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Further inequalities for log-convex functions related to Hermite-Hadamard result

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Abstract

Some unweighted and weighted inequalities of Hermite-Hadamard type for log-convex functions defined on real intervals are given.

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1. Introduction

A function $f : I \to (0, \infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log f$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality:

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}. \tag{1.1}$$

We note that if f and g are convex and g is increasing, then $g \circ f$ is convex; moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex, but the converse may not necessarily be true. This follows directly from (1.1) because, by the *arithmetic-geometric mean inequality*, we have

$$[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1-t) f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Let us recall the *Hermite-Hadamard inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{2}, \tag{1.2}$$

where $f : I \subseteq \mathbf{R} \to \mathbf{R}$ is a convex function on the interval I, $a, b \in I$ and $a < b$.

For related results, see [1]-[22], [25]-[29], [30]-[40] and [41]-[53].

Note that if we apply the above inequality for the log-convex functions $f : I \to (0, \infty)$, we have that

$$\ln\left[f\left(\frac{a+b}{2}\right)\right] \leq \frac{1}{b-a} \int_a^b \ln f(x)\, dx \leq \frac{\ln f(a) + \ln f(b)}{2}, \tag{1.3}$$

from which we get

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a} \int_a^b \ln f(x)\, dx\right] \leq \sqrt{f(a) f(b)} \tag{1.4}$$

that is an inequality of Hermite-Hadamard's type for log-convex functions.

By using simple properties of log-convex functions Dragomir and Mond proved in 1998 the following result [32].

Theorem 1. *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequality:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \sqrt{f(x)\, f(a+b-x)}dx \le \sqrt{f(a)\, f(b)}. \tag{1.5}$$

The inequality between the first and second term in (1.5) may be improved as follows [32]. A different upper bound for the middle term in (1.5) can be also provided.

Theorem 2. *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\, dx\right] \\ &\le \tfrac{1}{b-a}\textstyle\int_a^b \sqrt{f(x)\, f(a+b-x)}dx \\ &\le \tfrac{1}{b-a}\textstyle\int_a^b f(x)\, dx \le L(f(a), f(b)), \end{aligned} \tag{1.6}$$

where $L(p, q)$ is the logarithmic mean of the strictly positive real numbers p, q, i.e.,

$$L(p,q) := \frac{p-q}{\ln p - \ln q} \text{ if } p \neq q \text{ and } L(p,p) := p.$$

The last inequality in (1.6) was obtained in a different context in [43].
As shown in [59], the following result also holds:

Theorem 3. *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequalities:*

$$f\left(\frac{a+b}{2}\right) \le \left(\frac{1}{b-a}\int_a^b \sqrt{f(x)}dx\right)^2 \le \frac{1}{b-a}\int_a^b f(x)\, dx. \tag{1.7}$$

We define the *p-logarithmic mean* as

$$L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \text{with } a \neq b \\ a, \text{ if } a = b \end{cases}$$

for $p \neq 0, -1$ and $a, b > 0$.

In the recent work [28] we generalized the inequality (1.6) as follows:

Theorem 4. *Let $f : [a,b] \to (0,\infty)$ be a log-convex function on $[a,b]$. Then for any $p > 0$ we have the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \tag{1.8}$$

$$\leq \left(\tfrac{1}{b-a}\int_a^b f^p(x) f^p(a+b-x)\,dx\right)^{\frac{1}{2p}}$$

$$\leq \left(\tfrac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}}$$

$$\leq \begin{cases} [L_{2p-1}(f(a), f(b))]^{1-\frac{1}{2p}} [L(f(a), f(b))]^{\frac{1}{2p}}, & p \neq \frac{1}{2}; \\ L(f(a), f(b)), & p = \frac{1}{2}. \end{cases}$$

If $p \in \left(0, \frac{1}{2}\right)$, then we have

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \tag{1.9}$$

$$\begin{aligned} &\leq \left(\tfrac{1}{b-a}\int_a^b f^p(x) f^p(a+b-x)\,dx\right)^{\frac{1}{2p}} \\ &\leq \left(\tfrac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}} \leq \tfrac{1}{b-a}\int_a^b f(x)\,dx. \end{aligned}$$

Remark 1. *If we take in (1.8) $p = 1$, then we get*

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \tag{1.10}$$

$$\begin{aligned} &\leq \left(\tfrac{1}{b-a}\int_a^b f(x) f(a+b-x)\,dx\right)^{\frac{1}{2}} \\ &\leq \left(\tfrac{1}{b-a}\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}} \\ &\leq [A(f(a), f(b))]^{\frac{1}{2}} [L(f(a), f(b))]^{\frac{1}{2}}. \end{aligned}$$

If we take $p = \frac{1}{4}$ in (1.9), then we get

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \tag{1.11}$$

$$\leq \left(\tfrac{1}{b-a}\int_a^b \sqrt[4]{f(x)\,f(a+b-x)\,dx}\right)^2$$

$$\leq \left(\tfrac{1}{b-a}\int_a^b \sqrt{f(x)}dx\right)^2 \leq \tfrac{1}{b-a}\int_a^b f(x)\,dx.$$

This improves the inequality (1.7).

