LOCAL DEFECT-CORRECTION METHOD BASED ON MULTILEVEL DISCRETIZATION FOR STEKLOV EIGENVALUE PROBLEM

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Abstract. In this paper, we propose a local defect-correction method for solving the Steklov eigenvalue problem arising from the scalar second order positive definite partial differential equations based on the multilevel discretization. The objective is to avoid solving large-scale equations especially the large-scale Steklov eigenvalue problem whose computational cost increases exponentially. The proposed algorithm transforms the Steklov eigenvalue problem into a series of linear boundary value problems, which are defined in a multigrid space sequence, and a series of small-scale Steklov eigenvalue problems in a coarse correction space. Furthermore, we use the local defect-correction technique to divide the large-scale boundary value problems into small-scale subproblems. Through our proposed algorithm, we avoid solving large-scale Steklov eigenvalue problems. As a result, our proposed algorithm demonstrates significantly improved the solving efficiency. Additionally, we conduct numerical experiments and a rigorous theoretical analysis to verify the effectiveness of our proposed approach.

1. Introduction

Owing to continuous advancements in computer technology and computing technology, computational science and engineering has become the third approach for conducting scientific and engineering research after experimentation and theoretical analysis. Currently, increasing practical applications require efficient computational methods to cope with the increasing scale and difficulty of computing. In practice, the Steklov eigenvalue problem with eigenvalue parameters in boundary conditions is a significant problem in the current field of computational science and engineering. Steklov eigenvalue problems refer to eigenvalue problems for which the eigenvalue parameter appears in the (Robin type) boundary condition, and can in general be formulated for any partial differential equations. Numerous physical and engineering models, such as those involving the vibrations of pendulums [2], surface waves [7], the dynamics of liquids in moving containers [16, 24, 44], and the stability of mechanical oscillators immersed in viscous media [39], have been reduced to solve the Steklov eigenvalue problem. Besides, the non-selfadjoint Steklov eigenvalue problems have important applications in the inverse scattering theory to reconstruct the index of refraction of an inhomogeneous media [15, 54]. This is precisely because the Steklov eigenvalue problem has a broad range of vital applications in various fields

of engineering and physics. Therefore, research regarding this problem has crucial theoretical significance and significant application value.

Extensive research on the Steklov eigenvalue problem has been conducted owing to its wide range of applications. Currently, various research results have been obtained through algorithm design and theoretical analysis. Chatelin [17] and Ciarlet [19] analyzed the finite element method for the Steklov eigenvalue problems arising from the second order positive definite partial differential equations and obtained optimal error estimates. Based on the standard finite element error estimates obtained by the researchers mentioned above, many efficient algorithms can be analyzed. For instance, Andreev and Todorov [3] and Bramble and Osborn [11] analyzed the use of the conforming finite element method to solve the Steklov eigenvalue problem. Other examples of numerical experiments conducted to solve the Steklov eigenvalue problem can be found in the works of [4,5,26,27,30,32,43] which solve the Steklov eigenvalue problems arising from the scalar second order positive definite partial differential equations, and works of [8,38] which solve the Steklov eigenvalue problems arising from the fourth order positive definite partial differential equations and in the references cited therein.

The multigrid method was first proposed by Fedorenko based on the finite difference method in the 1960s. However, this method did not attract significant attention at that time. In the 1970s, scientists gradually began to pay attention to the multigrid method, thereby attracting a large number of researchers seeking to conduct further studies on algorithms and theories [12, 13, 48]. Currently, the multigrid method has resulted in the development of a complete theoretical system. The error order of the approximate solution obtained using the multigrid method is equivalent to the theoretical order determined through finite element discretization. However, the computational cost involved is only proportional to the unknowns in the discrete equation. The multigrid method is composed of two main components: Smoothing step on the current mesh and error correction step on the coarse mesh. The smoothing step can efficiently eliminate the high-frequency components of the error. Then the smooth part of the error can be corrected on the coarse mesh. There also exist studies on the application of the multigrid method to solve eigenvalue problems. Xu and Zhou [52] proposed a two-grid method for tackling eigenvalue problems. Their proposed method must solve an eigenvalue problem on a coarse mesh and a linear boundary value problem on a fine mesh. If the size of the coarse mesh is equal to the square root of the fine mesh, the optimal estimate can be derived. Based on the approach mentioned above, Lin and Xie [34] proposed a multilevel correction method for solving the eigenvalue problem. They extended the feature by which the two-grid method can only be corrected once to an arbitrary number of corrections. More detailed information regarding multilevel correction can be found in the works of [18, 31, 33, 46, 47, 49].

Over the recent years, the development of local defect-correction methods (or local and parallel methods) has progressed rapidly because its use is significantly convenient in large-scale scientific and engineering computing. This computational technique for solving linear elliptic equations was first proposed by Xu and Zhou [51]. The local defect-correction method is designed based on the understanding of the local and global properties of the finite element solution. The global behavior of a solution is mainly governed by low-frequency components while the local behavior is mainly governed by high-frequency components. The local defect-correction method uses a coarse mesh to approximate the low-frequency components and then uses a fine mesh to correct the resulted residue through some local procedures. To date, it has been applied to a variety of mathematical models, such as those presented in the works of [9,10,20–23,28,29,35–37,40–42,51,55–58], etc. Xu and Zhou [53] used a local defect-correction finite element algorithm to solve Laplace eigenvalue problem. Their technique was based on the two-grid finite element discretization scheme and the local defect correction technique for solving elliptic boundary value problems. For eigenvalue problems with Dirichlet boundary conditions, the algorithm has already been analyzed, see [33,50]. But the algorithm was not successfully extended to solve Steklov eigenvalue problems in a long time. This is because the local defect-correction technique will generate a series of local subdomains whose boundaries will remain in the interior of the overall computing domain Ω. We need to assemble the local solutions to form the final global solution. For eigenvalue problems with Dirichlet boundary condition, this is quite simple. But because Steklov eigenvalue problem has the variable in the boundary, thus the inner boundaries of these subdomains will cause many troubles, then this process can not be used any more for Steklov eigenvalue problem in a long time. Based on the approach mentioned above, Bi *et al.* [10] attempted to solve
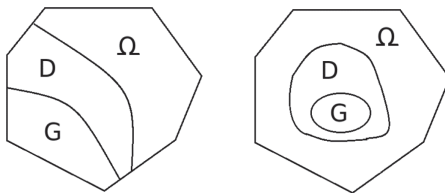
FIGURE 1. $G \subset\subset D \subset \Omega$.

the Steklov eigenvalue problem by combining two-grid discretization with a local defect correction algorithm. However, there exists a strict constrain on the mesh size ratio between coarse mesh and fine mesh for two-grid method. In this study, we design an efficient local defect-correction method for solving the Steklov eigenvalue problems from the scalar second order positive definite partial differential equations. We eliminate the effect of the boundaries of the subdomains based on some new local estimates. Our proposed method is based on the local defect-correction technique and the multilevel correction algorithm. The objective is to avoid solving large-scale equations, such as large-scale Steklov eigenvalue problems, whose computational cost increases exponentially with mesh refinement. Through our algorithm, we simply must solve some linear boundary value problems in a multigrid space sequence and some small-scale Steklov eigenvalue problems in a coarse correction space, whereby the dimensions are small and remain unchanged. Additionally, the linear boundary value problem defined in each level of the multigrid space sequence is solved using the local defect-correction technique. Because the main computational work of this algorithm is controlled by the linear boundary value problem, which can be solved efficiently using the local defect-correction technique, its efficiency in solving the Steklov eigenvalue problem can be significantly improved. Additionally, to verify the validity of the results of our theoretical analysis, we further develop the theoretical works of [10, 51, 53] to adapt these to our algorithmic framework.

The remainder of this paper is organized as follows: In the next section, we introduce the basic theory regarding the finite element method, the elliptic boundary value problem, and the local *a priori* error estimates. In Section 3, we introduce the Steklov eigenvalue problem to be solved in this study. In Section 4, we present the local defect-correction method based on multilevel discretization for solving the Steklov eigenvalue problem and the corresponding theoretical analysis. In Section 5, we describe the numerical experiments conducted to validate our theoretical analysis. Finally, we present the concluding remarks in the last section.

2. FINITE ELEMENT METHOD FOR SOLVING THE ELLIPTIC BOUNDARY VALUE PROBLEM

In this section, we introduce the basic notations and preliminary estimates of the finite element method. Ω denotes a bounded domain with a Lipschitz-continuous boundary in $\mathbb{R}^d (d \geq 1)$. $H^s(\Omega)$ denotes the standard Sobolev space [1], and $\| \cdot \|_{s,\Omega}$ and $\| \cdot \|_{s,\partial\Omega}$ denote the corresponding norms on Ω and $\partial\Omega$, respectively. In this study, we use C to denote a generic positive constant, which may be different at its different occurrences. For convenience, in this study, we use $x \lesssim y$ to denote $x \leq Cy$. For the three nested domains $G \subset D \subset \Omega$, we use $G \subset\subset D$ to denote $\mathrm{dist}(\partial D \backslash \partial\Omega, \partial G \backslash \partial\Omega) > 0$ (see Fig. 1).

In this section, we introduce the finite element method for solving the following elliptic boundary value problem with Neumann boundary condition:

$$\begin{cases} Lu := -\nabla \cdot (\mathcal{A}\nabla u) + \phi u = 0, & \text{in } \Omega, \\ (\mathcal{A}\nabla u) \cdot n = f, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\mathcal{A} \in (L^\infty(\Omega))^{d \times d}$ denotes a symmetric and uniformly positive definite matrix function, and ϕ denotes a nonnegative function bounded from above and below by positive constants.

The weak form of (2.1) is defined by: Find $u \in H^1(\Omega)$ such that

$$a(u, v) = b(f, v), \quad \forall v \in H^1(\Omega),$$

where

$$a(u, v) = \int_\Omega (\mathcal{A}\nabla u \cdot \nabla v + \phi u v)\, d\Omega, \quad b(f, v) = \int_{\partial\Omega} f v\, ds.$$

Obviously, $a(\cdot, \cdot)$ is a symmetric, continuous, and $H^1(\Omega)$-elliptic bilinear form.

To use the finite element method, we generate a shape-regular triangulation $\mathcal{T}_h(\Omega)$ for the computing domain Ω. We use h_K to denote the diameter of the mesh element $K \in \mathcal{T}_h(\Omega)$, and we use $h(x)$ to denote the diameter of the mesh element that includes x.

Based on the triangulation $\mathcal{T}_h(\Omega)$, we define a finite element space $S^h(\Omega)$ as follows:

$$S^h(\Omega) = \{v \in C(\bar{\Omega}) : v|_K \in \mathcal{P}_k, \forall K \in \mathcal{T}_h(\Omega)\}, \tag{2.2}$$

which is composed of piecewise polynomials whose degree is not greater than k. Then $S^h(\Omega) \subset H^1(\Omega)$.

