))d\kappa \\
&\quad = \left[\frac{(1-2\kappa)\Psi(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}\right]_0^1 + \frac{2}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}\int_0^1 \Psi(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))d\kappa \\
&\quad = -\frac{\Psi(\varrho_2) + \Psi(\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))}{2} + \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}\int_{\varrho_2}^{\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \Psi(x)dx.
\end{aligned}$$

This completes the proof. □

Theorem 9. *Suppose I° is an generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$ and $\Psi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\varrho_1, \varrho_2 \in I^\circ$ with $\varrho_1 < \varrho_2$ and Ψ' is integrable function on the interval $[\varrho_1, \varrho_2]$. Suppose $|\Psi'|$ is an n–poly generalized convex function of Raina type on $L[\varrho_1, \varrho_2]$, then*

$$\begin{aligned}
&\left|\frac{\Psi(\varrho_2) + \Psi(\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))}{2} - \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}\int_{\varrho_2}^{\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \Psi(x)dx\right| \\
&\quad \le \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{n}\sum_{s=1}^{n}\left[\frac{2^s(s^2+s+2)-2}{(s+1)(s+2)2^{s+1}}\right] A\Big(|\Psi'(\varrho_1)|, |\Psi'(\varrho_2)|\Big),
\end{aligned}$$

holds for $\kappa \in [0,1]$, where $A(.,.)$ is Arithmetic mean.

Proof. Suppose that $\varrho_1, \varrho_2 \in I^\circ$. Since I° is an generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, for any $\kappa \in [0,1]$, we have $\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) \in I^\circ$.

From Lemma 1, n–poly generalized convex function of Raina type of $|\Psi'|$ and properties of modulus, we have

$$\begin{aligned}
&\left|\frac{\Psi(\varrho_2) + \Psi(\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))}{2} - \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}\int_{\varrho_2}^{\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \Psi(x)dx\right| \\
&\le \left|\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2}\int_0^1 (1-2\kappa)\Psi'(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))d\kappa\right| \\
&\le \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2}\int_0^1 |1-2\kappa|\left(\frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s]|\Psi'(\varrho_1)| + \frac{1}{n}\sum_{s=1}^{n}[1-\kappa^s]|\Psi'(\varrho_2)|\right)d\kappa \\
&\le \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2n}\left(|\Psi'(\varrho_1)|\int_0^1 |1-2\kappa|\sum_{s=1}^{n}[1-(1-\kappa)^s]d\kappa + |\Psi'(\varrho_2)|\int_0^1 |1-2\kappa|\sum_{s=1}^{n}\Big([1-\kappa^s]d\kappa\right) \\
&\le \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2n}\left(|\Psi'(\varrho_1)|\sum_{s=1}^{n}\int_0^1 |1-2\kappa|[1-(1-\kappa)^s]d\kappa + |\Psi'(\varrho_2)|\sum_{s=1}^{n}\int_0^1 |1-2\kappa|[1-\kappa^s]d\kappa\right) \\
&\le \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2n}\left(|\Psi'(\varrho_1)|\sum_{s=1}^{n}\left[\frac{2^s(s^2+s+2)-2}{(s+1)(s+2)2^{s+1}}\right] + |\Psi'(\varrho_2)|\sum_{s=1}^{n}\left[\frac{2^s(s^2+s+2)-2}{(s+1)(s+2)2^{s+1}}\right]\right) \\
&\le \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{n}\sum_{s=1}^{n}\left[\frac{2^s(s^2+s+2)-2}{(s+1)(s+2)2^{s+1}}\right] A\Big(|\Psi'(\varrho_1)|, |\Psi'(\varrho_2)|\Big).
\end{aligned}$$

This completes the proof. □

Corollary 3. *If $n = 1$ in above Theorem 9, then*

$$\left|\frac{\Psi(\varrho_2) + \Psi(\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))}{2} - \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}\int_{\varrho_2}^{\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \Psi(x)dx\right| \le \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{4} A\Big(|\Psi'(\varrho_1)|, |\Psi'(\varrho_2)|\Big).$$

Corollary 4. *If we put $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) = \varrho_1 - \varrho_2$ in Theorem 9, we get inequality (4.1) in [9].*

Corollary 5. *If we put $n = 1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) = \varrho_1 - \varrho_2$ in Theorem 9, then we get Corollary 1 in [9].*

