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MODELING, SIMULATION AND OPTIMIZATION OF ELECTRORHEOLOGICAL FLUIDS

R.H.W. HOPPE[1,2] AND W.G. LITVINOV[2]

1. INTRODUCTION

Electrorheological fluids are concentrated suspensions of electrically polarizable particles of small size in the range of micrometers in non-conducting or semi-conducting liquids such as silicone oils. Under the influence of an outer electric field, the particles form chains along the field lines followed by a coalescence of the chains into columns in the plane orthogonal to the field due to short-ranged potentials arising from charge-density fluctuations. The formation of the chains is a process which happens in milliseconds, whereas the aggregation to columns occurs on a time scale that is larger by an order of magnitude. On a macroscopic scale, the chainlike and columnar structures have a significant impact on the rheological properties of the suspensions. In particular, the viscosity increases rapidly with increasing electric field strength in the direction perpendicular to the field. The fluid experiences a phase transition to a viscoplastic state, and the flow shows a pronounced anisotropic behavior. Under the influence of large stresses, the columns break into continuously fragmenting and aggregating volatile structures which tilt away from strict field alignment. As a result, the viscosity decreases and the fluid flow behaves less anisotropic. The electrorheological effect is reversible, i.e., the viscosity decreases for decreasing electric field strength such that for vanishing field strength the fluid behaves again like a Newtonian one. The fast response to an outer electric field and the reversibility of the effect make electrorheological fluids particularly attractive for all technical applications which require a controllable power transmission.

Although the discovery of the electrorheological effect is credited to WINSLOW [1947] (cf. also WINSLOW [1949, 1962]), it has already been observed experimentally by PRIESTLEY [1769] during the second half of the eighteenth century and by DUFF [1896] and QUINKE [1897] at the end of the nineteenth century. However, WINSLOW was the first scientist who conducted quantitative experiments on suspensions of silica gel particles in oils of low viscosity. He reported fibration parallel to the electric field with a solid-like behavior of the suspension at field strengths larger than $3kV/mm$. In his experiments, he also observed that the yield stress, i.e., when the shear stress is proportional to the shear rate, is proportional to the square of the electric field strength.

WINSLOW's work did not immediately launch tremendous research activities in the area of electrorheological fluids. In fact, it took roughly twenty to thirty more years, when the availability of modern, high-resolution measurement technology on one hand and more advanced and powerful computing facilities on the other hand enabled researchers to conduct detailed experimental studies and to perform extensive numerical simulations (see BLOCK and KELLY [1988], BLOCK et al. [1990], BÖSE [1998], BÖSE and TRENDLER [2001], CLERCX and BOSSIS [1993], CONRAD et al. [1991], DEINEGA and VINOGRADOV [1984], GAST and ZUKOSKI [1989], HANAOKA et al. [2002], INOUE and MANIWA [1995], KHUSID and ACRIVOS [1995], KIMURA et al. [1998],

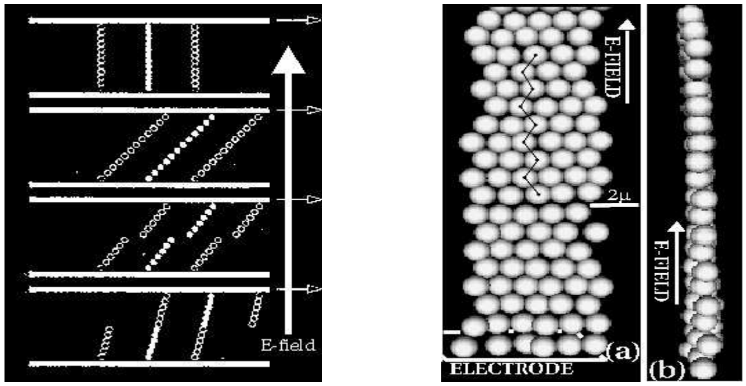

FIGURE 1.1. Formation of chains aligned with the field (left) and aggregation to sheets (right)

KLASS and MARTINEK [1967a,b] and KLINGENBERG et al. [1989], KLINGENBERG and ZUKOSKI [1990], LEMAIRE et al. [1992], MARSHALL et al. [1989], MOKEEV et al. [1992], RHEE et al. [2003], SHULMAN and NOSOV [1985], STANGROOM [1977, 1983], STANWAY et al. [1987], TAO and SUN [1991b], VOROBEVA et al. [1969], WHITTLE [1990], WEN et al. [2003], YU and WAN [2000], ZHAO et al. [2002]). The experimental work focused on the creation of the chainlike and columnar structures (see KLINGENBERG and ZUKOSKI [1990], MARTIN and ANDERSON [1996], MARTIN et al. [1998a], QI and WEN [2002]) (cf. Figure 1.1 (left)) up to the formation of sheets (cf. Figure 1.1 (right)) and body-centered tetragonal crystal lattices (see DASSANAYAKE et al. [2000]) (cf. Figure 1.2) as well as on the dynamics of the process (cf., e.g., ADOLF and GARINO [1995], FOULC et al. [1996], KLINGENBERG [1998], KLINGENBERG and ZUKOSKI [1990], KLINGENBERG et al. [2005], MARTIN et al. [1998b], PFEIL et al. [2002], TAM et al. [1997], UGAZ et al. [1994], WHITTLE et al. [1999], ZHAO and GAO [2001]). The measurements have been performed using, e.g., confocal scanning laser microscopy (DASSANAYAKE et al. [2000]), two-dimensional light scattering techniques (MARTIN et al. [1998b]), and nuclear magnetic resonance imaging (UGAZ et al. [1994]).

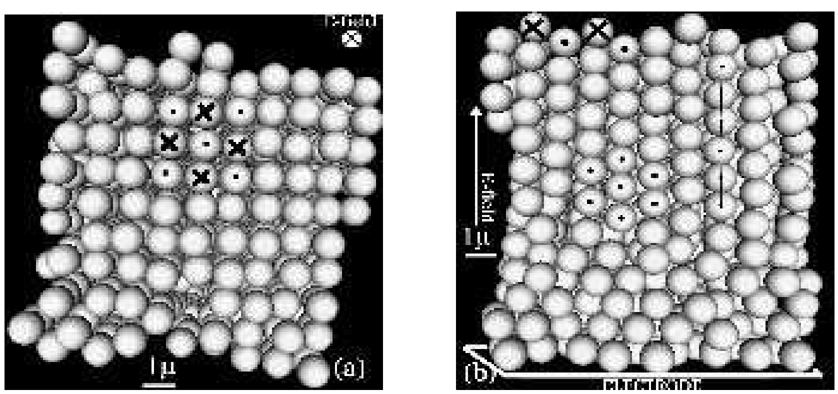

FIGURE 1.2. Body-centered tetragonal crystal lattice in the xy-plane (left) and the xz-plane (right)

The potential industrial applicability of electrorheological fluids in automotive applications (BAYER [1998], BUTZ and STRYK [2001], COULTER et al. [1993], FILISKO [1995], GARG and ANDERSON [2003], GAVIN [2001], GAVIN et al. [1996a,b], HARTSOCK et al. [1991], HOPPE et al. [2000], JANOCHA et al. [1996], LORD [1996], PEEL et al. [1996], SIMS et al. [1999], STANWAY et al. [1996], WEYENBERG et al. [1996], ZHAO et al. [2005]), aerospace applications (BERG and WELLSTEAD [1998], LOU et al. [2001], WERELEY et al. [2001]), food processing (DAUBERT et al. [1998]), geophysics (MAKRIS [1999], XU et al. [2000]), life sciences (KLEIN et al. [2004], LIU et al. [2005], MAVROIDIS et al. [2001], MONKMANN et al. [2003a,b], TAKASHIMA and SCHWAN [1985]), manufacturing (KIM et al. [2003]), military applications (DEFENSE UPDATE [2004]), and nondestructive testing (MAVROIDIS [2002]) caused the US Department of Energy to issue a research assessment of electrorheological fluids (DOE [1993]) and popular scientific journals such as Science and Nature to publish overview articles (HALSEY [1992], WHITTLE and BULLOUGH [1992]). Further references covering various aspects of experimental work, modeling efforts, and applications of electrorheological fluids can be found in BOSSIS [2002], HAO [2001], TAO and ROY [1995].

The experimental work was always accompanied by the development of physically consistent, mathematical models, their analysis, numerical simulations, and model validations on the basis of available data from measurements and simulations. Roughly speaking, one has to distinguish between microstructural models, which combine electrostatics (see JONES [1995]), microhydrodynamics (cf. KIM and KARRILA [1991]), and liquid state theory (see CACCANO et al. [1999]; cf. also LARSON [1999], LUKASZEWICZ [1999]), and macroscopic models based on continuum field theories (cf., e.g., RAJAGOPAL and TRUESDELL [2000], TRUESDELL and NOLL [1965], TRUESDELL and TOUPIN [1960]).

The simplest microscale models assume the electrorheological fluids to consist of monodisperse, neutrally buoyant hard dielectric spheres dispersed in a Newtonian continuous phase thus neglecting small conductivities in both phases, ionic impurities in the continuous phase, and triboelectric effects. Idealized electrostatic polarization methods obtain the electrostatic potential via Laplace's equation and compute the motion of the particles by Newton's equation which requires the proper specification of the total force exerted on a particle by taking into account the interparticle forces. Since the exact solution is unavailable and the computation of all possible interparticle forces is cumbersome, the system is simplified by the point-dipole approximation (see JONES [1995], KIM and KLINGENBERG [1997], PARTHASARATHY and KLINGENBERG [1996], PFEIL and KLINGENBERG [2004]) assuming that two spheres of the same size do not change their charge distributions. The resulting force equation only depends on the distance of the particles, the angle between them, the particle size, and on the properties of the induced electric field. The results of the model differ by an order of magnitude from experimentally available data, since the dipole moment of the particles enhances the polarization. This has been accounted for in PARTHASARATHY and KLINGENBERG [1996] by a modified point-dipole approximation and by providing multipole models (see CONRAD et al. [1991], CLERCX and BOSSIS [1993]) which are based on several electric field equations (up to four), whereas the particle interaction is performed for an N particle cluster allowing the consideration of particles in lattice structures such as body-centered tetragonal crystal lattices. The dipole-induced dipole model in YU and WAN [2000] represents a further development of the multipole models in so far as it admits spheres of different

sizes. Maxwell-Wagner polarization due to accumulated charges between the interface of the particles and the continuous phase has been incorporated in PARTHASARATHY and KLINGENBERG [1996] by assuming a point dipole model for this interfacial polarization. The Maxwell-Wagner model in KHUSID and ACRIVOS [1995] further acknowledges effects of the disturbance field between particles.
Microstructural models based on energy-type methods have been derived in BONNECAZE and BRADY [1992a,b]. They take into account hydrodynamic and electrostatic particle interactions using Stokesian dynamics and a model for the electrostatic energy. The latter one is determined from the capacitance matrix of the suspension. The models allow simulations of monolayers of particles for a wide range of the ratio of viscous to electrostatic forces as described by the Mason number. The macroscopic rheology can be deduced from the simulations. In accordance with experimental results, it shows that for large electric field strengths there is a pronounced Bingham-type behavior of the suspension with a dynamic yield stress that can be related to jumps in the electrostatic energy. Numerical simulations based on microscale models are typically of molecular dynamics type (cf, e.g., HU and CHEN [1998], MELROSE [1992], MELROSE and HAYES [1993], TAO and SUN [1991a], ZHAO and GAO [2001]) using methodologies from ALLEN and TILDESLEY [1983].

The microstructural features of electrorheological fluids have been used to derive models for a description of the macroscopic properties (cf. e.g., KLINGENBERG [1993], PARTHASARATHY et al. [1994], PARTHASARATHY and KLINGENBERG [1995a,b, 1999], PFEIL et al. [2003], SEE [1999, 2000], VERNESCU [2002], WANG and XIAO [2003]). On the other hand, macroscopic models have been obtained by phenomenological approaches within the framework of mixture theory (see RAJAGOPAL [1996], RAJAGOPAL et al. [1994]) and classical continuum mechanics (we refer to ATKIN et al. [1991] as one of the first attempts in this direction (cf. also ATKIN et al. [1999])). Since electrorheological fluids exhibit a Non-Newtonian flow behavior, significant efforts have been devoted to the derivation of appropriate constitutive equations relating the stress tensor to the rate of deformation tensor by taking into account the influence of the electric field. We mention the pioneering work by RAJAGOPAL and WINEMAN [1992, 1995] (see also ENGELMANN et al. [2000]) and the systematic treatment by RUZICKA [2000] providing a constitutive equation of power law type (see also BUSUIOC and CIORANESCU [2003], ECKART [2000], RAJAGOPAL and RUZICKA [2001]). Other continuum-based approaches try to incorporate micro- and mesoscale effects by using internal variables (DROUOT et al. [2002]), transverse isotropy (BRUNN and ABU-JDAYIL [1998, 2004]), polar theory (ECKART and SADIKI [2001]), and more general rate-type models (SADIKI and BALAN [2003]). In this contribution, we will adopt the constitutive laws that have been suggested, analyzed and validated in HOPPE and LITVINOV [2004] and LITVINOV and HOPPE [2005] for isothermal and non-isothermal electrorheological fluid flows which take into account the orientation of the velocity field of the flow with respect to the outer electric field.

The content of this chapter is as follows: In section 2, we are concerned with balance equations and constitutive laws for isothermal and non-isothermal electrorheological fluid flows and with the existence and/or uniqueness of solutions. In section 3, we deal with numerical methods both for steady and time-dependent fluid flows. Finally, in section 4 we present numerical simulation results for some selected electrorheological devices and also briefly address optimal design issues.

2. MATHEMATICAL MODELS FOR ELECTRORHEOLOGICAL FLUID FLOWS

In this section, we study balance equations and constitutive laws for isothermal and non-isothermal electrorheological fluid flows. After a general presentation in 2.1, in 2.2 we consider stationary isothermal fluid flows based on the extended Bingham-type models from HOPPE and LITVINOV [2004]. In particular, we shall be concerned with existence and/or uniqueness results for a regularized version in 2.2.1 and for the non-regularized model in 2.2.2. In 2.3, we deal with time-dependent problems, whereas 2.4 and 2.5 are devoted to the derivation of model equations for non-isothermal fluid flows and the discussion of the existence of solutions following the approach in LITVINOV and HOPPE [2005]. We refer to DUVAUT and LIONS [1976], GALDI [1994], GLOWINSKI [2004], LADYZHENSKAYA [1969], TEMAM [1979] for general aspects related to the mathematical modelling, the analysis and the numerical solution of fluid mechanical problems and to LITVINOV [2000] for a general treatment of optimization problems for nonlinear viscous fluids.

2.1. **Balance equations and constitutive laws for isothermal fluid flows.** We consider isothermal incompressible electrorheological fluid flows in $Q := \Omega \times (0,T), T \in \mathbb{R}_+$, where Ω is supposed to be a bounded Lipschitz domain in $\mathbb{R}^d, d = 2$ or $d = 3$. We denote by $u(x,t) = (u_1(x,t), \cdots, u_d(x,t))^T, (x,t) \in \bar{Q}$, and $p(x,t), (x,t) \in Q$, the velocity of the fluid and the pressure, whereas $E(x,t) = (E_1(x,t), \cdots, E_d(x,t))^T, (x,t) \in \bar{Q}$, stands for the electric field. We use the notation $u_t := \partial u/\partial t$ for the partial derivative of u with respect to time. Then, referring to $\rho \in \mathbb{R}_+$ as the density of the fluid, to $f : Q \to \mathbb{R}^d$ as a forcing term, and to $\sigma = \sigma(u,p,E)$ as the stress tensor, the balance equations (conservation of mass and momentum) are given by

$$\rho\Big(u_t + (u \cdot \nabla)u\Big) \ - \ \nabla \cdot \sigma \ = \ f \quad \text{in } Q \ , \tag{2.1a}$$

$$\nabla \cdot u \ = \ 0 \quad \text{in } Q \ , \tag{2.1b}$$

which have to be complemented by properly specified initial and boundary conditions and a constitutive law relating the stress tensor σ to the independent variables u, p and E. Neglecting magnetic fields, the electric field can be considered as quasi-static so that for each $t \in [0,T]$ the field $E(\cdot,t)$ can be computed by $E(\cdot,t) = -\nabla\psi(\cdot,t)$ as the gradient of an electric potential $\psi(\cdot,t)$ satisfying Laplace's equation

$$\nabla \cdot (\epsilon \nabla \psi(\cdot,t)) \ = \ 0 \quad \text{in } \Omega \ , \tag{2.2}$$

which also has to be complemented by appropriate boundary conditions. Here, ϵ stands for the dielectric permittivity.

For the discussion of the constitutive law, we further denote by

$$\varepsilon(u) \ = \ \frac{1}{2}\left(\nabla u + (\nabla u)^T\right) \tag{2.3}$$

the rate of deformation tensor (linearized strain tensor) and by

$$I(u) \ = \ \|\varepsilon(u)\|_F^2 \tag{2.4}$$

the second invariant of the rate of deformation tensor, where $\|\cdot\|_F$ stands for the Frobenius norm. For shear flows, we refer to $\tau = \tau(u,E)$ as the shear stress which is a field dependent function of the shear rate

$$\gamma \ = \ (2^{-1} I(u))^{1/2} \ . \tag{2.5}$$

In case of flow modes such as Couette flow or Poiseuille flow, where the electric field is perpendicular to the fluid velocity, constitutive equations of the form

$$\sigma \;=\; -pI \;+\; 2\varphi(I(u),|E|)\,\varepsilon(u)\;. \tag{2.6}$$

have been widely used. Here, $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ stands for a viscosity function depending on the second invariant of the rate of deformation tensor and the electric field strength. The most commonly used constitutive law for simple flow modes is that of a Bingham-type fluid ATKIN et al. [1991], FILISKO [1995], PARTHASARATHY and KLINGENBERG [1996], RHEE et al. [2003], STANWAY et al. [1996], WHITTLE, ATKIN and BULLOUGH [1995]. For stresses above a field dependent yield stress $\sigma_Y(E)$ the viscosity function φ is given by

$$\varphi(I(u),|E|) \;=\; \eta_0(E) \;+\; 2^{-1/2}\,\tau_0(E)\,I(u)^{-1/2}\;, \tag{2.7}$$

whereas $I(u) = 0$ for $|\sigma| \le \sigma_Y(E)$. Here, $\eta_0(E)$ is a field dependent constant and $\tau_0(E)$ denotes the shear stress for vanishing shear rate γ.

A related model, which can be viewed as some extension of the Bingham fluid model, is that of CASSON [1959]. For $|\sigma| > \sigma_Y(E)$, the viscosity function

$$\begin{aligned}\varphi(I(u),|E|) \;=&\; \eta_0(E) \;+\; 2^{-1/2}\,\tau_0(E)\,I(u)^{-1/2} \;+ \\ &+\; 2^{3/4}\,(\eta_0(E)\,\tau_0(E))^{1/2}\,I(u)^{-1/4}\end{aligned} \tag{2.8}$$

is used, whereas again $I(u) = 0$ for $|\sigma| \le \sigma_Y(E)$.