Motivated by the above results, we establish in this paper some new unweighted and weighted inequalities for log-convex functions, some of them improving earlier results. Applications for special means are also provided.

2. New Results

The following result holds.

Theorem 1. *Let $f : [a,b] \to (0,\infty)$ be a log-convex function. Then for every $t \in [0,1]$ we have*

$$\int_a^b f(x)\,dx \geq \int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx \tag{2.1}$$

$$\geq \begin{cases} \frac{1}{1-2t}\displaystyle\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du, \text{ if } t \neq \dfrac{1}{2} \\[2ex] (b-a)\,f\left(\frac{a+b}{2}\right), \text{ if } t = \frac{1}{2}. \end{cases}$$

Proof. The cases $t = 0, \frac{1}{2}, 1$ are obvious.

Assume that $t \in (0,1) \setminus \left\{\frac{1}{2}\right\}$. By the log-convexity of f we have

$$[f(x)]^{1-t}[f(a+b-x)]^t \geq f((1-t)x + t(a+b-x)) \tag{2.2}$$

$$= f((1-2t)x + t(a+b))$$

for any $x \in [a,b]$.

Integrating the inequality (2.2) over x on $[a,b]$ we have

$$\int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx \geq \int_a^b f((1-2t)x+t(a+b))\,dx.$$

Since $t \neq \frac{1}{2}$, then $u := (1-2t)x + t(a+b)$ is a change of variable with $du = (1-2t)\,dx$.

For $x = a$ we get $u = (1-t)a + tb$ and for $x = b$ we get $u = ta + (1-t)b$. Therefore

$$\int_a^b f((1-2t)x+t(a+b))\,dx = \frac{1}{1-2t}\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du$$

and the second inequality in (2.1) is proved.

By the Hölder integral inequality for $p = \frac{1}{1-t}$, $q = \frac{1}{t}$ we have

$$\begin{aligned}
&\textstyle\int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx \\
&\leq \left(\textstyle\int_a^b \left([f(x)]^{1-t}\right)^{\frac{1}{1-t}}dx\right)^{1-t}\left(\textstyle\int_a^b \left([f(a+b-x)]^{t}\right)^{\frac{1}{t}}dx\right)^{t} \\
&= \left(\textstyle\int_a^b f(x)\,dx\right)^{1-t}\left(\textstyle\int_a^b f(a+b-x)\,dx\right)^{t} \\
&= \left(\textstyle\int_a^b f(x)\,dx\right)^{1-t}\left(\textstyle\int_a^b f(x)\,dx\right)^{t} = \textstyle\int_a^b f(x)\,dx,
\end{aligned}$$

that proves the first inequality in (2.1). □

Corollary 1. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function. Then for every $t \in [0,1]\setminus\left\{\frac{1}{2}\right\}$ we have for $p>0$ that*

$$\left(\frac{1}{b-a}\int_a^b f^{2p}(x)\,dx\right)^{\frac{1}{2p}} \tag{2.3}$$

$$
\begin{aligned}
&\geq \left(\tfrac{1}{b-a}\int_a^b \left[f^{2p}(x)\right]^{1-t}\left[f^{2p}(a+b-x)\right]^{t}dx\right)^{\frac{1}{2p}}\\
&\geq \left(\tfrac{1}{(1-2t)(b-a)}\int\limits_{(1-t)a+tb}^{ta+(1-t)b} f^{2p}(u)\,du\right)^{\frac{1}{2p}}\\
&\geq \left(\tfrac{1}{(1-2t)(b-a)}\int\limits_{(1-t)a+tb}^{ta+(1-t)b} f^{p}(u)\,f^{p}(a+b-u)\,dx\right)^{\frac{1}{2p}}\\
&\geq \exp\left[\tfrac{1}{(1-2t)(b-a)}\int\limits_{(1-t)a+tb}^{ta+(1-t)b}\ln f(u)\,du\right]\geq f\left(\tfrac{a+b}{2}\right).
\end{aligned}
$$

Proof. Follows from Theorem 5 applied for the log-convex function f^{2p} with $p>0$ and by Theorem 4 applied for the interval $[(1-t)\,a+tb, ta+(1-t)\,b]$ when $t\in\left(0,\frac{1}{2}\right)$ or $[ta+(1-t)\,b,(1-t)\,a+tb]))$ when $t\in\left(\frac{1}{2},1\right)$). □