Given any subset $G \subset \Omega$, we use $S^h(G)$ and $\mathcal{T}_h(G)$ to denote the restriction of $S^h(\Omega)$ and $\mathcal{T}_h(\Omega)$ to G, and we define the following spaces:

$$H^1_\Gamma(G) = \{v \in H^1(G) : \mathrm{supp}\, v \subset\subset G\} \tag{2.3}$$

and

$$S^0_h(G) = \{v \in S^h(\Omega) : \mathrm{supp}\, v \subset\subset G\}. \tag{2.4}$$

In this paper, we use $G \subset\subset D$ to denote $\mathrm{dist}(\partial D\backslash\partial\Omega, \partial G\backslash\partial\Omega) > 0$ (see Fig. 1). Thus, for any $v \in H^1_\Gamma(G)$, v equals zero on the boundary $\partial G\backslash\partial\Omega$, and v may not equal zero on the boundary $\partial G \cap \partial\Omega$. Besides, $H^1_\Gamma(G)$ consists of all functions $v \in H^1(G)$ for which the extension by zero to $\Omega\backslash G$ is in $H^1(\Omega)$

Based on the works of [14, 19, 51, 53], we obtain the following fractional norm property for the finite element space.

Lemma 2.1. *For any subset $G \subset \Omega$, the following estimate holds true*

$$\inf_{v \in S^0_h(G)} \|w - v\|_{1,G} \lesssim \|w\|_{1/2, \partial G\backslash\partial\Omega}, \qquad \forall w \in S^h(G). \tag{2.5}$$

For the theoretical analysis, we introduce the following quantity:

$$\rho_\Omega(h) = \sup_{f \in L^2(\partial\Omega), \|f\|_{0,\partial\Omega}=1} \inf_{v_h \in S^h(\Omega)} \|Tf - v_h\|_{1,\Omega}, \tag{2.6}$$

where the operator $T : L^2(\partial\Omega) \to H^1(\Omega)$ by

$$a(Tf, v) = b(f, v), \qquad \forall v \in H^1(\Omega). \tag{2.7}$$

Similarly, we also introduce the following quantity:

$$r_\Omega(h) = \sup_{f \in L^2(\Omega), \|f\|_{0,\Omega}=1} \inf_{v_h \in S^h(\Omega)} \|T'f - v_h\|_{1,\Omega}, \tag{2.8}$$

where the operator $T' : L^2(\Omega) \to H^1(\Omega)$ by

$$a(T'f, v) = (f, v), \qquad \forall v \in H^1(\Omega). \tag{2.9}$$

Based on the finite element space, we define the projection operator $P_h : H^1(\Omega) \to S^h(\Omega)$ by

$$a(u - P_h u, v) = 0, \qquad \forall v \in S^h(\Omega). \tag{2.10}$$

We can then derive the following estimates for the projection operator.

Lemma 2.2. *The following estimates for the projection operator hold true*

$$\|(I - P_h)Tf\|_{1,\Omega} \lesssim \rho_\Omega(h)\|f\|_{0,\partial\Omega}, \qquad\qquad \forall f \in L^2(\partial\Omega),$$

$$\|(I - P_h)T'f\|_{1,\Omega} \lesssim r_\Omega(h)\|f\|_{0,\Omega}, \qquad\qquad \forall f \in L^2(\Omega),$$

$$\|u - P_h u\|_{0,\partial\Omega} \lesssim \rho_\Omega(h)\|u - P_h u\|_{1,\Omega}, \qquad\qquad \forall u \in H^1(\Omega)$$

$$\|u - P_h u\|_{0,\Omega} \lesssim r_\Omega(h)\|u - P_h u\|_{1,\Omega}, \qquad\qquad \forall u \in H^1(\Omega). \qquad (2.11)$$

Proof. From the definition of projection operator in (2.10), there holds

$$\|u - P_h u\|_{1,\Omega} \lesssim \inf_{v_h \in S^h} \|u - v_h\|_{1,\Omega}, \qquad \forall u \in H^1(\Omega). \qquad (2.12)$$

Denote $\hat{f} = f/\|f\|_{0,\partial\Omega}$, then we can derive

$$\begin{aligned}
\|(I - P_h)Tf\|_{1,\Omega} &= \|(I - P_h)T\hat{f}\|_{1,\Omega}\|f\|_{0,\partial\Omega} \\
&\lesssim \inf_{v_h \in S^h} \|T\hat{f} - v_h\|_{1,\Omega}\|f\|_{0,\partial\Omega} \\
&\lesssim \rho_\Omega(h)\|f\|_{0,\partial\Omega}.
\end{aligned}$$

Similarly, denote $\hat{f} = f/\|f\|_{0,\Omega}$, we can also derive

$$\begin{aligned}
\|(I - P_h)T'f\|_{1,\Omega} &= \|(I - P_h)T'\hat{f}\|_{1,\Omega}\|f\|_{0,\Omega} \\
&\lesssim \inf_{v_h \in S^h} \|T'\hat{f} - v_h\|_{1,\Omega}\|f\|_{0,\Omega} \\
&\lesssim r_\Omega(h)\|f\|_{0,\Omega}.
\end{aligned}$$

Then we derive the first two estimates. The left two estimates can also be proved easily through the Aubin–Nitsche technique. $\qquad\square$

Next, we further develop the works of [51, 53] to adapt to the Steklov eigenvalue problem to be solved in this study. For each $\Omega_0 \subset \Omega$, we assume that the finite element space used in this study satisfies the following conditions:

A.1. There exists $\gamma > 1$ such that

$$h_\Omega^\gamma \lesssim h(x), \qquad \forall x \in \Omega, \qquad (2.13)$$

with $h_\Omega = \max_{x \in \Omega} h(x)$.

A.2. Inverse Estimate. For any $v \in S^h(\Omega_0)$,

$$\|v\|_{1,\Omega_0} \lesssim \|h^{-1}v\|_{0,\Omega_0}. \qquad (2.14)$$

A.3. Superapproximation. For $G \subset \Omega$, let $\omega \in C^\infty(\bar{\Omega})$ with $(\text{supp } \omega \backslash (\partial G \cap \partial\Omega)) \subset\subset G$. Then for any $w \in S^h(G)$, there exists $v \in S_h^0(G)$ such that

$$\left\|h_G^{-1}(\omega w - v)\right\|_{1,G} \lesssim \|w\|_{1,G}. \qquad (2.15)$$

Let

$$a_0(u, v) = \int_\Omega (\mathcal{A}\nabla u \cdot \nabla v)\, d\Omega. \qquad (2.16)$$

From Lemma 3.1 of [51], we can prove the following lemmas:

Lemma 2.3. *Let $D \subset\subset \Omega_0 \subset \Omega$, $\omega \in C^\infty(\bar{\Omega})$ with $(\text{supp } \omega \backslash (\partial\Omega_0 \cap \partial\Omega)) \subset\subset \Omega_0$, then*

$$a_0(\omega w, \omega w) \leq 2a(w, \omega^2 w) + C\|w\|_{0,\Omega_0}^2, \qquad \forall w \in H^1(\Omega). \tag{2.17}$$

Lemma 2.4. *Suppose that $f \in L^2(\partial\Omega)$ and $D \subset\subset \Omega_0 \subset \Omega$. If $w \in S^h(\Omega_0)$ satisfies the following:*

$$a(w, v) = b(f, v), \qquad \forall v \in S_h^0(\Omega_0), \tag{2.18}$$

then we have the following local estimate

$$\|w\|_{1,D} \lesssim \|w\|_{0,\Omega_0} + \|f\|_{0,\partial\Omega \cap \partial\Omega_0}. \tag{2.19}$$

Proof. Let $p \geq 2\gamma - 1$ be an integer and Ω_j $(j = 1, 2, \cdots, p)$ be a domain sequence satisfying

$$D \subset\subset \Omega_p \subset\subset \Omega_{p-1} \subset\subset \cdots \subset\subset \Omega_1 \subset\subset \Omega_0.$$

Next, choose $D_1 \subset \Omega$ satisfying $D \subset\subset D_1 \subset\subset \Omega_p$, $\omega \in C^\infty(\bar{\Omega})$ such that $\omega = 1$ on $\bar{D}_1$ and $(\text{supp } \omega \backslash (\partial\Omega_p \cap \partial\Omega)) \subset\subset \Omega_p$. Then from A.3., there exists $v \in S_h^0(\Omega_p)$ such that

$$\left\|\omega^2 w - v\right\|_{1,\Omega_p} \lesssim h_{\Omega_0} \|w\|_{1,\Omega_p}. \tag{2.20}$$

Based on (2.20) and trace inequality, we can further derive

$$
\begin{aligned}
|b(f, v)| = \left|\int_{\partial\Omega} fv \, \mathrm{d}s\right| &= \left|\int_{\partial\Omega_p \cap \partial\Omega} fv \, \mathrm{d}s\right| \\
&\lesssim \|f\|_{0,\partial\Omega_p \cap \partial\Omega} \|v\|_{0,\partial\Omega_p \cap \partial\Omega} \\
&\lesssim \|f\|_{0,\partial\Omega_0 \cap \partial\Omega} \|v\|_{1,\Omega_p} \\
&\lesssim \|f\|_{0,\partial\Omega_0 \cap \partial\Omega} \big(h_{\Omega_0} \|w\|_{1,\Omega_p} + \|\omega w\|_{1,\Omega}\big).
\end{aligned} \tag{2.21}
$$

From (2.17), (2.18), (2.20) and (2.21), there holds

$$
\begin{aligned}
\|\omega w\|_{1,\Omega}^2 &\lesssim a_0(\omega w, \omega w) + \|w\|_{0,\Omega_0}^2 \lesssim a(w, \omega^2 w) + \|w\|_{0,\Omega_0}^2 \\
&= a(w, \omega^2 w - v) + \|w\|_{0,\Omega_0}^2 + b(f, v) \\
&\lesssim \|w\|_{1,\Omega_p} \|\omega^2 w - v\|_{1,\Omega_p} + \|w\|_{0,\Omega_0}^2 + \|f\|_{0,\partial\Omega \cap \partial\Omega_0}(h_{\Omega_0}\|w\|_{1,\Omega_p} + \|\omega w\|_{1,\Omega}) \\
&\lesssim h_{\Omega_0}\|w\|_{1,\Omega_p}^2 + \|w\|_{0,\Omega_0}^2 + \|f\|_{0,\partial\Omega \cap \partial\Omega_0}(h_{\Omega_0}\|w\|_{1,\Omega_p} + \|\omega w\|_{1,\Omega}) \\
&\lesssim \|f\|_{0,\partial\Omega \cap \partial\Omega_0}\|\omega w\|_{1,\Omega} + h_{\Omega_0}\|w\|_{1,\Omega_p}^2 + \|w\|_{0,\Omega_0}^2 + \|f\|_{0,\partial\Omega \cap \partial\Omega_0}^2.
\end{aligned}
$$

