

# **Interaction of Mechanics and Mathematics**

## **Series editor**

Lev Truskinovsky, Laboratoire de Mechanique des Solid, Palaiseau, France  
e-mail: [trusk@lms.polytechnique.fr](mailto:trusk@lms.polytechnique.fr)

### *About this Series*

The Interaction of Mechanics and Mathematics (IMM) series publishes advanced textbooks and introductory scientific monographs devoted to modern research in the wide area of mechanics. The authors are distinguished specialists with international reputation in their field of expertise. The books are intended to serve as modern guides in their fields and anticipated to be accessible to advanced graduate students. IMM books are planned to be comprehensive reviews developed to the cutting edge of their respective field and to list the major references.

### *Advisory Board*

D. Colton, USA  
R. Knops, UK  
G. DelPiero, Italy  
Z. Mroz, Poland  
M. Slemrod, USA  
S. Seelecke, USA  
L. Truskinovsky, France

IMM is promoted under the auspices of ISIMM (International Society for the Interaction of Mechanics and Mathematics).

More information about this series at <http://www.springer.com/series/5395>

Taeyoung Lee • Melvin Leok  
N. Harris McClamroch


# Global Formulations of Lagrangian and Hamiltonian Dynamics on Manifolds


A Geometric Approach to Modeling  
and Analysis



Springer

Taeyoung Lee   
The George Washington University  
Washington, District of Columbia, USA

Melvin Leok   
Department of Mathematics  
University of California, San Diego  
La Jolla, California, USA

N. Harris McClamroch   
Department of Aerospace Engineering  
The University of Michigan  
Ann Arbor, Michigan, USA

ISSN 1860-6245                      ISSN 1860-6253 (electronic)  
Interaction of Mechanics and Mathematics  
ISBN 978-3-319-56951-2            ISBN 978-3-319-56953-6 (eBook)  
DOI 10.1007/978-3-319-56953-6

Library of Congress Control Number: 2017938585

Mathematics Subject Classification (2010): 70-XX, 70-02, 70Exx, 70Gxx, 70Hxx

© Springer International Publishing AG 2018

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Printed on acid-free paper

This Springer imprint is published by Springer Nature  
The registered company is Springer International Publishing AG  
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

# Preface

This book is a tutorial on foundational geometric principles of Lagrangian and Hamiltonian dynamics and their application in studying important physical systems. As the title indicates, the emphasis is on describing Lagrangian and Hamiltonian dynamics in a form that enables global formulations and, where suitable mathematical tools are available, global analysis of dynamical properties. This emphasis on global descriptions, that is, descriptions that hold everywhere on the configuration manifold, as a means of determining global dynamical properties is in marked contrast to the most common approach in the literature on Lagrangian and Hamiltonian dynamics that makes use of local coordinates on the configuration manifold, thereby resulting in formulations that are typically limited to a small open subset of the configuration manifold. In this sense, the material that we introduce and develop represents a *significant conceptual departure* from the traditional methods of studying Lagrangian and Hamiltonian dynamics.

More specifically, this book differs from most of the traditional studies of analytical mechanics on Euclidean spaces, such as [13, 75]. Moreover, the global formulation of mechanics presented in this book should be distinguished from the geometric treatments that appear in [1, 10, 16, 25, 27, 37, 38, 39, 69, 70], which explicitly make use of local coordinates when illustrating the abstract formulation through specific examples. In contrast, we directly use the representations in the embedding space of the configuration manifold, without resorting to an atlas of coordinate charts. This allows us to obtain equations of motion that are globally valid and do not require changes of coordinates. This is particularly useful in constructing a compact and elegant form of Lagrangian and Hamiltonian mechanics for complex dynamical systems without algebraic constraints or coordinate singularities. This treatment is novel and unique, and it is the most important distinction and contribution of this monograph to the existing literature.

This book is the result of a research collaboration that began in 2005, when the first author initiated his doctoral research at the University of Michigan with the other two authors as his graduate advisers. That research program led to the completion of his doctoral degree and to numerous conference and journal publications.

The research plan, initiated in 2005, was based on our belief that there were advantages to be gained by the formulation, analysis, and computation of Lagrangian or Hamiltonian dynamics by explicitly viewing configurations of the system as elements of a manifold embedded in a finite-dimensional vector space. This viewpoint was not new in 2005, but we believed that the potential of this perspective had not been fully exploited in the research literature available at that time. This led us to embark on a long-term research program that would make use of powerful methods of variational calculus, differential geometry, and Lie groups for studying the dynamics of Lagrangian and Hamiltonian systems. Our subsequent research since 2005 confirms that there are important practical benefits to be gained by this perspective, especially for multi-body and other mechanical systems with dynamics that evolve in three dimensions.

This book arose from our research and the resulting publications in [21], [46, 47], and [49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 61, 62, 63] since 2005, but it goes substantially beyond this earlier work. During the writing of this book, we were motivated to consider many new issues that we had not previously studied; in this sense, all of Chapter 4 is new material. We also had many new insights and obtained new results that have not been previously published. Nevertheless, this book is intended to be a self-contained treatment containing many of the results of those publications plus new tutorial material to provide a unifying framework for Lagrangian and Hamiltonian dynamics on a manifold. As our research has progressed, we have come to realize the practical importance and effectiveness of this geometric perspective.

This book is not a complete treatment of Lagrangian and Hamiltonian dynamics; many important topics, such as geometric reduction, canonical transformations, Hamilton–Jacobi theory, Poisson geometry, and nonholonomic constraints, are not treated. These subjects are nicely covered in many excellent books [10, 37, 38, 39, 70]. All of these developments, as well as the development in this book, treat Lagrangian and Hamiltonian dynamics that are smooth in the sense that they can be described by differentiable vector fields. We note the important literature, summarized in [15], that treats non-smooth Lagrangian and Hamiltonian dynamics. A complete development of these topics, within the global geometric framework proposed in this book, remains to be accomplished.