The singular character of the viscosity function φ in the Bingham and Casson fluid models requires to formulate the equations of motion (2.1a),(2.1b) as variational inequalities. A possible way to circumvent the difficulties associated with the non-smooth behavior of the viscosity function is by regularization which in case of a Bingham model gives rise to

$$\varphi(I(u),|E|) \;=\; \eta_0(E) \;+\; 2^{-1/2}\,\tau_0(E)\,(\kappa + I(u))^{-1/2}\;. \tag{2.9}$$

Here, κ stands for a positive regularization parameter. For the Casson model (2.8), one may use an analogous regularization. The implications of using the classical models and the regularized models will be discussed in a more general context later in this section.

Other frequently used constitutive equations for non-Newtonian fluids assume a power law behavior (SIGNIER et al. [1999]). For electrorheological fluids, this leads to a viscosity function φ of the form

$$\varphi(I(u),|E|) \;=\; \begin{cases} m(E)\,\gamma_0^{n(E)-1}\;, \gamma \le \gamma_0(E) \\ m(E)\,\gamma^{n(E)-1}\;, \gamma > \gamma_0(E) \end{cases}\;, \tag{2.10}$$

where $m(E), n(E)$ are field dependent material parameters and $\gamma_0(E)$ stands for a field dependent shear rate. Regularizations of the power law model can be used as well. In this case, the viscosity function (2.10) is replaced by

$$\varphi(I(u),|E|) \;=\; m(E)\,(\kappa + \gamma^2)^{(n(E)-1)/2}\;,\; \kappa > 0\;. \tag{2.11}$$

We note that in case of steady shear flows in axially symmetric geometrical configurations the use of the previously mentioned models in the equations of motion (2.1a),(2.1b) leads to scalar nonlinear equations that can be solved semi-analytically. However, a serious drawback of the models is that the electric field strength $|E|$ occurs as a parameter in the constitutive laws thus assuming a homogeneous distribution of the electric field. This assumption is justified for simple flows in geometrical settings, where the flow occurs between conventionally shaped electrodes at small distance from each other (cf. subsections

4.1 and 4.2), but due to experimental evidence does not hold true for more general configurations (cf. e.g., ABU-JDAYIL [1996], ABU-JDAYIL and BRUNN [1995, 1996, 1997, 2002] and EDAMURA and OTSUBO [2004], GEORGIADES [2003], OTSUBO [1997], OTSUBO and EDAMURA [1998, 1999]).

One of the first systematic approaches towards a general phenomenological model based on continuum field theories has been undertaken by RAJAGOPAL and WINEMAN in RAJAGOPAL and WINEMAN [1992] (cf. also RAJAGOPAL and WINEMAN [1995]), where the constitutive law is assumed to be of the form

$$\begin{aligned} \sigma \; = \; & - pI \; + \; \alpha_2 \, E \otimes E \; + \; \alpha_3 \, \varepsilon(u) \; + \; \alpha_4 \, \varepsilon^2(u) \; + \\ & + \; \alpha_5 \, (\varepsilon(u) E \otimes E + E \otimes \varepsilon(u) E) \; + \; \alpha_6 \, (\varepsilon^2 E \otimes E + E \otimes \varepsilon(^2 u) E) \; . \end{aligned} \tag{2.12}$$

Here, $\otimes$ denotes the tensor product and $\alpha_i = \alpha_i(I_1, \cdots, I_6), 2 \leq i \leq 6$, are scalar functions of the six invariants

$$I_1 := \text{tr}(EE^T) \; , \; I_2 := \text{tr}(\varepsilon(u)) \; , \; I_3 := \text{tr}(\varepsilon^2(u)) \; , \; I_4 := \text{tr}(\varepsilon^3(u)) \; ,$$
$$I_5 := \text{tr}(\varepsilon(u) E \otimes E) \; , \; I_6 := \text{tr}(\varepsilon^2(u) E \otimes E) \; ,$$

where tr stands for the trace of a matrix.
Motivated by RAJAGOPAL and WINEMAN [1992, 1995], an extended Bingham-type fluid model

$$\sigma = -pI + \eta_0 \varepsilon(u) + \gamma |\varepsilon(u) E|^{-1} |E| \Big(\varepsilon(u) E \otimes E + E \otimes \varepsilon(u) E \Big) \tag{2.13}$$

has been used in ENGELMANN et al. [2000], HOPPE and MAZURKEVICH [2001], HOPPE et al. [2000] in combination with a potential equation for the electric potential ψ ($E = -\nabla \psi$) to provide numerical simulations of steady electrorheological fluid flows.

In the spirit of RAJAGOPAL and WINEMAN [1992, 1995], RUZICKA [2000] has developed a model that takes into account the interaction between the electric field and the fluid flow (see also RAJAGOPAL and RUZICKA [1996, 2001]). The constitutive equation is of power law type

$$\begin{aligned} \sigma = & - pI \; + \; \gamma_1 \Big((1 + |\varepsilon(u)|^2)^{(r-1)/2} - 1 \Big) \; E \otimes E \; + \\ & + \; (\gamma_2 + \gamma_3 |E|^2)(1 + |\varepsilon(u)|^2)^{(r-2)/2} \varepsilon(u) \; + \\ & + \; \gamma_4 (1 + |\varepsilon(u)|^2)^{(r-2)/2} \Big(\varepsilon(u) E \otimes E + E \otimes \varepsilon E \Big) \; , \end{aligned} \tag{2.14}$$

where $\gamma_i, 1 \leq i \leq 4$, are constants and $r : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function of $|E|^2$ satisfying

$$1 \; < \; r_\infty \; \leq \; r(|E|^2) \; \leq \; r_0 \; . \tag{2.15}$$

Here, r_0 and r_∞ are the constants

$$r_0 := \lim_{|E|^2 \to 0} r(|E|^2) \quad , \quad r_\infty := \lim_{|E|^2 \to \infty} r(|E|^2) \; .$$

As far as the electric field E is concerned, the quasi-static form of Maxwell's equations ERINGEN and MAUGIN [1989], LANDAU and LIFSHITZ [1984] can be used such that E can be computed via the gradient of an electric potential satisfying an elliptic boundary value problem.
Due to the power law (2.14), the existence of weak solutions of the equations of motion (2.1a),(2.1b) both in the case of steady and time-dependent flows has to be studied within the framework of generalized Lebesgue and generalized Sobolev spaces (for related work see also FREHSE, MALEK and STEINHAUER [1997], LITVINOV [1982],

MALEK, NECAS and RUZICKA [1996], MALEK and RAJAGOPAL [2007], MALEK, RAJAGOPAL and RUZICKA [1995]).
A further development of Ruzicka's approach by means of an extended Casson model has been studied in ECKART [2000].

Motivated by experimental evidence (CECCIO and WINEMAN [1994], SHULMAN and NOSOV [1985]), in HOPPE and LITVINOV [2004] a constitutive law

$$\sigma \ = \ -pI \ + \ 2\varphi(I(u),|E|,\mu(u,E))\varepsilon(u) \ , \tag{2.16}$$

has been suggested where the viscosity function $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ additionally depends on the orientation of the electric field E with respect to the velocity u of the fluid flow as described by a function $\mu : S_1^d \times S_1^d \to [0,1]$ with S_1^d denoting the d-dimensional unit sphere. We refer to $\hat{u}$ as the velocity of the electrode. Then, for $u - \hat{u} \neq 0$ and $E \neq 0$ the function $\mu : S_1^d \times S_1^d \to [0,1]$ is defined according to

$$\mu(u,E) \ := \ \frac{u-\hat{u}}{|u-\hat{u}|} \cdot \frac{E}{|E|} \ , \tag{2.17}$$

where $\cdot$ stands for the Euclidean inner product in $\mathbb{R}^d$. The function $\mu = \mu(u,E)$ is an invariant which is independent of the choice of the reference frame and the motion of the frame with respect to the electrode. For a further discussion we refer to HOPPE and LITVINOV [2004].

Shear rate γ [per sec]	Shear stress (Pa) 0.0 V/mm	1.5 kV/mm	2.0 kV/mm	2.5 kV/mm	3.0 kV/mm
1.0 $\times 10^2$	30.2	563.0	979.0	1360.0	1720.0
2.0 $\times 10^2$	48.0	650.0	1070.0	1500.0	1900.0
4.0 $\times 10^2$	69.3	695.0	1140.0	1600.0	2030.0
6.0 $\times 10^2$	83.5	700.0	1170.0	1640.0	2070.0
8.0 $\times 10^2$	100.0	712.0	1180.0	1670.0	2110.0
1.0 $\times 10^3$	110.0	723.0	1200.0	1676.0	2140.0
1.2 $\times 10^3$	115.0	727.0	1210.0	1686.0	2160.0
1.4 $\times 10^3$	120.0	731.0	1220.0	1693.0	2180.0
1.6 $\times 10^3$	225.0	735.0	1240.0	1696.0	2190.0
1.8 $\times 10^3$	230.0	740.0	1250.0	1706.0	2200.0
2.0 $\times 10^3$	235.0	743.0	1254.0	1710.0	2210.0

TABLE 1. Experimental data (shear stress - shear rate dependence) at various electric field strengths for the commercially available electrorheological fluid RHEOBAY TP AI 3565 (from BAYER [1997a])

For specific electrorheological fluids, the viscosity function φ has to be determined based on experimental data for the relationship $\tau = \tau(\gamma)$ between the shear stress τ and the shear rate γ. For various electric field strengths, these data are usually available at discrete points $\gamma_i \in [\gamma_{min}, \gamma_{max}], 0 \leq i \leq N$, with $0 < \gamma_{min} < \gamma_{max} < \infty$ (cf. Table 1). Complete cubic spline interpolands are then used for the construction of flow curves in $[\gamma_{min}, \gamma_{max}]$ (cf. Figure 2.3), and the flow curves are continuously extended to (γ_{max}, ∞) by straight lines $\tau(\gamma) = a_1 + a_2\gamma$ with coefficients $a_i, 1 \leq i \leq 2$, depending on $|E|$

and $\mu(u, E)$. The extension to $[0, \gamma_{min})$ can de done such that either $\tau(0) = \tau_0 \neq 0$ or $\tau(0) = 0$. In the former case, the viscosity function takes the form

$$\varphi(I(u), |E|, \mu(u, E)) \; = \; b(|E|, \mu(u, E)) I(u)^{-1/2} \; + \; c(I(u), |E|, \mu(u, e)) \; , \tag{2.18}$$

where $b(|E|, \mu(u, E)) = 2^{-1/2}\tau_0$ and $c : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ is a continuous function.

REMARK 2.1. *The viscosity function φ as given by* (2.18) *represents an extended Bingham-type fluid model (cf.* (2.7)*. Due to its singular behavior for $I(u) = 0$, the equations of motion* (2.1a),(2.1b) *have to be formulated as variational inequalities (see subsection* 2.2.2 *below).*

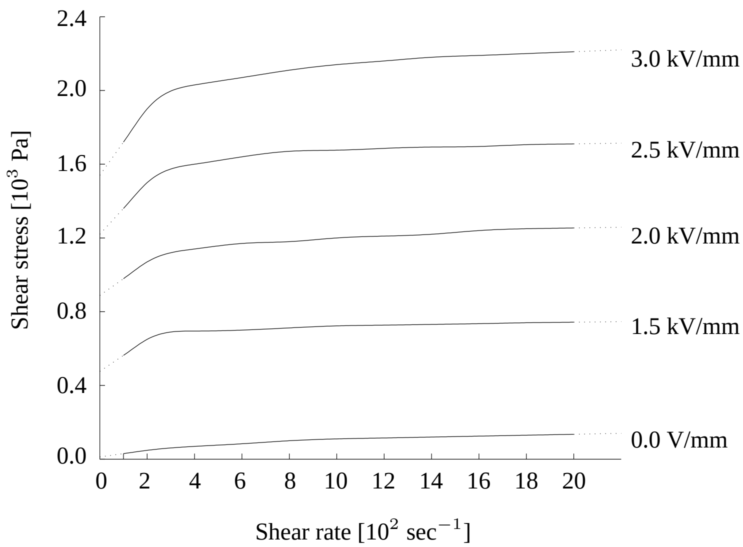

FIGURE 2.3. Flow curves generated by cubic spline interpolands based on the experimental data from Table 1 showing the effect of the field strength (50Hz, AC) and the shear rate γ on the shear stress τ at 40^{o}C.

On the other hand, if the flow curves are extended to $[0, \gamma_{min})$ such that $\tau = 0$ for $\gamma = 0$, the viscosity function can be written as

$$\varphi(I(u), |E|, \mu(u, E)) \; = \; b(|E|, \mu(u, E))(\kappa + I(u))^{-1/2} + c(I(u), |E|, \mu(u, e)) \; , \tag{2.19}$$

where $0 < \kappa \ll 1$ and $b : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$, $c : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ are continuous functions.

REMARK 2.2. *The viscosity function φ of the form* (2.19) *can be interpreted as an extension of the regularized Bingham fluid model* (2.9)*.*

As far as the functions b, c in (2.18) and (2.19) are concerned, we assume that the following conditions are satisfied:

$(\mathbf{A_1})$ c is a continuous function of its arguments, i.e., $c \in C(\mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1])$, and there exist positive constants $c_i, 1 \leq i \leq 2$, such that for all $(y_1, y_2, y_3) \in \mathbb{R}_+ \times \mathbb{R}_+ \times [0, 1]$ there holds

$$c_1 \; \leq \; c(y_1, y_2, y_3) \; \leq \; c_2 \; .$$

Moreover, for fixed $(y_2, y_3) \in \mathbb{R}_+ \times [0,1]$, the function $c(\cdot, y_2, y_3) : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, i.e., $c(\cdot, y_2, y_3) \in C^1(\mathbb{R}_+)$, and there exist positive constants $c_i, 3 \leq i \leq 4$, such that for all $y_1 \in \mathbb{R}_+$ there holds

$$c(y_1, y_2, y_3) \; + \; 2 \, \frac{\partial c}{\partial y_1}(y_1, y_2, y_3) \; \geq \; c_3 \; ,$$
$$|\frac{\partial c}{\partial y_1}(y_1, y_2, y_3)| \; y_1 \; \leq \; c_4 \; .$$

$(\mathbf{A_2})$ b is a continuous function of its arguments, i.e., $b \in C(\mathbb{R}_+ \times [0,1])$, and there exists a positive constant c_5 such that for all $(y_1, y_2) \in \mathbb{R}_+ \times [0,1]$ there holds

$$0 \; \leq \; b(y_1, y_2) \; \leq \; c_5 \; .$$

REMARK 2.3. *The first condition in* $(\mathbf{A_1})$ *and condition* $(\mathbf{A_2})$ *imply that for the models* (2.18) *and* (2.19) *the viscosity function* φ *is bounded from below by a positive constant and that for the regularized Bingham-type model* (2.19) *the viscosity function* φ *is bounded from above as well, whereas* $\varphi(I(u), |E|, \mu(u, E)) \to +\infty$ *as* $I(u) \to 0$ *for the extended Bingham-type model* (2.18)*.*
The second condition in $(\mathbf{A_1})$ *implies that for fixed values of* $|E|$ *and* $\mu(u, E)$ *the derivative of the function* $I(v) \longmapsto G(v) := 4(\varphi(I(v), |E|, \mu(u, E)))^2 I(v)$ *is positive, where* $G(v)$ *is the second invariant of the stress deviator. The physical meaning of this condition is that in case of shear flow the shear stress increases with increasing shear rate.*
The third condition in $(\mathbf{A_1})$ *imposes a restriction on the function* $\partial c / \partial y_1$ *for large values of* y_1 *which reflects the experimentally observable behavior of electrorheological fluids that their structure is destroyed at large shear rates.*

On the basis of the assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$, existence and uniqueness results for steady and time-dependent isothermal incompressible electrorheological fluid flows will be established in the subsequent subsections 2.2 and 2.3 relying on the theory of monotone operators (BREZIS [1973], BROWDER [1968], LIONS [1969], MINTY [1962], VAINBERG [1964], VISIK [1962], ZEIDLER [1990]).

We note that under some weaker monotonicity assumptions, an existence result has been derived in DREYFUSS and HUNGERBUEHLER [2004] using the theory of Young measures (see, e.g., VALADIER [1994]). We further refer to DREYFUSS and HUNGERBUEHLER [2004].

Since the macroscopic behavior of electrorheological fluids is largely determined by physical processes occurring on a microscale, a natural approach to develop physically consistent macroscopic models is to use homogenization techniques within a multiscale framework. Such an approach has been undertaken in VERNESCU [2002] (cf. also BANKS et al. [1999] for a similar approach in case of magnetorheological fluids).

2.2. **Boundary value problems for steady isothermal incompressible fluid flows based on regularized Bingham-type flow models.** We adopt standard notation from Lebesgue and Sobolev space theory (cf., e.g., ADAMS [1975], GRISVARD [1985], LIONS and MAGENES [1968]). In particular, for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, we refer to $L^p(\Omega)^d, 1 \leq p \leq \infty$, as the Lebesgue spaces with norms $\| \cdot \|_{p,\Omega}$ and denote by $(\cdot,\cdot)_{0,\Omega}$ the inner product in $L^2(\Omega)^d$. The spaces $W^{m,p}(\Omega)^d, m \in \mathbb{N}$, stand for the Sobolev spaces with norms $\| \cdot \|_{m,p,\Omega}$, whereas $W^{-m,q}(\Omega)^d, 1/p + 1/q = 1, 1 \leq p < \infty$, and $W^{m-1/p,p}(\Gamma)^d, \Gamma := \partial\Omega$, refer to their dual and trace spaces, respectively. For $\Sigma \subseteq \Gamma$, the space $W_{0,\Sigma}^{m-p,p}(\Omega)^d$ denotes the space of functions $v \in W^{m,p}(\Omega)^d$ with vanishing trace on Σ, i.e., $v|_\Sigma = 0$, and $W_{00}^{m-1/p,p}(\Sigma)^d$ is the space of functions $\psi \in W^{m-1/p,p}(\Gamma)^d$ such that $\psi = v|_\Sigma$ for some $v \in W^{m,p}(\Omega)^d$ with $v|_{\Gamma\setminus\Sigma} = 0$. Furthermore, we refer to $H(\mathrm{div};\Omega) := \{v \in L^2(\Omega)^d | \nabla \cdot v \in L^2(\Omega)\}$ and $H(\mathrm{curl};\Omega) := \{v \in L^2(\Omega)^d | \nabla \times v \in L^2(\Omega)^d\}$, if $d \geq 3$, and $H(\mathrm{curl};\Omega) := \{v \in L^2(\Omega)^2 | \nabla \times v \in L^2(\Omega)\}$, if $d = 2$, as the Hilbert spaces of square integrable vector-valued functions with square integrable divergence and rotation, respectively, equipped with the standard graph norm. We denote by $H(\mathrm{div}^0;\Omega)$ and $H(\mathrm{curl}^0;\Omega)$ the subspaces $H(\mathrm{div}^0;\Omega) := \{v \in H(\mathrm{div};\Omega) | \nabla \cdot v = 0\}$ and $H(\mathrm{curl}^0;\Omega) := \{v \in H(\mathrm{curl};\Omega) | \nabla \times v = 0\}$.

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ with boundary $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N, \Gamma_D \cap \Gamma_N = \emptyset$, and functions

$$f \in L^2(\Omega)^d \quad , \quad g \in L^2(\Gamma_N)^d \quad , \quad u^D \in W^{1/2,2}(\Gamma_D)^d \; , \tag{2.20}$$

we consider the following boundary value problem for steady, incompressible, isothermal electrorheological fluid flows under the Stokes approximation, i.e., we ignore inertial forces,

$$\nabla \cdot \sigma \;=\; f \quad \text{in}\, \Omega \; , \tag{2.21a}$$

$$\nabla \cdot u \;=\; 0 \quad \text{in}\, \Omega \; , \tag{2.21b}$$

$$u \;=\; u^D \quad \text{on}\, \Gamma_D \times (0,T) \; , \tag{2.21c}$$

$$\nu \cdot \sigma \;=\; g \quad \text{on}\, \Gamma_N \; , \; , \tag{2.21d}$$

where the stress tensor σ is supposed to satisfy one of the constitutive equations from the previous subsection.