If we take $p=1$ in (2.3), then we get

$$
\left(\frac{1}{b-a}\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}}\geq\left(\frac{1}{b-a}\int_a^b\left[f^2(x)\right]^{1-t}\left[f^2(a+b-x)\right]^{t}dx\right)^{\frac{1}{2}}
$$

(2.4)

$$
\begin{aligned}
&\geq \left(\tfrac{1}{(1-2t)(b-a)}\int\limits_{(1-t)a+tb}^{ta+(1-t)b} f^{2}(u)\,du\right)^{\frac{1}{2}}\\
&\geq \left(\tfrac{1}{(1-2t)(b-a)}\int\limits_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,f(a+b-u)\,dx\right)^{\frac{1}{2}}\\
&\geq \exp\left[\tfrac{1}{(1-2t)(b-a)}\int\limits_{(1-t)a+tb}^{ta+(1-t)b}\ln f(u)\,du\right]\geq f\left(\tfrac{a+b}{2}\right).
\end{aligned}
$$

If we take $p=\frac{1}{2}$ in (2.3), then we get

$$
\frac{1}{b-a}\int_a^b f(x)\,dx \tag{2.5}
$$

$$\geq \frac{1}{b-a}\int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx$$
$$\geq \frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du$$
$$\geq \frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} \sqrt{f(u)f(a+b-u)}dx$$
$$\geq \exp\left[\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u)\,du\right] \geq f\left(\frac{a+b}{2}\right).$$

Corollary 2. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function. Then for every $t\in[0,1]\setminus\left\{\frac{1}{2}\right\}$ we have for $p\in\left(0,\frac{1}{2}\right)$ that*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \tag{2.6}$$
$$\geq \frac{1}{b-a}\int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx$$
$$\geq \frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du$$
$$\geq \left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f^{2p}(u)\,du\right)^{\frac{1}{2p}}$$
$$\geq \left(\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f^{p}(u)\,f^{p}(a+b-u)\,du\right)^{\frac{1}{2p}}$$
$$\geq \exp\left[\frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u)\,du\right] \geq f\left(\frac{a+b}{2}\right).$$

Follows by Theorem 5 and Theorem 4 for $p\in\left(0,\frac{1}{2}\right)$.

If we take $p=\frac{1}{4}$ in (2.6), then we get

$$\frac{1}{b-a}\int_a^b f(x)\,dx \tag{2.7}$$

$$\begin{aligned}
&\geq \tfrac{1}{b-a}\int_a^b [f(x)]^{1-t}[f(a+b-x)]^t\,dx \\
&\geq \tfrac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du \\
&\geq \left(\tfrac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} \sqrt{f(u)}du\right)^2 \\
&\geq \left(\tfrac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} \sqrt[4]{f(u)\,f(a+b-u)}du\right)^2 \\
&\geq \exp\left[\tfrac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} \ln f(u)\,du\right] \geq f\left(\tfrac{a+b}{2}\right).
\end{aligned}$$

If we use the inequality

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq \frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)\,du, \tag{2.8}$$

where $t \neq \frac{1}{2}$, for the log-convex function $f:[a,b]\to(0,\infty)$, $f(x)=\frac{1}{x}$, then we get

$$\frac{\ln b-\ln a}{b-a} \geq \frac{\ln(ta+(1-t)b)-\ln((1-t)a+tb)}{(ta+(1-t)b)-(((1-t)a+tb))}$$

which, in terms of logarithmic mean, is

$$L(ta+(1-t)b,(1-t)a+tb) \geq L(a,b) \tag{2.9}$$

for any $t\in[0,1]\setminus\left\{\frac{1}{2}\right\}$.

When $t=\frac{1}{2}$ the inequality (2.9) becomes $A(a,b)\geq L(a,b)$ that is also true.

For $q\neq 0,-1$ define the L_q-*logarithmic mean* as

$$L_q(a,b) := \begin{cases} \left(\frac{b^{q+1}-a^{q+1}}{(q+1)(b-a)}\right)^{\frac{1}{q}} \text{ if } b\neq a, \\ \\ a \text{ if } b=a. \end{cases}$$

If we use the inequality (2.8) for the log-convex function $f:[a,b]\to(0,\infty)$, $f(x)=\frac{1}{x^p}$, with $p>0$, $p\neq 1$, then we have for any $t\in[0,1]\setminus\left\{\frac{1}{2}\right\}$ that

$$\frac{1}{b-a}\int_a^b x^{-p}dx \geq \frac{1}{(1-2t)(b-a)}\int_{(1-t)a+tb}^{ta+(1-t)b} u^{-p}du,$$

i.e.