Remark 7. Under the assumption of Theorem 9, if we take $\sigma = (1,1,...)$ with $\rho = \alpha$, $\lambda = 1$, we get the following inequality involving classical Mittag–Leffler function

$$\left|\frac{\Psi(\varrho_2) + \Psi(\varrho_2 + E_\alpha(\varrho_1 - \varrho_2))}{2} - \frac{1}{E_\alpha(\varrho_1 - \varrho_2)} \int_{\varrho_2}^{\varrho_2 + E_\alpha(\varrho_1 - \varrho_2)} \Psi(x)dx\right| \leq \frac{E_\alpha(\varrho_1 - \varrho_2)}{n} \sum_{s=1}^{n} \left[\frac{2^s(s^2+s+2)-2}{(s+1)(s+2)2^{s+1}}\right] A\Big(|\Psi'(\varrho_1)|, |\Psi'(\varrho_2)|\Big).$$

Theorem 10. *Suppose I° is an generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$ and $\Psi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\varrho_1, \varrho_2 \in I^\circ$ with $\varrho_1 < \varrho_2, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and Ψ' is integrable function on the interval $[\varrho_1, \varrho_2]$. Suppose $|\Psi'|^q$ is an n–poly generalized convex function of Raina type on $L[\varrho_1, \varrho_2]$, then*

$$\left|\frac{\Psi(\varrho_2) + \Psi(\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))}{2} - \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \int_{\varrho_2}^{\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \Psi(x)dx\right| \leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\Big(|\Psi'(\varrho_1)|^q, |\Psi'(\varrho_2)|^q\Big),$$

holds for $\kappa \in [0,1]$, where A(x,y) is Arithmetic mean.

Proof. Suppose that $\varrho_1, \varrho_2 \in I^\circ$. Since I° is an generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, for any $\kappa \in [0,1]$, we have $\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) \in I^\circ$.

From Lemma 1, Hölder's integral inequality, n–poly generalized convex function of Raina type of $|\Psi'|^q$ and properties of modulus, we have

$$\begin{aligned}
&\left|\frac{\Psi(\varrho_2) + \Psi(\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))}{2} - \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \int_{\varrho_2}^{\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \Psi(x)dx\right| \\
&\leq \left|\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2} \int_0^1 (1-2\kappa)\Psi'(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))d\kappa\right| \\
&\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2} \left(\int_0^1 |1-2\kappa|^p\right)^{\frac{1}{p}} \left(\int_0^1 |\Psi'(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))|^q d\kappa\right)^{\frac{1}{q}} \\
&\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{|\Psi'(\varrho_1)|^q}{n} \int_0^1 \sum_{s=1}^{n}[1-(1-\kappa)^s]d\kappa + \frac{|\Psi'(\varrho_2)|^q}{n} \int_0^1 \sum_{s=1}^{n}[1-\kappa^s]d\kappa\right)^{\frac{1}{q}} \\
&\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\Big(|\Psi'(\varrho_1)|^q, |\Psi'(\varrho_2)|^q\Big).
\end{aligned}$$

This completes the proof. □

Corollary 6. *If we put $n = 1$ in Theorem 10, then*

$$\left|\frac{\Psi(\varrho_2) + \Psi(\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))}{2} - \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \int_{\varrho_2}^{\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \Psi(x)dx\right| \leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2} \left(\frac{1}{p+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\Big(|\Psi'(\varrho_1)|, |\Psi'(\varrho_2)|\Big).$$

Corollary 7. *If we put $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) = \varrho_1 - \varrho_2$ in Theorem 10, then we get inequality (4.2) in [9].*

Corollary 8. *If we put $n = 1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) = \varrho_1 - \varrho_2$ in Theorem 10, then we get Corollary 2 in [9].*

Remark 8. Under the assumption of Theorem 10, if we take $\sigma = (1, 1, ...)$ with $\rho = \alpha$, $\lambda = 1$, we get the following inequality involving classical Mittag–Leffler function

$$\left|\frac{\Psi(\varrho_2) + \Psi(\varrho_2 + E_\alpha(\varrho_1 - \varrho_2))}{2} - \frac{1}{E_\alpha(\varrho_1 - \varrho_2)} \int_{\varrho_2}^{\varrho_2 + E_\alpha(\varrho_1 - \varrho_2)} \Psi(x)dx\right|$$
$$\leq \frac{E_\alpha(\varrho_1 - \varrho_2)}{2} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{2}{n}\sum_{s=1}^{n} \frac{s}{s+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(|\Psi'(\varrho_1)|^q, |\Psi'(\varrho_2)|^q\right).$$