As far as the electric field E is concerned, we assume that the boundary Γ features n pairs of electrodes and counter-electrodes occupying open subsets $\Gamma_i^e, \Gamma_i^c \subset \Gamma, \Gamma_i^e \cap \Gamma_i^c = \emptyset, 1 \leq i \leq n, n \in \mathbb{N}$, with voltages U_i applied to the electrodes Γ_i^e. Since we assume the electric field E to be quasi-static, it satisfies $E \in H(\mathrm{curl}^0;\Omega)$ and $\epsilon E \in H(\mathrm{div}^0;\Omega)$, where ϵ stands for the electric permittivity. Hence, there exists an electric potential $\psi \in W^{1,2}(\Omega)$ satisfying the elliptic boundary value problem

$$\nabla \cdot (\epsilon \nabla \psi) \;=\; 0 \quad \text{in}\, \Omega \; , \tag{2.22a}$$

$$\psi \;=\; \begin{cases} U_i & \text{on}\ \Gamma_i^e \\ 0 & \text{on}\ \Gamma_i^c \end{cases} \; , 1 \leq i \leq n \; , \tag{2.22b}$$

$$\nu \cdot \epsilon \nabla \psi \;=\; 0 \quad \text{on}\, \Gamma \setminus \bigcup_{i=1}^n \overline{(\Gamma_i^e \cup \Gamma_i^c)} \; . \tag{2.22c}$$

Since the coupling between the electric field and the fluid is supposed to be unilateral, the boundary value problem (2.22a)-(2.22c) can be solved beforehand.

THEOREM 2.1. *Assume $U_i \in W_{00}^{1/2,2}(\Gamma_i^e), 1 \leq i \leq n$, and $\epsilon = (\epsilon_{ij})_{i,j=1}^d, \epsilon_{ij} \in L^\infty(\Omega)$, $1 \leq i,j \leq d$, such that for almost all $x \in \Omega$*

$$\sum_{i,j=1}^d \epsilon_{ij}(x)\xi_i\xi_j \;\geq\; \alpha|\xi|^2 \quad , \quad \xi \in \mathbb{R}^d \; , \; \alpha > 0 \; .$$

Then, the boundary value problem (2.22a)-(2.22c) admits a unique weak solution $\theta \in W_{0,\Gamma^c}^{1,2}(\Omega), \Gamma^c := \bigcup_{i=1}^n \Gamma_i^c$.

Proof. Due to the assumption on the voltages U_i there exists $\tilde{\theta} \in W^{1,2}(\Omega)$ such that $\tilde{\theta}|_{\Gamma_i^e} = U_i$ and $\tilde{\theta}|_{\Gamma_i^c} = 0, 1 \leq i \leq n$. Defining $a(v,w) := \int_\Omega \epsilon \nabla v \cdot \nabla w dx, v, w \in V :=$

$W^{1,2}_{0,\tilde{\Gamma}}(\Omega), \tilde{\Gamma} := \bigcup_{i=1}^n (\Gamma_i^e \cup \Gamma_i^c)$, the V-ellipticity of the bilinear form $a(\cdot,\cdot)$ implies the existence and uniqueness of $\hat{\theta} \in V$ satisfying

$$a(\hat{\theta}, v) \;=\; -a(\tilde{\theta}, v) \quad , \quad v \in V \; .$$

Then, $\theta = \hat{\theta} + \tilde{\theta}$ is the unique weak solution of (2.22a)-(2.22c). □

2.2.1. *The regularized extended Bingham fluid model.* We study the existence and uniqueness of a solution of the boundary value problem (2.21a)-(2.21d) for the electrorheological fluid model (2.19) with regularization parameter κ. We show that a weak solution of (2.21a)-(2.21d) satisfies a system of variational equations of saddle point type and establish an existence result by means of appropriate Galerkin approximations in finite dimensional subspaces of the underlying function spaces. To this end, we set

$$X \;:=\; W^{1,2}_{0,\Gamma_D}(\Omega)^d \quad , \quad V \;:=\; X \cap H(\text{div}^0;\Omega) \tag{2.23}$$

and denote by $\tilde{u} \in W^{1,2}(\Omega)^d \cap H(\text{div}^0;\Omega)$ the function with trace $\tilde{u}|_{\Gamma_D} = u^D$. Moreover, we introduce a functional $J_\kappa : X \times X \to \mathbb{R}, \kappa \in \mathbb{R}_+$, and an operator $L : X \to X^*$ according to

$$J_\kappa(v,w) \;:=\; 2 \int\limits_\Omega c(|E|, \mu(\tilde{u}+v,E))(\kappa + I(\tilde{u}+w))^{1/2} \, dx \; , \tag{2.24a}$$

$$\langle L(v), w\rangle \;:=\; 2 \int\limits_\Omega b(I((\tilde{u}+v), |E|, \mu(\tilde{u}+v,E))\varepsilon(\tilde{u}+v) : \varepsilon(w) \, dx \; , \tag{2.24b}$$

where $\langle\cdot,\cdot\rangle$ stands for the dual pairing between X^* and X.
For $\kappa > 0$, the functional J_κ is Gâteaux differentiable on X with respect to the second argument. Indeed, the partial Gâteaux derivative $\frac{\partial J_\kappa}{\partial w}(v,\cdot) \in \mathcal{L}(X,X^*), v \in X$, is given by

$$\begin{aligned} &\langle \frac{\partial J_\kappa}{\partial w}(v,w), z\rangle \;= \\ &2 \int\limits_\Omega c(|E|, \mu(\tilde{u}+v,E))(\kappa + I(\tilde{u}+w))^{-1/2} \varepsilon(\tilde{u}+w) : \varepsilon(z) \, dx \; , \; w,z \in X \; . \end{aligned} \tag{2.25}$$

We further define an operator $M_\kappa; X \times X \to X^*, \kappa > 0$, by

$$M_\kappa(v,v) \;:=\; \frac{\partial J_\kappa}{\partial w}(v,v) \;+\; L(v) \quad , \quad v \in X \; . \tag{2.26}$$

We consider the problem: Find $v \in V$ such that

$$\langle M_\kappa(v,v), z\rangle \;=\; \langle f+g, z\rangle \quad , \quad z \in V \; , \tag{2.27}$$

where we formally view $f+g$ as an element of X^*. We will refer to $u = \tilde{u} + v$ as a weak solution of (2.21a)-(2.21d). If a pair (u,p) is a solution of (2.21a)-(2.21d), by Green's formula it can be easily seen that $v = u - \tilde{u}$ solves (2.27). We can state (2.27) equivalently as a system of variational equations of saddle point type, if we couple the incompressibility condition by means of a Lagrange multiplier in $L^2(\Omega)$. Denoting by $B \in \mathcal{L}(X, L^2(\Omega))$ the divergence operator, i.e., $Bv = \nabla \cdot v, v \in X$, this leads to the following system: Find $(v,p) \in X \times L^2(\Omega)$ such that

$$\langle M_\kappa(v,v), z\rangle \;-\; \langle B^* p, z\rangle \;=\; \langle f+g, z\rangle \; , \; z \in X \; , \tag{2.28a}$$

$$(Bv, q)_{0,\Omega} \;=\; 0 \; , \; q \in L^2(\Omega) \; . \tag{2.28b}$$

LEMMA 2.1. *Let $v \in V$ be a solution of (2.27). Then, there exists a unique $p \in L^2(\Omega)$ such that* (2.28a),(2.28a) *holds true. Conversely, if $(v,p) \in X \times L^2(\Omega)$ is a solution of* (2.28a),(2.28a), *then the pair $(\tilde{u}+v,p)$ satisfies (2.27). Moreover, if v,p and $\tilde{u}$ are smooth functions, then $(\tilde{u}+v,p)$ solves* (2.28a),(2.28b).

Proof. The proof follows readily from the properties of the divergence operator B. In particular, denoting by $V^\perp$ the orthogonal complement of V in X and by V^0 the polar set

$$V^0 \;:=\; \{\ell \in X^* \mid \langle \ell, w\rangle = 0\;,\; w \in V\}\;,$$

the operator B is an isomorphism from $V^\perp$ onto $L^2(\Omega)$, whereas its adjoint B^* is an isomorphism from $L^2(\Omega)$ onto V^0 (see BELONOSOV and LITVINOV [1996] and Lemma 6.1.1 in LITVINOV [2000]). We note that the case $B : H_0^1(\Omega)^d \to L_0^2(\Omega)$ has been addressed, e.g., in BREZZI and FORTIN [1991], GIRAULT and RAVIART [1986], LADYZHENSKAYA and SOLONNIKOV [1976]. □

The existence of a solution $(u,p) \in X \times L^2(\Omega)$ of (2.28a),(2.28b) will be shown by a Galerkin approximation with respect to sequences $\{X_n\}_{\mathbb{N}}$ and $\{Q_n\}_{\mathbb{N}}$ of finite dimensional subspaces that are limit dense in X and $L^2(\Omega)$, i.e.,

$$\lim_{n\to\infty} \inf_{v_n \in X_n} \|v - v_n\|_X \;=\; 0 \quad , \quad v \in X\;, \tag{2.29a}$$

$$\lim_{n\to\infty} \inf_{\mu_n \in Q_n} \|p - p_n\|_{0,\Omega} \;=\; 0 \quad , \quad p \in L^2(\Omega)\;. \tag{2.29b}$$

We refer to $B_n \in \mathcal{L}(X_n, Q_n^*), n \in \mathbb{N}$, as the discrete divergence operator

$$(B_n v_n, p_n)_{0,\Omega} \;:=\; \int_\Omega p_n \nabla \cdot v_n \, dx \quad , \quad v_n \in X_n\;,\; p_n \in Q_n\;, \tag{2.30}$$

and assume that for each $n \in \mathbb{N}$ the discrete LBB-condition

$$\inf_{p_n \in Q_n} \sup_{v_n \in X_n} \frac{(B_n v_n, p_n)_{0,\Omega}}{\|v_n\|_X \|p_n\|_{0,\Omega}} \;\geq\; \beta > 0 \tag{2.31}$$

is satisfied. As can be easily established, under the above assumption the discrete divergence operators $B_n, n \in \mathbb{N}$, inherit the properties of their continuous counterpart B.

LEMMA 2.2. *Assume that $\{X_n\}_{\mathbb{N}}$ and $\{Q_n\}_{\mathbb{N}}$ are finite dimensional subspaces $X_n \subset X, n \in \mathbb{N}$, and $Q_n \subset L^2(\Omega), n \in \mathbb{N}$,. Moreover, let $B_n, n \in \mathbb{N}$, be the discrete divergence operator as given by* (2.30) *and suppose that the discrete LBB-condition* (2.31) *holds true. Then, B_n is an isomorphism from $(Ker(B_n))^\perp$ onto Q_n^* and B_n^* is an isomorphism from Q_n onto the polar set $(Ker(B_n))^0$ such that*

$$\|B_n\| \;\leq\; \beta^{-1} \quad , \quad \|(B_n^*)^{-1}\| \;\leq\; \beta^{-1} \quad , \quad n \in \mathbb{N}\;. \tag{2.32}$$

We consider the following approximating system of finite dimensional variational equations: Find $(v_n, p_n) \in X_n \times Q_n, n \in \mathbb{N}$, such that

$$\langle M_\kappa(v_n, v_n), z_n\rangle \;-\; \langle B_n^* p_n, z_n\rangle \;=\; \langle f + g, z_n\rangle\;,\; z_n \in X_n\;, \tag{2.33a}$$

$$(B_n v_n, q_n)_{0,\Omega} \;=\; 0\;,\; q_n \in Q_n\;. \tag{2.33b}$$

The main result of this subsection states the solvability of the system (2.33a),(2.33b) for each $n \in \mathbb{N}$ and the existence of a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that the associated sequence $\{(v_n, p_n)\}_{\mathbb{N}'}$ of solutions converges to a pair $(v,p) \in X \times L^2(\Omega)$ which solves (2.28a),(2.28b).

THEOREM 2.2. *Assume that the conditions* $(\mathbf{A_1}),(\mathbf{A_2})$ *are fulfilled and* f, g, u^d *satisfy* (2.20). *Further, let* $\{X_n\}_{\mathbb{N}}$ *and* $\{Q_n\}_{\mathbb{N}}$ *be nested sequences of finite dimensional subspaces* $X_n \subset X, n \in \mathbb{N}$, *and* $Q_n \subset L^2(\Omega), n \in \mathbb{N}$, *i.e.,*

$$X_n \subset X_{n+1} \quad , \quad Q_n \subset Q_{n+1} \quad , \quad n \in \mathbb{N} \,, \tag{2.34}$$

that are limit dense in X *and* $L^2(\Omega)$ *and suppose that the discrete LBB-condition* (2.31) *holds true. Then, for any* $\kappa > 0$ *and* $n \in \mathbb{N}$ *there exists a solution* $(v_n, p_n) \in X_n \times Q_n$ *of the discrete saddle point problem* (2.33a),(2.33b). *Moreover, there exist a subsequence* $\mathbb{N}' \subset \mathbb{N}$ *and a pair* $(v,p) \in X \times L^2(\Omega)$ *such that*

$$v_n \rightharpoonup v \quad \text{in } X \quad (\mathbb{N}' \ni n \to \infty) \,, \tag{2.35a}$$

$$p_n \to p \quad \text{in } L^2(\Omega) \quad (\mathbb{N}' \ni n \to \infty) \,. \tag{2.35b}$$

The pair $(v,p) \in X \times L^2(\Omega)$ *is a solution of* (2.28a),(2.28b).

Theorem 2.2 will be proved by a series of Lemmas which enable us to deduce the existence of a bounded sequence $\{(u_n, p_n)\}_{\mathbb{N}}$ of solutions of (2.33a),(2.33b) and to pass to the limit.

For $z = (\tilde{z}, z_1, z_2)$ with $\tilde{z} \in W^{1,2}(\Omega), z_1 \in L^2_+(\Omega)$ and $z_2 \in L^\infty(\Omega), z_2(x) \in [0,1]$ f.a.a. $x \in \Omega$, we define $L_z : X \to X^*$ as the operator

$$\langle L_z(v), w \rangle \;:=\; 2 \int_\Omega b(I(v+\tilde{z}), z_1, z_2)\varepsilon(v+\tilde{z}) : \varepsilon(w) \, dx \;,\; v, w \in X \,. \tag{2.36}$$

LEMMA 2.3. *Under the assumption* $(\mathbf{A_1})$, *the operator* L_z *as given by* (2.36) *is a continuous, strongly monotone operator from* X *into* X^*. *In particular, for* $v, w \in X$ *there holds*

$$\|L_z(v) - L_z(w)\|_{X^*} \;\leq\; C_L \, \|v - w\|_X \,, \tag{2.37a}$$

$$\langle L_z(v) - L_z(w), v - w \rangle \;\geq\; \gamma_L \, \|v - w\|_X^2 \,, \tag{2.37b}$$

where $C_L := (2c_2 + 4c_4)$ *and* $\gamma_L := 2min(c_1, c_3)$ *with* $c_i, 1 \leq i \leq 4$, *from* $(\mathbf{A_1})$.

Proof. For $v, w \in X$ we set $q := v - w$ and consider the function $\tau : [0,1] \to \mathbb{R}$ which for an arbitrarily, but fixed chosen $h \in X$ is given by

$$\tau(t) \;:=\; \int_\Omega b(I(\tilde{z}+w+tq), z_1, z_2)\varepsilon(\tilde{z}+w+tq) : \varepsilon(h) \, dx \;,\; t \in [0,1] \,.$$

Obviously, τ satisfies

$$\tau(1) \;-\; \tau(0) \;=\; \frac{1}{2} \, \langle L_z(v) - L_z(w), h \rangle \,.$$

Since $\tau \in C^1([0,T])$, classical calculus tells us that for some $\xi \in (0,1)$

$$\tau(1) \;=\; \tau(0) \;+\; \frac{d\tau}{dt}(\xi) \,,$$

where $(d\tau/dt)(\xi)$ is given by

$$\begin{gathered}\frac{d\tau}{dt}(\xi) \;=\; \int_\Omega \Big(b(I(\tilde{z}+w+\xi q), z_1, z_2)\varepsilon(q) : \varepsilon(h) \,+ \\ 2\frac{\partial b}{\partial y_1}(I(\tilde{z}+w+\xi q), z_1, z_2)(\varepsilon(\tilde{z}+w+\xi q) : \varepsilon(q))(\varepsilon(\tilde{z}+w+\xi q) : \varepsilon(h)) \Big) \, dx \,.\end{gathered} \tag{2.38}$$

In view of the inequality

$$|\varepsilon(\tilde{z}+w+\xi q):\varepsilon(q)| \ \le \ (I(\tilde{z}+w+\xi q))^{1/2}(I(q))^{1/2}$$

and taking $(\mathbf{A_1})(i)$ and $(\mathbf{A_1})(iii)$ into account, (2.37a) can be easily deduced.
On the other hand, we define $\eta : \mathbb{R}_+ \times \Omega \to \mathbb{R}_-$ by

$$\eta(\alpha,x) \ := \ (\frac{\partial b}{\partial y_1}(\alpha,z_1(x),z_2(x))^- \quad , \quad \alpha\in\mathbb{R}_+ \ , \ x\in\Omega \ .$$

Then, if we set $\alpha := I(\tilde{z}+w+\xi q)$ and choose $h = q$ in (2.38), we obtain

$$\frac{d\tau}{dt}(\xi) \ \ge \ \int\limits_\Omega \Big(b(I(\tilde{z}+w+\xi q),z_1,z_2)I(q) +$$
$$2g(\alpha,z_1(x),z_2(x))(\varepsilon(\tilde{z}+w+\xi q):\varepsilon(q))^2\Big) \ dx \ \ge \ \min(c_1,c_3) \ \|q\|_X^2 \ ,$$

which proves (2.37b). The continuity of the operator L_z follows from the continuity of the Nemytskii operator. □

In view of the representation of the partial Gâteaux derivative $\partial J_\kappa/\partial w$ by (2.25) and assumption $(\mathbf{A_2})$, for a given function

$$\chi \in U \ := \ \{\vartheta\in L_+^\infty(\Omega) \mid \vartheta(x)\le c_5 \text{ f.a.a. } x\in\Omega\}$$

and $\tilde{v}\in W^{1,2}(\Omega)$ we define an operator $S_\kappa : U\times X \to X^*, \kappa>0$, according to

$$\langle S_\kappa(\chi,v),w\rangle \ := \ \int\limits_\Omega \chi(\kappa+I(\tilde{v}+v))^{-1/2}\varepsilon(\tilde{v}+v):\varepsilon(w) \ dx \ , \ v,w\in X \ . \tag{2.39}$$