Then we have

$$\|\omega w\|_{1,\Omega} \lesssim h_{\Omega_0}^{\frac{1}{2}}\|w\|_{1,\Omega_p} + \|w\|_{0,\Omega_0} + \|f\|_{0,\partial\Omega \cap \partial\Omega_0}. \tag{2.22}$$

Because we have $\omega = 1$ in D, from (2.22), we can derive the following estimate

$$\|w\|_{1,D} \lesssim h_{\Omega_0}^{\frac{1}{2}}\|w\|_{1,\Omega_p} + \|w\|_{0,\Omega_0} + \|f\|_{0,\partial\Omega \cap \partial\Omega_0}. \tag{2.23}$$

For $D \subset\subset \Omega_p$, we derive the estimate (2.23). Similarly, since $\Omega_j \subset\subset \Omega_{j-1}$ for $j = 1, 2, \cdots, p$, we can deduce the following estimate using the same way as that for (2.23):

$$\|w\|_{1,\Omega_j} \lesssim h_{\Omega_0}^{\frac{1}{2}}\|w\|_{1,\Omega_{j-1}} + \|w\|_{0,\Omega_0} + \|f\|_{0,\partial\Omega \cap \partial\Omega_0}, \quad j = 1, 2, \cdots, p. \tag{2.24}$$

Combining (2.23), (2.24), A1 and A.2, we can get the following estimate

$$
\begin{aligned}
\|w\|_{1,D} &\lesssim h_{\Omega_0}^{\frac{p+1}{2}}\|w\|_{1,\Omega_0} + \|w\|_{0,\Omega_0} + \|f\|_{0,\partial\Omega \cap \partial\Omega_0} \\
&\lesssim h_{\Omega_0}^{\frac{p+1}{2}}\|h^{-1}w\|_{0,\Omega_0} + \|w\|_{0,\Omega_0} + \|f\|_{0,\partial\Omega \cap \partial\Omega_0} \\
&\lesssim \|w\|_{0,\Omega_0} + \|f\|_{0,\partial\Omega \cap \partial\Omega_0},
\end{aligned}
$$

where the Neumann series is adopted to obtain the above estimate. This completes the proof. $\qquad\square$

3. Finite element method for solving the Steklov eigenvalue problem

In this study, we consider the following Steklov eigenvalue problem:

$$\begin{cases} -\nabla \cdot (\mathcal{A}\nabla u) + \phi u = 0, & \text{in } \Omega, \\ (\mathcal{A}\nabla u) \cdot n = \lambda u, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

The weak form of (3.1) is as follows: Find $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$ such that $b(u, u) = 1$ and

$$a(u, v) = \lambda b(u, v), \quad \forall v \in H^1(\Omega). \tag{3.2}$$

From the work of [6, 17], we know that the Steklov eigenvalue problem (3.2) has an eigenvalue sequence, as follows:

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots, \tag{3.3}$$

and the associated eigenfunction sequence:

$$u_1, u_2, \cdots, u_k, \cdots, \tag{3.4}$$

which satisfies

$$b(u_i, u_j) = \delta_{ij}.$$

Based on the finite element method, we should solve the discrete approximation of (3.2) as follows: Find $(\bar{\lambda}_h, \bar{u}_h) \in \mathbb{R} \times S^h(\Omega)$ such that $b(\bar{u}_h, \bar{u}_h) = 1$ and

$$a(\bar{u}_h, v_h) = \bar{\lambda}_h b(\bar{u}_h, v_h), \quad \forall v_h \in S^h(\Omega). \tag{3.5}$$

From the work of [6, 17], we know that (3.5) also has an eigenvalue sequence, which is expressed as follows:

$$0 < \bar{\lambda}_{1,h} \le \bar{\lambda}_{2,h} \le \cdots \le \bar{\lambda}_{k,h} \le \cdots \le \bar{\lambda}_{N_h,h}, \tag{3.6}$$

and it has a corresponding eigenfunction sequence, which is expressed as follows:

$$\bar{u}_{1,h}, \bar{u}_{2,h}, \cdots, \bar{u}_{k,h}, \cdots, \bar{u}_{N_h,h}, \tag{3.7}$$

where $b(\bar{u}_{i,h}, \bar{u}_{j,h}) = \delta_{ij}$, $1 \le i, j \le N_h$ (N_h denotes the dimension of $S^h(\Omega)$).

We then define the eigenfunction set with respect to the eigenvalue λ_i in the following way:

$$M(\lambda_i) = \left\{ w \in H^1(\Omega), w \text{ is an eigenfunction of (3.2), which corresponds to } \lambda_i, \ \|w\|_{0,\partial\Omega} = 1 \right\}. \tag{3.8}$$

From the works of [6, 17], we obtain the following error estimates for the Steklov eigenvalue problem (3.2).

Lemma 3.1. *For any eigenpair approximation $(\bar{\lambda}_{i,h}, \bar{u}_{i,h})$ of (3.5), there exists an eigenfunction u_i of the Steklov eigenvalue problem (3.2), which corresponds to λ_i, such that $\|u_i\|_{0,\partial\Omega} = 1$ and*
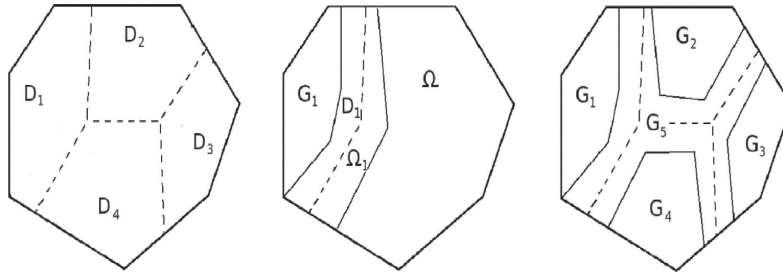
$$\|u_i - \bar{u}_{i,h}\|_{1,\Omega} \lesssim \delta_h(\lambda_i), \tag{3.9}$$
$$\|u_i - \bar{u}_{i,h}\|_{0,\partial\Omega} \lesssim \rho_\Omega(h)\delta_h(\lambda_i), \tag{3.10}$$
$$|\lambda_i - \bar{\lambda}_{i,h}| \lesssim \delta_h^2(\lambda_i), \tag{3.11}$$

where

$$\delta_h(\lambda_i) = \sup_{w \in M(\lambda_i)} \inf_{v \in S^h(\Omega)} \|w - v\|_{1,\Omega}.$$

FIGURE 2. $G \subset\subset D \subset \Omega$.

4. LOCAL DEFECT-CORRECTION METHOD BASED ON MULTILEVEL DISCRETIZATION FOR SOLVING THE STEKLOV EIGENVALUE PROBLEM

In this section, we introduce the local defect-correction method based on multilevel discretization for solving the Steklov eigenvalue problem. To describe the algorithm, we need to construct a multigrid mesh sequence. $\mathcal{T}_H$ represents a coarse triangulation of the domain Ω. We then construct an initial mesh $\mathcal{T}_{h_1}$ which can be chosen as $\mathcal{T}_H$ or as a uniform refinement of $\mathcal{T}_H$. We can then construct a mesh sequence $\mathcal{T}_{h_k}$, $k = 2, 3, \cdots, n$, where we obtain $\mathcal{T}_{h_k}$ from $\mathcal{T}_{h_{k-1}}$ through a one-time uniform refinement, which means that all the mesh elements of $\mathcal{T}_{h_{k-1}}$ are refined at the same time. We then obtain a mesh sequence that satisfies the following:

$$\mathcal{T}_H(\Omega) \subset \mathcal{T}_{h_1}(\Omega) \subset \cdots \subset \mathcal{T}_{h_k}(\Omega) \subset \mathcal{T}_{h_{k+1}}(\Omega) \subset \cdots \subset \mathcal{T}_{h_n}(\Omega),$$

Based on the mesh sequence, we can then construct the corresponding finite element space sequence that satisfies the following:

$$S^H(\Omega) \subset S^{h_1}(\Omega) \subset \cdots \subset S^{h_k}(\Omega) \subset S^{h_{k+1}}(\Omega) \subset \cdots \subset S^{h_n}(\Omega). \tag{4.1}$$

In this section, we first elucidate how to execute the algorithm in one level of the finite element space, after which we propose a complete algorithm for the multigrid space sequence.

4.1. One step of the local defect-correction method

In this subsection, we demonstrate how to perform the local defect-correction method in the finite element space $S^{h_{k+1}}(\Omega)$. Based on the coarsest triangulation $\mathcal{T}_H(\Omega)$, we divide Ω into a number of disjoint subdomains $D_1, \cdots, D_m$ such that $\bigcup_{j=1}^{m} \bar{D}_j = \bar{\Omega}$, $D_i \cap D_j = \emptyset$, after which we enlarge and reduce each D_j to obtain Ω_j and G_j, which both align with $\mathcal{T}_H(\Omega)$. We then derive a sequence of subdomains $G_j \subset\subset D_j \subset \Omega_j \subset \Omega$, $j = 1, \cdots, m$ and $G_{m+1} = \Omega \setminus (\bigcup_{j=1}^{m} \bar{G}_j)$ (see Fig. 2).

In this study, the decomposition is assumed to satisfy the following:

$$\sum_{j=1}^{m} \|v\|_{l,\Omega_j}^2 \lesssim \|v\|_{l,\Omega}^2 \quad \text{and} \quad \sum_{j=1}^{m} \|v\|_{l,\partial\Omega_j \cap \partial\Omega}^2 \lesssim \|v\|_{l,\partial\Omega}^2. \tag{4.2}$$

Next, we briefly discuss the hidden coefficient of (4.2). Whether the coefficient depends on m or not, the theoretical analysis is still valid for the presented algorithm. But for a given problem in practice, it is easy to guarantee that the coefficient is bounded by a constant as long as we control the enlarged regions appropriately. (4.2) is trivially satisfied with constant m. But this is a quite rough estimate or a result for the extreme case $\Omega_j = \Omega$. In practice, we will not use such a strategy because if Ω_j is too large, the solving efficiency will be deteriorated.

If $D_j \subset \Omega_j$, then there exist many strategies to derive Ω_j. Though the theoretical analysis still holds true if the coefficient is m, we suggest to slightly enlarge D_j such that (4.2) can be bounded by a constant which is

independent of m, which can be done by controlling the enlarged regions appropriately. Especially, if we just choose $\Omega_j = D_j$, then (4.2) holds true with the coefficient equaling 1.

We construct a domain decomposition to transform the large-scale equation into some small-scale subproblems. We construct $G_j \subset\subset D_j \subset \Omega_j$ because when we solve the subproblem in Ω_j, we will adopt the zero boundary condition which is different from the exact solution, so we need to choose the value of approximate solution defined in the inner domain G_j as the approximation. In the theoretical analysis, each subproblem is analyzed independently, then the final global solution is a combination of these subproblems. Similar to the classical domain decomposition method, the increase of m will lead to the improvement of solving efficiency.