Theorem 11. *Suppose I° is an generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$ and $\Psi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\varrho_1, \varrho_2 \in I^\circ$ with $\varrho_1 < \varrho_2, q \geq 1$, and Ψ' is integrable function on the interval $[\varrho_1, \varrho_2]$. Suppose $|\Psi'|^q$ is an n–poly generalized convex function of Raina type on $L[\varrho_1, \varrho_2]$, then*

$$\left|\frac{\Psi(\varrho_2) + \Psi(\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))}{2} - \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \int_{\varrho_2}^{\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \Psi(x)dx\right|$$
$$\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2} \left(\frac{1}{2}\right)^{1-\frac{2}{q}} \left(\frac{1}{n}\sum_{s=1}^{n} \left[\frac{2^s(s^2+s+2) - 2}{(s+1)(s+2)2^{s+1}}\right]\right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(|\Psi'(\varrho_1)|^q, |\Psi'(\varrho_2)|^q\right),$$

holds for $\kappa \in [0, 1]$, where $A(.,.)$ is Arithmetic mean.

Proof. Suppose that $\varrho_1, \varrho_2 \in I^\circ$. Since I° is an generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, for any $\kappa \in [0, 1]$, we have $\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) \in I^\circ$.

Assume that $q > 1$. From Lemma 1, power mean inequality, n–poly generalized convex function of Raina type of $|\Psi'|^q$ and properties of modulus, we have

$$\begin{aligned}
&\left|\frac{\Psi(\varrho_2) + \Psi(\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))}{2} - \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \int_{\varrho_2}^{\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \Psi(x)dx\right| \\
&\leq \left|\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2} \int_0^1 (1-2\kappa)\Psi'(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))d\kappa\right| \\
&\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2} \left(\int_0^1 |1-2\kappa|d\kappa\right)^{1-\frac{1}{q}} \left(\int_0^1 |1-2\kappa||\Psi'(\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))|^q d\kappa\right)^{\frac{1}{q}} \\
&\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\int_0^1 |1-2\kappa| \left[\frac{1}{n}\sum_{s=1}^{n}[1-(1-\kappa)^s]|\Psi'(\varrho_1)|^q + \frac{1}{n}\sum_{s=1}^{n}[1-t^s]|\Psi'(\varrho_2)|^q\right] d\kappa\right)^{\frac{1}{q}} \\
&\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\frac{|\Psi'(\varrho_1)|^q}{n} \int_0^1 |1-2\kappa| \sum_{s=1}^{n}[1-(1-\kappa)^s]d\kappa + \frac{|\Psi'(\varrho_2)|^q}{n} \int_0^1 |1-2\kappa| \sum_{s=1}^{n}[1-\kappa^s]d\kappa\right)^{\frac{1}{q}} \\
&= \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{2} \left(\frac{1}{2}\right)^{1-\frac{2}{q}} \left(\frac{1}{n}\sum_{s=1}^{n} \left[\frac{2^s(s^2+s+2) - 2}{(s+1)(s+2)2^{s+1}}\right]\right)^{\frac{1}{q}} A^{\frac{1}{q}} \left(|\Psi'(\varrho_1)|^q, |\Psi'(\varrho_2)|^q\right).
\end{aligned}$$

For $q = 1$, we use the estimates from the proof of Theorem 9, which also follow step by step the above estimates. This completes the proof. □

Corollary 9. *If we put $n = 1$ in Theorem 11, then*

$$\left|\frac{\Psi(\varrho_2) + \Psi(\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))}{2} - \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \int_{\varrho_2}^{\varrho_2 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)} \Psi(x)dx\right|$$
$$\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)}{4} A^{\frac{1}{q}} \left[|\Psi'(\varrho_1)|^q, |\Psi'(\varrho_2)|^q\right].$$

Corollary 10. *If we put $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) = \varrho_1 - \varrho_2$ in Theorem 11, we get inequality (4.3) in [9].*

Corollary 11. *If we put $n = 1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) = \varrho_1 - \varrho_2$ in Theorem 11, then we get Corollary 4 in [9].*

Remark 9. Under the assumption of Theorem 11, if we take $\sigma = (1,1,...)$ with $\rho = \alpha$, $\lambda = 1$, we get the following inequality involving classical Mittag–Leffler function

$$\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_2+E_\alpha(\varrho_1-\varrho_2))}{2}-\frac{1}{E_\alpha(\varrho_1-\varrho_2)}\Psi(x)dx\right|$$
$$\leq \frac{E_\alpha(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{s=1}^{n}\left[\frac{2^s(s^2+s+2)-2}{(s+1)(s+2)2^{s+1}}\right]\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|\Psi'(\varrho_1)|^q,|\Psi'(\varrho_2)|^q\right).$$