The following manifolds, which naturally arise as configuration manifolds for Lagrangian and Hamiltonian systems, are of primary importance in our subsequent development. The standard linear vector spaces of two- and three-dimensional vectors are denoted by  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , endowed with the usual dot product operation; the cross product operation is also fundamental in  $\mathbb{R}^3$ . As

usual,  $\mathbb{R}^n$  denotes the linear space of ordered real  $n$ -tuples. All translations of subspaces in  $\mathbb{R}^n$ , e.g., lines, planes, and hyperplanes, are examples of embedded manifolds. The unit sphere in two dimensions is denoted by  $S^1$ ; it is a one-dimensional manifold embedded in  $\mathbb{R}^2$ ; similarly, the unit sphere in three dimensions is denoted by  $S^2$ ; it is a two-dimensional manifold embedded in  $\mathbb{R}^3$ . The Lie group of orthogonal transformations in three dimensions is denoted by  $SO(3)$ . The Lie group of homogeneous transformations in three dimensions is denoted by  $SE(3)$ . Each of these Lie groups has an additional structure based on a group operation, which in each case corresponds to matrix multiplication. Finally, products of the above manifolds also commonly arise as configuration manifolds.

All of the manifolds that we consider are embedded in a finite-dimensional vector space. Hence, the geometry of these manifolds can be described using mathematical tools and operations in the embedding vector space. Although we are only interested in Lagrangian and Hamiltonian dynamics that evolve on such an embedded manifold, it is sometimes convenient to extend the dynamics to the embedding vector space. In fact, most of the results in the subsequent chapters can be viewed from this perspective.

It is important to justify our geometric assumption that the configurations constitute a manifold for Lagrangian and Hamiltonian systems. First, manifolds can be used to encode certain types of important motion constraints that arise in many mechanical systems; such constraints may arise from restrictions on the allowed motion due to physical restrictions. A formulation in terms of manifolds is a direct encoding of the constraints and does not require the use of additional holonomic constraints and associated Lagrange multipliers. Second, there is a beautiful theory of embedded manifolds, including Lie group manifolds, that can be brought to bear on the development of geometric mechanics in this context. It is important to recognize that configurations, as elements in a manifold, may often be described and analyzed in a globally valid way that does not require the use of local charts, coordinates, or parameters that may lead to singularities or ambiguities in the representation. We make extensive use of Euclidean frames in  $\mathbb{R}^3$  and associated Euclidean coordinates in  $\mathbb{R}^3$ ,  $\mathbb{R}^n$ , and  $\mathbb{R}^{n \times n}$ , but we do *not* use coordinates to describe the configuration manifolds. In this sense, this geometric formulation is said to be *coordinate-free*. Third, this geometric formulation turns out to be an efficient way to formulate, analyze, and compute the kinematics, dynamics, and their temporal evolution on the configuration manifold. This representational efficiency has a major practical advantage for many complex dynamical systems that has not been widely appreciated by the applied scientific and engineering communities. The associated cost of this efficiency is the requirement to make use of the well-developed mathematical machinery of manifolds, calculus on manifolds, and Lie groups.

We study dynamical systems that can be viewed as Lagrangian or Hamiltonian systems. Under appropriate assumptions, such dynamical systems are conservative in the sense that the Hamiltonian, which oftentimes coincides

with the total energy of the system, is conserved. This is an ideal assumption but a very useful one in many applications. Although our main attention is given to dynamical systems that are conservative, many of the results can be extended to dissipative dynamical systems and to dynamical systems with inputs.

There are two basic requirements to make use of the Lagrangian perspective in obtaining the equations of motion. Based on the physical properties of the dynamical system, it is first necessary to select the set of possible configurations of the system and to identify the set of all configurations  $M$  as a manifold. The second requirement is to develop a Lagrangian function  $L : TM \rightarrow \mathbb{R}^1$  which is a real-valued function defined on the tangent bundle  $TM$  of the configuration manifold and satisfying certain assumptions. The Lagrangian function is the difference of the kinetic energy of the system and the potential energy of the system. It is assumed that the reader has sufficient background to construct the kinetic energy function and the potential energy function; we do not go into detail on the basic physics to construct these energy functions. Rather, numerous specific examples of Lagrangian and Hamiltonian systems are introduced and used to illustrate the concepts.

*Hamilton's variational principle* is the fundamental basis for the theory of Lagrangian and Hamiltonian dynamics. The action integral is the integral of the Lagrangian function over a fixed time period. Along a motion of the system, a specific value of the action integral is induced. Small variations of the system motion, which are consistent with the configuration manifold but not necessarily possible motions of the system, induce variations in the value of the action integral. Hamilton's variational principle states that these variations in the value of the action integral are necessarily of higher than first order for arbitrarily small variations about any system motion. In other words, the directional or Gateaux derivative of the action integral vanishes for all allowable variations of the system motion. Using methods of variational calculus where variations are introduced in terms of a small scalar parameter, this principle leads to Euler–Lagrange equations which characterize all possible system motions.

Hamilton's equations of motion are obtained by introducing the Legendre transformation that is a mapping from the tangent bundle of the configuration manifold to the cotangent bundle of the configuration manifold. A Hamiltonian function is introduced, and Hamilton's equations are obtained using a phase space version of Hamilton's variational principle. Methods of variational calculus are used to express the dynamics on the cotangent bundle of the configuration manifold.

It is admitted that some of the derivations are lengthy and the details and formulas are sometimes complicated. However, most of the formulations of Lagrangian and Hamiltonian dynamics on specific configuration manifolds, considered in this book, are relatively simple and elegant. Consequently, their application to the formulation of the dynamics of mass particles, rigid bodies, deformable bodies, and multi-body systems follows a relatively straight-



forward pattern that is, in fact, both more general and simpler than the traditional formulations that make use of local coordinates.