Assuming that an eigenpair approximation $(\lambda_{h_k}, u_{h_k}) \in \mathbb{R} \times S^{h_k}(\Omega)$ has been obtained, we demonstrate how to obtain a more accurate approximation $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathbb{R} \times S^{h_{k+1}}(\Omega)$ in Algorithm 4.1.

Algorithm 4.1. One Step of the local defect-correction method.

(1) Solve the following linear boundary value problem in each subdomain: Find $e_{h_{k+1}}^j \in S_{h_{k+1}}^0(\Omega_j)$, $j = 1, 2, \cdots, m$ such that

$$a\left(e_{h_{k+1}}^j, v_{h_{k+1}}\right) = \lambda_{h_k} b\left(u_{h_k}, v_{h_{k+1}}\right) - a\left(u_{h_k}, v_{h_{k+1}}\right), \qquad \forall v_{h_{k+1}} \in S_{h_{k+1}}^0(\Omega_j). \tag{4.3}$$

Set $\tilde{u}_{h_{k+1}}^j = u_{h_k} + e_{h_{k+1}}^j \in S^{h_{k+1}}(\Omega_j)$.

(2) Solve the following boundary value problem in G_{m+1}: Find $\tilde{u}_{h_{k+1}}^{m+1} \in S^{h_{k+1}}(G_{m+1})$ such that $\tilde{u}_{h_{k+1}}^{m+1}|_{\partial G_j \cap \partial G_{m+1}} = \tilde{u}_{h_{k+1}}^j, j = 1, \cdots, m$ and

$$a\left(\tilde{u}_{h_{k+1}}^{m+1}, v_{h_{k+1}}\right) = \lambda_{h_k} b\left(u_{h_k}, v_{h_{k+1}}\right), \qquad \forall v_{h_{k+1}} \in S_{h_{k+1}}^0(G_{m+1}). \tag{4.4}$$

(3) Construct $\tilde{u}_{h_{k+1}} \in S^{h_{k+1}}(\Omega)$ such that $\tilde{u}_{h_{k+1}} = \tilde{u}_{h_{k+1}}^j$ in G_j, $j = 1, \cdots, m+1$.

(4) Define a new space $S^{H,h_{k+1}} = S^H(\Omega) + \text{span}\{\tilde{u}_{h_{k+1}}\}$ and solve the following small-scale Steklov eigenvalue problem: Find $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathbb{R} \times S^{H,h_{k+1}}(\Omega)$ such that $b(u_{h_{k+1}}, u_{h_{k+1}}) = 1$ and

$$a\left(u_{h_{k+1}}, v_{H,h_{k+1}}\right) = \lambda_{h_{k+1}} b\left(u_{h_{k+1}}, v_{H,h_{k+1}}\right), \qquad \forall v_{H,h_{k+1}} \in S^{H,h_{k+1}}(\Omega). \tag{4.5}$$

Summarize the above four steps into

$$\left(\lambda_{h_{k+1}}, u_{h_{k+1}}\right) = \text{Correction}\left(S^H(\Omega), \lambda_{h_k}, u_{h_k}, S^{h_{k+1}}(\Omega)\right).$$

Next, we can prove rigorously that the new approximate eigenpair $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathbb{R} \times S^{h_{k+1}}(\Omega)$ obtained by Algorithm 4.1 is more accurate than the given approximation $(\lambda_{h_k}, u_{h_k}) \in \mathbb{R} \times S^{h_k}(\Omega)$.

Theorem 4.2. *Assume the given eigenpair approximation* $(\lambda_{h_k}, u_{h_k}) \in \mathbb{R} \times S^{h_k}(\Omega)$ *satisfies*

$$\|u - u_{h_k}\|_{1,\Omega} \lesssim \varepsilon_{h_k}(\lambda), \tag{4.6}$$

$$\|u - u_{h_k}\|_{0,\partial\Omega} \lesssim \rho_\Omega(H)\varepsilon_{h_k}(\lambda), \tag{4.7}$$

$$|\lambda - \lambda_{h_k}| \lesssim \varepsilon_{h_k}^2(\lambda). \tag{4.8}$$

If Assumptions A.1–A.3 hold, then the new approximation $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathbb{R} \times S^{h_{k+1}}(\Omega)$ *obtained using Algorithm 4.1 satisfies*

$$\left\|u - u_{h_{k+1}}\right\|_{1,\Omega} \lesssim \varepsilon_{h_{k+1}}(\lambda), \tag{4.9}$$

$$\left\|u - u_{h_{k+1}}\right\|_{0,\partial\Omega} \lesssim \rho_\Omega(H)\varepsilon_{h_{k+1}}(\lambda), \tag{4.10}$$

$$|\lambda - \lambda_{h_{k+1}}| \lesssim \varepsilon_{h_{k+1}}^2(\lambda), \tag{4.11}$$

where $\varepsilon_{h_{k+1}}(\lambda) := (\rho_\Omega(H) + r_\Omega(H))\varepsilon_{h_k}(\lambda) + \varepsilon_{h_k}^2(\lambda) + \delta_{h_{k+1}}(\lambda).$

Proof. From the error estimate (3.9) presented in Lemma 3.1 and triangle inequality, there holds

$$\|u - u_{h_{k+1}}\|_{1,\Omega} \lesssim \|u - \tilde{u}_{h_{k+1}}\|_{1,\Omega}$$
$$\leq \|u - P_{h_{k+1}}u\|_{1,\Omega} + \|\tilde{u}_{h_{k+1}} - P_{h_{k+1}}u\|_{1,\Omega}, \tag{4.12}$$

and

$$\|\tilde{u}_{h_{k+1}} - P_{h_{k+1}}u\|_{1,\Omega}^2 = \sum_{j=1}^{m}\|\tilde{u}_{h_{k+1}}^j - P_{h_{k+1}}u\|_{1,G_j}^2 + \|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1,G_{m+1}}^2, \tag{4.13}$$

where the hidden constant in (4.12) depends on the exact eigenpair.

Next, we will divide the proof into four parts. In *Part 1*, we will prove the estimate for $\sum_{j=1}^{m}\|\tilde{u}_{h_{k+1}}^j - P_{h_{k+1}}u\|_{1,G_j}^2$. In *Part 2*, we will prove the estimate for $\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}}u\|_{1,G_{m+1}}^2$. In *Part 3*, we will prove the estimate for $\|\tilde{u}_{h_{k+1}} - P_{h_{k+1}}u\|_{1,\Omega}$. In *Part 4*, we will give the final conclusion based on the above three parts and the standard finite element error estimates.

Part 1. From (2.10), (3.2) and (4.3),we have

$$a\left(\tilde{u}_{h_{k+1}}^j - P_{h_{k+1}}u, v\right) = b(\lambda_{h_k}u_{h_k} - \lambda u, v), \qquad \forall v \in S_{h_{k+1}}^0(\Omega_j), \quad j = 1, 2, \cdots, m.$$

Then from Lemma 2.4, we can derive

$$\left\|\tilde{u}_{h_{k+1}}^j - P_{h_{k+1}}u\right\|_{1,G_j} \lesssim \left\|\tilde{u}_{h_{k+1}}^j - P_{h_{k+1}}u\right\|_{0,\Omega_j} + \|\lambda_{h_k}u_{h_k} - \lambda u\|_{0,\partial\Omega\cap\partial\Omega_j}$$
$$\lesssim \left\|\tilde{u}_{h_{k+1}}^j - u_{h_k}\right\|_{0,\Omega_j} + \|u_{h_k} - P_{h_{k+1}}u\|_{0,\Omega_j} + \|\lambda_{h_k}u_{h_k} - \lambda u\|_{0,\partial\Omega\cap\partial\Omega_j}$$
$$\lesssim \left\|e_{h_{k+1}}^j\right\|_{0,\Omega_j} + \|u_{h_k} - P_{h_{k+1}}u\|_{0,\Omega_j} + \|\lambda_{h_k}u_{h_k} - \lambda u\|_{0,\partial\Omega\cap\partial\Omega_j}, \tag{4.14}$$

where the hidden coefficient depends on the coefficient $\mathcal{A}$ and ϕ, and is independent of k and mesh size. We will estimate the first term $\left\|e_{h_{k+1}}^j\right\|_{0,\Omega_j}$ of (4.14) based on Aubin–Nitsche technique: Given any $\psi \in L^2(\Omega_j)$, there exists $w \in H_\Gamma^1(\Omega_j)$ such that

$$a(v, w) = (v, \psi), \qquad \forall v \in H_\Gamma^1(\Omega_j). \tag{4.15}$$

The associated discrete equations can be defined by: Given any $\psi \in L^2(\Omega_j)$, there exist $w_H^j \in S_H^0(\Omega_j), w_{h_{k+1}}^j \in S_{h_{k+1}}^0(\Omega_j)$ such that

$$a\left(v_H, w_H^j\right) = (v_H, \psi), \qquad\qquad \forall v_H \in S_H^0(\Omega_j), \tag{4.16}$$

$$a\left(v_{h_{k+1}}, w_{h_{k+1}}^j\right) = \left(v_{h_{k+1}}, \psi\right), \qquad\qquad \forall v_{h_{k+1}} \in S_{h_{k+1}}^0(\Omega_j). \tag{4.17}$$

Meanwhile, we know the following standard finite element error estimates hold true

$$\left\|w - w_{h_{k+1}}^j\right\|_{1,\Omega_j} \lesssim r_{\Omega_j}(h_{k+1})\|\psi\|_{0,\Omega_j}, \tag{4.18}$$

$$\left\|w - w_H^j\right\|_{1,\Omega_j} \lesssim r_{\Omega_j}(H)\|\psi\|_{0,\Omega_j}. \tag{4.19}$$

From (2.10), (3.2), (3.5), (4.3), (4.16), (4.17) and $S_H^0(\Omega_j) \subset S_{h_k}^0(\Omega_j) \subset S_{h_{k+1}}^0(\Omega_j)$, we have

$$\left(e_{h_{k+1}}^j, \psi\right) = a\left(e_{h_{k+1}}^j, w_{h_{k+1}}^j\right)$$

$$= b\Big(\lambda_{h_k} u_{h_k}, w_{h_{k+1}}^j\Big) - a\Big(u_{h_k}, w_{h_{k+1}}^j\Big)$$

$$= b\Big(\lambda_{h_k} u_{h_k} - \lambda u, w_{h_{k+1}}^j\Big) + a\Big(P_{h_{k+1}} u - u_{h_k}, w_{h_{k+1}}^j\Big)$$

$$= b\Big(\lambda_{h_k} u_{h_k} - \lambda u, w_{h_{k+1}}^j - w_H^j\Big) + b\Big(\lambda_{h_k} u_{h_k} - \lambda u, w_H^j\Big) + a\Big(P_{h_{k+1}} u - u_{h_k}, w_{h_{k+1}}^j\Big)$$

$$= b\Big(\lambda_{h_k} u_{h_k} - \lambda u, w_{h_{k+1}}^j - w_H^j\Big) + a\Big(P_{h_{k+1}} u - u_{h_k}, w_{h_{k+1}}^j - w_H^j\Big). \tag{4.20}$$