LEMMA 2.4. *Under the assumption* $(\mathbf{A_2})$, *for an arbitrarily, but fixed chosen* $\chi\in U$, *the operator* $S_\kappa(\chi,\cdot),\kappa>0$, *with* S_κ *as given by* (2.39) *is a continuous, monotone operator from* X *into* X^*. *In particular, there holds*

$$\|S_\kappa(\chi,v)-S_\kappa(\chi,w)\|_{X^*} \ \le \ 2c_5\kappa^{-1/2} \ \|v-w\|_X \ , \ v,w\in X \ , \tag{2.40a}$$

$$\|S_\kappa(\chi,v)\|_{X^*} \ \le \ \Big(\int\limits_\Omega \chi^2 \ dx\Big)^{1/2} \ , \ v\in X \ . \tag{2.40b}$$

Proof. We set $v_1 := \tilde{v}+v, w_1 := \tilde{v}+w$ and define $\varphi_\kappa : \mathbb{R}_+\to\mathbb{R}_+, \kappa>0$, by

$$\varphi_\kappa(y) \ := \ \frac{1}{2} \ \chi \ (\kappa+y)^{-1/2} \ , \ y\in\mathbb{R}_+ \ . \tag{2.41}$$

Then, if we take

$$|\varepsilon(v_1):\varepsilon(w_1)| \ \le \ (I(v_1))^{1/2}(I(w_1))^{1/2}$$

into account, it follows that

(2.42)

$$\langle S_\kappa(\chi,v)-S_\kappa(\chi,w),v-w\rangle \ = \ \langle S_\kappa(\chi,v)-S_\kappa(\chi,w),v_1-w_1\rangle \ =$$
$$= \ 2\int\limits_\Omega \Big(\varphi_\kappa(I(v_1))I(v_1)+\varphi_\kappa(I(w_1))I(w_1) -$$
$$- \ (\varphi_\kappa(I(v_1))+\varphi_\kappa(I(w_1)))\varepsilon(v_1):\varepsilon(w_1)\Big)dx \ \ge$$
$$\ge \ 2\int\limits_\Omega \Big(\varphi_\kappa(I(v_1))(I(v_1))^{1/2}-\varphi_\kappa(I(w_1))(I(w_1))^{1/2}\Big)\Big((I(v_1))^{1/2}-I(w_1)^{1/2}\Big) \ dx \ .$$

Now, for the function φ_κ from (2.41) one easily finds

$$\varphi_\kappa(y) + 2\frac{d\varphi_\kappa}{dy}(y)y \;=\; \frac{1}{2}\chi(\kappa+y)^{-1/2}\Big(1-(\kappa+y)^{-1}y\Big) \;>\; 0\;,\; y\in\mathbb{R}_+\;. \tag{2.43}$$

Considering $\psi(z) := \varphi_\kappa(z^2)z$, we have $(d\psi/dz)(z) = \varphi_\kappa(z^2) + 2(d\varphi_\kappa)(z^2)z^2$ which is the left-hand side in (2.43) for $z^2 = y$. It follows that ψ is a monotonously increasing function, and (2.42) implies the monotonicity of the operator $S_\kappa(\chi,\cdot)$. The boundedness (2.40b) of $S_\kappa(\chi,\cdot)$ is an immediate consequence of

$$\begin{aligned}
&|\langle S_\kappa(\chi,v),w\rangle| \;\le\\
&\le\; \int_\Omega \chi(\kappa+(I(\tilde v+v))^{-1/2}(I(\tilde v+v))^{1/2}(I(w))^{1/2}\;dx \;\le\\
&\le\; \Big(\int_\Omega \chi^2\;dx\Big)^{1/2}\;\|w\|_X\;.
\end{aligned}$$

Finally, in view of

$$\varphi_\kappa(y) \;\le\; \frac{1}{2}c_5\kappa^{-1/2} \quad,\quad |\frac{d\varphi_\kappa}{dy}(y)|y \;\le\; \frac{1}{4}c_5\kappa^{-1/2} \quad,\quad y\in\mathbb{R}_+\;,$$

the estimate (2.40a) can be deduced as in the proof of Lemma 2.3. □

COROLLARY 2.1. *Under the assumptions of Lemma* 2.4 *assume that* $\{v_n\}_\mathbb{N}$ *is a sequence of elements* $v_n\in X, n\in\mathbb{N}$, *and* $v\in X$ *such that*

$$\begin{aligned}
v_n &\to v \quad \textit{in } X \quad (n\to\infty)\;,\\
v_n &\to v \quad \textit{a.e. in } \Omega \quad (n\to\infty)\;,\\
\nabla v_n &\to \nabla v \quad \textit{a.e. in } \Omega \quad (n\to\infty)\;.
\end{aligned} \tag{2.44}$$

Moreover, suppose that $\{\chi_n\}_\mathbb{N}$ *is a sequence of elements* $\chi_n\in U, n\in\mathbb{N}$, *such that for some* $\chi\in U$ *there holds*

$$\chi_n \to \chi \quad \textit{a.e. in } \Omega \quad (n\to\infty)\;. \tag{2.45}$$

Then, for any $\kappa>0$ *we have*

$$S_\kappa(\chi_n,v_n) \to S_\kappa(\chi,v) \quad \textit{in } X^* \quad (n\to\infty)\;. \tag{2.46}$$

Proof. Straightforward estimation from above yields

$$\begin{aligned}
&\|S_\kappa(\chi_n,v_n) - S_\kappa(\chi,v)\|_{X^*} \;\le\\
&\le\; \|S_\kappa(\chi_n,v_n) - S_\kappa(\chi_n,v)\|_{X^*} \;+\; \|S_\kappa(\chi_n,v) - S_\kappa(\chi,v)\|_{X^*}\;.
\end{aligned} \tag{2.47}$$

Due to (2.45), the second term on the right-hand side in (2.47) tends to zero as $n\to\infty$. As far as the first term is concerned, for $w\in X$ we have

$$\begin{aligned}
\langle S_\kappa(\chi_n,v_n) - S_\kappa(\chi_n,v),w\rangle \;=\; \int_\Omega \chi_n\Big((\kappa+I(\tilde v+v_n))^{-1/2}\varepsilon(v_n-v):\varepsilon(w) \;+\\
+\; ((\kappa+I(\tilde v+v_n))^{-1/2} - (\kappa+I(\tilde v+v))^{-1/2})\varepsilon(\tilde v+v):\varepsilon(w)\Big)\;dx\;,
\end{aligned}$$

from which we deduce

$$\begin{aligned}
(2.48)\quad & \|S_\kappa(\chi_n, v_n) - S_\kappa(\chi_n, v)\|_{X^*} \le \\
& \le \underbrace{\Big(\int_\Omega \chi_n^2 (\kappa + I(\tilde{v} + v_n))^{-1} I(v_n - v)\, dx \Big)^{1/2}}_{=:\, I_1} + \\
& + \underbrace{\Big(\int_\Omega \chi_n^2 ((\kappa + I(\tilde{v} + v_n))^{-1/2} - (\kappa + I(\tilde{v} + v))^{-1/2})^2 I(\tilde{v} + v)\, dx \Big)^{1/2}}_{=:\, I_2} .
\end{aligned}$$

In view of the uniform boundedness of the sequence $\{\chi_n\}_{\mathbb{N}}$ and (2.44), obviously $I_1 \to 0$ as $n \to \infty$. On the other hand, (2.44) also implies

$$I(\tilde{v} + v_n) \to I(\tilde{v} + v) \quad (n \to \infty) ,$$

whence $I_2 \to 0$ as $n \to \infty$ by the Lebesgue theorem. Consequently, the first term on the right-hand side in (2.47) tends to zero as $n \to \infty$ which allows to conclude. □

We are now in a position to provide the proof of Theorem 2.2.

Proof of Theorem 2.2. If $(v_n, p_n) \in X_n \times Q_n, n \in \mathbb{N}$, is a solution of (2.33a),(2.33b), then $v_n \in \mathrm{Ker}(B_n)$ and

$$(2.49)\qquad \langle M_\kappa(v_n, v_n), w_n \rangle + \langle L(v_n), w_n \rangle = \langle f + g, w_n \rangle \ , \ w_n \in \mathrm{Ker}(B_n) .$$

By assumption $(\mathbf{A_2})$, for $\kappa > 0$ and $w \in X$ we have

$$\begin{aligned}
(2.50)\quad & |\langle \frac{\partial J_\kappa}{\partial w}(w, w), w \rangle| = \\
& = 2 \, | \int_\Omega c(|E|, \mu(\tilde{u} + w, E))(\kappa + I(\tilde{u} + w))^{-1/2} \varepsilon(\tilde{u} + w) : \varepsilon(w)\, dx | \le \\
& \le 2 \int_\Omega c(|E|, \mu(\tilde{u} + w))(I(w))^{1/2}\, dx \le 2c_5 |\Omega|^{1/2} \, \|w\|_X .
\end{aligned}$$

If we take assumption $(\mathbf{A_1})$ as well as (2.20) and (2.50) into account, it follows that for some $C_1 \in \mathbb{R}_+$

$$\varrho(w) := \langle M_\kappa(w, w), w \rangle - \langle f + g, w \rangle \ge \|w\|_X \Big(2c_1 \|w\|_X - C_1 \Big) ,$$

whence

$$\varrho(w) \ge 0 \quad \text{for} \quad \|w\|_X \ge r := C_1/(2c_1) .$$

Then, the corollary of Brouwer's fixed point theorem in GAJEWSKI et al. [1974] implies the existence of a solution $v_n \in \mathrm{Ker}(B_n)$ of (2.49) which satisfies

$$(2.51)\qquad \|v_n\|_X \le r \quad , \quad \|L(v_n)\|_{X^*} \le C_2 \quad , \quad n \in \mathbb{N} ,$$

for some constant $C_2 > 0$. Now, for $\ell \in X^*$ let $\ell_n := \ell|_{X_n}, n \in \mathbb{N}$. Then, $\ell_n \in X_n^*$ and in view of (2.49) we have

$$\ell_n(M_\kappa(v_n, v_n) - (f + g)) \in \mathrm{Ker}(B_n)^0 .$$

By means of Lemma 2.2 we deduce the existence of a unique $p_n \in Q_n$ such that

$$B_n^* p_n = \ell_n(M_\kappa(v_n, v_n) - (f + g))$$

and the pair $(v_n, p_n) \in X_n \times Q_n$ solves (2.33a),(2.33b). Taking advantage of assumption $(\mathbf{A_2})$, (2.20),(2.51) and Lemmas 2.2 and 2.4 we obtain the boundedness of the sequence $\{p_n\}_{\mathbb{N}}$, i.e., with some $C_3 > 0$ there holds

$$\|p_n\|_{0,\Omega} \le C_3 \quad , \quad n \in \mathbb{N} . \tag{2.52}$$

Due to (2.51) and (2.52) there exist a subsequence $\mathbb{N}' \subset \mathbb{N}$ and elements $v^* \in X, p^* \in L^2(\Omega)$ as well as $\ell_1^*, \ell_2^* \in X^*$ such that

$$v_n \rightharpoonup v^* \quad \text{in } X \quad (\mathbb{N}' \ni n \to \infty) , \tag{2.53a}$$

$$v_n \to v^* \quad \text{in } L^2(\Omega) \quad (\mathbb{N}' \ni n \to \infty) , \tag{2.53b}$$

$$v_n \to v^* \quad \text{a.e. in } \Omega \quad (\mathbb{N}' \ni n \to \infty) , \tag{2.53c}$$

$$p_n \to p^* \quad \text{in } L^2(\Omega) \quad (\mathbb{N}' \ni n \to \infty) , \tag{2.53d}$$

$$L(v_n) \rightharpoonup \ell_1^* \quad \text{in } X^* \quad (\mathbb{N}' \ni n \to \infty) , \tag{2.53e}$$

$$\frac{\partial J_\kappa}{\partial w}(v_n, v_n) \rightharpoonup \ell_2^* \quad \text{in } X^* \quad (\mathbb{N}' \ni n \to \infty) . \tag{2.53f}$$

In view of (2.29a),(2.29b) and (2.53a) as well as (2.53d)-(2.53f) we pass to the limit in (2.33a),(2.33b) and obtain

$$\langle \ell_2^* + \ell_1^* - B^* p^*, w\rangle = \langle f + g, w\rangle , \ w \in X , \tag{2.54a}$$

$$(\nabla \cdot v^*, q)_{0,\Omega} = 0 , \ q \in L^2(\Omega) . . \tag{2.54b}$$

We note that the action of operator L can be written as $L(v) = L(w,w), w \in X$, where the mapping $(w,z) \longmapsto L(w,z)$ is from $X \times X$ into X^* according to

$$\langle L(w,z), h\rangle := 2 \int_\Omega b(I(\tilde{u}+z), |E|, \mu(\tilde{u}+w, E))\varepsilon(\tilde{u}+z) : \varepsilon(h)\, dx , \ h \in X .$$

For $n \in \mathbb{N}'$ we define $\hat{\ell}_n \in X^*$ by

$$\begin{aligned} \hat{\ell}_n(w) \ :=\ & \langle \frac{\partial J_\kappa}{\partial w}(v_n, v_n) + L(v_n, v_n) - \\ & - \ (\frac{\partial J_\kappa}{\partial w}(v_n, w) + L(u_n, v)), v_n - w\rangle , \ w \in X . \end{aligned} \tag{2.55}$$

The previous results show

$$\hat{\ell}_n(w) \ge 0 \quad , \quad w \in X , \ n \in \mathbb{N}' . \tag{2.56}$$

On the other hand, observing

$$\begin{aligned} & \|\frac{\partial J_\kappa}{\partial w}(v_n, w) - \frac{\partial J_\kappa}{\partial w}(v^*, w)\|_{X^*} \le \\ & \le 2\Big(\int_\Omega (c(|E|, \mu(\tilde{u}+v_n, E)) - c(|E|, \mu(\tilde{u}+v^*, E)))^2\, dx\Big)^{1/2} , \end{aligned}$$

assumption $(\mathbf{A_2})$ in combination with (2.53b),(2.53c) and the Lebesgue theorem yield

$$\frac{\partial J_\kappa}{\partial w}(v_n, w) \to \frac{\partial J_\kappa}{\partial w}(v^*, w) \quad \text{in } X^* \quad (\mathbb{N}' \ni n \to \infty) . \tag{2.57}$$

In a similar way, we obtain

$$L(v_n, w) \to L(v^*, w) \quad \text{in } X^* \quad (\mathbb{N}' \ni n \to \infty) . \tag{2.58}$$

Taking $(B_n v_n, p_n)_{0,\Omega} = 0$ into account, (2.33a) and (2.53a),(2.53d) imply

$$\langle M_\kappa(v_n,v_n),v_n\rangle = \langle f+g,v_n\rangle \to \langle f+g,v^*\rangle \quad (\mathbb{N}'\ni n\to\infty)\ , \tag{2.59a}$$

$$\langle M_\kappa(v_n,v_n),w\rangle \to \langle B^*\lambda^*,w\rangle + \ + \ \langle f+g,w\rangle \quad (\mathbb{N}'\ni n\to\infty)\ ,\ w\in X\ . \tag{2.59b}$$

Consequently, passing to the limit in (2.55) and observing (2.54a),(2.54b) as well as (2.56)-(2.58),(2.59a),(2.59b), it follows that

$$\Big(\langle f+g,v^*-w\rangle - \ - \ \langle \frac{\partial J_\kappa}{\partial w}(v^*,w) + L(v^*,w) - B^*p^*, v^*-w\rangle\Big) \ \ge\ 0 \quad , \quad w\in X\ .$$

We choose $v = u^* - \tau z$ where $\tau > 0$ and $z \in X$. The limit process $\tau \to 0$ results in

$$\Big(\langle f+g,z\rangle - \langle M_\kappa(v^*,v^*) - B^*p^*, z\rangle\Big) \ \ge\ 0\ . \tag{2.60}$$

Since this inequality holds true for all $z \in X$, we may replace z by $-z$ and deduce equality in (2.60). We have thus shown that the pair $(v^*, p^*) \in X \times L^2(\Omega)$ solves (2.28a),(2.28b). □

For further existence results in case of stationary electrorheological fluid flows and for studies of the regularity of solutions we refer to ETTWEIN and RUZICKA [2002] and to ACERBI and MINGIONE [2002], BILDHAUER and FUCHS [2004].

With regard to the uniqueness of a solution of (2.28a),(2.28b) we refer to HOPPE and LITVINOV [2004]. We also note that electrorheological fluid flows under conditions of slip on the boundary have been studied in HOPPE et al. [2006] and LITVINOV [2007].

2.2.2. *The extended Bingham-type electrorheological fluid model.* We deal now with the solution of the boundary value problem (2.21a)-(2.21d) for an extended Bingham-type electrorheological fluid model (cf. (2.18)) with viscosity function

$$\varphi(I(u),|E|,\mu(u,E)) \ =\ b(|E|,\mu(u,E))I(u)^{-1/2} \ +\ c(|E|,\mu(u,E))\ . \tag{2.61}$$

We assume that the function b in (2.61) satisfies $(\mathbf{A_2})$, whereas the function c is subject to the following assumption:

$(\mathbf{A_1})'$ $c : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ is a continuous, strictly positive, and uniformly bounded function, i.e., $c \in C(\mathbb{R}_+ \times [0,1])$, and there exist constants $c_8 > 0$ and $c_9 > 0$ such that

$$c_8 \ \le\ c(z_1,z_2) \ \le\ c_9\ ,\ z_1,z_2 \in \mathbb{R}_+ \times [0,1]\ .$$

We formulate (2.21a)-(2.21d) as a variational inequality of the second kind (cf., e.g., GLOWINSKI et al. [1981]). To this end, we denote by $\tilde u \in W^{1,2}(\Omega)^d \cap H(\mathrm{div}^0;\Omega)$ the function with trace $\tilde u|_{\Gamma_D} = u^D$. Moreover, we introduce a functional $J : X \times X \to \mathbb{R}$ and an operator $L : X \to X^*$ according to

$$J(v,w) \ :=\ 2\int_\Omega b(|E|,\mu(\tilde u + v,E))I(\tilde u + w)^{1/2}\ dx\ , \tag{2.62a}$$

$$\langle L(v),w\rangle \ :=\ 2\int_\Omega c(|E|,\mu(\tilde u + v,E))\varepsilon(\tilde u + v) : \varepsilon(w)\ dx\ , \tag{2.62b}$$

where $\langle\cdot,\cdot\rangle$ stands for the dual pairing between X^* and X.
For the constitutive equation (2.61), problem (2.27) can be written as the following variational inequality:

Find $v \in V$ such that for all $w \in V$ there holds

$$J(v,w) - J(v,v) + \langle L(v), w-v\rangle \;\geq\; \langle f+g, w-v\rangle \;. \tag{2.63}$$

The function $u = \tilde{u} + v$, where $v \in V$ is a solution of (2.63), is called a weak solution of (2.21a)-(2.21d) for the constitutive equation (2.61).