This book presents a unifying framework for this geometric perspective that we intend to be accessible to a wide audience. In concrete terms, the book is intended to achieve the following objectives:

- Study the geometric formulations of dynamical equations of motion for Lagrangian and Hamiltonian systems that evolve on a configuration manifold using variational methods.
- Express theoretical results in a global geometric form that does not require local charts or coordinates for the configuration manifold.
- Demonstrate simple methods for the analysis of solution properties.
- Present numerous illustrations of theory and analysis for the dynamics of multiple interacting particles and of rigid and deformable bodies.
- Identify theoretical and analytical benefits to be gained by the proposed treatment of geometric mechanics.

The book is also intended to set the stage for a treatment of computational issues associated with Lagrangian and Hamiltonian dynamics that evolve on a configuration manifold. In particular, the material in this book can be extended to obtain a framework for computational aspects of Lagrangian and Hamiltonian dynamics that achieve the analogous objectives:

- Study the geometric formulations of discrete-time dynamical equations of motion for Lagrangian and Hamiltonian systems that evolve on an embedded configuration manifold using discrete-time variational methods.
- Develop discrete-time versions of Lagrangian and Hamiltonian dynamics; these are referred to as geometric variational integrators to reflect the configuration manifold for the problems considered.
- Demonstrate the benefits of these discrete-time dynamics as a computational approximation of the continuous-time Lagrangian or Hamiltonian dynamics.
- Express computational dynamics in a global geometric form that does not require local charts.
- Present numerous computational illustrations for the dynamics of multiple interacting particles, and of rigid and deformable bodies.
- Identify computational benefits to be gained by the proposed treatment of geometric mechanics.

Computational developments for Lagrangian and Hamiltonian dynamics, following the above prescription, lead to computational algorithms that are not based on the discretization of differential equations on a manifold, but are based on the discretization of variational principles on a manifold. The above computational approach has been developed in [46, 50, 51, 54]. A symbolic approach to obtaining differential equations on a manifold has been proposed in [9], without addressing computational issues.

This book is written for a general audience of mathematicians, engineers, and physicists who have a basic knowledge of classical Lagrangian and Hamiltonian dynamics. Some background in differential geometry would be helpful to the reader, but it is not essential as arguments in the book make primary use of basic differential geometric concepts that are introduced in the book. Hence, our hope is that the material in this book is accessible to a wide range of readers.

In this book, Chapter 1 provides a summary of mathematical material required for the subsequent development; in particular, manifolds and Lie groups are introduced. Chapter 2 then introduces kinematics relationships for ideal particles, rigid bodies, multi-bodies, and deformable bodies, expressed in terms of differential equations that evolve on a configuration manifold.

Chapter 3 treats the classical approach to variational mechanics where the configurations lie in an open set of a vector space  $\mathbb{R}^n$ . This is standard material, but the presentation provides a development that is followed in subsequent chapters. Chapters 4 and 5 develop the fundamental results for Lagrangian and Hamiltonian dynamics when the configuration manifold  $(S^1)^n$  is the product of  $n$  copies of the one-sphere in  $\mathbb{R}^2$  (in Chapter 4) and the configuration manifold  $(S^2)^n$  is the product of  $n$  copies of the two-sphere in  $\mathbb{R}^3$  (Chapter 5). The geometries of these two configuration manifolds are exploited in the developments, especially the definitions of variations. Chapter 6 introduces the geometric approach for rigid body rotation in three dimensions using configurations in the Lie group  $\text{SO}(3)$ . The development follows Chapter 3, Chapter 4, and Chapter 5, except that the variations are carefully defined to be consistent with the Lie group structure of  $\text{SO}(3)$ . Chapter 7 introduces the geometric approach for rigid body rotation and translation in three dimensions using configurations in the Lie group  $\text{SE}(3)$ . The development reflects the fact that the variations are defined to be consistent with the Lie group structure of  $\text{SE}(3)$ . The results in Chapters 3–7 are developed using only well-known results from linear algebra and elementary properties of orthogonal matrices and skew-symmetric matrices; minimal knowledge of differential geometry or Lie groups is required, and all of it is introduced in the book.

Chapter 8 makes use of the notation and formalism of differential geometry and Lie groups. This mathematical machinery enables the development of Lagrangian and Hamiltonian dynamics with configurations that lie in an arbitrary differentiable manifold, an arbitrary matrix Lie group, or an arbitrary homogeneous manifold (a manifold that is transitive with respect to a Lie group action). The power of this mathematical formalism is that it allows a relatively straightforward development that follows the variational calculus approach of the previous chapters and it results in a simple abstract statement of the results in each case. The development, however, does require a level of abstraction and some knowledge of differential geometry and Lie groups.

Chapter 9 makes use of the prior results to treat the dynamics of various multi-body systems; Chapter 10 treats the dynamics of various deformable multi-body systems. In each of these example illustrations, the equations of motion are obtained in several different forms. The equations of motion are used to study conservation properties and equilibrium properties in each example illustration. The book concludes with two appendices that provide brief summaries of fundamental lemmas of the calculus of variations and procedures for linearization of a vector field on a manifold in a neighborhood of an equilibrium solution.

Numerous examples of mechanical and multi-body systems are developed in the text and introduced in the end of chapter problems. These examples form a core part of the book, since they illustrate the way in which the developed theory can be applied in practice. Many of these examples are classical, and they are studied in the existing literature using local coordinates; some of the examples are apparently novel. Various multi-body examples, involving pendulums, are introduced, since these provide good illustrations for the theory. The books [6, 29] include many examples developed using local coordinates.