Taking $\psi = e_{h_{k+1}}^j$ in (4.20), combining (4.18), (4.19) and (4.20), we have

$$\left\|\tilde{u}_{h_{k+1}}^j - u_{h_k}\right\|_{0,\Omega_j}^2 = \Big(e_{h_{k+1}}^j, e_{h_{k+1}}^j\Big) = b\Big(\lambda_{h_k} u_{h_k} - \lambda u, w_{h_{k+1}}^j - w_H^j\Big) + a\Big(P_{h_{k+1}} u - u_{h_k}, w_{h_{k+1}}^j - w_H^j\Big)$$

$$\lesssim \|\lambda_{h_k} u_{h_k} - \lambda u\|_{0,\partial\Omega\cap\partial\Omega_j} \left\|w_{h_{k+1}}^j - w_H^j\right\|_{0,\partial\Omega\cap\partial\Omega_j} + \|P_{h_{k+1}} u - u_{h_k}\|_{1,\Omega_j} \left\|w_{h_{k+1}}^j - w_H^j\right\|_{1,\Omega_j}$$

$$\lesssim \left(\left\|w - w_{h_{k+1}}^j\right\|_{1,\Omega_j} + \left\|w - w_H^j\right\|_{1,\Omega_j}\right)\left(\|P_{h_{k+1}} u - u_{h_k}\|_{1,\Omega_j} + \|\lambda_{h_k} u_{h_k} - \lambda u\|_{0,\partial\Omega\cap\partial\Omega_j}\right)$$

$$\lesssim r_{\Omega_j}(H)\left\|\tilde{u}_{h_{k+1}}^j - u_{h_k}\right\|_{0,\Omega_j}\left(\|P_{h_{k+1}} u - u_{h_k}\|_{1,\Omega_j} + \|\lambda_{h_k} u_{h_k} - \lambda u\|_{0,\partial\Omega\cap\partial\Omega_j}\right),$$

which yields

$$\left\|\tilde{u}_{h_{k+1}}^j - u_{h_k}\right\|_{0,\Omega_j} \lesssim r_{\Omega_j}(H)\Big(\|P_{h_{k+1}} u - u_{h_k}\|_{1,\Omega_j} + \|\lambda_{h_k} u_{h_k} - \lambda u\|_{0,\partial\Omega\cap\partial\Omega_j}\Big). \tag{4.21}$$

Here, the hidden coefficient is inherited from (4.18) which depends on the coefficient $\mathcal{A}$ and ϕ, but is independent of k and mesh size.

Combining (4.14) and (4.21), we can derive the following estimate

$$\left\|\tilde{u}_{h_{k+1}}^j - P_{h_{k+1}} u\right\|_{1,G_j} \lesssim r_{\Omega_j}(H)\|P_{h_{k+1}} u - u_{h_k}\|_{1,\Omega_j} + \|P_{h_{k+1}} u - u_{h_k}\|_{0,\Omega_j} + \|\lambda_{h_k} u_{h_k} - \lambda u\|_{0,\partial\Omega\cap\partial\Omega_j}. \tag{4.22}$$

Part 2. In this part, we come to estimate $\left|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u\right|_{1,G_{m+1}}$. Based on (2.10), (3.2) and (4.4), there holds

$$a\Big(\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u, v_{h_{k+1}}\Big) = b(\lambda_{h_k} u_{h_k} - \lambda u, v_{h_{k+1}}), \qquad \forall v_{h_{k+1}} \in S_{h_{k+1}}^0(G_{m+1}). \tag{4.23}$$

Let $a_{G_{m+1}}(\cdot, \cdot)$ denote the restriction of $a(\cdot, \cdot)$ on G_{m+1}. For any $v \in S_{h_{k+1}}^0(G_{m+1})$, there holds

$$\left\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u\right\|_{1,G_{m+1}}^2 \lesssim a_{G_{m+1}}\Big(\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u, \tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u\Big)$$

$$\lesssim a_{G_{m+1}}\Big(\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u, \tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u - v\Big) + b(\lambda_{h_k} u_{h_k} - \lambda u, v)$$

$$\lesssim \left\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u\right\|_{1,G_{m+1}} \left\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u - v\right\|_{1,G_{m+1}}$$

$$\quad + \|\lambda_{h_k} u_{h_k} - \lambda u\|_{0,\partial\Omega\cap\partial G_{m+1}} \|v\|_{0,\partial\Omega\cap\partial G_{m+1}}$$

$$\lesssim \left\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u\right\|_{1,G_{m+1}} \left\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u - v\right\|_{1,G_{m+1}}$$

$$\quad + \|\lambda_{h_k} u_{h_k} - \lambda u\|_{0,\partial\Omega\cap\partial G_{m+1}} \|v\|_{1,G_{m+1}}$$

$$\lesssim \left\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u\right\|_{1,G_{m+1}} \inf_{\psi\in S_{h_{k+1}}^0(G_{m+1})} \left\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u - \psi\right\|_{1,G_{m+1}}$$

$$\quad + \|\lambda_{h_k} u_{h_k} - \lambda u\|_{0,\partial\Omega\cap\partial G_{m+1}}\left(\left\|\tilde{u}_{h_{k+1}}^{m+1} - P_{h_{k+1}} u\right\|_{1,G_{m+1}}\right.$$

$$+ \inf_{\psi \in S^0_{h_{k+1}}(G_{m+1})} \left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u - \psi \right\|_{1, G_{m+1}} \Bigg). \tag{4.24}$$

Further using Lemma 2.1 and trace theorem, (4.24) can be written as

$$
\begin{aligned}
\left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u \right\|^2_{1, G_{m+1}} &\lesssim \left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u \right\|_{1, G_{m+1}} \left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u \right\|_{1/2, \partial G_{m+1} \setminus \partial \Omega} \\
&\quad + \left\| \lambda_{h_k} u_{h_k} - \lambda u \right\|_{0, \partial \Omega \cap \partial G_{m+1}} \left(\left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u \right\|_{1, G_{m+1}} \right. \\
&\quad \left. + \left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u \right\|_{1/2, \partial G_{m+1} \setminus \partial \Omega} \right) \\
&\lesssim \left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u \right\|_{1, G_{m+1}} \left(\left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u \right\|_{1/2, \partial G_{m+1} \setminus \partial \Omega} \right. \\
&\quad \left. + \left\| \lambda_{h_k} u_{h_k} - \lambda u \right\|_{0, \partial \Omega \cap \partial G_{m+1}} \right) \\
&\quad + \left\| \lambda_{h_k} u_{h_k} - \lambda u \right\|^2_{0, \partial \Omega \cap \partial G_{m+1}} + \left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u \right\|^2_{1/2, \partial G_{m+1} \setminus \partial \Omega}, \tag{4.25}
\end{aligned}
$$

where the hidden coefficient depends on the coefficient $\mathcal{A}$, ϕ and Ω, but is independent of k and mesh size.
Set

$$
\begin{aligned}
x &:= \left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u \right\|_{1, G_{m+1}}, \\
m &:= \left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u \right\|_{1/2, \partial G_{m+1} \setminus \partial \Omega} + \left\| \lambda_{h_k} u_{h_k} - \lambda u \right\|_{0, \partial \Omega \cap \partial G_{m+1}}, \\
n &:= \left\| \lambda_{h_k} u_{h_k} - \lambda u \right\|^2_{0, \partial \Omega \cap \partial G_{m+1}} + \left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u \right\|^2_{1/2, \partial G_{m+1} \setminus \partial \Omega}.
\end{aligned}
$$

Then (4.25) can be simplified to

$$x^2 \leq Cmx + Cn,$$

so we obtain

$$x \leq \frac{Cm + \sqrt{C^2 m^2 + 4Cn}}{2} \lesssim m + \sqrt{n}. \tag{4.26}$$

Since

$$
\begin{aligned}
\left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u \right\|^2_{\frac{1}{2}, \partial G_{m+1} \setminus \partial \Omega} &\lesssim \sum_{j=1}^{m} \left\| \tilde{u}^{j}_{h_{k+1}} - P_{h_{k+1}} u \right\|^2_{\frac{1}{2}, \partial G_j} \\
&\lesssim \sum_{j=1}^{m} \left\| \tilde{u}^{j}_{h_{k+1}} - P_{h_{k+1}} u \right\|^2_{1, G_j}. \tag{4.27}
\end{aligned}
$$

Combining (4.26) and (4.27), we can derive

$$\left\| \tilde{u}^{m+1}_{h_{k+1}} - P_{h_{k+1}} u \right\|^2_{1, G_{m+1}} \lesssim \sum_{j=1}^{m} \left\| \tilde{u}^{j}_{h_{k+1}} - P_{h_{k+1}} u \right\|^2_{1, G_j} + \left\| \lambda_{h_k} u_{h_k} - \lambda u \right\|^2_{0, \partial \Omega \cap \partial G_{m+1}}, \tag{4.28}$$

where the hidden coefficient depends on the coefficient $\mathcal{A}$, ϕ and Ω, but is independent of k and mesh size.

Part 3. From (4.2), (4.13), (4.22) and (4.28), we obtain

$$\left\|\tilde{u}_{h_{k+1}} - P_{h_{k+1}}u\right\|_{1,\Omega}^2 \lesssim \sum_{j=1}^m \left(r_{\Omega_j}^2(H)\left\|P_{h_{k+1}}u - u_{h_k}\right\|_{1,\Omega_j}^2 + \left\|P_{h_{k+1}}u - u_{h_k}\right\|_{0,\Omega_j}^2 \right) + \left\|\lambda_{h_k}u_{h_k} - \lambda u\right\|_{0,\partial\Omega}^2$$

$$\lesssim r_\Omega^2(H)\left\|P_{h_{k+1}}u - u_{h_k}\right\|_{1,\Omega}^2 + \left\|P_{h_{k+1}}u - u_{h_k}\right\|_{0,\Omega}^2 + \left\|\lambda_{h_k}u_{h_k} - \lambda u\right\|_{0,\partial\Omega}^2$$

$$\lesssim r_\Omega^2(H)\|u_{h_k} - u\|_{1,\Omega}^2 + r_\Omega^2(H)\left\|u - P_{h_{k+1}}u\right\|_{1,\Omega}^2 + \left\|P_{h_{k+1}}u - u_{h_k}\right\|_{0,\Omega}^2$$

$$+ |\lambda - \lambda_{h_k}|^2 + \|u_{h_k} - u\|_{0,\partial\Omega}^2. \tag{4.29}$$