We will prove the existence of a solution $v \in V$ of (2.63) via an approximation of J by the functional $J_\kappa : X \times X \to \mathbb{R}, \kappa \in \mathbb{R}_+$, as given by (2.24a), i.e., for a sequence $\{\kappa_n\}_{\mathbb{N}}$ of regularization parameters $\kappa_n > 0, n \in \mathbb{N}$, with $\kappa_n \to 0$ as $n \to \infty$ we consider the variational problem:

Find $v_{\kappa_n} \in V$ such that for all $w \in V$ there holds

$$\langle \frac{\partial J_{\kappa_n}}{\partial v}(v_{\kappa_n}, v_{\kappa_n}), w\rangle + \langle L(v_{\kappa_n}), w\rangle \;=\; \langle f+g, w\rangle \;. \tag{2.64}$$

We further consider the related saddle point problem:

Find $(v_{\kappa_n}, p_{\kappa_n}) \in X \times L^2(\Omega)$ such that for all $w \in X$ and $q \in L^2(\Omega)$ there holds

$$\langle \frac{\partial J_{\kappa_n}}{\partial w}(v_{\kappa_n}, v_{\kappa_n}), w\rangle + \langle L(v_{\kappa_n}), w\rangle - \langle B^* p_{\kappa_n}, w\rangle \;=\; \langle f+g, w\rangle \;, \tag{2.65a}$$

$$(B v_{\kappa_n}, q)_{0,\Omega} \;=\; 0 \;. \tag{2.65b}$$

The existence result partially relies on the following result about functionals $\Psi : U \times X \to \mathbb{R}_+$ of the form

$$\Psi(h,w) \;:=\; \int\limits_\Omega h I(w)^{1/2}\, dx \quad , \quad h \in U \;,\; w \in X \;.$$

Here, $U := \{h \in L^\infty(\Omega) | 0 \leq h(x) \leq c_1 0 \text{ a.e. in } \Omega\}$ for some $c_1 0 > 0$.

LEMMA 2.5. *For an arbitrarily chosen, but fixed $h \in U$, the functional $\Psi(h,\cdot) : X \to \mathbb{R}_+$ is a continuous convex functional. Moreover, for any sequence $\{h_n\}_{\mathbb{N}}$ of elements $h_n \in U, n \in \mathbb{N}$, and any sequence $\{w_n\}_{\mathbb{N}}$ of elements $w_n \in X, n \in \mathbb{N}$, such that for $n \to \infty$*

$$h_n \to h \quad \text{a.e. in } \Omega \quad , \quad w_n \rightharpoonup w \quad \text{in } X \;, \tag{2.66}$$

there holds

$$\liminf_{n\to\infty} \Psi(h_n, w_n) \;\geq\; \Psi(h,w) \;.$$

Proof. Assume $w_n \rightharpoonup w$ in X. In view of

$$\int\limits_\Omega h I(w_n - w)^{1/2}\, dx \;\leq\; \Big(\int\limits_\Omega h^2\, dx\Big)^{1/2} \Big(\int\limits_\Omega I(w_n - w)\, dx\Big)^{1/2} \;,$$

for $n \to \infty$ we have

$$\int\limits_\Omega h I(w_n - w)^{1/2}\, dx \;\to\; 0 \;,$$

$$\int\limits_\Omega h I(w_n - w)^{1/2}\, dx \;\geq\; |\int\limits_\Omega h I(w_n)^{1/2}\, dx - \int\limits_\Omega h I(w)^{1/2}\, dx| \;,$$

whence

$$\Psi(h, u_n) \;\to\; \Psi(h,w) \;,$$

which proves the continuity of $\Psi(h,\cdot)$. For $\lambda \in [0,1]$ and $u,v \in X$ there holds

$$I(\lambda u + (1-\lambda)v) \;=\; I(\lambda u) + 2\lambda(1-\lambda)\,\varepsilon(u) : \varepsilon(v) + I(1-\lambda)v) \;\leq \\ \leq\; \Big(\lambda I(u)^{1/2} + (1-\lambda) I(v)^{1/2}\Big)^2 \;,$$

which implies

$$\Psi(h, \lambda u + (1-\lambda)v) \;= \\ =\; \int_\Omega h I(\lambda u + (1-\lambda)v)^{1/2}\, dx \;\leq\; \lambda \Psi(h,u) + (1-\lambda)\Psi(h,v)\;,$$

and thus proves the convexity of $\Psi(h,\cdot)$. We have

$$\Psi(h_n, w_n) \;=\; \int_\Omega \Big(h I(w_n)^{1/2} + (h_n - h) I(w_n)^{1/2}\Big)\, dx\;, \tag{2.67a}$$

$$|\int_\Omega (h_n - h) I(w_n)^{1/2}\, dx| \;\leq\; \|h_n - h\|_{0,\Omega}\, \|w_n\|_X\;. \tag{2.67b}$$

Due to (2.66) the right-hand side in (2.67b) goes to zero as $n \to \infty$ and hence, the convexity and the continuity of $\Psi(h,\cdot)$ as well as (2.67a),(2.67b) imply

$$\liminf_{n\to\infty} \Psi(h_n, w_n) \;=\; \liminf_{n\to\infty} \Psi(h, w_n) \;\geq\; \Psi(h,w)\;,$$

which completes the proof of the lemma. □

THEOREM 2.3. *Assume that the conditions* $(\mathbf{A_1})', (\mathbf{A_2})$ *are fulfilled and* f, g, u^D *satisfy* (2.20)*. Then, for each* $n \in \mathbb{N}$ *there exist a solution* $v_{\kappa_n} \in V$ *of* (2.64) *and a function* $p_{\kappa_n} \in L^2(\Omega)$ *such that the pair* $(v_{\kappa_n}, p_{\kappa_n})$ *solves the saddle point system* (2.65a),(2.65b). *Moreover, there exist a subsequence* $\mathbb{N}' \subset \mathbb{N}$ *and a function* $v \in V$ *such that*

$$v_{\kappa_n} \rightharpoonup v \quad \text{in } X \quad (\mathbb{N}' \ni n \to \infty)\;, \tag{2.68a}$$

$$v_{\kappa_n} \to v \quad \text{in } L^2(\Omega)^d \quad (\mathbb{N}' \ni n \to \infty)\;. \tag{2.68b}$$

The function v *satisfies* (2.63)*. Further, if* $I(\tilde{u} + v) \neq 0$ *a.e. in* Ω*, the functional*

$$w \longmapsto J(v,w)\;,\; w \in V\;,$$

is Gâteaux-differentiable at the point v *and there exists a function* $p \in L^2(\Omega)$ *such that for all* $w \in X$ *there holds*

$$\langle \frac{\partial J}{\partial v}(v,v), w\rangle + \langle L(v), w\rangle - \langle B^* p, w\rangle \;=\; \langle f + g, w\rangle\;.$$

Proof. Theorem 2.2 yields both the existence of $v_{\kappa_n} \in V$ satisfying (2.64) as well as the existence of $p_{\kappa_n} \in L^2(\Omega)$ such that the pair $(v_{\kappa_n}, p_{\kappa_n})$ solves (2.65a),(2.65b). Moreover, it follows from the proof of Theorem 2.2 that the sequence $\{v_{\kappa_n}\}_{\mathbb{N}}$ is bounded in V. Consequently, there exist a subsequence $\mathbb{N}' \subset \mathbb{N}$ and a function $v \in V$ such that (2.68a),(2.68b) hold true. In view of Lemma 2.4, for $w \in V$ the functional $v \longmapsto J_{\kappa_n}(w,v)$ is convex, whence

$$J_{\kappa_n}(v_{\kappa_n}, w) - J_{\kappa_n}(v_{\kappa_n}, v_{\kappa_n}) + \langle L(v_{\kappa_n}), w - v_{\kappa_n}\rangle - \langle f + g, w - v_{\kappa_n}\rangle \;= \\ =\; -\langle \frac{\partial J_{\kappa_n}}{\partial v}(v_{\kappa_n}, v_{\kappa_n}), w - v_{\kappa_n}\rangle + J_{\kappa_n}(v_{\kappa_n}, w) - J_{\kappa_n}(v_{\kappa_n}, v_{\kappa_n}) \;\geq\; 0\;. \tag{2.69}$$

Assumption $(\mathbf{A_1})'$, (2.68b) and the Lebesgue theorem imply that for $\mathbb{N}' \ni n \to \infty$

$$c(|E|, \mu(\tilde{u} + v_{\kappa_n}, E))\varepsilon(v) \to c(|E|, \mu(\tilde{u} + v, E))\varepsilon(w) \quad \text{in } L^2(\Omega)\;, \tag{2.70}$$

whence by (2.68a)

$$\langle L(v_{\kappa_n}), w\rangle \to \langle L(v), w\rangle \ . \tag{2.71}$$

We define

$$M_{\kappa_n}^{(1)} \ := \ 2\int\limits_\Omega c(|E|, \mu(\tilde u + v_{\kappa_n}, E))\varepsilon(\tilde u) : \varepsilon(v_{\kappa_n})\ dx \ , \tag{2.72a}$$

$$M_{\kappa_n}^{(2)} \ := \ 2\int\limits_\Omega c(|E|, \mu(\tilde u + v_{\kappa_n}, E))I(v_{\kappa_n})\ dx \ , \tag{2.72b}$$

such that

$$\langle L(v_{\kappa_n}), v_{\kappa_n}\rangle \ = \ M_{\kappa_n}^{(1)} \ + \ M_{\kappa_n}^{(2)} \ . \tag{2.73}$$

Since (2.70) also holds true with w replaced by $\tilde u$, (2.68a) implies that for $\mathbb{N}' \ni n \to \infty$

$$M_{\kappa_n}^{(1)} \to 2\int\limits_\Omega c(|E|, \mu(\tilde u + v, E))\varepsilon(\tilde u) : \varepsilon(v)\ dx \ . \tag{2.74}$$

On the other hand, assumption $(\mathbf{A_1})'$ and (2.68b) imply that for any $w \in L^2(\Omega)$ and $\mathbb{N}' \ni n \to \infty$ there holds

$$(c(|E|, \mu(\tilde u + v_{\kappa_n}, E)))^{1/2} w \to (c(|E|, \mu(\tilde u + v, E)))^{1/2} w \quad \text{in } L^2(\Omega) \ .$$

Consequently, (2.68a) gives

$$\int\limits_\Omega (c(|E|, \mu(\tilde u + v_{\kappa_n}, E)))^{1/2}\varepsilon(v_{\kappa_n}) w\ dx \to \int\limits_\Omega (c(|E|, \mu(\tilde u + v, E)))^{1/2}\varepsilon(v) w\ dx \ .$$

whence

$$(c(|E|, \mu(\tilde u + v_{\kappa_n}, E)))^{1/2}\varepsilon(v_{\kappa_n}) \rightharpoonup (c(|E|, \mu(\tilde u + v, E)))^{1/2}\varepsilon(v) \ . \tag{2.75}$$

In view of (2.72b), (2.75) yields

$$\liminf_{\mathbb{N}' \ni\, n \to \infty} M_{\kappa_n}^{(2)} \ \geq \ 2\int\limits_\Omega c(|E|, \mu(\tilde u + v, E) I(v)\ dx \ , \tag{2.76}$$

and hence, (2.73),(2.74) and (2.76) imply

$$\liminf_{\mathbb{N}' \ni\, n \to \infty} \langle L(v_{\kappa_n}), v_{\kappa_n}\rangle \ \geq \ \langle L(v), v\rangle \ . \tag{2.77}$$

The Lebesgue theorem and (2.68b) also show that for $\mathbb{N}' \ni n \to \infty$ there holds

$$J_{\kappa_n}(v_{\kappa_n}, w) \to J(v, w) \ . \tag{2.78}$$

We have

$$J_{\kappa_n}(v_{\kappa_n}, v_{\kappa_n}) \ = \ J_{\kappa_n}(v, v_{\kappa_n}) \ + \ 2\int\limits_\Omega (b_{\kappa_n} - b_0)(\kappa_n + I_{\kappa_n})^{1/2}\ dx \ , \tag{2.79}$$

where

$$b_{\kappa_n} \ := \ b(|E|, \mu(\tilde u + v_{\kappa_n}, E)) \quad , \quad b_0 \ := \ b(|E|, \mu(\tilde u + v, E)) \ ,$$
$$I_{\kappa_n} \ := \ I(\tilde u + v_{\kappa_n}) \quad , \quad I_0 \ := \ I(\tilde u + v) \ .$$

In view of

$$|\int\limits_\Omega (b_{\kappa_n} - b_0)(\kappa_n + I_{\kappa_n})^{1/2}\ dx| \ \leq \ \Big(\int\limits_\Omega (\kappa_n + I_{\kappa_n})\ dx\Big)^{1/2}\ \Big(\int\limits_\Omega |b_{\kappa_n} - b_0|^2\ dx\Big)^{1/2} \ ,$$

$(\mathbf{A_2})$ and (2.68a),(2.68b) imply that for $\mathbb{N}' \ni n \to \infty$

$$\int_\Omega (b_{\kappa_n} - b_0)(\kappa_n + I_{\kappa_n})^{1/2}\, dx \to 0\ . \tag{2.80}$$

Since $J_{\kappa_n}(v, v_{\kappa_n}) \geq J(v, v_{\kappa_n})$, we have

$$\lim_{\mathbb{N}' \ni n \to \infty}\inf J_{\kappa_n}(v, v_{\kappa_n}) \ \geq\ \lim_{\mathbb{N}' \ni n \to \infty}\inf J(v, v_{\kappa_n})\ . \tag{2.81}$$

Lemma 2.5 and (2.68a),(2.68b) give

$$\lim_{\mathbb{N}' \ni n \to \infty}\inf J(v, v_{\kappa_n}) \ \geq\ J(v, v)\ . \tag{2.82}$$

Now, combining (2.79)-(2.82) results in

$$\lim_{\mathbb{N}' \ni n \to \infty}\inf J_{\kappa_n}(v_{\kappa_n}, v_{\kappa_n}) \ \geq\ J(v, v)\ . \tag{2.83}$$

(2.65b) and (2.68a) show $v \in V$, whereas (2.69),(2.71),(2.77),(2.78) and (2.83) imply (2.63). Finally, if $I(\tilde{u} + v) \neq 0$, it is easy to verify the existence of $p \in L^2(\Omega)$such that (2.65a),(2.65b) hold true. □

2.3. **Initial-boundary value problems for isothermal incompressible electrorheological fluid flows.** For $\bar{I} := [0, T] \subset \mathbb{R}_+$ and a closed subspace $V \subset H^1(\Omega)^d$ we refer to $L^2(I; V)$ as the space of functions $v : \bar{Q} \to \mathbb{R}^d, \bar{Q} := I \times \overline{\Omega}$, with $v(t, \cdot) \in V$ f.a.a. $t \in I$ with norm $\|v\|_{L^2(I;V)} := (\int_I \|v(t, \cdot)\|^2_{1,\Omega} dt)^{1/2}$.

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ with boundary $\Gamma = \partial\Omega$, we refer to V and H as the function spaces

$$V \ :=\ \{v \in H^1_0(\Omega)^d \mid \nabla \cdot v = 0\} \quad , \quad H \ :=\ \{w \in L^2(\Omega)^d \mid \nabla \cdot w = 0\}\ .$$

Then, given functions

$$f \in L^2(I; H^{-s}(\Omega)^d \quad , \quad u^0 \in H\ , \tag{2.84}$$

where $s = 1$ for $d = 2$ and $s = 3/2$ for $d = 3$, we consider the following initial-boundary value problem for incompressible, isothermal electrorheological fluid flows

$$\rho(u_t + (u \cdot \nabla)u) \ -\ \nabla \cdot \sigma \ =\ f \quad \text{in } Q\ , \tag{2.85a}$$

$$\nabla \cdot u \ =\ 0 \quad \text{in } Q\ , \tag{2.85b}$$

$$u \ =\ 0 \quad \text{on } \Gamma \times (0, T)\ , \tag{2.85c}$$

$$u(\cdot, 0) \ =\ u^0 \quad \text{in } \Omega\ . \tag{2.85d}$$

Here, the stress tensor σ is supposed to satisfy either the constitutive law (2.18) or (2.19).

In case of the regularized extended Bingham fluid model (2.19), we introduce a nonlinear operator $A_\kappa : V \to V^*$ according to

$$A_\kappa(u) \ :=\ (u \cdot \nabla)u \ +\ M_\kappa(u, u)\ , \tag{2.86}$$

where $M_\kappa(\cdot, \cdot)$ is given as in (2.26) with $\tilde{u} = 0$. We are looking for a weak solution

$$u \in L^2(I; V)\ ,\ u_t \in L^2(I; H^{-s}(\Omega)^d)$$

of (2.85a)-(2.85d) such that for all $v \in L^2(I;V)$ and $w \in H$

$$\int_0^T \langle \rho u_t, v\rangle \, dt + \int_0^T \langle A_\kappa(u), v\rangle \, dt \;=\; \int_0^T \langle f, v\rangle \, dt \;, \tag{2.87a}$$

$$(u(\cdot,0), w)_{0,\Omega} \;= (u^0, w)_{0,\Omega} \;. \tag{2.87b}$$

THEOREM 2.4. *Assume that* $(\mathbf{A_1}),(\mathbf{A_2})$ *and* (2.84) *hold true, Then, the initial-boundary value* (2.85a)-(2.85d) *admits a weak solution.*

Proof. We provide a constructive existence proof by means of a Galerkin approximation with respect to a sequence $\{V_n\}_{\mathbb{N}}$ of finite dimensional subspaces $V_n \subset V, n \in \mathbb{N}$, that are limit dense in V. We assume $V_n = \text{span}\{\varphi_n^{(1)}, \cdots, \varphi_n^{(N_n)}\}$ and look for a solution

$$u_n(t) \;=\; \sum_{i=1}^{N_n} \gamma_n^{(i)}(t)\, \varphi_n^{(i)} \tag{2.88}$$

of the problem

$$(\rho \frac{du_n}{dt}, \varphi_n^{(i)})_{0,\Omega} + \langle A_\kappa(u_n), \varphi_n^{(i)}\rangle \;= \langle f, \varphi_n^{(i)}\rangle \;,\; 1 \le i \le N_n \;, \tag{2.89a}$$

$$u_n(0) \;= P_n u^0 \;, \tag{2.89b}$$

where $P_n : H \to V_n$ is the L^2 orthogonal projection onto V_n. We note that (2.89a),(2.89b) represents an initial-value problem for a system of first order ordinary differential equations. The assumptions $(\mathbf{A_1}),(\mathbf{A_2})$, guarantee the existence of a solution. Moreover, it follows that the sequences $\{u_n\}_{\mathbb{N}}$ and $\{A_\kappa(u_n)\}_{\mathbb{N}}$ are bounded in $L^p(I;V)$ and $L^2(I;H^{-s}(\Omega))$, respectively. Consequently, there exist a subsequence $\mathbb{N}' \subset \mathbb{N}$ and functions $u \in L^2(I;V)$ and $\ell^* \in L^2(I;H^{-s}(\Omega))$ such that

$$\begin{aligned} u_n &\rightharpoonup u^* \quad \text{in } L^2(I;V) \quad (\mathbb{N}' \ni n \to \infty) \;, \\ A_\kappa(u_n) &\rightharpoonup \ell^* \quad \text{in } L^2(I;H^{-s}(\Omega)) \quad (\mathbb{N}' \ni n \to \infty) \;. \end{aligned}$$

Arguments from the theory of parabolic partial differential equations (cf., e.g., LIONS [1969]) show that for $\varphi \in C_0^\infty(I;V)$ there holds

$$- \int_0^T (\rho u, \varphi_t)_{0,\Omega} \, dt + \int_0^T \langle A_\kappa(u), \varphi\rangle \, dt \;=\; \int_0^T \langle f, \varphi\rangle \, dt \;,$$

which gives $u \in L^2(I;V), u_t \in L^2(I;H^{-s}(\Omega))$ and implies that (2.87a) holds true, since $C_0^\infty(I;V)$ is dense in $L^2(I;V)$. A similar reasoning based on an appropriate choice of a test function allows to deduce $u(\cdot,0) = u^0$. □

We note that a generalization of Theorem 2.4 to the case of inhomogeneous Dirichlet data $u = u^D$ on $\Sigma \times I$ can be found in LITVINOV [2004].