This book could form the basis for a graduate-level course in applied mathematics, classical physics, or engineering. For students with some prior background in differential geometry, a course could begin with the theoretical material in Chapter 8 and then cover applications in Chapters 3–7 and 9–10 as time permits. For students with primary interest in the applications, the course could treat the topics in the order presented in the book, covering the theoretical topics in Chapter 8 as time permits. This book is also intended for self-study; these two paths through the material in the book may aid readers in this category.

In conclusion, the authors are excited to share our perspective on “global formulations of Lagrangian and Hamiltonian dynamics on manifolds” with a wide audience. We welcome feedback about theoretical issues the book introduces, the practical value of the proposed perspective, and indeed any aspect of this book.

TAEYOUNG LEE  
Washington, DC

MELVIN LEOK  
La Jolla, CA

N. HARRIS MCCLAMROCH  
Ann Arbor, MI

January, 2017

# Acknowledgments

This book provides our perspective on geometric mechanics, and it depends on the foundations provided by many scholars and researchers. Although this book is somewhat unique in its geometric perspective, we recognize the long history of Lagrangian and Hamiltonian dynamics and the many persons over the last two centuries who have contributed to its current status as a central subject area in mathematics, physics, and engineering.

The three authors would like to acknowledge several persons who have had great influence on our understanding of geometric mechanics. Our professional colleagues Tony Bloch, Dennis Bernstein, Jessy Grizzle, John Baillieul, Naomi Leonard, Francesco Bullo, and P. S. Krishnaprasad have had significant influence through their outstanding research and personal interactions.

TL wishes to thank Harris McClamroch, Melvin Leok, and Youdan Kim for their contributions to his inspiration and knowledge and especially for showing how to enjoy life in academia while being respected by peers and students. He personally dedicates this book to his wife Rachel with love, and he wishes that this book would help his two-year-old son Nathan soon to understand that the trains that he loves most wouldn't have been built without math.

ML dedicates this book to his wife Lina, for her love, support, and patience; to his children Richard and Catherine, for constantly reminding him that the world is full of curious and wonderful things; and to the memory of his adviser Jerry Marsden, for teaching him by example that mathematics at its best is both beautiful and useful.

Finally, NHM would like to acknowledge the contributions that his doctoral students Danwei Wang, Mahmut Reyhanoglu, Hariharan Krishnan, Ilya Kolmanovsky, Chunlei Rui, Sangbum Cho, Jinglai Shen, Amit Sanyal, Nalin Chaturvedi, and Taeyoung Lee have made to his understanding of geometric mechanics. His doctoral adviser, Lyle Clark at the University of Texas in the

1960s, did not know anything of the modern treatment of geometric mechanics, but his insight into and passion for analytical mechanics had a profound influence.

All three of us have been inspired by the seminal contributions of Professor Jerry Marsden through his publications and his personal example. His deep mathematical and physical insights and the clarity of his written publications and his oral presentations have inspired us and motivated us in the writing of this book.

The authors gratefully acknowledge support from the National Science Foundation.

TL  
ML  
NHM

# Contents

<b>1</b>	<b>Mathematical Background</b>	<b>1</b>
1.1	Vectors and Matrices	1
1.1.1	Vector Spaces	2
1.1.2	Symmetric and Skew-Symmetric Matrices	4
1.1.3	Vector Operations in $\mathbb{R}^2$	5
1.1.4	Vector Operations in $\mathbb{R}^3$	5
1.1.5	Orthogonal Matrices on $\mathbb{R}^3$	7
1.1.6	Homogeneous Matrices as Actions on $\mathbb{R}^3$	8
1.1.7	Identities Involving Vectors, Orthogonal Matrices, and Skew-Symmetric Matrices	10
1.1.8	Derivative Functions	11
1.2	Manifold Concepts	11
1.2.1	Manifolds	12
1.2.2	Tangent Vectors, Tangent Spaces and Tangent Bundles	13
1.2.3	Cotangent Vectors, Cotangent Spaces, and Cotangent Bundles	14
1.2.4	Intersections and Products of Manifolds	15
1.2.5	Examples of Manifolds, Tangent Bundles, and Cotangent Bundles	16
1.2.6	Lie Groups and Lie Algebras	25
1.2.7	Homogeneous Manifolds	26
1.3	Vector Fields on a Manifold	27
1.3.1	Vector Fields on a Manifold that Arise from Differential Equations	28
1.3.2	Vector Fields on a Manifold that Arise from Differential-Algebraic Equations	29
1.3.3	Linearized Vector Fields	32
1.3.4	Stability of an Equilibrium	33

1.3.5	Examples of Vector Fields .....	34
1.3.6	Geometric Integrators .....	42
1.4	Covector Fields on a Manifold .....	43
1.5	Problems .....	43
<b>2</b>	<b>Kinematics .....</b>	<b>53</b>
2.1	Multi-Body Systems .....	53
2.2	Euclidean Frames .....	54
2.3	Kinematics of Ideal Mass Particles .....	55
2.4	Rigid Body Kinematics .....	56
2.5	Kinematics of Deformable Bodies .....	57
2.6	Kinematics on a Manifold .....	58
2.7	Kinematics as Descriptions of Velocity Relationships .....	59
2.7.1	Translational Kinematics of a Particle on an Inclined Plane .....	59
2.7.2	Translational Kinematics of a Particle on a Hyperbolic Paraboloid .....	60
2.7.3	Rotational Kinematics of a Planar Pendulum .....	62
2.7.4	Rotational Kinematics of a Spherical Pendulum .....	64
2.7.5	Rotational Kinematics of a Double Planar Pendulum .....	65
2.7.6	Rotational Kinematics of a Double Spherical Pendulum .....	67
2.7.7	Rotational Kinematics of a Planar Pendulum Connected to a Spherical Pendulum .....	68
2.7.8	Kinematics of a Particle on a Torus .....	70
2.7.9	Rotational Kinematics of a Free Rigid Body .....	73
2.7.10	Rotational and Translational Kinematics of a Rigid Body Constrained to a Fixed Plane .....	75
2.7.11	Rotational and Translational Kinematics of a Free Rigid Body .....	76
2.7.12	Translational Kinematics of a Rigid Link with Ends Constrained to Slide Along a Straight Line and a Circle in a Fixed Plane .....	78
2.7.13	Rotational and Translational Kinematics of a Constrained Rigid Rod .....	80
2.8	Problems .....	82
<b>3</b>	<b>Classical Lagrangian and Hamiltonian Dynamics .....</b>	<b>89</b>
3.1	Configurations as Elements in $\mathbb{R}^n$ .....	89
3.2	Lagrangian Dynamics on $\mathbb{R}^n$ .....	90
3.2.1	Lagrangian Function .....	90
3.2.2	Variations on $\mathbb{R}^n$ .....	90
3.2.3	Hamilton's Variational Principle .....	91
3.2.4	Euler-Lagrange Equations .....	92