Next, we come to estimate $\left\|P_{h_{k+1}}u - u_{h_k}\right\|_{0,\Omega}^2$ involved in (4.29),

$$\left\|P_{h_{k+1}}u - u_{h_k}\right\|_{0,\Omega} \leq \left\|P_{h_{k+1}}u - u\right\|_{0,\Omega} + \left\|u - P_{H,h_k}u\right\|_{0,\Omega} + \left\|P_{H,h_k}u - u_{h_k}\right\|_{0,\Omega}$$

$$\lesssim r_\Omega(h_{k+1})\delta_{h_{k+1}} + r_\Omega(H)\|u - P_{H,h_k}u\|_{1,\Omega} + \|P_{H,h_k}u - u_{h_k}\|_{1,\Omega}$$

$$\lesssim r_\Omega(h_{k+1})\delta_{h_{k+1}} + r_\Omega(H)\|u - u_{h_k}\|_{1,\Omega} + \|P_{H,h_k}u - u_{h_k}\|_{1,\Omega}, \tag{4.30}$$

where the projection operator $P_{H,h_k} : H^1(\Omega) \to S^{H,h_k}$ is defined by

$$a(u - P_{H,h_k}u, v) = 0, \qquad \forall v \in S^{H,h_k}. \tag{4.31}$$

Based on (4.31), we can derive the following estimate

$$\|P_{H,h_k}u - u_{h_k}\|_{1,\Omega}^2 \lesssim a(P_{H,h_k}u - u_{h_k}, P_{H,h_k}u - u_{h_k})$$

$$= a(u - u_{h_k}, P_{H,h_k}u - u_{h_k})$$

$$= b(\lambda u - \lambda_{h_k}u_{h_k}, P_{H,h_k}u - u_{h_k})$$

$$\lesssim \left(\|u - u_{h_k}\|_{0,\partial\Omega} + |\lambda - \lambda_{h_k}| \right)\|P_{H,h_k}u - u_{h_k}\|_{1,\Omega}, \tag{4.32}$$

that is

$$\|P_{H,h_k}u - u_{h_k}\|_{1,\Omega} \lesssim \|u - u_{h_k}\|_{0,\partial\Omega} + |\lambda - \lambda_{h_k}|$$

$$\lesssim \rho_\Omega(H)\|u - u_{h_k}\|_{1,\Omega} + |\lambda - \lambda_{h_k}|. \tag{4.33}$$

From (4.29), (4.30) and (4.33), we have

$$\left\|\tilde{u}_{h_{k+1}} - P_{h_{k+1}}u\right\|_{1,\Omega} \lesssim (\rho_\Omega(H) + r_\Omega(H))\|u_{h_k} - u\|_{1,\Omega} + r_\Omega(H)\left\|u - P_{h_{k+1}}u\right\|_{1,\Omega}$$

$$+ |\lambda - \lambda_{h_k}| + r_\Omega(h_{k+1})\delta_{h_{k+1}}(\lambda). \tag{4.34}$$

where the hidden coefficient depends on the coefficient $\mathcal{A}$, ϕ, Ω and the exact eigenpair, but is independent of k and mesh size.

Part 4. Then from (4.8), (4.12) and (4.34), there holds

$$\|u - \tilde{u}_{h_{k+1}}\|_{1,\Omega} \lesssim \|u - P_{h_{k+1}}u\|_{1,\Omega} + |\lambda - \lambda_{h_k}| + (\rho_\Omega(H) + r_\Omega(H))\|u_{h_k} - u\|_{1,\Omega} + r_\Omega(h_{k+1})\delta_{h_{k+1}}(\lambda)$$

$$\lesssim (\rho_\Omega(H) + r_\Omega(H))\varepsilon_{h_k}(\lambda) + \varepsilon_{h_k}^2(\lambda) + \delta_{h_{k+1}}(\lambda)$$

$$\lesssim \varepsilon_{h_{k+1}},$$

where $\varepsilon_{h_{k+1}} := (\rho_\Omega(H) + r_\Omega(H))\varepsilon_{h_k}(\lambda) + \varepsilon_{h_k}^2(\lambda) + \delta_{h_{k+1}}(\lambda)$, and the hidden coefficient depends on the coefficient $\mathcal{A}$, ϕ, Ω and the exact eigenpair, but is independent of k and mesh size.

From Lemma 3.1, we have

$$\left\| u - u_{h_{k+1}} \right\|_{1,\Omega} \lesssim \left\| u - \tilde{u}_{h_{k+1}} \right\|_{1,\Omega} \lesssim \varepsilon_{h_{k+1}}. \tag{4.35}$$

Since

$$\tilde{\rho}_\Omega(H) = \sup_{f \in L^2(\partial\Omega), \|f\|_{0,\partial\Omega}=1} \inf_{v_h \in S^{H,h_{k+1}}} \|Tf - v_h\|_{1,\Omega} \le \rho_\Omega(H),$$

further using Lemma 3.1 and (4.35), we can obtain

$$\left\| u - u_{h_{k+1}} \right\|_{0,\partial\Omega} \lesssim \tilde{\rho}_\Omega(H) \left\| u - u_{h_{k+1}} \right\|_{1,\Omega}$$
$$\le \rho_\Omega(H) \left\| u - u_{h_{k+1}} \right\|_{1,\Omega},$$

and

$$\left| \lambda - \lambda_{h_{k+1}} \right| \lesssim \left\| u - u_{h_{k+1}} \right\|_{1,\Omega}^2 \lesssim \varepsilon_{h_{k+1}}^2(\lambda).$$

Then we complete the proof.

$\square$

4.2. Local defect-correction method based on multilevel discretization

Based on Algorithm 4.1, we come to propose the local defect-correction method based on multilevel discretization for solving the Steklov eigenvalue problem in this subsection. Suppose $\mathcal{T}_{h_k}(\Omega)$ is obtained from $\mathcal{T}_{h_{k-1}}(\Omega)$ through regular refinement such that the mesh sizes satisfy $h_k = \frac{1}{q} h_{k-1}$, $k \ge 2$, and the following relationship holds true for a p-order finite element method:

$$\delta_{h_k}(\lambda) \approx \frac{1}{q^p} \delta_{h_{k-1}}(\lambda), \quad q > 1. \tag{4.36}$$

Based on Algorithm 4.1, we can obtain the following local defect-correction algorithm.

Algorithm 4.3. Local defect-correction method based on multilevel discretization.

(1) Find $(\lambda_{h_1}, u_{h_1}) \in \mathbb{R} \times S^{h_1}(\Omega)$ such that $b(u_{h_1}, u_{h_1}) = 1$ and

$$a(u_{h_1}, v_{h_1}) = \lambda_{h_1} b(u_{h_1}, v_{h_1}), \qquad \forall v_{h_1} \in V_{h_1}.$$

(2) For $k = 1, \cdots, n-1$, we obtain a new eigenpair approximation $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathbb{R} \times S^{h_{k+1}}(\Omega)$ through:

$$(\lambda_{h_{k+1}}, u_{h_{k+1}}) = \text{Correction}(S^H(\Omega), \lambda_{h_k}, u_{h_k}, S^{h_{k+1}}(\Omega)).$$

End For.

Finally, we obtain $(\lambda_{h_n}, u_{h_n}) \in \mathbb{R} \times S^{h_n}(\Omega)$ in the finest space.

Theorem 4.4. *If Assumptions A.1–A.3 hold, there exists an eigenfunction $u \in M(\lambda)$ such that the resultant eigenpair approximation (λ_{h_n}, u_{h_n}) of Algorithm 4.3 has the following error estimate:*

$$\|u - u_{h_n}\|_{1,\Omega} \lesssim \delta_{h_n}(\lambda), \tag{4.37}$$
$$\|u - u_{h_n}\|_{0,\partial\Omega} \lesssim \rho_\Omega(H) \delta_{h_n}(\lambda), \tag{4.38}$$
$$|\lambda - \lambda_{h_n}| \lesssim \delta_{h_n}^2(\lambda), \tag{4.39}$$

when the mesh size of $\mathcal{T}_H$ is small enough such that $Cq^p(\rho_\Omega(H) + r_\Omega(H)) < 1$ for a mesh independent constant C which only depends on the coefficient $\mathcal{A}$, ϕ, Ω and the exact eigenpair.

Proof. Since we solve the Steklov eigenvalue problem directly in the initial space V_{h_1}, using Lemma 3.1, there exists an eigenfunction $u \in M(\lambda)$ such that

$$\|u - u_{h_1}\|_{1,\Omega} \lesssim \delta_{h_1}(\lambda), \tag{4.40}$$

$$\|u - u_{h_1}\|_{0,\partial\Omega} \lesssim \rho_\Omega(h_1)\delta_{h_1}(\lambda), \tag{4.41}$$

$$|\lambda - \lambda_{h_1}| \lesssim \delta_{h_1}^2(\lambda). \tag{4.42}$$

Let $\varepsilon_{h_1}(\lambda) := \delta_{h_1}(\lambda)$. From Theorem 4.2, the following estimates hold for $1 \leq k \leq n-1$:

$$\begin{aligned} \varepsilon_{h_{k+1}}(\lambda) &\lesssim (\rho_\Omega(H) + r_\Omega(H))\varepsilon_{h_k}(\lambda) + \varepsilon_{h_k}^2(\lambda) + \delta_{h_{k+1}}(\lambda) \\ &\lesssim (\rho_\Omega(H) + r_\Omega(H))\varepsilon_{h_k}(\lambda) + \delta_{h_{k+1}}(\lambda). \end{aligned} \tag{4.43}$$

Using (4.43) iteratively, we have

$$\begin{aligned} \varepsilon_{h_n}(\lambda) &\lesssim (\rho_\Omega(H) + r_\Omega(H))\varepsilon_{h_{n-1}}(\lambda) + \delta_{h_n}(\lambda) \\ &\lesssim (\rho_\Omega(H) + r_\Omega(H))^2\varepsilon_{h_{n-2}}(\lambda) + (\rho_\Omega(H) + r_\Omega(H))\delta_{h_{n-1}}(\lambda) + \delta_{h_n}(\lambda) \\ &\lesssim \sum_{k=1}^n (\rho_\Omega(H) + r_\Omega(H))^{n-k}\delta_{h_k}(\lambda). \end{aligned} \tag{4.44}$$

From Theorem 4.2, (4.36) and (4.44)

$$\begin{aligned} \|u - u_{h_n}\|_{1,\Omega} \lesssim \varepsilon_{h_n}(\lambda) &\lesssim \sum_{k=1}^n (\rho_\Omega(H) + r_\Omega(H))^{n-k}\delta_{h_k}(\lambda) \\ &\lesssim \sum_{k=1}^n (q^p(\rho_\Omega(H) + r_\Omega(H)))^{n-k}\delta_{h_n}(\lambda) \\ &\lesssim \frac{\delta_{h_n}(\lambda)}{1 - q^p(\rho_\Omega(H) + r_\Omega(H))} \lesssim \delta_{h_n}(\lambda), \end{aligned}$$

which is just the desired result (4.37). The remaining two results (4.38) and (4.39) can be obtained directly from Lemma 3.1. $\qquad\square$