On the other hand, if we consider the extended Bingham fluid model based on the viscosity function (2.18), we have to deal with a strongly nonlinear parabolic variational inequality. Adopting the notation from subsection 2.2.2, we are looking for a weak solution $u \in L^2(I;V), u_t \in L^2(I;H^{-s}(\Omega))$ of (2.85a)-(2.85d) in the sense that for all

$v \in L^2(I;V)$ and $w \in H$ there holds

(2.90a)
$$\int_0^T \langle \rho(u_t, v-u\rangle \, dt \;+\; \int_0^T \langle (u\cdot\nabla)u, v-u\rangle \, dt \;+$$
$$\int_0^T (J(u,v) - J(u,u)) \, dt + \int_0^T \langle L(u), v-u\rangle \, dt \;\geq\; \int_0^T \langle f, v-u\rangle \, dt \;,$$

(2.90b)
$$(u(\cdot,0), w)_{0,\Omega} \;=\; (u^0, w)_{0,\Omega} \;.$$

THEOREM 2.5. *Assume that* $(\mathbf{A_1})'', (\mathbf{A_2})$ *and* (2.84) *hold true. Then, the variational inequality* (2.90a),(2.90b) *has a solution* $u \in L^2(I;V), u_t \in L^2(I;H^{-s}(\Omega))$.

Proof. We choose $\{\kappa_n\}_{\mathbb{N}}$ as a null sequence of positive regularization parameters. For each $n \in \mathbb{N}$, Theorem 2.4 guarantees the existence of a weak solution u_n of (2.85a)-(2.85d) with respect to the regularized extended Bingham fluid model (2.19) (with κ replaced by κ_n). The boundedness of the sequence $\{u_n\}_{\mathbb{N}}$ in $L^2(I;V)$ infers the existence of a subsequence $\mathbb{N}' \subset \mathbb{N}$ and of a function $u \in L^2(I;V)$ such that $u_n \rightharpoonup u(\mathbb{N}' \ni n \to \infty)$ in $L^2(I;V)$. Passing to the limit as in the proof of Theorem 2.3 allows to conclude. □

2.4. **Balance equations and constitutive laws for non-isothermal incompressible electrorheological fluid flows.** Non-isothermal flows of non-Newtonian fluids have been studied in a series of papers mostly in the engineering literature with respect to industrially relevant applications. Various laws of the temperature dependence of the viscosity have been assumed, e.g., a hyperbolic law for the variation of the viscosity or a Reynolds-type relation. A rigorous mathematical analysis of non-isothermal flow in a Bingham fluid can be found in DUVAUT and LIONS [1971].

As far as electrorheological fluids are concerned, it is well-known by experimental evidence that their operational behavior exhibits a dependence on the temperature (cf. BENDERSKAIA et al. [1980], TABATABAI [1993], ZHIZKIN [1986]). Figure 2.4 displays the temperature dependence of the shear stress (left) and of the current density (right) for a polyurethane based electrorheological fluid under different operational conditions, i.e., electric field strengths. Mathematical models for non-isothermal electrorheological fluid flows based on a power law constitutive equation have been studied in RUZICKA [2000] (cf. also ECKART and SADIKI [2001], SADIKI and BALAN [2003]).

Here, we follow the approach in LITVINOV and HOPPE [2005]. We assume a general dependence of the viscosity function on the temperature θ and consider the following constitutive equation between the stress tensor σ and the rate of strain tensor ε

(2.91)
$$\sigma \;=\; -pI \;+\; 2\varphi(I(u), |E|, \mu(u,E), \theta)\varepsilon(u) \;.$$

As in subsection 2.1, u and p stand for the velocity and pressure of the fluid flow, $I(u)$ is the second invariant of the rate of strain tensor, E refers to the electric field, and $\mu(u,E)$ is the square of the cosine of the angle between the velocity and the electric field.

The equations of motion and the incompressibility condition for the fluid flow have to be completed by a thermodynamical balance equation which can be deduced from the energy conservation law

$$e_t \;+\; u\cdot\nabla e \;=\; \sigma : \varepsilon(u) \;-\; \nabla\cdot q \;+\; f_2 \;,$$

where e denotes the specific internal energy, q is the heat flux vector and f_2 stands for a volumetric heat source/sink. As constitutive equations we assume the linear Fourier law

$$e \;=\; \rho c\theta \quad , \quad q \;=\; -k\nabla\theta \;,$$

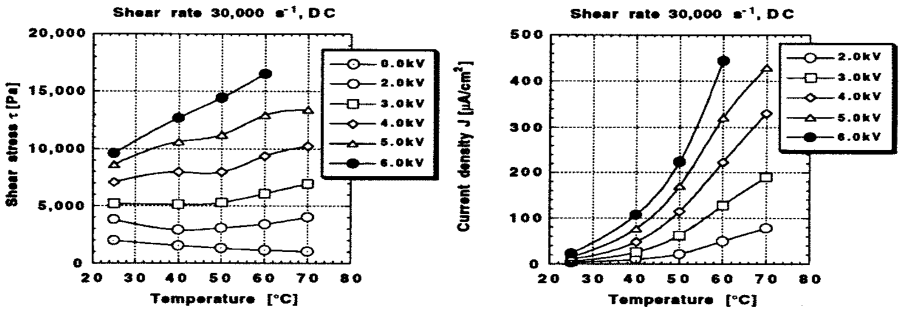

FIGURE 2.4. Temperature dependence of the shear stress (left) and the current density (right) in electrorheological fluids (from BAYER [1997a])

where ρ, c and k refer to the density, the specific heat, and the thermal conductivity. We are thus led to the following coupled system in $Q := \Omega \times (0,T)$

$$\rho(u_t + (u\cdot\nabla)u) \;-\; \nabla\cdot\sigma \;=\; f_1 \;, \tag{2.92a}$$

$$\nabla\cdot u \;=\; 0 \;, \tag{2.92b}$$

$$\rho c(\theta_t + u\cdot\nabla\theta) - k\Delta\theta - 2\varphi(I(u),|E|,\mu(u,E),\theta)I(u) \;=\; f_2 \;, \tag{2.92c}$$

where f_1 is a volumetric force on the fluid. The equations have to completed by appropriate initial and boundary conditions that will be discussed in detail in the subsequent subsections.

REMARK 2.4. *We note that the impact of the electrical conductivity in the thermal balance equation* (2.92c) *has been neglected, since electrorheological fluids are electrically nonconducting.*

As far as the viscosity function φ is concerned, we will assume that the following condition is satisfied:

$(\mathbf{T_1})$ φ is a continuous function of its arguments, i.e., $\varphi \in C(\mathbb{R}_+^2 \times [0,1] \times \mathbb{R})$. For fixed $(y_2,y_3,y_4) \in \mathbb{R}_+ \times [0,1] \times \mathbb{R}$ the function $\varphi(\cdot,y_2,y_3,y_4)$ is continuously differentiable in $\mathbb{R}_+$, i.e., $\varphi(\cdot,y_2,y_3,y_4) \in C^1(\mathbb{R}_+)$. There exist positive constants $c_i, 1 \le i \le 4$, such that

$$c_2 \;\ge\; \varphi(y_1,y_2,y_3,y_4) \;\ge\; c_1 \;,$$

$$\varphi(y_1,y_2,y_3,y_4) \;+\; 2\frac{\partial\varphi}{\partial y_1}(y_1,y_2,y_3,y_4) \;\ge\; c_3 \;,$$

$$\frac{\partial\varphi}{\partial y_1}(y_1,y_2,y_3,y_4)|\; y_1 \;\le\; c_4 \;.$$

The first condition in $(\mathbf{T_1})$ requires non-vanishing viscosity for vanishing shear rate and thus does not include Bingham-type electrorheological flow models. However, as in subsection 2.1 we may consider viscosity functions of the form

$$\begin{aligned} &\varphi(I(u),|E|,\mu(u,E),\theta) \;= \\ &=\; b(|E|,\mu(u,E),\theta)(\kappa + I(u))^{-1/2} \;+\; c((I(u),|E|,\mu(u,E),\theta) \;, \end{aligned} \tag{2.93}$$

where $\kappa \geq 0$, the function c is supposed to satisfy $(\mathbf{T_1})$, and the function b is subject to the assumption

$(\mathbf{T_2})$ b is a continuous function of its arguments, i.e., $b \in C(\mathbb{R}_+ \times [0,1] \times \mathbb{R})$. There exists a positive constant c_5 such that

$$c_5 \;\geq\; b(y_1, y_2, y_3) \;\geq\; 0\;.$$

The case $\kappa = 0$ in (2.93) refers to a generalized Bingham-type model for non-isothermal electrorheological fluid flows, whereas $\kappa > 0$ can be interpreted as a regularization thereof.
The physical relevance of these assumptions with respect to the fluid flow has been discussed in subsection 2.2.

We consider the following modification of the thermal balance equation (2.92c) which gives rise to a non-local model:

$$\rho c(\theta_t + u \cdot \nabla\theta) - k\Delta\theta - 2\varphi(I(u)), |E|, \mu(u,E), \theta) I(P_\beta(u)) \;=\; f_2\;. \tag{2.94}$$

Here $P_\beta \in \mathcal{L}(W^{1,2}(\Omega)^d, C^\infty(\overline{\Omega})^d), \beta > 0$, is the regularization operator

$$(P_\beta(v))(x) \;:=\; \int\limits_{\mathbb{R}^d} \omega_\beta(|x - x'|)\; (P_E(v))(x')\; dx'\;,\; x \in \overline{\Omega}\;,\; v \in W^{1,2}(\Omega)\;, \tag{2.95}$$

where $P_E \in \mathcal{L}(W^{1,2}(\Omega)^d, W^{1,2}(\mathbb{R}^d))$ is an extension operator and $\omega_\beta \in C_+^\infty(\mathbb{R}_+)$ with $\mathrm{supp}(\omega_\beta) \subset [0,\beta]$ and $\int_{\mathbb{R}^d} \omega_\beta(|x|)dx = 1$.

REMARK 2.5. *The physical interpretation of the regularization operator P_β in the thermal balance equation* (2.94) *is that the dissipation of energy at a point $x \in \Omega$ only depends on the rate of strain tensor in a small vicinity of the point. We note that non-local models agree remarkably well with atomistic theories and experimental observations (cf., e.g.,* ERINGEN [2002]).

2.5. **Boundary value problems for steady non-isothermal incompressible electrorheological fluid flows.** We consider steady, non-isothermal, incompressible electrorheological fluid flow and assume $\Omega \subset \mathbb{R}^d$ to be a bounded Lipschitz domain with boundary Γ such that $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N, \Gamma_D \cap \Gamma_N = \emptyset$. We further suppose

$$\begin{aligned} f_1 \in L^2(\Omega)^d\;,\; f_2 \in L^2(\Omega)\;,\; g \in L^2(\Gamma_N)^d\;, \\ u^D \in W^{1/2,2}(\Gamma_D)^d\;,\; \theta^D \in W^{1/2,2}(\Gamma) \end{aligned} \tag{2.96}$$

to be given functions and consider the following boundary value problem

$$\nabla \cdot \sigma \;=\; f_1 \quad , \quad \nabla \cdot u \;=\; 0 \qquad \text{in}\, \Omega\;, \tag{2.97a}$$

$$-\chi\Delta\theta + u \cdot \nabla\theta + 2\varrho\varphi(I(u), |E|, \mu(u,E), \theta) I(u) \;=\; f_2 \quad \text{in}\, \Omega\;, \tag{2.97b}$$

$$u \;=\; u^D \quad \text{on}\, \Gamma_D\;, \tag{2.97c}$$

$$\nu \cdot \sigma \;=\; g \quad \text{on}\, \Gamma_N\;,\;, \tag{2.97d}$$

$$\theta \;=\; \theta^D \quad \text{on}\, \Gamma\;, \tag{2.97e}$$

where $\chi = (\rho c)^{-1}\kappa$ and $\varrho = (\rho c)^{-1}$. As in subsection 2.2 we assume a unilateral coupling between the electric field E and the flow field, i.e., we suppose that E is given by means of an electrical potential ψ which satisfies the boundary value problem (2.22a)-(2.22c).

We study the existence of a weak solution of (2.97a)-(2.97e) where the velocity is supposed to be in $W^{1,2}(\Omega)^d \cap H(\mathrm{div}^0;\Omega)$, the pressure p in $L^2(\Omega)$, and the temperature θ in

$W^{1,r}(\Omega)$ with $1 < r < 2$ for $d = 2$ and $1 < r < 3/2$ for $d = 3$. In order to accommodate the inhomogeneous Dirichlet boundary data (2.97c),(2.97e), we define $\tilde{u} \in W^{1,2}(\Omega)^d$ and $\tilde{\theta} \in W^{1,r}(\Omega)$ such that $\tilde{u}|_{\Gamma_D} = u^D$ and $\tilde{\theta}|_\Gamma = \theta^D$. We set

$$X \ := \ W^{1,2}_{0,\Gamma_D}(\Omega)^d \cap H(\mathrm{div}^0;\Omega) \quad , \quad \|v\|_X \ := \ \Big(\int_\Omega (I(v))^2 \, dx \Big)^{1/2}$$

and consider the operators

$$N : X \times W^{1,r}_{0,\Gamma}(\Omega) \to X^* \quad , \quad A : X \times W^{1,r}_{0,\Gamma}(\Omega) \to W^{-1,s}(\Omega) \ , \ s = \frac{r}{r-1} \ ,$$

which are defined according to

$$\langle N(v,\zeta), w \rangle \ := \ 2 \int_\Omega \varphi(I(\tilde{u}+v), |E|, \mu(\tilde{u}+v,E), \tilde{\theta}+\zeta) \varepsilon(\tilde{u}+v) : \varepsilon(w) \, dx \ , \ , \tag{2.98a}$$

$$\langle A(w,\zeta), \xi \rangle \ := \ \chi^{-1} \int_\Omega \Big((\tilde{\theta}+\zeta)(\tilde{u}+w) \cdot \nabla \xi + \ + \ 2\varrho\varphi(I(\tilde{u}+w), |E|, \mu(\tilde{u}+w), \tilde{\theta}+\zeta) I(\tilde{u}+w)\xi \Big) \, dx \ . \tag{2.98b}$$

Here, $\langle \cdot,\cdot \rangle$ refers to the dual product between X^* and X in (2.98a) and to the dual product between $W^{-1,s}(\Omega)$ and $W^{1,r}_{0,\Gamma}(\Omega)$ in (2.98b). For the ease of exposition, we will use the same notation. The correct meaning will always follow easily from the context.
Moreover, we refer to $B \in \mathcal{L}(X, L^2(\Omega))$ as the divergence operator $Bv = \nabla \cdot v, v \in X$,.
We consider the following system of variational equations:
Find $(v,p,\theta) \in X \times L^2(\Omega) \times W^{1,r}_{0,\Gamma}(\Omega)$ such that

$$\langle N(v,\theta), w \rangle - \langle B^* p, w \rangle \ = \ \langle f_1 + g, w \rangle \ , w \in X \tag{2.99a}$$

$$(Bv, q)_{0,\Omega} \ = \ 0 \ , \ q \in L^2(\Omega) \ , \tag{2.99b}$$

$$(\nabla\theta, \nabla\zeta)_{0,\Omega} - \langle A(v,\theta), \zeta \rangle \ = \ (f_3, \zeta)_{0,\Omega} \ , \ \zeta \in W^{1,s}_{0,\Gamma}(\Omega) \ , \tag{2.99c}$$

where $(f_3,\zeta)_{0,\Omega} := (f_2,\zeta)_{0,\Omega} - (\nabla\tilde{\theta}, \nabla\zeta)_{0,\Omega}$. For notational convenience, we denote by θ both the solution of (2.97a)-(2.97e) and (2.99a)-(2.99c). It will be clear from the context which one is considered.

LEMMA 2.6. *Assume that (u,p,θ) is a classical solution of* (2.97a)-(2.97e). *Then, the triple $(u-\tilde{u}, p, \theta-\tilde{\theta})$ solves* (2.99a)-(2.99c). *Conversely, if (v,p,θ) is a sufficiently smooth solution of* (2.99a)-(2.99c), *then the triple $(\tilde{u}+v, p, \tilde{\theta}+\theta)$ solves* (2.97a)-(2.97c) *in the classical sense.*

Proof. The assertions are easily verified by Green's formula. □

We will prove the existence of a solution of the system (2.99a)-(2.99c) by an approximation involving the regularization operator P_β from (2.95). For that purpose, we introduce the operator

$$A_\beta : X \times W^{1,2}_{0,\Gamma}(\Omega) \to W^{-1,2}(\Omega) \ ,$$

which is given by means of

$$\langle A_\beta(w,\zeta),\xi\rangle \;:=\; \chi^{-1}\int\limits_\Omega \Big((\tilde\theta+\zeta)(\tilde u+w)\cdot\nabla\xi \;+ \; 2\varrho\varphi(I(\tilde u+w),|E|,\mu(\tilde u+w),\tilde\theta+\zeta)I(P_\beta(\tilde u+w))\xi\Big)\,dx\;. \tag{2.100}$$

Here, $\langle\cdot,\cdot\rangle$ stands for the dual product between $W^{-1,2}(\Omega)$ and $W^{1,2}_{0,\Gamma}(\Omega)$. The associated boundary value problem reads as follows:

Find $(v,p,\theta)\in X\times L^2(\Omega)\times W^{1,2}_{0,\Gamma}(\Omega)$ such that

$$\langle N(v,\theta),w\rangle - \langle B^*p,w\rangle \;=\; \langle f_1+g,w\rangle\;, w\in X \tag{2.101a}$$

$$(Bv,q)_{0,\Omega} \;=\; 0\;,\; q\in L^2(\Omega)\;, \tag{2.101b}$$

$$(\nabla\theta,\nabla\zeta)_{0,\Omega} - \langle A_\beta(v,\theta),\zeta\rangle \;=\; (f_3,\zeta)_{0,\Omega}\;,\; \zeta\in W^{1,2}_{0,\Gamma}(\Omega)\;. \tag{2.101c}$$

THEOREM 2.6. *Suppose that* $(\mathbf{T_1})$,(2.96) *are satisfied and* $E\in L^4(\Omega)$*. Then, for any* $\beta>0$ *there exists a solution* (v_β,p_β) *of* (2.101a)-(2.101c) *and there exist constants* $C_i>0, 1\le i\le 2$, *such that*

$$\|v_\beta\|_X \;\le\; C_1 \quad , \quad \|p_\beta\|_{0,\Omega} \;\le\; C_2\;,\; b\in(0,a)\;,\; a>0\;. \tag{2.102}$$

Proof. We refer to LITVINOV and HOPPE [2005]. □

We will now address the existence of a solution of the system (2.99a)-(2.99c). We define an operator $\Lambda_2: V\to\mathcal{L}(W^{1,r}_0(\Omega),W^{-1,s}(\Omega))$ according to

$$\langle\Lambda_2(v)\zeta,\xi\rangle \;:=\; \chi^{-1}\int\limits_\Omega \zeta(\tilde u+v)\cdot\nabla\xi\,dx\;, \tag{2.103}$$

where $v\in V,\zeta\in W^{1,r}_0(\Omega)$ and $\xi\in W^{1,s}_0(\Omega)$. We consider the auxiliary problem:
Find $\bar\theta\in W^{1,r}_0(\Omega)$ such that

$$\langle\nabla\bar\theta,\nabla\xi\rangle \;-\; \langle\Lambda_2(v)\bar\theta,\xi\rangle \;=\; 0 \quad , \quad \xi\in W^{1,s}_0(\Omega)\;. \tag{2.104}$$

Under these prerequisites, we now assume $\{\beta_n\}_{\mathbb{N}}$ to be a sequence of regularization parameters $\beta_n\in\mathbb{R}_+, n\in\mathbb{N}$, such that $\beta_n\to 0$ as $n\to\infty$ and further suppose that $\{(v_n,p_n,\theta_n)\}_{\mathbb{N}}$ is an associated sequence of solutions $(v_n,p_n,\theta_n)\in X\times L^2(\Omega)\times W^{1,2}_0(\Omega), n\in\mathbb{N}$, of the system (2.101a)-(2.101c) whose existence is guaranteed under the assumptions of Theorem 2.6.