3.3	Hamiltonian Dynamics on $\mathbb{R}^n$ . . . . .	94
3.3.1	Legendre Transformation and the Hamiltonian . . . . .	95
3.3.2	Hamilton's Equations and Euler–Lagrange Equations . . . . .	95
3.3.3	Hamilton's Phase Space Variational Principle . . . . .	97
3.3.4	Hamilton's Equations . . . . .	98
3.4	Flow Properties of Lagrangian and Hamiltonian Dynamics . . . . .	99
3.4.1	Energy Properties . . . . .	100
3.4.2	Cyclic Coordinates, Conserved Quantities, and Classical Reduction . . . . .	101
3.4.3	Symplectic Property . . . . .	104
3.5	Lagrangian and Hamiltonian Dynamics with Holonomic Constraints . . . . .	105
3.6	Lagrange–d'Alembert Principle . . . . .	108
3.7	Classical Particle Dynamics . . . . .	110
3.7.1	Dynamics of a Particle in Uniform, Constant Gravity . . . . .	110
3.7.2	Dynamics of a Particle, Constrained to an Inclined Plane, in Uniform, Constant Gravity . . . . .	112
3.7.3	Dynamics of a Particle, Constrained to a Hyperbolic Paraboloid, in Uniform, Constant Gravity . . . . .	115
3.7.4	Keplerian Dynamics of a Particle in Orbit . . . . .	117
3.7.5	Dynamics of a Particle Expressed in a Rotating Euclidean Frame . . . . .	120
3.8	Problems . . . . .	123
<b>4</b>	<b>Lagrangian and Hamiltonian Dynamics on <math>(S^1)^n</math> . . . . .</b>	<b>131</b>
4.1	Configurations as Elements in $(S^1)^n$ . . . . .	131
4.2	Kinematics on $(S^1)^n$ . . . . .	132
4.3	Lagrangian Dynamics on $(S^1)^n$ . . . . .	133
4.3.1	Hamilton's Variational Principle in Terms of $(q, \dot{q})$ . . . . .	133
4.3.2	Euler–Lagrange Equations in Terms of $(q, \dot{q})$ . . . . .	135
4.3.3	Hamilton's Variational Principle in Terms of $(q, \omega)$ . . . . .	138
4.3.4	Euler–Lagrange Equations in Terms of $(q, \omega)$ . . . . .	139
4.4	Hamiltonian Dynamics on $(S^1)^n$ . . . . .	142
4.4.1	Hamilton's Phase Space Variational Principle in Terms of $(q, \mu)$ . . . . .	142
4.4.2	Hamilton's Equations in Terms of $(q, \mu)$ . . . . .	144
4.4.3	Hamilton's Phase Space Variational Principle in Terms of $(q, \pi)$ . . . . .	148
4.4.4	Hamilton's Equations in Terms of $(q, \pi)$ . . . . .	149
4.5	Linear Approximations of Dynamics on $(S^1)^n$ . . . . .	151
4.6	Dynamics of Systems on $(S^1)^n$ . . . . .	152
4.6.1	Dynamics of a Planar Pendulum . . . . .	152



4.6.2	Dynamics of a Particle Constrained to a Circular Hoop That Rotates with Constant Angular Velocity ..	157
4.6.3	Dynamics of Two Elastically Connected Planar Pendulums .....	163
4.6.4	Dynamics of a Double Planar Pendulum .....	169
4.6.5	Dynamics of a Particle on a Torus .....	175
4.6.6	Dynamics of a Furuta Pendulum .....	181
4.6.7	Dynamics of a Three-Dimensional Revolute Joint Robot .....	187
4.7	Problems .....	195
<b>5</b>	<b>Lagrangian and Hamiltonian Dynamics on <math>(S^2)^n</math> .....</b>	<b>207</b>
5.1	Configurations as Elements in $(S^2)^n$ .....	207
5.2	Kinematics on $(S^2)^n$ .....	208
5.3	Lagrangian Dynamics on $(S^2)^n$ .....	209
5.3.1	Hamilton's Variational Principle in Terms of $(q, \dot{q})$ ....	209
5.3.2	Euler–Lagrange Equations Expressed in Terms of $(q, \dot{q})$ .....	212
5.3.3	Hamilton's Variational Principle in Terms of $(q, \omega)$ ...	214
5.3.4	Euler–Lagrange Equations in Terms of $(q, \omega)$ .....	216
5.4	Hamiltonian Dynamics on $(S^2)^n$ .....	222
5.4.1	Hamilton's Phase Space Variational Principle in Terms of $(q, \mu)$ .....	222
5.4.2	Hamilton's Equations in Terms of $(q, \mu)$ .....	224
5.4.3	Hamilton's Phase Space Variational Principle in Terms of $(q, \pi)$ .....	228
5.4.4	Hamilton's Equations in Terms of $(q, \pi)$ .....	229
5.5	Linear Approximations of Dynamics on $(S^2)^n$ .....	232
5.6	Dynamics on $(S^2)^n$ .....	233
5.6.1	Dynamics of a Spherical Pendulum .....	233
5.6.2	Dynamics of a Particle Constrained to a Sphere That Rotates with Constant Angular Velocity .....	238
5.6.3	Dynamics of a Spherical Pendulum Connected to Three Elastic Strings .....	243
5.6.4	Dynamics of Two Elastically Connected Spherical Pendulums .....	250
5.6.5	Dynamics of a Double Spherical Pendulum .....	255
5.7	Problems .....	261
<b>6</b>	<b>Lagrangian and Hamiltonian Dynamics on <math>SO(3)</math> .....</b>	<b>273</b>
6.1	Configurations as Elements in the Lie Group $SO(3)$ .....	274
6.2	Kinematics on $SO(3)$ .....	275
6.3	Lagrangian Dynamics on $SO(3)$ .....	276
6.3.1	Hamilton's Variational Principle .....	276