Theorem 12. *Suppose I° is an generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$ and $\Psi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\varrho_1, \varrho_2 \in I^\circ$ with $\varrho_1 < \varrho_2, q > 1, \frac{1}{p}+\frac{1}{q} = 1$ and Ψ' is integrable function on the interval $[\varrho_1, \varrho_2]$. Suppose $|\Psi'|^q$ is an n–poly generalized convex function of Raina type on $L[\varrho_1, \varrho_2]$, then*

$$\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))}{2}-\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\Psi(x)dx\right|$$
$$\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}\right)^{\frac{1}{q}}$$
$$+\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}\right)^{\frac{1}{q}},$$

holds for $\kappa \in [0,1]$.

Proof. Suppose that $\varrho_1, \varrho_2 \in I^\circ$. Since I° is an generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, for any $\kappa \in [0,1]$, we have $\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2) \in I^\circ$.

From Lemma 1, Hölder–İşcan integral inequality, n–poly generalized convex function of Raina type of $|\Psi'|^q$ and properties of modulus, we have

$$\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))}{2}-\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\Psi(x)dx\right|$$
$$\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\int_0^1|1-2\kappa||\Psi'(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))|d\kappa$$
$$\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\int_0^1(1-\kappa)|1-2\kappa|^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^1(1-\kappa)|\Psi'(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))|^q d\kappa\right)^{\frac{1}{q}}$$
$$+\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\int_0^1\kappa|1-2\kappa|^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\kappa|\Psi'(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))|^q d\kappa\right)^{\frac{1}{q}}$$
$$\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}\int_0^1(1-\kappa)[1-(1-\kappa)^s]d\kappa+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}\int_0^1(1-\kappa)[1-\kappa^s]d\kappa\right)^{\frac{1}{q}}$$
$$+\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}\int_0^1\kappa[1-(1-\kappa)^s]d\kappa+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}\int_0^1\kappa[1-\kappa^s]dt\right)^{\frac{1}{q}}$$
$$\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}\right)^{\frac{1}{q}}$$
$$+\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 12. *If we put $n = 1$ in Theorem 12, then*

$$\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))}{2}-\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\Psi(x)dx\right|$$
$$\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|\Psi'(\varrho_1)|^q}{3}+\frac{2|\Psi'(\varrho_2)|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2|\Psi'(\varrho_1)|^q}{3}+\frac{|\Psi'(\varrho_2)|^q}{3}\right)^{\frac{1}{q}}\right].$$

Corollary 13. *If we put $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)=\varrho_1-\varrho_2$ in Theorem 12, we get inequality (4.4) in [9].*

Corollary 14. *If we put $n = 1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)=\varrho_1-\varrho_2$ in Theorem 12, then we get Corollary 5 in [9].*

Remark 10. Under the assumption of Theorem 12, if we take $\sigma = (1, 1, ...)$ with $\rho = \alpha$, $\lambda = 1$, we get the following inequality involving classical Mittag–Leffler function

$$\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_2+E_\alpha(\varrho_1-\varrho_2))}{2}-\frac{1}{E_\alpha(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+E_\alpha(\varrho_1-\varrho_2)}\Psi(x)dx\right|$$
$$\leq \frac{E_\alpha(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}\right)^{\frac{1}{q}}$$
$$+\frac{E_\alpha(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}\frac{s(s+3)}{2(s+1)(s+2)}+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}\frac{s}{2(s+2)}\right)^{\frac{1}{q}}.$$

Theorem 13. *Suppose I° is an generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$ and $\Psi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\varrho_1, \varrho_2 \in I^\circ$ with $\varrho_1 < \varrho_2, q \geq 1$ and suppose that $\Psi' \in L[\varrho_1, \varrho_2]$. If $|\Psi'|^q$ is an n–poly generalized convex function of Raina type on $L[\varrho_1, \varrho_2]$, then*

$$\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))}{2}-\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\Psi(x)dx\right|$$
$$\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}k_1(s)+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}k_2(s)\right)^{\frac{1}{q}}$$
$$+\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}k_2(s)+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}k_1(s)\right)^{\frac{1}{q}},$$

holds for $\kappa \in [0, 1]$, where

$$k_1(s)=\int_0^1(1-\kappa)|1-2\kappa|[1-(1-\kappa)^s]d\kappa=\int_0^1\kappa|1-2\kappa|[1-\kappa^s]d\kappa=\frac{(s^2+s+2)2^s-2}{2^{s+2}(s+2)(s+3)},$$
$$k_2(s)=\int_0^1\kappa|1-2\kappa|[1-(1-\kappa)^s]d\kappa=\int_0^1(1-\kappa)|1-2\kappa|[1-\kappa^s]d\kappa=\frac{(s+5)(s^2+s+2)2^s-2}{2^{s+2}(s+1)(s+2)(s+3)}.$$