THEOREM 2.7. *Assume that* $\Omega\subset\mathbb{R}^d, d=2$ *or* $d=3$ *is a bounded* C^3*-domain. Further, suppose that the conditions* $(\mathbf{T_1})$ *and* (2.96) *hold true and the variational equation* (2.104) *is only trivially solvable. For a null sequence* $\{\beta_n\}_{\mathcal{N}}$ *of positive regularization parameters let* $\{(v_n,p_n,\theta_n)\}_{\mathbb{N}}$ *be the associated sequence of solutions* $(v_n,p_n,\theta_n)\in X\times L^2(\Omega)\times W^{1,2}_0(\Omega), n\in\mathbb{N}$, *of the system* (2.101a)-(2.101c)*. Then, there exist a subsequence* $\mathbb{N}^*\subset\mathbb{N}$ *and a triple* $(v,p,\theta)\in X\times L^2(\Omega)\times W^{1,r}_0(\Omega)$ *such that for* $\mathbb{N}^*\ni n\to\infty$

$$v_n\to v \quad \text{in } X\;, \tag{2.105a}$$

$$p_n\to p \quad \text{in } L^2(\Omega)\;, \tag{2.105b}$$

$$\theta_n\to\theta \quad \text{in } W^{1,r}_0(\Omega)\;. \tag{2.105c}$$

The triple (v,p,θ) *is a solution of the system* (2.99a)-(2.99c)*.*

Proof. We refer to LITVINOV and HOPPE [2005]. □

3. NUMERICAL SOLUTION OF ELECTRORHEOLOGICAL FLUID FLOWS

This section is devoted to the numerical solution of stationary and time-dependent, isothermal and non-isothermal electrorheological fluid flows. We shall begin in 3.1 with steady-state isothermal problems with emphasis on nonlinear Uzawa-type algorithms in 3.1.1 as well as augmented Lagrangian methods in 3.1.2. This includes the construction of preconditioners based on approximate inverses of the Stokes operator which will be the subject of 3.1.3. An augmented Lagrangian approach particularly suited for non-regularized Bingham models shall be considered in 3.1.4. Time-dependent problems shall be taken care of in 3.2, and in 3.3 we shall address non-isothermal fluid flows. We refer to CROCHET [1984], ELMAN, SILVESTER and WATHEN [2005], GLOWINSKI [2004], GUNZBURGER [1989], HUANG [1998], THOMASSET [1981], TUREK [1999] with regard to a general presentation of numerical solution techniques for Newtonian and non-Newtonian fluid flows.

3.1. **Steady-state isothermal incompressible flow problems.** As we have seen in subsection 2.2 (cf. Theorem 2.2), steady isothermal, incompressible electrorheological fluid flows with a regularized viscosity function can be approximated by finite dimensional nonlinear saddle point problems of the form:

Find $(v_n, p_n) \in X_n \times Q_n$ such that

$$\langle S_n(v_n), w_n\rangle \;-\; \langle B_n^* p_n, w_n\rangle \;=\; \langle f+g, w_n\rangle \;,\; w_n \in X_n \;, \tag{3.1a}$$

$$(B_n v_n, q_n)_{0,\Omega} \;=\; 0 \;,\; q_n \in Q_n \;, \tag{3.1b}$$

where $X_n \subset X := W^{1,2}_{0,\Gamma_D}(\Omega)$ and $Q_n \subset L^2(\Omega), n \in \mathbb{N}$, are finite dimensional subspaces, $S_n(u_n) := M_\kappa(u_n, u_n)$ with $M_\kappa : X \times X \to X^*$ being the nonlinear operator given by (2.26), and B_n refers to the discrete divergence operator (2.30). We assume that the pairs $(X_n, Q_n), n \in \mathbb{N}$, satisfy the discrete LBB-condition (2.31).
Since the nonlinear operator S_n admits an inverse S_n^{-1}, the discrete velocity field v_n can be formally eliminated from (3.1a),(3.1b) which gives rise to

$$B_n S_n^{-1}(B_n^* p_n + f_n + g_n) \;=\; 0 \;. \tag{3.2}$$

REMARK 3.1. *In the linear regime, the linear operator $B_n S_n^{-1} B_n^*$ is called the Schur complement and* (3.2) *is referred to as the Schur complement system.*

All numerical techniques for the solution of (3.1a),(3.1b) are nonlinear versions of methods that have been developed for linear saddle point problems, i.e., when the operator S in (3.1a) is a linear operator. The most popular numerical schemes are Uzawa-type algorithms and those based on the augmented Lagrangian approach (cf., e.g., CAO [2003], FORTIN and GLOWINSKI [1983], GLOWINSKI [1984, 2004], GLOWINSKI and LE TALLEC [1989], LIN and CAO [2006]). In the nonlinear regime, these methods are outer-inner iterative schemes where the outer iteration takes care of the saddle point structure of the problem and the inner iteration is devoted to the nonlinear problem associated with the operator S.

3.1.1. *Nonlinear Uzawa-type algorithms.* The nonlinear Uzawa algorithm can be formally derived as a damped nonlinear Richardson iteration with damping parameter $\tau > 0$ applied to (3.2):

Given $p_n^{(0)} \in Q_n$, compute $p_n^{(\nu)} \in Q_n, \nu \in \mathbb{N}$, according to

$$p_n^{(\nu+1)} \;=\; p_n^{(\nu)} \;-\; \tau B_n S_n^{-1}(B_n^* p_n^{(\nu)} + f_n + g_n) \quad , \quad \nu \in \mathbb{N}_0 \;. \tag{3.3}$$

Of course, we are interested in iterates $u_n^{(\nu)}$ for the discrete velocity field as well which can be obtained by means of (3.1a). Thus we arrive at the following standard form of the nonlinear Uzawa algorithm:

Nonlinear Uzawa algorithm:
Given $(v_n^{(0)}, p_n^{(0)}) \in X_n \times Q_n$ and $\tau > 0$, compute $(v_n^{(\nu)}, p_n^{(\nu)}) \in X_n \times Q_n, \nu \in \mathbb{N}$, as the solution of

$$\langle S_n(v_n^{(\nu+1)}, w_n\rangle - \langle B_n^* p_n^{(\nu)}, w_n\rangle \;=\; \langle f + g, w_n\rangle \;,\; w_n \in X_n \;, \tag{3.4a}$$

$$(p_n^{(\nu+1)} - p_n^{(\nu)}, q_n)_{0,\Omega} \;=\; -\tau (B_n v_n^{(\nu+1)}, q_n)_{0,\Omega} \;,\; q_n \in Q_n \;. \tag{3.4b}$$

THEOREM 3.1. *Let* $(v_n, p_n) \in X_n \times Q_n$ *be the solution of* (3.1a),(3.1b) *and suppose that* $\{(v_n^{(\nu)}, p_n^{(\nu)})\}_{\mathbb{N}}$ *is the sequence of iterates generated by the nonlinear Uzawa algorithm* (3.4a),(3.4b). *Assume* $\tau < 2\gamma_L \beta^2$ *with* γ_L *as in Lemma* 2.3 *and* β *from Lemma* 2.2. *Then, for* $\nu \to \infty$ *there holds*

$$v_n^{(\nu)} \to v_n \quad \textit{in } X \quad , \quad p_n^{(\nu)} \to p_n \quad \textit{in } L^2(\Omega) \;.$$

Proof. We set $e_v^{(\nu)} := v_n^{(n))} - v_n$ and $e_p^{(\nu)} := p_n^{(\nu)} - p_n$. If we subtract (3.1a) from (3.4a) and (3.1b) from (3.4b), we obtain

$$\langle S_n(v_n^{(\nu+1)}) - S_n(v_n), w_n\rangle \;=\; \langle B_n^* e_p^{(\nu)}, w_n\rangle \;,\; w_n \in X_n \;, \tag{3.5a}$$

$$(e_p^{(\nu+1)} - e_p^{(\nu)}, q_n)_{0,\Omega} \;=\; -\tau (B_n e_v^{(\nu+1)}, q_n)_{0,\Omega} \;,\; q_n \in Q_n \;. \tag{3.5b}$$

We choose $w_n = 2e_u^{(\nu+1)}$ in (3.5a) and $q_n = 2e_p^{(\nu+1)}$ in (3.5b). Then, multiplying (3.5a) by 2τ and adding it to (3.5b) yields

$$\begin{aligned} &\|e_p^{(\nu+1)}\|_{0,\Omega}^2 \;+\; \|e_p^{(\nu+1)} - e_p^{(\nu)}\|_{0,\Omega}^2 \;-\; \|e_p^{(\nu)}\|_{0,\Omega}^2 \;+ \\ &+\; 2\tau\, \langle S_n(v_n^{(\nu+1)}) - S_n(v_n), e_v^{(\nu+1)}\rangle \;=\; 2\tau\, (B_n e_v^{(\nu+1)}, e_p^{(\nu)} - e_p^{(\nu+1)})_{0,\Omega} \;. \end{aligned}$$

The results of subsection 2.2 imply

$$\begin{aligned} &\|e_p^{(\nu+1)}\|_{0,\Omega}^2 \;+\; \|e_p^{(\nu+1)} - e_p^{(\nu)}\|_{0,\Omega}^2 \;-\; \|e_p^{(\nu)}\|_{0,\Omega}^2 \;+ \\ &+\; 2\tau\, \gamma_L\, \|e_v^{(\nu+1)}\|_X^2 \;\le\; 2\frac{\tau}{\beta}\, \|e_v^{(\nu+1)}\|_X\, \|e_p^{(\nu+1)} - e_p^{(\nu)}\|_{0,\Omega} \;, \end{aligned}$$

and hence, Young's inequality gives

$$\|e_p^{(\nu+1)}\|_{0,\Omega}^2 \;-\; \|e_p^{(\nu)}\|_{0,\Omega}^2 \;+\; \tau\, (2\gamma_L - \frac{\tau}{\beta^2})\, \|e_v^{(\nu+1)}\|_X^2 \;\le\; 0 \;. \tag{3.6}$$

We deduce from (3.6) that the sequence $\{\|e_p^{(\nu)}\|_{0,\Omega}^2\}_{\mathbb{N}}$ is convergent which in turn gives us $e_v^{(\nu)} \to 0$ as $\nu \to \infty$. Moreover, we have

$$\|S_n(v_n^{(\nu+1)}) - S_n(v_n)\|_{X^*} \to 0 \quad \text{as} \quad \nu \to \infty \;. \tag{3.7}$$

On the other hand, in view of (3.5a) and Lemma 2.2 it follows that

$$\|S_n(v_n^{(\nu+1)}) - S_n(v_n)\|_{X^*} \;=\; \|B_n^* e_p^{(\nu)}\|_{X*} \;\ge\; \beta\, \|e_p^{(\nu)}\|_{0,\Omega} \;.$$

Hence. (3.7) tells us that $e_p^{(\nu)} \to 0$ as $\nu \to \infty$. □

It is well-known from the theory of linear iterative schemes that the convergence can be significantly improved by preconditioning (cf., e.g., BANK et al. [1990], BRAMBLE et al. [1997], ELMAN [2002], ELMAN and GOLUB [1994], ELMAN and SILVESTER

[1996], KLAWONN [1998], RUSTEN and WINTHER [1992]). In terms of the Richardson iteration (3.3), we may use

$$p_n^{(\nu+1)} \; = \; p_n^{(\nu)} \; + \; P_n^{-1} B_n S_n^{-1} (B_n^* p_n^{(\nu)} + f_n + g_n) \quad , \quad \nu \in \mathbb{N}_0 \; ,$$

with a preconditioner $P_n : Q_n \to Q_n$ which is assumed to be a linear symmetric positive operator. This leads to the preconditioned nonlinear Uzawa algorithm:

Preconditioned nonlinear Uzawa algorithm:

Let $P_n : Q_n \to Q_n$ be a linear symmetric positive operator. Then, given $(v_n^{(0)}, p_n^{(0)}) \in X_n \times Q_n$, compute $(v_n^{(\nu)}, p_n^{(\nu)}) \in X_n \times Q_n, \nu \in \mathbb{N}$, as the solution of

$$\text{(3.8a)} \qquad \langle S_n(v_n^{(\nu+1)}, w_n\rangle - \langle B_n^* p_n^{(\nu)}, w_n\rangle \; = \; \langle f + g, w_n\rangle \; , \; w_n \in X_n \; ,$$

$$\text{(3.8b)} \qquad (p_n^{(\nu+1)} - p_n^{(\nu)}, q_n)_{0,\Omega} \; = \; -(P_n^{-1} B_n v_n^{(\nu+1)}, q_n)_{0,\Omega} \; , \; q_n \in Q_n \; .$$

REMARK 3.2. *The preconditioned nonlinear Uzawa algorithm contains the standard form* (3.4a),(3.4b) *as a special case as can be readily seen by choosing $P_n = \tau I_n, \tau > 0$, with I_n denoting the identity on Q_n.*

A major problem in the practical realization of the algorithm (3.8a),(3.8b) is that it requires the solution of a nonlinear problem. This issue is usually taken care of by an approximation $\tilde{S}_n$ of S_n. We will discuss feasible choices of $\tilde{S}_n$ in subsection 3.1.3. Since in this case we do not solve (3.8a),(3.8b) exactly, the resulting scheme is referred to as a preconditioned inexact nonlinear Uzawa algorithm:

Preconditioned inexact nonlinear Uzawa algorithm:

Let $\tilde{S}_n^{-1}$ be an approximate inverse of S_n^{-1} and assume that $P_n : Q_n \to Q_n$ is a linear symmetric positive operator. Then, given $(v_n^{(0)}, p_n^{(0)}) \in X_n \times Q_n$, compute $(v_n^{(\nu)}, p_n^{(\nu)}) \in X_n \times Q_n, \nu \in \mathbb{N}$, as the solution of

$$\text{(3.9a)} \qquad \langle \tilde{S}_n(v_n^{(\nu+1)}, w_n\rangle - \langle B_n^* p_n^{(\nu)}, w_n\rangle \; = \; \langle f + g, w_n\rangle \; , \; w_n \in X_n \; ,$$

$$\text{(3.9b)} \qquad (p_n^{(\nu+1)} - p_n^{(\nu)}, q_n)_{0,\Omega} \; = \; -(P_n^{-1} B_n v_n^{(\nu+1)}, q_n)_{0,\Omega} \; , \; q_n \in Q_n \; .$$

In case of a linear symmetric positive definite operator S_n, the convergence of preconditioned inexact nonlinear Uzawa algorithms has been analyzed in BRAMBLE et al. [1997], ELMAN and GOLUB [1994]. As can be expected, it requires some conditions on the approximate inverse $\tilde{S}_n^{-1}$ and on the preconditioner P_n.

3.1.2. *Augmented Lagrangian methods.* As we already know from subsection 2.2.1, the nonlinear saddle point problem (3.1a),(3.1b) results from the constrained minimization problem

$$\min_{v_n \in V_n} \Big(J_\kappa(v_n, v_n) + \langle L(v_n), v_n\rangle \Big) \; ,$$

where $V_n := X_n \cap H(\text{div}^0; \Omega)$ and $J_\kappa : X \times X \to \mathbb{R}$ and $L : X \to X^*$ are given by (2.24a)(2.24b), if we couple the constraints $B_n v_n = 0$ by Lagrange multipliers $p_n \in Q_n$. An alternative is to use penalty methods

$$\min_{v_n \in X_n} \Big(J_\kappa(v_n, v_n)) + \langle L(v_n), v_n\rangle + r(B_n v_n, B_n v_n)_{0,\Omega} \Big) \; ,$$

where the constraints are taken care of by a penalty term with penalty parameter $r > 0$. The disadvantage with penalty methods is that the penalty parameter r usually has to be chosen quite large which has a negative impact on the condition of the resulting algebraic system.

The augmented Lagrangian techniques combine the previous approaches in such a way that they work sufficiently well for a moderate choice of the penalty parameter. A convergence analysis in the symmetric case is given in FORTIN and GLOWINSKI [1983], GLOWINSKI and LE TALLEC [1989], whereas the nonsymmetric case has been addressed in AWANOU [2005].