6.3.2	Euler–Lagrange Equations: General Form	278
6.3.3	Euler–Lagrange Equations: Quadratic Kinetic Energy	280
6.4	Hamiltonian Dynamics on $\text{SO}(3)$	281
6.4.1	Hamilton’s Phase Space Variational Principle	281
6.4.2	Hamilton’s Equations: General Form	282
6.4.3	Hamilton’s Equations: Quadratic Kinetic Energy	284
6.5	Linear Approximations of Dynamics on $\text{SO}(3)$	284
6.6	Dynamics on $\text{SO}(3)$	285
6.6.1	Dynamics of a Freely Rotating Rigid Body	285
6.6.2	Dynamics of a Three-Dimensional Pendulum	288
6.6.3	Dynamics of a Rotating Rigid Body in Orbit	293
6.6.4	Dynamics of a Rigid Body Planar Pendulum	298
6.7	Problems	304
<b>7</b>	<b>Lagrangian and Hamiltonian Dynamics on <math>\text{SE}(3)</math></b>	<b>313</b>
7.1	Configurations as Elements in the Lie Group $\text{SE}(3)$	313
7.2	Kinematics on $\text{SE}(3)$	314
7.3	Lagrangian Dynamics on $\text{SE}(3)$	316
7.3.1	Hamilton’s Variational Principle	316
7.3.2	Euler–Lagrange Equations: General Form	318
7.3.3	Euler–Lagrange Equations: Quadratic Kinetic Energy	320
7.4	Hamiltonian Dynamics on $\text{SE}(3)$	322
7.4.1	Hamilton’s Phase Space Variational Principle	322
7.4.2	Hamilton’s Equations: General Form	323
7.4.3	Hamilton’s Equations: Quadratic Kinetic Energy	325
7.5	Linear Approximations of Dynamics on $\text{SE}(3)$	327
7.6	Dynamics on $\text{SE}(3)$	327
7.6.1	Dynamics of a Rotating and Translating Rigid Body	327
7.6.2	Dynamics of an Elastically Supported Rigid Body	331
7.6.3	Dynamics of a Rotating and Translating Rigid Dumbbell Satellite in Orbit	336
7.7	Problems	339
<b>8</b>	<b>Lagrangian and Hamiltonian Dynamics on Manifolds</b>	<b>347</b>
8.1	Lagrangian Dynamics on a Manifold	348
8.1.1	Variations on the Tangent Bundle $TM$	348
8.1.2	Lagrangian Variational Conditions	349
8.1.3	Euler–Lagrange Equations on $TM$	351
8.1.4	Extension of the Lagrangian Vector Field from $TM$ to $T\mathbb{R}^n$	352
8.2	Hamiltonian Dynamics on a Manifold	353
8.2.1	Legendre Transformation and the Hamiltonian	353
8.2.2	Variations on the Cotangent Bundle $T^*M$	353
8.2.3	Hamilton’s Phase Space Variational Principle	354

8.2.4	Hamilton's Equations on $T^*M$ .....	354
8.2.5	Invariance of the Hamiltonian .....	357
8.2.6	Extension of the Hamiltonian Vector Field from $T^*M$ to $T^*\mathbb{R}^n$ .....	358
8.3	Lagrangian and Hamiltonian Dynamics on Products of Manifolds .....	359
8.3.1	Lagrangian and Hamiltonian Dynamics on a Product of Linear Manifolds .....	360
8.3.2	Lagrangian and Hamiltonian Dynamics on $(S^1)^n$ .....	362
8.3.3	Lagrangian and Hamiltonian Dynamics on $(S^2)^n$ .....	364
8.4	Lagrangian and Hamiltonian Dynamics Using Lagrange Multipliers .....	367
8.5	Lagrangian and Hamiltonian Dynamics on $SO(3)$ .....	369
8.6	Lagrangian and Hamiltonian Dynamics on a Lie Group .....	371
8.6.1	Additional Material on Lie Groups and Lie Algebras ..	371
8.6.2	Variations on a Lie Group .....	372
8.6.3	Euler–Lagrange Equations .....	373
8.6.4	Legendre Transformation and Hamilton's Equations ..	376
8.6.5	Hamilton's Phase Space Variational Principle .....	377
8.6.6	Reassessment of Results in the Prior Chapters .....	379
8.7	Lagrangian and Hamiltonian Dynamics on a Homogeneous Manifold .....	381
8.7.1	Additional Material on Homogeneous Manifolds .....	381
8.7.2	A Lifting Process .....	382
8.7.3	Euler–Lagrange Equations .....	382
8.7.4	Reassessment of Results in the Prior Chapters .....	384
8.8	Lagrange–d'Alembert Principle .....	387
8.9	Problems .....	388
<b>9</b>	<b>Rigid and Multi-Body Systems .....</b>	<b>399</b>
9.1	Dynamics of a Planar Mechanism .....	400
9.1.1	Euler–Lagrange Equations .....	402
9.1.2	Hamilton's Equations .....	403
9.1.3	Conservation Properties .....	404
9.1.4	Equilibrium Properties .....	404
9.2	Dynamics of a Horizontally Rotating Pendulum on a Cart ...	405
9.2.1	Euler–Lagrange Equations .....	405
9.2.2	Hamilton's Equations .....	407
9.2.3	Conservation Properties .....	408
9.2.4	Equilibrium Properties .....	408
9.3	Dynamics of a Connection of a Planar Pendulum and Spherical Pendulum .....	409
9.3.1	Euler–Lagrange Equations .....	410
9.3.2	Hamilton's Equations .....	412