4.3. Estimate of computational work

In Algorithm 4.3, we use a local defect-correction scheme to solve the Steklov eigenvalue problem. As a result, solving the Steklov eigenvalue problem requires almost the same work as solving the corresponding linear boundary value problem. In this subsection, we aim to present the computation work of Algorithm 4.3. Let

$$N_k^j = \dim S_{h_k}^0(\Omega_j) \quad \text{and} \quad N_k = \dim S^{h_k}(\Omega) \quad \text{for} \quad k = 1, \cdots n, \ j = 1, \cdots, m+1.$$

The following then holds

$$N_k^j \approx \left(\frac{1}{q}\right)^{d(n-k)} N_n^j \quad \text{and} \quad N_k^j \approx \left(\frac{N_k}{m}\right) \quad \text{for} \quad k = 1, \cdots, n, \ j = 1, \cdots, m+1. \tag{4.45}$$

Theorem 4.5. *We assume that solving the Steklov eigenvalue problem in the coarsest spaces $S^H(\Omega)$ and $S^{h_1}(\Omega)$ requires work $O(M_H)$ and $O(M_{h_1})$, respectively, and solving the boundary value problem in each space $S_{h_k}^0(\Omega_j)$ requires work $O(N_k^j)$, where $k = 1, \cdots, n$, $j = 1, \cdots, m+1$. If Assumptions A.1–A.3 hold, then the computational work of each computing node involved in Algorithm 4.3 can then be controlled using $O(N_n/m + M_H \log N_n + M_{h_1})$. Furthermore, the computational work becomes $O(N_n/m)$ provided that $M_H \ll N_n/m, M_{h_1} \leq N_n/m$.*

Proof. Let us use W_k to denote the computational work of Algorithm 4.3 in finite element space $S^{h_k}(\Omega)$. Then we have

$$W_1 = O(M_{h_1}) \quad \text{and} \quad W_k = O(N_k/m + M_H) \quad \text{for} \quad k \geq 2. \tag{4.46}$$

Using (4.45) and (4.46), the total work of Algorithm 4.3 can be estimated as follows:

$$\begin{aligned}
\text{Total work} &= O\left(M_{h_1} + \sum_{k=2}^{n} (N_k/m + M_H) \right) \\
&= O\left(\sum_{k=2}^{n} N_k/m + (n-1)M_H + M_{h_1} \right) \\
&= O\left(\sum_{k=2}^{n} \left(\frac{1}{q}\right)^{d(n-k)} N_n/m + (n-1)M_H + M_{h_1} \right) \\
&= O(N_n/m + M_H \log N_n + M_{h_1}).
\end{aligned} \tag{4.47}$$

Then we derive the desired result. Additionally, if $M_H \ll N_n/m$, $M_{h_1} \leq N_n/m$, equation (4.47) can be controlled by $O(N_n/m)$. $\qquad\square$

5. NUMERICAL RESULT

In this section, we propose two numerical experiments to demonstrate the efficiency of Algorithm 4.3.

5.1. Example 1

In the first example, using Algorithm 4.3, we solve the following Steklov eigenvalue problem in the computing domain $\Omega = (0,1)^2$: Find $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$ such that

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega, \\ \nabla u \cdot n = \lambda u, & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

In the first step, we divide the computing domain Ω into four disjoint subdomains D_1, D_2, D_3, D_4 which satisfy $\cup_{j=1}^4 \bar{D}_j = \bar{\Omega}$, $D_i \cap D_j = \emptyset$. Herein, we set $D_1 = (0.0, 0.5) \times (0.0, 0.5)$, $D_2 = (0.5, 1.0) \times (0.0, 0.5)$, $D_3 = (0.0, 0.5) \times (0.5, 1.0)$, $D_4 = (0.5, 1.0) \times (0.5, 1.0)$. We then set G_j and Ω_j satisfying $G_j \subset\subset D_j \subset \Omega_j \subset \Omega$. Herein, we set $\Omega_1 = (0.0, 0.625) \times (0.0, 0.625)$, $\Omega_2 = (0.375, 1.0) \times (0.0, 0.625)$, $\Omega_3 = (0.0, 0.625) \times (0.375, 1.0)$, $\Omega_4 = (0.375, 1.0) \times (0.375, 1.0)$, $G_1 = (0.0, 0.375) \times (0.0, 0.375)$, $G_2 = (0.625, 1.0) \times (0.0, 0.375)$, $G_3 = (0.0, 0.375) \times (0.625, 1.0)$, $G_4 = (0.625, 1.0) \times (0.625, 1.0)$, and $G_5 = \Omega \backslash (\cup_{j=1}^4 \bar{G}_j)$.

In this example, we investigate the first eigenvalue. We use the linear finite element space which comprises piecewise linear polynomials on a nested multigrid mesh sequence. The mesh sequence is produced through uniform refinement. Therefore, the refinement index is $q = 2$. In our numerical experiment, the coarsest space $S_H(\Omega)$ is the same as the initial space $S_{h_1}(\Omega)$. The corresponding mesh size is set to $H = h_1 = 1/8$. The initial mesh is presented in Figure 3.

In this experiment, both Algorithm 4.3 and the direct finite element method (*i.e.* solve the Steklov eigenvalue problem directly in the final finite element space) are used to solve the Steklov eigenvalue problem (5.1). The corresponding numerical error estimates are presented in Figure 3, which shows that Algorithm 4.3 can derive an optimal estimate similar to that derived using the direct finite element method.

5.2. Example 2

In the second example, we solve the following Steklov eigenvalue problem by Algorithm 4.3 in the computing domain $\Omega = (0,1)^3$: Find $(\lambda, u) \in \mathbb{R} \times H^1(\Omega)$ such that

$$\begin{cases} -\nabla \cdot (\mathcal{A}\nabla u) + \phi u = 0, & \text{in } \Omega, \\ (\mathcal{A}\nabla u) \cdot n = \lambda u, & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$
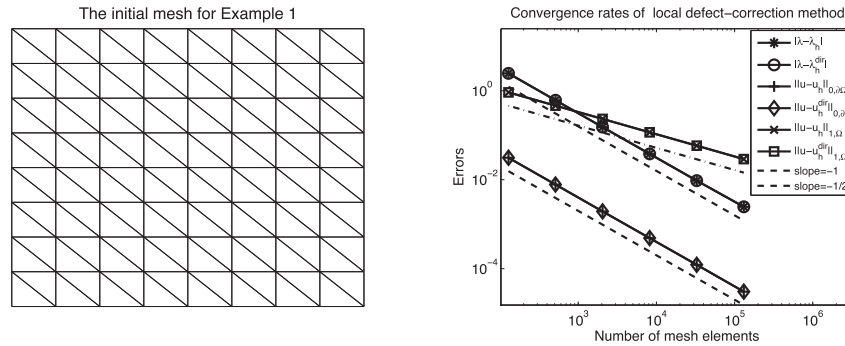
FIGURE 3. The initial mesh (*left*) and error estimates (*right*) of Algorithm 4.3 for Example 1.

where

$$\mathcal{A} = \begin{pmatrix} 1 + \left(x_1 - \frac{1}{2}\right)^2 & \left(x_1 - \frac{1}{2}\right)\left(x_2 - \frac{1}{2}\right) & \left(x_1 - \frac{1}{2}\right)\left(x_3 - \frac{1}{2}\right) \\ \left(x_1 - \frac{1}{2}\right)\left(x_2 - \frac{1}{2}\right) & 1 + \left(x_2 - \frac{1}{2}\right)^2 & \left(x_2 - \frac{1}{2}\right)\left(x_3 - \frac{1}{2}\right) \\ \left(x_1 - \frac{1}{2}\right)\left(x_3 - \frac{1}{2}\right) & \left(x_2 - \frac{1}{2}\right)\left(x_3 - \frac{1}{2}\right) & 1 + \left(x_3 - \frac{1}{2}\right)^2 \end{pmatrix},$$

and

$$\phi = e^{\left(x_1 - \frac{1}{2}\right)\left(x_2 - \frac{1}{2}\right)\left(x_3 - \frac{1}{2}\right)}.$$

In the first step, we divide the computing domain Ω into eight disjoint subdomains $D_1, \cdots, D_8$ which satisfy $\cup_{j=1}^{8} \bar{D}_j = \bar{\Omega}$, $D_i \cap D_j = \emptyset$. Herein, we set $D_1 = (0.0, 0.5) \times (0.0, 0.5) \times (0.0, 0.5)$, $D_2 = (0.5, 1.0) \times (0.0, 0.5) \times (0.0, 0.5)$, $D_3 = (0.0, 0.5) \times (0.5, 1.0) \times (0.0, 0.5)$, $D_4 = (0.5, 1.0) \times (0.5, 1.0) \times (0.0, 0.5)$, $D_5 = (0.0, 0.5) \times (0.0, 0.5) \times (0.5, 1.0)$, $D_6 = (0.5, 1.0) \times (0.0, 0.5) \times (0.5, 1.0)$, $D_7 = (0.0, 0.5) \times (0.5, 1.0) \times (0.5, 1.0)$, $D_8 = (0.5, 1.0) \times (0.5, 1.0) \times (0.5, 1.0)$. Then set G_j and Ω_j satisfying $G_j \subset\subset D_j \subset \Omega_j \subset \Omega$. Herein, we set $\Omega_1 = (0.0, 0.625) \times (0.0, 0.625) \times (0.0, 0.625)$, $\Omega_2 = (0.375, 1.0) \times (0.0, 0.625) \times (0.0, 0.625)$, $\Omega_3 = (0.0, 0.625) \times (0.375, 1.0) \times (0.0, 0.625)$, $\Omega_4 = (0.375, 1.0) \times (0.375, 1.0) \times (0.0, 0.625)$, $\Omega_5 = (0.0, 0.625) \times (0.0, 0.625) \times (0.375, 1.0)$, $\Omega_6 = (0.375, 1.0) \times (0.0, 0.625) \times (0.375, 1.0)$, $\Omega_7 = (0.0, 0.625) \times (0.375, 1.0) \times (0.375, 1.0)$, $\Omega_8 = (0.375, 1.0) \times (0.375, 1.0) \times (0.375, 1.0)$, $G_1 = (0.0, 0.375) \times (0.0, 0.375) \times (0.0, 0.375)$, $G_2 = (0.625, 1.0) \times (0.0, 0.375) \times (0.0, 0.375)$, $G_3 = (0.0, 0.375) \times (0.625, 1.0) \times (0.0, 0.375)$, $G_4 = (0.625, 1.0) \times (0.625, 1.0) \times (0.0, 0.375)$, $G_5 = (0.0, 0.375) \times (0.0, 0.375) \times (0.625, 1.0)$, $G_6 = (0.625, 1.0) \times (0.0, 0.375) \times (0.625, 1.0)$, $G_7 = (0.0, 0.375) \times (0.625, 1.0) \times (0.625, 1.0)$, $G_8 = (0.625, 1.0) \times (0.625, 1.0) \times (0.625, 1.0)$ and $G_9 = \Omega \backslash \left(\cup_{j=1}^{8} \bar{G}_j \right)$.