$$\frac{b^{-p+1}-a^{-p+1}}{(1-p)(b-a)} \geq \frac{(ta+(1-t)b)^{-p+1}-((1-t)a+tb)^{-p+1}}{((1-p))[(ta+(1-t)b)-(((1-t)a+tb))]},$$

which can be written as

$$L_{-p}^{-p}(a,b) \geq L_{-p}^{-p}(ta+(1-t)b,(1-t)a+tb).$$

Therefore we have

$$L_{-p}(ta+(1-t)b,(1-t)a+tb) \geq L_{-p}(a,b) \tag{2.10}$$

for any $p>0$, $p \neq 1$ and for any $t \in [0,1] \setminus \left\{\frac{1}{2}\right\}$. The case $t=\frac{1}{2}$ reduces to the known inequality $A(a,b) \geq L_{-p}(a,b)$ for any $p>0$.

3. Weighted Inequalities

We have the following generalized weighted version of the inequality (1.5).

Theorem 1. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function. If $w:[a,b]\to[0,\infty)$ is integrable and $\int_a^b w(x)\,dx>0$, then*

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x)\,f^p(x)\,f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \leq \sqrt{f(a)\,f(b)} \tag{3.1}$$

for any $p>0$.

In particular, we have

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x)\,f(x)\,f(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}} \leq \sqrt{f(a)\,f(b)}. \tag{3.2}$$

Proof. We know that, see [32] or [35, p. 198], if g is log-convex, then

$$g\left(\frac{a+b}{2}\right) \le \sqrt{g(x)\,g(a+b-x)} \le \sqrt{g(a)\,g(b)} \tag{3.3}$$

for any $x \in [a,b]$.

For any $p > 0$ the function f^{2p} is log-convex and by (3.3) we have

$$f^{2p}\left(\frac{a+b}{2}\right) \le f^p(x)\,f^p(a+b-x) \le f^p(a)\,f^p(b) \tag{3.4}$$

for any $x \in [a,b]$.

If we multiply (3.4) by $w(x) \ge 0$ and integrate, then we get

$$f^{2p}\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx \le \int_a^b w(x)\,f^p(x)\,f^p(a+b-x)\,dx$$
$$\le f^p(a)\,f^p(b)\int_a^b w(x)\,dx$$

namely

$$f^{2p}\left(\frac{a+b}{2}\right) \le \frac{\int_a^b w(x)\,f^p(x)\,f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx} \le f^p(a)\,f^p(b)\,. \tag{3.5}$$

Taking the power $\frac{1}{2p}$ in (3.5) we get the desired result (3.1). □

We also have the inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{\int_a^b w(x)\sqrt{f(x)\,f(a+b-x)}dx}{\int_a^b w(x)\,dx} \le \sqrt{f(a)\,f(b)}, \tag{3.6}$$

that is a weighted version of (1.5).

If we take $p = \frac{1}{4}$ in (3.1), then we get

$$f\left(\frac{a+b}{2}\right) \le \left(\frac{\int_a^b w(x)\sqrt[4]{f(x)\,f(a+b-x)}dx}{\int_a^b w(x)\,dx}\right)^2 \le \sqrt{f(a)\,f(b)}. \tag{3.7}$$

Using Jensen's inequality for the power $p \ge 1$ ($p \in (0,1)$), namely

$$\left(\frac{\int_a^b w(x)\,g(x)\,dx}{\int_a^b w(x)\,dx}\right)^p \le (\ge)\,\frac{\int_a^b w(x)\,g^p(x)\,dx}{\int_a^b w(x)\,dx},$$

we can start the following more precise result:

Corollary 3. *Let $f : [a,b] \to (0,\infty)$ be a log-convex function and $w : [a,b] \to [0,\infty)$ be integrable and $\int_a^b w(x)\,dx > 0$.*

If $p \geq 1$, then

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x) f(x) f(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}} \leq \left(\frac{\int_a^b w(x) f^p(x) f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \leq \sqrt{f(a) f(b)}. \tag{3.8}$$

If $p \in (0,1)$, then

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b w(x) f^p(x) f^p(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2p}} \leq \left(\frac{\int_a^b w(x) f(x) f(a+b-x)\,dx}{\int_a^b w(x)\,dx}\right)^{\frac{1}{2}} \leq \sqrt{f(a) f(b)}. \tag{3.9}$$

Remark 2. *Let $f : [a,b] \to (0,\infty)$ be a log-convex function. We observe that if we take in (3.1) $w(x) = f^{-p}(a+b-x)$, $p > 0$, then we get*

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b f^p(x)\,dx}{\int_a^b f^{-p}(x)\,dx}\right)^{\frac{1}{2p}} \leq \sqrt{f(a) f(b)} \tag{3.10}$$

for any $p > 0$.