9.3.3	Conservation Properties . . . . .	413
9.3.4	Equilibrium Properties . . . . .	413
9.4	Dynamics of a Spherical Pendulum on a Cart . . . . .	415
9.4.1	Euler–Lagrange Equations . . . . .	416
9.4.2	Hamilton’s Equations . . . . .	418
9.4.3	Conservation Properties . . . . .	419
9.4.4	Equilibrium Properties . . . . .	420
9.5	Dynamics of a Rotating Rigid Body with Appendage . . . . .	421
9.5.1	Euler–Lagrange Equations . . . . .	421
9.5.2	Hamilton’s Equations . . . . .	423
9.5.3	Conservation Properties . . . . .	424
9.5.4	Equilibrium Properties . . . . .	424
9.6	Dynamics of a Three-Dimensional Pendulum on a Cart . . . . .	425
9.6.1	Euler–Lagrange Equations . . . . .	426
9.6.2	Hamilton’s Equations . . . . .	428
9.6.3	Conservation Properties . . . . .	429
9.6.4	Equilibrium Properties . . . . .	430
9.7	Dynamics of Two Rigid Bodies Constrained to Have a Common Material Point . . . . .	430
9.7.1	Euler–Lagrange Equations . . . . .	431
9.7.2	Hamilton’s Equations . . . . .	433
9.7.3	Conservation Properties . . . . .	434
9.7.4	Equilibrium Properties . . . . .	435
9.8	Dynamics of a Rotating and Translating Rigid Body with an Appendage . . . . .	435
9.8.1	Euler–Lagrange Equations . . . . .	436
9.8.2	Hamilton’s Equations . . . . .	439
9.8.3	Conservation Properties . . . . .	440
9.8.4	Equilibrium Properties . . . . .	441
9.9	Dynamics of a Full Body System . . . . .	441
9.9.1	Euler–Lagrange Equations . . . . .	442
9.9.2	Hamilton’s Equations . . . . .	444
9.9.3	Conservation Properties . . . . .	445
9.9.4	Equilibrium Properties . . . . .	446
9.9.5	Relative Full Body Dynamics . . . . .	446
9.10	Dynamics of a Spacecraft with Reaction Wheel Assembly . . . . .	446
9.10.1	Euler–Lagrange Equations . . . . .	448
9.10.2	Hamilton’s Equations . . . . .	453
9.10.3	Conservation Properties . . . . .	454
9.10.4	Equilibrium Properties . . . . .	455
9.11	Dynamics of a Rotating Spacecraft and Control Moment Gyroscope . . . . .	456
9.11.1	Euler–Lagrange Equations . . . . .	457

9.11.2	Hamilton's Equations	462
9.11.3	Conservation Properties	463
9.12	Dynamics of Two Quad Rotors Transporting a Cable-Suspended Payload	464
9.12.1	Euler–Lagrange Equations	465
9.12.2	Hamilton's Equations	470
9.12.3	Conservation Properties	470
9.12.4	Equilibrium Properties	470
9.13	Problems	471
<b>10</b>	<b>Deformable Multi-Body Systems</b>	<b>485</b>
10.1	Infinite-Dimensional Multi-Body Systems	485
10.2	Dynamics of a Chain Pendulum	486
10.2.1	Euler–Lagrange Equations	487
10.2.2	Hamilton's Equations	489
10.2.3	Comments	490
10.2.4	Conservation Properties	491
10.2.5	Equilibrium Properties	491
10.3	Dynamics of a Chain Pendulum on a Cart	492
10.3.1	Euler–Lagrange Equations	493
10.3.2	Hamilton's Equations	496
10.3.3	Comments	498
10.3.4	Conservation Properties	499
10.3.5	Equilibrium Properties	499
10.4	Dynamics of a Free-Free Chain	501
10.4.1	Euler–Lagrange Equations	502
10.4.2	Hamilton's Equations	504
10.4.3	Conservation Properties	506
10.4.4	Equilibrium Properties	507
10.5	Dynamics of a Fixed-Free Elastic Rod	507
10.5.1	Euler–Lagrange Equations	508
10.5.2	Hamilton's Equations	511
10.5.3	Conservation Properties	512
10.5.4	Equilibrium Properties	513
10.6	Problems	513
<b>A</b>	<b>Fundamental Lemmas of the Calculus of Variations</b>	<b>521</b>
A.1	Fundamental Lemma of Variational Calculus on $\mathbb{R}^n$	521
A.2	Fundamental Lemma of the Calculus of Variations on an Embedded Manifold	522
A.3	Fundamental Lemma of Variational Calculus on a Lie Group	522

<b>B</b>	<b>Linearization as an Approximation to Lagrangian</b>	
	<b>Dynamics on a Manifold</b> .....	523
B.1	Linearization on $TS^1$ .....	524
B.2	Linearization on $TS^2$ .....	526
B.3	Linearization on $TSO(3)$ .....	527
	<b>References</b> .....	529
	<b>Index</b> .....	535