In this example, we investigate the first eigenvalue. We also use the linear finite element space on the mesh sequence that produced through uniform refinement, so the refinement index $q = 2$. In our numerical experiment, the coarsest space $S_H(\Omega)$ is the same as the initial space $S_{h_1}(\Omega)$. The corresponding mesh size is set to be $H = h_1 = 1/8$. The initial mesh is presented in Figure 4.

In this experiment, both Algorithm 4.3 and the direct finite element method are used to solve the Steklov eigenvalue problem (5.2). The corresponding numerical error estimates are presented in Figure 5, which shows that Algorithm 4.3 can derive an optimal estimate similar to that derived using the direct finite element method.

In this example, we present the computational time of Algorithm 4.3 to demonstrate the efficiency. In order to show the efficiency of Algorithm 4.3, we also test the direct finite element method, the two-grid method for Steklov eigenvalue problems designed in [45] and the the full multigrid method for Steklov eigenvalue problems designed in [49]. The corresponding results are depicted in Figure 5. Figure 5 intuitively shows that Algorithm 4.3 has a linear complexity, making it significantly more advantageous than the direct finite element method, the two-grid method and the full multigrid method.

In addition, we also test Algorithm 4.3 for the 10 smallest eigenvalues. Figure 6 demonstrates the corresponding error estimates and computational time for Algorithm 4.3, the direct finite element method, the two-grid
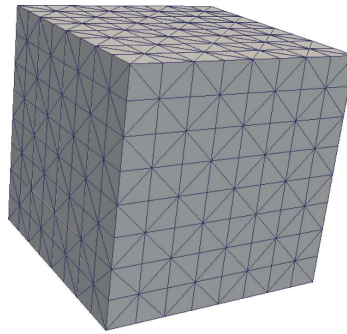
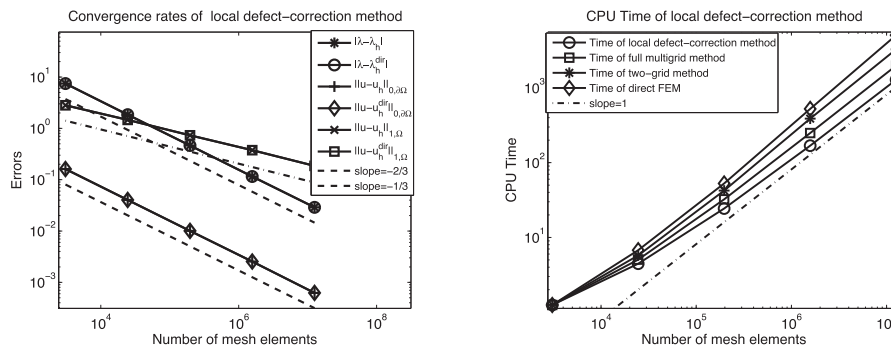FIGURE 4. The initial mesh of Algorithm 4.3 for Example 2.

FIGURE 5. Errors (*left*) and computational time (*right*) of Algorithm 4.3 for Example 2.
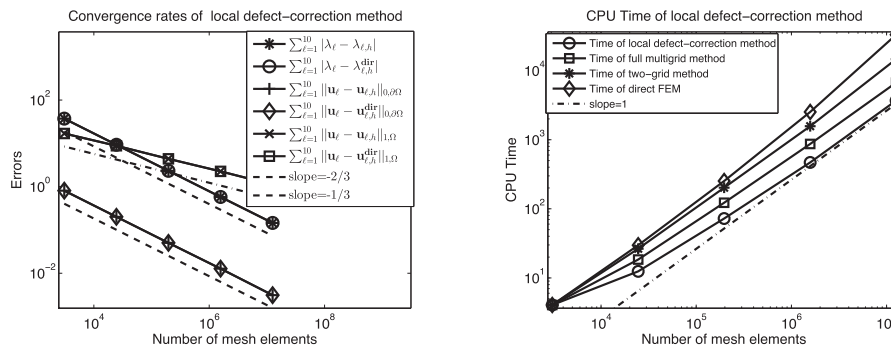

FIGURE 6. Errors (*left*) and computational time (*right*) of Algorithm 4.3 for Example 2.

method [45] and the full multigrid method [49], which shows that Algorithm 4.3 can still work for multiple eigenvalues. Besides, Algorithm 4.3 can derive the optimal error estimates with the linear complexity and Algorithm 4.3 is more efficient than other adopted algorithms. For this example with mesh size 10^7, our method is about 10 times faster than the direct finite element method, 4 times faster than the two-grid method and 2 times faster than the full multigrid method.

6. Concluding remark

In this study, we design a local defect-correction method based on multilevel discretization for solving the Steklov eigenvalue problem arising from the scalar second order positive definite partial differential equations. It is well known that solving large-scale Steklov eigenvalue problems directly in the finite element space is quite time-consuming. Through the novel algorithm presented in this study, solving efficiency can be improved using two approaches. The first approach is to avoid solving large-scale Steklov eigenvalue problem by transforming it into linear boundary value problems in a multigrid space sequence and small-scale Steklov eigenvalue problems in a low-dimensional correction space. The second approach involves decomposing the linear boundary value problem into small-scale equations through the local defect-correction technique. Rigorous theoretical analysis are proposed in this paper and some numerical results are presented to support our theoretical results.

As we can see in our numerical experiments, Algorithm 4.3 can be extended to solve the multiple eigenvalues. For completeness, we give the following local defect-correction method to solve the multiple eigenvalue λ with multiplicity q, *i.e.* $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+q-1}$. Similarly, we first give a type of one correction step for the given eigenpair approximations $\{\lambda_{\ell,h_k}, u_{\ell.h_k}\}_{\ell=i}^{i+q-1}$.

Algorithm 6.1. One Step of the local defect-correction method for multiple eigenvalues.

(1) For $\ell = i, \cdots, i+q-1$, solve the following linear boundary value problem in each subdomain: Find $e_{\ell,h_{k+1}}^j \in S_{h_{k+1}}^0(\Omega_j)$, $j = 1, 2, \cdots, m$ such that

$$a\left(e_{\ell,h_{k+1}}^j, v_{h_{k+1}}\right) = \lambda_{\ell,h_k} b\left(u_{\ell,h_k}, v_{h_{k+1}}\right) - a\left(u_{\ell,h_k}, v_{h_{k+1}}\right), \qquad \forall v_{h_{k+1}} \in S_{h_{k+1}}^0(\Omega_j). \tag{6.1}$$

Set $\tilde{u}_{\ell,h_{k+1}}^j = u_{\ell,h_k} + e_{\ell,h_{k+1}}^j \in S^{h_{k+1}}(\Omega_j)$, $\ell = i, \cdots, i+q-1$.

(2) For $\ell = i, \cdots, i+q-1$, solve the following boundary value problem in G_{m+1}: Find $\tilde{u}_{\ell,h_{k+1}}^{m+1} \in S^{h_{k+1}}(G_{m+1})$ such that $\tilde{u}_{\ell,h_{k+1}}^{m+1}|_{\partial G_j \cap \partial G_{m+1}} = \tilde{u}_{\ell,h_{k+1}}^j, j = 1, \cdots, m$ and

$$a\left(\tilde{u}_{\ell,h_{k+1}}^{m+1}, v_{h_{k+1}}\right) = \lambda_{\ell,h_k} b\left(u_{\ell,h_k}, v_{h_{k+1}}\right), \qquad \forall v_{h_{k+1}} \in S_{h_{k+1}}^0(G_{m+1}). \tag{6.2}$$

(3) Construct $\tilde{u}_{\ell,h_{k+1}} \in S^{h_{k+1}}(\Omega)$ such that $\tilde{u}_{\ell,h_{k+1}} = \tilde{u}_{\ell,h_{k+1}}^j$ in G_j, $\ell = i, \cdots, i+q-1$, $j = 1, \cdots, m+1$.

(4) Define a new finite element space $S^{H,h_{k+1}} = S^H(\Omega) + \mathrm{span}\{\tilde{u}_{i,h_{k+1}}, \cdots, \tilde{u}_{i+q-1,h_{k+1}}\}$ and solve the following small-scale Steklov eigenvalue problem: Find $\left(\lambda_{\ell,h_{k+1}}, u_{\ell,h_{k+1}}\right) \in \mathbb{R} \times S^{H,h_{k+1}}(\Omega)$, $\ell = i, \cdots, i+q-1$, such that $b\left(u_{\ell,h_{k+1}}, u_{\ell,h_{k+1}}\right) = 1$ and

$$a\left(u_{\ell,h_{k+1}}, v_{H,h_{k+1}}\right) = \lambda_{\ell,h_{k+1}} b\left(u_{\ell,h_{k+1}}, v_{H,h_{k+1}}\right), \qquad \forall v_{H,h_{k+1}} \in S^{H,h_{k+1}}(\Omega). \tag{6.3}$$

Summarize the above four steps into

$$\left\{\lambda_{\ell,h_{k+1}}, u_{\ell,h_{k+1}}\right\}_{\ell=i}^{i+q-1} = \mathrm{Correction}\left(S^H(\Omega), \{\lambda_{\ell,h_k}, u_{\ell,h_k}\}_{\ell=i}^{i+q-1}, S^{h_{k+1}}(\Omega)\right).$$

Based on Algorithm 6.1, we can obtain the following local defect-correction algorithm based on multilevel discretization for multiple eigenvalues.

Algorithm 6.2. Local defect-correction method based on multilevel discretization.

(1) Find $(\lambda_{\ell,h_1}, u_{\ell,h_1}) \in \mathbb{R} \times S^{h_1}(\Omega)$, $\ell = i, \cdots, i+q-1$, such that $b(u_{\ell,h_1}, u_{\ell,h_1}) = 1$ and

$$a(u_{\ell,h_1}, v_{h_1}) = \lambda_{\ell,h_1} b(u_{\ell,h_1}, v_{h_1}), \qquad \forall v_{h_1} \in V_{h_1}.$$

(2) For $k = 1, \cdots, n-1$, we obtain the new eigenpair approximation $\left(\lambda_{\ell,h_{k+1}}, u_{\ell,h_{k+1}}\right) \in \mathbb{R} \times S^{h_{k+1}}(\Omega)$ through:

$$\left\{\lambda_{\ell,h_{k+1}}, u_{\ell,h_{k+1}}\right\}_{\ell=i}^{i+q-1} = \text{Correction}\left(S^H(\Omega), \{\lambda_{\ell,h_k}, u_{\ell,h_k}\}_{\ell=i}^{i+q-1}, S^{h_{k+1}}(\Omega)\right).$$

End For.

Finally, we obtain $(\lambda_{\ell,h_n}, u_{\ell,h_n}) \in \mathbb{R} \times S^{h_n}(\Omega)$, $\ell = i, \cdots, i+q-1$ in the finest space.

We can also give the error analysis for Algorithm 6.2 in a similar way to that used in Section 4 based on the conclusions for multiple eigenvalues [6,18].

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