In particular, we have the inequalities

$$f\left(\frac{a+b}{2}\right) \leq \left(\frac{\int_a^b f(x)\,dx}{\int_a^b \frac{1}{f(x)}dx}\right)^{\frac{1}{2}} \leq \sqrt{f(a) f(b)}, \tag{3.11}$$

and

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b \sqrt{f(x)}dx}{\int_a^b \frac{1}{\sqrt{f(x)}}dx} \leq \sqrt{f(a) f(b)}. \tag{3.12}$$

If we take in (3.10) $f(x) = x^{-1}$, $x \in [a,b] \subset (0,\infty)$, which is log-convex, then we have

$$\left(\frac{a+b}{2}\right)^{-1} \le \left(\frac{\int_a^b x^{-p}dx}{\int_a^b x^p dx}\right)^{\frac{1}{2p}} \le \left(\sqrt{f(a)\,f(b)}\right)^{-1} \tag{3.13}$$

Observe that

$$\begin{aligned}\frac{\int_a^b x^{-p}dx}{\int_a^b x^p dx} &= \frac{\frac{1}{b-a}\int_a^b x^{-p}dx}{\frac{1}{b-a}\int_a^b x^p dx}\\ &= \frac{[L_{-p}(a,b)]^{-p}}{[L_p(a,b)]^p}\end{aligned}$$

and by (3.13) we have

$$\left(\frac{a+b}{2}\right)^{-1} \le \left(\frac{1}{[L_p(a,b)]\,[L_{-p}(a,b)]}\right)^{\frac{1}{2}} \le \left(\sqrt{f(a)\,f(b)}\right)^{-1}$$

that is equivalent to

$$G(a,b) \le \sqrt{[L_p(a,b)]\,[L_{-p}(a,b)]} \le A(a,b)\,.$$

This can be also written as

$$G(a,b) \le G(L_p(a,b)\,, L_{-p}(a,b)) \le A(a,b)\,. \tag{3.14}$$

If we take in the first inequality (3.14) $p = 1$, then we get

$$G^2(a,b) \le A(a,b)\,L(a,b)\,. \tag{3.15}$$

We have the following weighted version of (1.4).

Theorem 2. *Let $f : [a,b] \to (0,\infty)$ be a log-convex function. If $w : [a,b] \to [0,\infty)$ is integrable and $\int_a^b w(x)\,dx > 0$, then*

$$\begin{aligned} & f\left(\frac{\int_a^b w(x)\,x dx}{\int_a^b w(x)\,dx}\right) \\ \le\ & \exp\left(\frac{\int_a^b w(x)\ln f(x)dx}{\int_a^b w(x)dx}\right) \\ \le\ & [f(b)]^{\frac{1}{b-a}\left(\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}-a\right)}[f(a)]^{\frac{1}{b-a}\left(b-\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}\right)} \\ \le\ & \tfrac{1}{b-a}\left[\left(\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}-a\right)f(b) + \left(b-\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}\right)f(a)\right]. \end{aligned} \tag{3.16}$$

Proof. Since $\ln f$ is convex, then by Jensen's inequality we have

$$\ln f\left(\frac{\int_a^b w(x)\,x dx}{\int_a^b w(x)\,dx}\right) \le \frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}. \tag{3.17}$$

Taking the exponential in (3.17) we get the first inequality in (3.16).

Since $\ln f$ is convex, then

$$\ln f(x) = \ln f\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}a\right) \le \frac{x-a}{b-a}\ln f(b) + \frac{b-x}{b-a}\ln f(a) \tag{3.18}$$

for any $x \in [a,b]$.

By taking the weighted integral mean in (3.18) we get

$$\begin{aligned} &\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx} \\ &\le \tfrac{1}{b-a}\left[\left(\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx} - a\right)\ln f(b) + \left(b - \frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}\right)\ln f(a)\right] \\ &= \ln\left([f(b)]^{\frac{1}{b-a}\left(\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx} - a\right)}[f(a)]^{\frac{1}{b-a}\left(b-\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}\right)}\right). \end{aligned} \tag{3.19}$$

By taking the exponential in (3.19), we get the second inequality in (3.16).