Proof. Suppose that $\varrho_1, \varrho_2 \in I^\circ$. Since I° is a generalized convex set with respect to $\mathcal{F}^{\sigma}_{\rho,\lambda}$, for any $\kappa \in [0, 1]$, we have $\varrho_2 + \kappa\, \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2) \in I^\circ$.

Assume that $q > 1$. From Lemma 1, improved power-mean integral inequality, n–poly generalized convex function of Raina type of $|\Psi'|^q$ and properties of modulus, we have

$$\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))}{2}-\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\Psi(x)dx\right|$$
$$\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\int_0^1|1-2\kappa||\Psi'(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))|d\kappa$$
$$\leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\int_0^1(1-\kappa)|1-2\kappa|d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1(1-\kappa)|1-2\kappa||\Psi'(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))|^q d\kappa\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&+\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\int_0^1(\kappa|1-2\kappa|d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1\kappa|1-2\kappa||\Psi'(\varrho_2+\kappa\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))|^q d\kappa\right)^{\frac{1}{q}}\\
&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\\
&\times\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}\int_0^1(1-\kappa)|1-2\kappa|[1-(1-\kappa)^s]d\kappa+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}\int_0^1(1-\kappa)|1-2\kappa|[1-\kappa^s]d\kappa\right)^{\frac{1}{q}}\\
&+\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}\int_0^1\kappa|1-2\kappa|[1-(1-\kappa)^s]d\kappa+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}\int_0^1\kappa|1-2\kappa|[1-\kappa^s]d\kappa\right)^{\frac{1}{q}}\\
&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}\frac{(s^2+s+2)2^s-2}{2^{s+2}(s+2)(s+3)}+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}\frac{(s+5)[(s^2+s+2)2^s-2]}{2^{s+2}(s+1)(s+2)(s+3)}\right)^{\frac{1}{q}}\\
&+\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}\frac{(s+5)[(s^2+s+2)2^s-2]}{2^{s+2}(s+1)(s+2)(s+3)}+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}\frac{(s^2+s+2)2^s-2}{2^{s+2}(s+2)(s+3)}\right)^{\frac{1}{q}}\\
&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}k_1(s)+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}k_2(s)\right)^{\frac{1}{q}}\\
&+\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}k_2(s)+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}k_1(s)\right)^{\frac{1}{q}}.
\end{aligned}
$$

For $q=1$, we use the estimates from the proof of Theorem 9, which also follow step by step the above estimates. This completes the proof. □

Corollary 15. *If we put $n=1$ in Theorem 13, then*

$$
\begin{aligned}
&\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2))}{2}-\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}\Psi(x)dx\right|\\
&\leq\frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)}{8}\left[\left(\frac{|\Psi'(\varrho_1)|^q}{4}+\frac{3|\Psi'(\varrho_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|\Psi'(\varrho_1)|^q}{4}+\frac{|\Psi'(\varrho_2)|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Corollary 16. *If we put $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)=\varrho_1-\varrho_2$ in Theorem 13, we get inequality (4.5) in [9].*

Corollary 17. *If we put $n=1$ and $\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1-\varrho_2)=\varrho_1-\varrho_2$ in Theorem 13, then we get Corollary 6 in [9].*

Remark 11. Under the assumption of Theorem 13, if we take $\sigma=(1,1,...)$ with $\rho=\alpha$, $\lambda=1$, we get the following inequality involving classical Mittag–Leffler function

$$
\begin{aligned}
&\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_2+E_\alpha(\varrho_1-\varrho_2))}{2}-\frac{1}{E_\alpha(\varrho_1-\varrho_2)}\int_{\varrho_2}^{\varrho_2+E_\alpha(\varrho_1-\varrho_2)}\Psi(x)dx\right|\\
&\leq\frac{E_\alpha(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}k_1(s)+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}k_2(s)\right)^{\frac{1}{q}}\\
&+\frac{E_\alpha(\varrho_1-\varrho_2)}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|\Psi'(\varrho_1)|^q}{n}\sum_{s=1}^{n}k_2(s)+\frac{|\Psi'(\varrho_2)|^q}{n}\sum_{s=1}^{n}k_1(s)\right)^{\frac{1}{q}}.
\end{aligned}
$$