# List of Symbols

## Vectors and Matrices

$\mathbb{R}^n$	Vector space of $n$ -tuples of real numbers over the real field
$\mathbb{R}^{n \times m}$	Vector space of matrices with $n$ rows and $m$ columns over the real field
$I_{n \times n}$	$n \times n$ identity matrix
$e_1, \dots, e_n$	Standard basis vectors in $\mathbb{R}^n$
$x \cdot y$	Inner or dot product of vectors in $\mathbb{R}^n$
$x \times y$	Cross product of vectors in $\mathbb{R}^3$
$\ x\ $	Norm of vector in $\mathbb{R}^n$
$A^T$	Matrix transpose
$A^{-1}$	Matrix inverse
$\text{trace}(A)$	Trace of a matrix
$\mathcal{N}(A)$	Null space of a matrix $A$
$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$2 \times 2$ skew-symmetric matrix
$S(x)$	Isomorphism between $\mathbb{R}^3$ and $\mathfrak{so}(3)$ , or cross product operator on $\mathbb{R}^3$ , that is, $S(x)y = x \times y$ and $S(x)^T = -S(x)$
$\mathcal{S}(\omega, v) = \mathcal{S} \left( \begin{bmatrix} \omega \\ v \end{bmatrix} \right)$	Isomorphism between $\mathbb{R}^6$ and $\mathfrak{se}(3)$ , defined as $\mathcal{S}(\omega, v) = \begin{bmatrix} S(\omega) & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 4}$

## Manifolds

$\text{GL}(n)$	The set of $n \times n$ real nonsingular matrices
$M$	Differentiable, embedded manifold
$S^1$	Unit sphere manifold embedded in $\mathbb{R}^2$

$\mathbb{S}^2$	Unit sphere manifold embedded in $\mathbb{R}^3$
$\text{SO}(3)$	Special orthogonal group, that is, the set of all $3 \times 3$ orthogonal matrices with determinant $+1$
$\text{SE}(3)$	Special Euclidean group
$(M)^n$	Manifold, that is, the $n$ -fold product $M \times \cdots \times M$
$\mathbb{T}_x M$	Tangent space of the manifold $M$ at $x \in M$
$\mathbb{T}M$	Tangent bundle of the manifold $M$
$\mathbb{T}_x^* M$	Cotangent space of the manifold $M$ at $x \in M$
$\mathbb{T}^* M$	Cotangent bundle of the manifold $M$
$P(x) : \mathbb{R}^n \rightarrow \mathbb{T}_x M$	Orthogonal projection for $x \in M$
$f : M \rightarrow \mathbb{R}^n$	Vector-valued function defined on a manifold $M$
$\frac{\partial f(x)}{\partial x} = D_x f(x)$	Derivative function
$x(t, t_0, x_0) \in M$	Solution of initial-value problem at $t$ corresponding to initial time $t_0$ and initial-value $x_0 \in M$ associated with a vector field on a manifold $M$
$F_{t,t_0} : M \rightarrow M$	Flow map associated with a vector field on a manifold $M$

## Mechanics

$q$	Configuration vector in $\mathbb{R}^n$ or $\mathbb{S}^1$ or $\mathbb{S}^2$
$R$	Configuration in $\text{SO}(3)$ , referred to as an attitude or rotation matrix
$(R, x)$	Configuration in $\text{SE}(3)$
$\omega \in \mathbb{T}_q \mathbb{S}^1$	Scalar angular velocity associated with configuration $q \in \mathbb{S}^1$
$\omega \in \mathbb{T}_q \mathbb{S}^2$	Angular velocity vector associated with configuration $q \in \mathbb{S}^2$
$\omega \in \mathbb{T}_R \text{SO}(3)$	Angular velocity vector associated with configuration $R \in \text{SO}(3)$
$\Omega \in \mathbb{T}_R \text{SO}(3)$	Angular velocity vector associated with configuration $R \in \text{SO}(3)$
$\mathfrak{G}$	Action integral
$\delta(\cdot)$	Infinitesimal variation
$T : \mathbb{T}M \rightarrow \mathbb{R}^1$	Kinetic energy function
$U : M \rightarrow \mathbb{R}^1$	Potential energy function
$L : \mathbb{T}M \rightarrow \mathbb{R}^1$	Lagrangian function, $L(q, \dot{q}) = T(q, \dot{q}) - U(q)$
$E : \mathbb{T}M \rightarrow \mathbb{R}^1$	Energy function, $E(q, \dot{q}) = T(q, \dot{q}) + U(q)$
$\lambda_1, \dots, \lambda_m$	Lagrange multipliers
$L^a : \mathbb{T}M \times \mathbb{R}^m \rightarrow \mathbb{R}^1$	Augmented Lagrangian function
$\tilde{L} : \mathbb{T}M \rightarrow \mathbb{R}^1$	Modified Lagrangian function
$p, \pi, \Pi, \mu$	Conjugate momentum
$H : \mathbb{T}^* M \rightarrow \mathbb{R}^1$	Hamiltonian function
$H^a : \mathbb{T}^* M \times \mathbb{R}^m \rightarrow \mathbb{R}^1$	Augmented Hamiltonian function
$\tilde{H} : \mathbb{T}^* M \rightarrow \mathbb{R}^1$	Modified Hamiltonian function



$R(q, \dot{q})$	Routhian function
$\Omega_{can}$	Symplectic form
$\mathbb{J}$	Symplectic matrix
$\int_{\mathcal{B}}$	Integral over material points that constitute a rigid body $\mathcal{B}$

## Lie Groups and Lie Algebras

$G$	Lie group
$\mathfrak{g}$	Lie algebra
$L_h : G \rightarrow G$	Left translation map
$R_h : G \rightarrow G$	Right translation map
$\exp : \mathfrak{g} \rightarrow G$	Exponential map
$\text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g}$	Adjoint operator
$\text{ad}_\xi : \mathfrak{g} \rightarrow \mathfrak{g}$	Adjoint operator
$\text{Ad}_g^* : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$	Coadjoint operator
$\text{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$	Coadjoint operator