The last part of (3.16) follows by the weighted geometric mean-arithmetic mean inequality. □

Remark 3. *If we take $w(x) = 1$, $x \in [a,b]$ in the first two inequalities (3.16), we recapture (1.4).*

If we take $w(x) = \frac{1}{x}$, $x \in [a,b] \subset (0,\infty)$ in (3.16), then we get

$$\begin{aligned} f(L(a,b)) &\le \exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(x)}{x}dx\right) \\ &\le [f(b)]^{\frac{L(a,b)-a}{b-a}}[f(a)]^{\frac{b-L(a,b)}{b-a}} \\ &\le \frac{(L(a,b)-a)f(b)+(b-L(a,b))f(a)}{b-a}, \end{aligned} \tag{3.20}$$

where $L(a,b)$ is the logarithmic mean, i.e.

$$L(a,b)=\begin{cases}\frac{b-a}{\ln b-\ln a} & \text{if } a\neq b\\ a & \text{if } a=b.\end{cases}$$

If we take $w(x)=\frac{1}{x^2}$, $x\in[a,b]\subset(0,\infty)$ in (3.16), then we get

$$f\left(\frac{G^2(a,b)}{L(a,b)}\right)\leq\exp\left(\frac{ab}{b-a}\int_a^b\frac{\ln f(x)}{x^2}dx\right) \tag{3.21}$$

$$\begin{aligned}&\leq [f(b)]^{\frac{1}{b-a}\left(\frac{G^2(a,b)}{L(a,b)}-a\right)}[f(a)]^{\frac{1}{b-a}\left(b-\frac{G^2(a,b)}{L(a,b)}\right)}\\ &\leq \tfrac{1}{b-a}\left[\left(\tfrac{G^2(a,b)}{L(a,b)}-a\right)f(b)+\left(b-\tfrac{G^2(a,b)}{L(a,b)}\right)f(a)\right].\end{aligned}$$

We also have the alternative result:

Theorem 3. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function. If $w:[a,b]\to[0,\infty)$ is integrable and $\int_a^b w(x)\,dx>0$,*
then

$$f\left(\frac{\int_a^b w(x)\,xdx}{\int_a^b w(x)\,dx}\right) \tag{3.22}$$

$$\begin{aligned}&\leq \exp\left(\frac{\int_a^b w(x)\ln f(x)dx}{\int_a^b w(x)dx}\right)\\ &\leq \frac{\int_a^b w(x)f(x)dx}{\int_a^b w(x)dx}\leq\left(\frac{[f(a)]^b}{[f(b)]^a}\right)^{\frac{1}{b-a}}\frac{\int_a^b w(x)\left(\frac{f(b)}{f(a)}\right)^{\frac{x}{b-a}}dx}{\int_a^b w(x)dx}\\ &\leq \frac{1}{b-a}\left[\left(\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}-a\right)f(b)+\left(b-\frac{\int_a^b xw(x)dx}{\int_a^b w(x)dx}\right)f(a)\right].\end{aligned}$$

Proof. Using Jensen's inequality for the exponential function we have

$$\begin{aligned}\exp\left(\frac{\int_a^b w(x)\ln f(x)dx}{\int_a^b w(x)dx}\right) &\leq \frac{\int_a^b w(x)\exp(\ln f(x))dx}{\int_a^b w(x)dx}\\ &= \frac{\int_a^b w(x)f(x)dx}{\int_a^b w(x)dx}\end{aligned}$$

and the second inequality in (3.22) is proved.

From (3.18) and the arithmetic mean - geometric mean inequality we have

$$f(x) \le [f(b)]^{\frac{x-a}{b-a}} [f(a)]^{\frac{b-x}{b-a}} = \left(\frac{[f(a)]^b}{[f(b)]^a}\right)^{\frac{1}{b-a}} \left(\frac{f(b)}{f(a)}\right)^{\frac{x}{b-a}} \le \frac{x-a}{b-a} f(b) + \frac{b-x}{b-a} f(a) \tag{3.23}$$

for any $x \in [a,b]$.

By taking the weighted integral mean in (3.23) we get

$$\frac{\int_a^b w(x) f(x)\, dx}{\int_a^b w(x)\, dx} \le \left(\frac{[f(a)]^b}{[f(b)]^a}\right)^{\frac{1}{b-a}} \frac{\int_a^b w(x) \left(\frac{f(b)}{f(a)}\right)^{\frac{x}{b-a}} dx}{\int_a^b w(x)\, dx}$$

$$\le \frac{1}{b-a}\left[\left(\frac{\int_a^b x w(x)\, dx}{\int_a^b w(x)\, dx} - a\right) f(b) + \left(b - \frac{\int_a^b x w(x)\, dx}{\int_a^b w(x)\, dx}\right) f(a)\right]$$

and the last part of (3.22) is proved. □

Remark 4. *If we take $w(x) = 1$, $x \in [a,b]$ in (3.22), then we have*

$$f\left(\frac{a+b}{2}\right) \le \exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\, dx\right) \tag{3.24}$$

$$\le \frac{1}{b-a}\int_a^b f(x)\, dx \le L(f(a), f(b)) \le \frac{f(a)+f(b)}{2}.$$