6. Applications

In this section, we recall the following special means of two positive number ϱ_1,ϱ_2 with $\varrho_1<\varrho_2$:

1. The arithmetic mean
$$A=A(\varrho_1,\varrho_2)=\frac{\varrho_1+\varrho_2}{2}.$$
2. The geometric mean
$$G=G(\varrho_1,\varrho_2)=\sqrt{\varrho_1\varrho_2}.$$

3. The harmonic mean
$$H = H(\varrho_1, \varrho_2) = \frac{2\varrho_1\varrho_2}{\varrho_1 + \varrho_2}.$$

Proposition 1. *Let* $0 < \varrho_1 < \varrho_2$, *then*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)(\varrho_2 + \frac{1}{2}\,\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2)) \leq \frac{\mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2) + 2\varrho_2}{2} \leq A(\varrho_1, \varrho_2)\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}. \tag{10}$$

Proof. If $\Psi(\varrho) = \varrho$ for $\varrho > 0$ in above Theorem 8, then we obtained the inequality (10). □

Remark 12. Under the assumption of Proposition 1, if we take $\sigma = (1, 1, ...)$ with $\rho = \alpha,\ \lambda = 1$, we get the following inequality involving classical Mittag–Leffler function

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)(\varrho_2 + \frac{1}{2}E_{\alpha}(\varrho_1 - \varrho_2)) \leq \frac{E_{\alpha}(\varrho_1 - \varrho_2) + 2\varrho_2}{2} \leq A(\varrho_1, \varrho_2)\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}.$$

Proposition 2. *Let* $\varrho_1, \varrho_2 \in (0, 1]$ *with* $\varrho_1 < \varrho_2$, *then*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\ln G(\varrho_1, \varrho_2) \leq \frac{1}{\varrho_2(\varrho_2 + \ \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))} \leq \ln(\varrho_2 + \frac{1}{2}\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}. \tag{11}$$

Proof. If $\Psi(\varrho) = -\ln\varrho$ for $x \in (0, 1]$ in above Theorem 8, then we obtained the inequality (11). □

Remark 13. Under the assumption of Proposition 2, if we take $\sigma = (1, 1, ...)$ with $\rho = \alpha,\ \lambda = 1$, we get the following inequality involving classical Mittag–Leffler function

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\ln G(\varrho_1, \varrho_2) \leq \frac{1}{\varrho_2(\varrho_2 + E_{\alpha}(\varrho_1 - \varrho_2))} \leq \ln(\varrho_2 + \frac{1}{2}E_{\alpha}(\varrho_1 - \varrho_2))\frac{1}{n}\sum_{s=1}^{n}\frac{s}{s+1}.$$

Proposition 3. *Let* $0 < \varrho_1 < \varrho_2$, *then*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\frac{1}{(\varrho_2 + \frac{1}{2}\ \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))^2} \leq \frac{1}{\varrho_2(\varrho_2 + \ \mathcal{F}^{\sigma}_{\rho,\lambda}(\varrho_1 - \varrho_2))} \leq \frac{1}{H(\varrho_1^2, \varrho_2^2)}\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}. \tag{12}$$

Proof. If $\Psi(\varrho) = \frac{1}{\varrho^2}$ for $\varrho \in \mathbb{R} \setminus \{0\}$ in above Theorem 8, then we obtained the inequality (12). □

Remark 14. Under the assumption of Proposition 3, if we take $\sigma = (1, 1, ...)$ with $\rho = \alpha,\ \lambda = 1$, we get the following inequality involving classical Mittag–Leffler function

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)\frac{1}{(\varrho_2 + \frac{1}{2}E_{\alpha}(\varrho_1 - \varrho_2))^2} \leq \frac{1}{\varrho_2(\varrho_2 + E_{\alpha}(\varrho_1 - \varrho_2))} \leq \frac{1}{H(\varrho_1^2, \varrho_2^2)}\frac{2}{n}\sum_{s=1}^{n}\frac{s}{s+1}.$$

7. Conclusion

In this paper, we have introduced a new family of convex functions namely n–poly generalized convex functions of Raina type. We established a new version of Hermite–Hadamard type inequality and some of its refinements. We believe that this new family of convex functions will have very immeasurable and chasmic research in this mesmerizing and absorbing field of inequalities and will inspire interested readers.

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