If we take $w(x) = \frac{1}{x}$, $x \in [a,b] \subset (0,\infty)$ in (3.22), then we get

$$f(L(a,b)) \le \exp\left(\frac{1}{\ln b - \ln a}\int_a^b \frac{\ln f(x)}{x} dx\right) \tag{3.25}$$

$$\begin{aligned} &\le \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x} dx \\ &\le \left(\frac{[f(a)]^b}{[f(b)]^a}\right)^{\frac{1}{b-a}} \frac{\int_a^b \frac{1}{x}\left(\frac{f(b)}{f(a)}\right)^{\frac{x}{b-a}} dx}{\ln b - \ln a} \\ &\le \frac{(L(a,b)-a) f(b) + (b - L(a,b)) f(a)}{b-a}. \end{aligned}$$

4. Inequalities for Symmetric Weights

We say that the weight $w : [a,b] \to [0,\infty)$ is *symmetric* on $[a,b]$ if

$$w(a+b-x) = w(x) \text{ for all } x \in [a,b].$$

It is well known that if $f:[a,b]\to\mathbf{R}$ is convex and $w:[a,b]\to[0,\infty)$ is integrable and symmetric on $[a,b]$, then the *Fejér inequality* holds

$$f\left(\frac{a+b}{2}\right)\le\frac{\int_a^b w(x)\,f(x)\,dx}{\int_a^b w(x)\,dx}\le\frac{f(a)+f(b)}{2}. \tag{4.1}$$

If $f:[a,b]\to(0,\infty)$ is a log-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ is integrable and symmetric on $[a,b]$, then by (4.1) we have

$$\ln f\left(\frac{a+b}{2}\right)\le\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\le\frac{\ln f(a)+\ln f(b)}{2},$$

which is equivalent to

$$f\left(\frac{a+b}{2}\right)\le\exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right)\le\sqrt{f(a)\,f(b)}. \tag{4.2}$$

Theorem 1. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ be integrable and symmetric on $[a,b]$. Then*

$$f\left(\frac{a+b}{2}\right)\le\exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right) \tag{4.3}$$

$$\le\frac{\int_a^b w(x)\sqrt{f(x)f(a+b-x)}dx}{\int_a^b w(x)dx}\le\frac{\int_a^b w(x)f(x)dx}{\int_a^b w(x)dx}.$$

Proof. By Jensen's integral inequality for the exponential we have

$$\exp\left(\frac{\int_a^b w(x)\ln\sqrt{f(x)\,f(a+b-x)}dx}{\int_a^b w(x)\,dx}\right) \tag{4.4}$$

$$\begin{aligned}&\le\frac{\int_a^b w(x)\exp\left(\ln\sqrt{f(x)f(a+b-x)}\right)dx}{\int_a^b w(x)dx}\\&=\frac{\int_a^b w(x)\sqrt{f(x)f(a+b-x)}dx}{\int_a^b w(x)dx}.\end{aligned}$$

Observe, by the symmetry of w, that

$$\begin{aligned}
&\int_a^b w(x)\ln\sqrt{f(x)f(a+b-x)}dx\\
&= \tfrac{1}{2}\left[\int_a^b w(x)\ln f(x)\,dx + \int_a^b w(x)\ln f(a+b-x)\,dx\right]\\
&= \tfrac{1}{2}\left[\int_a^b w(x)\ln f(x)\,dx + \int_a^b w(a+b-x)\ln f(a+b-x)\,dx\right]\\
&= \int_a^b w(x)\ln f(x)\,dx
\end{aligned}$$

since, obviously

$$\int_a^b w(a+b-x)\ln f(a+b-x)\,dx = \int_a^b w(x)\ln f(x)\,dx.$$

By (4.4) we then get the second inequality in (4.3).

By Cauchy-Buniakovski-Schwarz integral inequality we also have

$$\begin{aligned}
&\int_a^b w(x)\sqrt{f(x)f(a+b-x)}dx\\
&\le \sqrt{\int_a^b w(x)f(x)\,dx}\sqrt{\int_a^b w(x)f((a+b-x))\,dx}\\
&= \sqrt{\int_a^b w(x)f(x)\,dx}\sqrt{\int_a^b w(x)f(x)\,dx}\\
&= \int_a^b w(x)f(x)\,dx,
\end{aligned}$$

which proves the third inequality in (4.3). □

The above inequality (4.2) may be generalized as follows by replacing f with f^{2p} for $p>0$.

Corollary 4. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ be integrable and symmetric on $[a,b]$. Then for any $p>0$ we have*

$$f\left(\frac{a+b}{2}\right) \le \exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right) \tag{4.5}$$

$$\leq \left(\frac{\int_a^b w(x)f^p(x)f^p(a+b-x)dx}{\int_a^b w(x)dx}\right)^{\frac{1}{2p}}$$

$$\leq \left(\frac{\int_a^b w(x)f^{2p}(x)dx}{\int_a^b w(x)dx}\right)^{\frac{1}{2p}}.$$

Remark 5. *We observe that for $p \geq 1$ we have*

$$f\left(\frac{a+b}{2}\right) \leq \exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right) \tag{4.6}$$

$$\begin{aligned}
&\leq \left(\frac{\int_a^b w(x)f(x)f(a+b-x)dx}{\int_a^b w(x)dx}\right)^{\frac{1}{2}} \\
&\leq \left(\frac{\int_a^b w(x)f^p(x)f^p(a+b-x)dx}{\int_a^b w(x)dx}\right)^{\frac{1}{2p}} \\
&\leq \left(\frac{\int_a^b w(x)f^{2p}(x)dx}{\int_a^b w(x)dx}\right)^{\frac{1}{2p}}
\end{aligned}$$

and for $p \in (0,1)$

$$f\left(\frac{a+b}{2}\right) \leq \exp\left(\frac{\int_a^b w(x)\ln f(x)\,dx}{\int_a^b w(x)\,dx}\right) \tag{4.7}$$

$$\begin{aligned}
&\leq \left(\frac{\int_a^b w(x)f^p(x)f^p(a+b-x)dx}{\int_a^b w(x)dx}\right)^{\frac{1}{2p}} \\
&\leq \left(\frac{\int_a^b w(x)f(x)f(a+b-x)dx}{\int_a^b w(x)dx}\right)^{\frac{1}{2}} \\
&\leq \left(\frac{\int_a^b w(x)f^2(x)dx}{\int_a^b w(x)dx}\right)^{\frac{1}{2}}.
\end{aligned}$$

Finally, we have:

Theorem 2. *Let $f:[a,b]\to(0,\infty)$ be a log-convex function on $[a,b]$ and $w:[a,b]\to[0,\infty)$ be integrable and symmetric on $[a,b]$. Then for any $p>0$ we have*

$$f\left(\frac{a+b}{2}\right)\le\left(\frac{\int_a^b f^p(x)\,w(x)\,dx}{\int_a^b\frac{w(x)dx}{f^p(x)}}\right)^{\frac{1}{2p}}\le\sqrt{f(a)\,f(b)}. \tag{4.8}$$

Proof. From (3.4) we have

$$f^{2p}\left(\frac{a+b}{2}\right)\frac{1}{f^p(a+b-x)}\le f^p(x)\le f^p(a)\,f^p(b)\,\frac{1}{f^p(a+b-x)} \tag{4.9}$$

for any $x\in[a,b]$.

If we multiply by $w(x)\ge 0$ and integrate on $[a,b]$, then we get

$$f^{2p}\left(\tfrac{a+b}{2}\right)\int_a^b\tfrac{w(x)dx}{f^p(a+b-x)}$$
$$\le\int_a^b f^p(x)\,w(x)\,dx$$
$$\le f^p(a)\,f^p(b)\int_a^b\tfrac{w(x)dx}{f^p(a+b-x)}.$$

Since, by symmetry of w we have

$$\int_a^b\frac{w(x)\,dx}{f^p(a+b-x)}=\int_a^b\frac{w(a+b-x)\,dx}{f^p(a+b-x)}=\int_a^b\frac{w(x)\,dx}{f^p(x)},$$

which implies that

$$f^{2p}\left(\tfrac{a+b}{2}\right)\int_a^b\tfrac{w(x)dx}{f^p(x)}$$
$$\le\int_a^b f^p(x)\,w(x)\,dx$$
$$\le f^p(a)\,f^p(b)\int_a^b\tfrac{w(x)dx}{f^p(x)}.$$

and the inequality (4.8) is proved. □

Remark 6. *If we write the inequality (4.8) for the log-convex function $f:[a,b]\subset(0,\infty)\to(0,\infty)$, $f(x)=\frac{1}{x}$, then we have for $p>0$*

$$\left(\frac{a+b}{2}\right)^{-1}\le\left(\frac{\int_a^b x^{-p}w(x)\,dx}{\int_a^b x^p w(x)\,dx}\right)^{\frac{1}{2p}}\le\left(\sqrt{ab}\right)^{-1},$$

that is equivalent to

$$G(a,b) \leq \left(\frac{\int_a^b x^p w(x)\, dx}{\int_a^b x^{-p} w(x)\, dx} \right)^{\frac{1}{2p}} \leq A(a,b), \tag{4.10}$$

for any symmetric integrable weight $w : [a,b] \to [0,\infty)$.

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