Augmented Lagrangian algorithm:
Given $(v_n^{(0)}, p_n^{(0)}) \in X_n \times Q_n$ and $r, \rho > 0$, compute $(v_n^{(\nu)}, p_n^{(\nu)}) \in X_n \times Q_n, \nu \in \mathbb{N}$, such that for $(w_n, q_n) \in X_n \times Q_n$ there holds

$$\langle S_n(v_n^{(\nu+1)}, w_n\rangle - \langle B_n^* p_n^{(\nu)}, w_n\rangle + r(B_n v_n^{(\nu+1)}, B_n w_n)_{0,\Omega} \ = \ \langle f+g, w_n\rangle \ , \tag{3.10a}$$

$$(p_n^{(\nu+1)} - p_n^{(\nu)}, q_n)_{0,\Omega} + \rho(B_n v_n^{(\nu+1)}, q_n)_{0,\Omega} \ = \ 0 \ . \tag{3.10b}$$

THEOREM 3.2. *Let* $(v_n, p_n) \in X_n \times Q_n$ *be the unique solution of* (3.1a),(3.1b) *and let* $\{(v_n^{(\nu)}, p_n^{(\nu)})\}_{\mathbb{N}}$ *be the sequence of iterates generated by the augmented Lagrangian algorithm* (3.10a),(3.10b). *Then, under the assumption* $\rho < 2r$ *for* $\nu \to \infty$ *there holds*

$$v_n^{(\nu)} \to v_n \quad \textit{in } X \quad , \quad p_n^{(\nu)} \to p_n \quad \textit{in } L^2(\Omega) \ .$$

Proof. The convergence result can be verified using a similar reasoning as in the proof of Theorem 3.1. Setting $e_v^{(\nu)} := v_n^{(n))} - v_n$ and $e_p^{(\nu)} := p_n^{(\nu)} - p_n$, it follows from (3.1a),(3.1b) and (3.10a),(3.10b) that for $w_n \in X_n$ and $q_n \in Q_n$ there holds

$$\langle S_n(v_n^{(\nu+1)} - S_n(v_n), w_n\rangle \ + \ r(Be_v^{(\nu+1)}, Bw_n)_{0,\Omega} \ = \ \langle B_n^* e_p^{(\nu)}, w_n\rangle \ , \tag{3.11a}$$

$$(e_p^{(\nu+1)} - e_p^{(\nu)}, q_n)_{0,\Omega} \ = \ -\rho(B_n e_v^{(\nu+1)}, q_n)_{0,\Omega} \ . \tag{3.11b}$$

With $w_n = 2e_v^{(\nu+1)}, q_n = 2e_p^{(\nu+1)})$ in (3.11a),(3.11b) and the results of subsection 2.2 as well as Young's inequality we obtain

$$\|e_p^{(\nu+1)}\|_{0,\Omega}^2 \ - \ \|e_p^{(\nu)}\|_{0,\Omega}^2 \ + \ 2\rho\,\gamma\,\|e_v^{(\nu+1)}\|_X^2 \ + \ \rho\,(2r-\tau)\,\|Be_v^{(\nu+1)}\|_X^2 \ \le \ 0 \ ,$$

from which we first deduce the convergence of $\{\|e_p^{(\nu)}\|_{0,\Omega}^2\}_{\mathbb{N}}$ and then

$$e_v^{(\nu)} \to 0 \quad \text{in } X \quad (\nu \to \infty) \ , \tag{3.12a}$$

$$Be_v^{(\nu)} \to 0 \quad \text{in } L^2(\Omega) \quad (\nu \to \infty) \ , \tag{3.12b}$$

Now, (3.11a) and Lemma 2.2 result in

$$\|S_n(v_n^{(\nu+1)}) - S_n(v_n) + rB_n^* B_n e_v^{(\nu+1)}\|_{X^*} \ = \ \|B_n^* e_p^{(\nu)}\|_{X*} \ \ge \ \beta\,\|e_p^{(\nu)}\|_{0,\Omega} \ .$$

Hence, (3.12a),(3.12b) and the continuity of S_n imply $e_p^{(\nu)} \to 0$ as $\nu \to \infty$. □

As in the case of the nonlinear Uzawa algorithm, in practical computations we replace S_n in (3.10a) by some appropriate approximation $\tilde{S}_n$. This leads to the inexact augmented Lagrangian algorithm

Inexact augmented Lagrangian algorithm:
Let $\tilde{S}_n$ be an approximation of S_n. Then, given $(v_n^{(0)}, p_n^{(0)}) \in X_n \times Q_n$ and $r, \rho > 0$, compute $(v_n^{(\nu)}, p_n^{(\nu)}) \in X_n \times Q_n, \nu \in \mathbb{N}$, such that for $(w_n, q_n) \in X_n \times Q_n$ there holds

$$\langle \tilde{S}_n(v_n^{(\nu+1)}), w_n\rangle - \langle B_n^* p_n^{(\nu)}, w_n\rangle + r(B_n v_n^{(\nu+1)}, B_n w_n)_{0,\Omega} \ = \ \langle f+g, w_n\rangle \ , \tag{3.13a}$$

$$(p_n^{(\nu+1)} - p_n^{(\nu)}, q_n)_{0,\Omega} + \rho(B_n v_n^{(\nu+1)}, q_n)_{0,\Omega} \ = \ 0 \ . \tag{3.13b}$$

The convergence of the inexact augmented Lagrangian algorithm requires that $\tilde{S}_n^{-1}$ provides a sufficiently good approximation of S_n^{-1} which also affects the choice of the parameters r and ρ.

REMARK 3.3. *More efficient preconditioners can be constructed in the framework of multigrid techniques (cf.* HACKBUSCH [1985]*) with respect to a hierarchy of discretizations and/or domain decomposition methods (cf.* QUARTERONI and VALLI [1999] *and* TOSELLI and WIDLUND [2005]*) relying on overlapping or non-overlapping decompositions of the computational domain. However, we are not aware of any scientific contributions where such approaches have been applied to the numerical solution of electrorheological fluid flows.*

3.1.3. *Construction of approximate inverses.* There is a wide variety of possible approximates inverses $\tilde{S}_n^{-1}$ of S_n^{-1} for the realization of the inexact nonlinear Uzawa algorithm (3.9a),(3.9b) and the inexact augmented Lagrangian algorithm (3.13a),(3.13b), among them the Picard iteration, fixed point techniques and Newton-type methods.
We recall that the operator S_n in (3.8a) and (3.11a) can be formally written as $S_n(v_n) = \hat{S}_n(v_n, v_n)$ where $\hat{S}_n : X \times X \to X^*$ is given by

$$\langle \hat{S}_n(v_n, w_n), z_n\rangle := 2 \int\limits_\Omega \Big(b(|E|, x)(\kappa + I(\tilde{u} + v_n))^{-1/2} \varepsilon(\tilde{u} + w_n) : \varepsilon(z_n) + c(I(\tilde{u} + v_n), |E|, x)\varepsilon(\tilde{u} + w_n) : \varepsilon(z_n)\Big)\, dx \, . \tag{3.14}$$

Then, for a given $f_n \in X_n^*$ the solution of the nonlinear variational equation

$$\langle S_n(v_n), z_n\rangle = \langle f_n, z_n\rangle \quad , \quad z_n \in X_n \, , \tag{3.15}$$

can be obviously reformulated as

$$\langle \hat{S}_n(v_n, v_n), z_n\rangle = \langle f_n, z_n\rangle \quad , \quad z_n \in X_n \, . \tag{3.16}$$

We first consider a Picard-type iteration (cf. MOORE and CLOUD [2007]) which in the Russian literature is also known as the Birger-Kachanov method (cf. FUCIK et al. [1973]).
Picard iteration
Given $v_n^{(0)} \in X_n$, compute $v_n^{(\nu)}, \nu \in \mathbb{N}$, as the solution of the linear variational equation

$$\langle \hat{S}_n(v_n^{(\nu)}, v_n^{(\nu+1)}), z_n\rangle = \langle f_n, z_n\rangle \quad , \quad z_n \in X_n \, , \; \nu \in \mathbb{N}_0 \, . \tag{3.17}$$

THEOREM 3.3. *Let $v_n \in X_n$ be the solution of* (3.15) *and $\{v_n^{(\nu)}\}_{\mathbb{N}}$ be the sequence of iterates $v_n^{(\nu)} \in X_n, \nu \in \mathbb{N}$, generated by the Picard iteration* (3.17)*. Then, under the assumptions $(\mathbf{A_1}), (\mathbf{A_2})$ and for $\kappa > 0$, there holds*

$$v_n^{(\nu)} \to v_n \quad \text{in } X \quad (\nu \to \infty) \, .$$

Proof. We refer to FUCIK et al. [1973], MOORE and CLOUD [2007]. □

We will not consider the issue how well the inverse $\tilde{S}_n^{-1}$ associated with the Picard iteration (3.17) approximates S_n^{-1} in order to access the convergence of the inexact nonlinear Uzawa algorithm or the inexact augmented Lagrangian algorithm, but instead address this question in the framework of a fixed point iteration:
We introduce $A : X \to X^*$ as a linear, continuous self-adjoint coercive operator, i.e., we assume that for $v, w \in X$

$$\langle Av, w\rangle = \langle Aw, v\rangle \, , \tag{3.18a}$$

$$|\langle Av, w\rangle| \le C_A \, \|v\|_X \, \|w\|_X \, , \tag{3.18b}$$

$$\langle Av, v\rangle \ge \gamma_A \, \|v\|_X^2 \, . \tag{3.18c}$$

Hence, $\|\cdot\|_A := \langle A\cdot, \cdot\rangle^{1/2}$ defines a norm on X which is equivalent to the $\|\cdot\|_X$-norm and the $\|\cdot\|_{1,2,\Omega}$-norm. We refer to $\|\cdot\|_{A^*}$ as the associated norm on the dual space X^*. Hence, the operator S_n retains its properties with respect to the $\|\cdot\|_A$- and the $\|\cdot\|_{A^*}$-norm. In particular, for $w_n, z_n \in X_n$ there holds

$$\|S_n(w_n) - S_n(z_n)\|_{A^*} \leq C_S \|w_n - z_n\|_A , \tag{3.19a}$$

$$\langle S_n(w_n) - S_n(z_n), w_n - z_n \rangle \geq \gamma_S \|w_n - z_n\|_A^2 . \tag{3.19b}$$

Setting $A_n := A|_{X_n}$, for the solution of (3.15) we consider the following fixed point iteration:

Fixed point iteration
Given $v_n^{(0)} \in X_n$ and $t \in \mathbb{R}_+$, compute $v_n^{(\nu)} \in X_n, \nu \in \mathbb{N}$, as the solution of

$$\langle A_n v_n^{(\nu+1)}, z_n \rangle = \langle A_n v_n^{(\nu)}, z_n \rangle - t\Big(\langle S_n(v_n^{(\nu)}), z_n \rangle - \langle f_n, z_n \rangle \Big) , \; z_n \in X_n . \tag{3.20}$$

THEOREM 3.4. *Let $v_n \in X_n$ be the unique solution of* (3.15)*. Assume that the operator $A \in \mathcal{L}(X, X^*)$ satisfies* (3.18a)-(3.18c) *and that assumptions $(\mathbf{A_1}), (\mathbf{A_2})$ hold true. Then, for $\kappa > 0$ and $t \in (0, 2\gamma_S C_S^{-2})$ the linear problem* (3.20) *has a unique solution $v_n^{(\nu+1)} \in X_n$, and there holds*

$$\|v_n^{(\nu)} - v_n\|_A \leq \frac{k(t)^\nu}{1 - k(t)} \|S_n(v_n^{(0)}) - f_n\|_{A^*} \quad , \quad \nu \in \mathbb{N} , \tag{3.21}$$

where

$$k(t) = (1 - 2\gamma_S t + C_S^2 t^2)^{1/2} < 1 . \tag{3.22}$$

The optimal value is

$$k_{opt} = k(t_{opt}) = (1 - \gamma_S^2 C_S^{-2})^{1/2} \quad , \quad t_{opt} = \gamma_S C_S^{-2} .$$

Proof. We denote by $J : X^* \to X$ the Riesz operator. Then, the iteration (3.20) amounts to the computation of a fixed point of the operator $T_n(t) : X_n \to X_n$ given by

$$T_n(t)(w_n) := w_n - t\, J(S_n(w_n) - f_n) \quad , \quad w_n \in X_n . \tag{3.23}$$

Taking (3.19a),(3.19b) and the isometry of J into account, from (3.23) we deduce

$$\begin{aligned} &\|T_n(t)(w_n) - T_n(t)(z_n)\|_A^2 = \|w_n - z_n - t\, J(S_n(w_n) - S_n(z_n))\|_A^2 = \\ &= \|w_n - z_n\|_A^2 - 2t\, \langle S_n(w_n) - S_n(z_n), w_n - z_n \rangle + t^2\, \|S_n(w_n) - S_n(z_n)\|_{A^*}^2 \leq \\ &\leq \|w_n - z_n\|_A^2 - 2t\, \gamma_S\, \|w_n - z_n\|_A^2 + t^2\, C_S^2\, \|w_n - z_n\|_A^2 = k(t)^2\, \|w_n - z_n\|_A^2 . \end{aligned}$$

Hence, the assertion follows from the Banach fixed point theorem. □

REMARK 3.4. *Some comments are in order with regard to an appropriate choice of the finite dimensional subspaces X_n and Q_n. In the framework of finite element approximations based on simplicial and/or quadrilateral triangulations of the computational domain, for incompressible Stokes and Navier-Stokes type fluid flow problems various families of finite elements have been suggested. The Taylor-Hood P_k/P_{k-1}-elements, $k \in \mathbb{N}$, and its generalizations have become the most popular choice in applications. For a thorough presentation and discussion including the discrete inf-sup condition we refer to* BRAESS [2007], BREZZI and FORTIN [1991].

3.1.4. *An augmented Lagrangian approach for an extended Bingham fluid model.* In case of the extended Bingham fluid model based on the viscosity function (2.18), the fluid flow is described by the nonlinear variational inequality of the second kind (2.63). Hence, appropriate numerical methods for such variational inequalities have to be provided (cf., e.g., GLOWINSKI et al. [1981]). We present here an augmented Lagrangian approach relying on a mixed formulation of the problem that has been used in ENGELMANN et al. [2000] for the computation of electrorheological fluid flows obeying the constitutive law (2.13). The motivation for the mixed formulation is that the nonlinearity and non-smoothness of the problem is confined to the gradients of the components of the velocity. Hence, introducing $p = \nabla u$ as additional unknowns and using a $P1/P0$ finite element discretization of (u, p) boils down the global nonlinear problem to a sequence of local, low-dimensional nonlinear problems that can be easily solved. For simplicity we restrict ourselves to a problem setting with full rotational symmetry where $E = E_r(r,z)e_r + E_z(r,z)e_z$ and $u = u(r,z)e_\vartheta$ with e_r, e_ϑ and e_z denoting the unit vectors in a cylindrical coordinate system. The incompressibility condition is then automatically satisfied.

Based on the constitutive law (2.13), the steady state $u \in V := W^{1,2}_{0,\Gamma_D}(\Omega)$ of the electrorheological fluid flow corresponds to the minimizer of the global energy

$$J(u) \;=\; \inf_{v \in V} J(v) \; . \tag{3.24}$$

Here, $J : V \to \mathbb{R}$ stands for the energy functional

$$J(v) \;:=\; \gamma \int\limits_\Omega |E||E \cdot \nabla u| r \; dr \; dz \;+\; \frac{1}{2}\eta \int\limits_\Omega |\nabla u|^2 r \; dr \; dz \; \ell(v) \; , \; v \in V \; , \tag{3.25}$$

where $\ell : V \to \mathbb{R}$ comprises volume and surface forces according to

$$\ell(v) \;:=\; \langle f + g, v \rangle \quad , \quad v \in V \; .$$

We introduce $p = \nabla u \in L^2(\Omega)^2$ as additional unknowns and couple the constraint $p = \nabla u$ both by a Lagrangian multiplier $\lambda \in L^2(\Omega)^2$ and by a penalty term with penalty parameter $\tau > 0$ which gives rise to the saddle point problem:

Find $(u, p, \lambda) \in V \times L^2(\Omega)^2 \times L^2(\Omega)^2$ such that

$$L^{(\tau)}(u, p, \lambda) \;=\; \inf_{v,q} \sup_{\mu} L^{(\tau)}(v, q, \mu) \; , \tag{3.26}$$

where the augmented Lagrangian $L^{(\tau)}(\cdot,\cdot,\cdot)$ is given by

$$\begin{aligned} L^{(\tau)}(v, q, \mu) \;:=&\; \gamma \int\limits_\Omega |E||E \cdot p| r \; dr \; dz \;+\; \frac{1}{2}\eta \int\limits_\Omega |p|^2 r \; dr \; dz \;+ \\ &+ \int\limits_\Omega \mu \cdot (p - \nabla u) \; dr \; dz \;+\; \frac{1}{2}\tau \int\limits_\Omega |p - \nabla u|^2 \; dr \; dz - \ell(v) \; . \end{aligned}$$

For a simplicial triangulation $\mathcal{T}_h(\Omega)$ of the computational domain Ω, we use a $P1/P0$ discretization $(u_h, p_h) \in V_h \times W_h^2$ of (u, p) where V_h stands for the standard finite element space of continuous piecewise linear finite elements and W_h for the linear space of elementwise constants. If an approximation of the electric field E is obtained based on a $P1$ approximation, we define $E_h \in W_h$ locally as the elementwise integral mean of that approximation. Consequently, the discrete minimization problem amounts to the computation of $(u_h, p_h, \lambda_h) \in V_h \times W_h^2 \times W_h^2$ such that

$$L^{(\tau)}(u_h, p_h, \lambda_h) \;=\; \inf_{v_h, q_h} \sup_{\mu_h} L^{(\tau)}(v_h, q_h, \mu_h) \; , \tag{3.27}$$

where E in the definition of $L^{(\tau)}(\cdot,\cdot,\cdot)$ has to be replaced by E_h.
The minimization problem (3.27) is solved iteratively by an operator splitting technique where each iteration step requires the solution of a global quadratic minimization problem and local, i.e., elementwise nonlinear minimization problems along with appropriate updates of the discrete Lagrangian multipliers λ_h. In particular, given sequences $\{\rho_n\}_{\mathbb{N}}$ and $\{\tau_n\}_{\mathbb{N}}$ of update parameters $\rho_n \in \mathbb{R}_+$ and penalty parameters $\tau_n \in \mathbb{R}_+, n \in \mathbb{N}$, as well as start vectors $(p_h^{(0)}, \lambda_h^{(1)}) \in W_h^2 \times W_h^2$, an iteration consists of the following two steps:

Step 1: Compute $u_h^{(n)} \in V_h$ as the solution of the global quadratic minimization problem

$$L^{(\tau_n)}(u_h^{(n)}, p_h^{(n-1)}, \lambda_h^{(n)}) \;=\; \inf_{v_h \in V_h} L^{(\tau)}(v_h, p_h^{(n-1)}, \lambda_h^{(n)}) \tag{3.28}$$

and update the multiplier according to

$$\lambda_h^{(n+1/2)} \;=\; \lambda_h^{(n)} \;+\; \rho_n(\nabla u_h^{(n)} - p_h^{(n-1)}) \;. \tag{3.29}$$

Step 2: Compute $p_h^{(n)} \in W_h^2$ as the solution of

$$L^{(\tau_n)}(u_h^{(n)}, p_h^{(n)}, \lambda_h^{(n+1/2)}) \;=\; \inf_{q_h \in W_h^2} L^{(\tau)}(u_h^{(n)}, q_h, \lambda_h^{(n+1/2)}) \tag{3.30}$$

and update the multiplier according to

$$\lambda_h^{(n+1)} \;=\; \lambda_h^{(n+1/2)} \;+\; \rho_n(\nabla u_h^{(n)} - p_h^{(n)}) \;. \tag{3.31}$$

The minimization problem (3.28) requires the solution of a linear algebraic system where the coefficient matrix corresponds to the stiffness matrix associated with the $P1$ approximation of the Laplacian $-\Delta$. On the other hand, the minimization problem (3.30) reduces to the simultaneous solution of the elementwise minimization problems: For each $T \in \mathcal{T}_h(\Omega)$ compute $p_h^{(n)}|_T \in P_0(T)^2$ such that

$$J_T^{(\tau_n)}(p_h^{(n)}|_T) \;=\; \inf_{q_h^T \in P_0(T)^2} J_T^{(\tau_n)}(q_h^T) \;, \tag{3.32}$$

where the functional $J_T^{(\tau_n)} : P_0(T)^2 \to \mathbb{R}$ is given by

$$J_T^{(\tau_n)}(q_h^T) \;:=\; L^{(\tau_n)}(u_h^{(n)}|_T, q_h^T, \lambda_h^{(n+1/2)}) \;.$$

The local minimization problems (3.32) give rise to two-dimensional variational inequalities which can be solved analytically.

3.2. **Evolutionary isothermal incompressible flow problems.** We consider the discretization of initial-boundary value problems for time-dependent incompressible isothermal electrorheological fluid problems (2.1a),(2.1b) by a difference approximation in time and by the Galerkin method in space using finite dimensional subspaces $X_n \subset X := W^{1,2}_{0,\Gamma_D}$ and $Q_n \subset L^2(\Omega), n \in \mathbb{N}$ as in the previous subsection 3.1. For discretization in time we refer to

$$\bar{I}_k \;:=\; \{t_m = mk \mid 0 \le m \le M \;,\; k := T/M\} \;,\; M \in \mathbb{N} \;, \tag{3.33}$$

as a uniform partition of the time interval $[0,T]$ of step size k and approximate the time derivative $u_t(\cdot,t)$ in $t \in \bar{I}_k$ by the forward and backward difference quotients $\partial_k^{\pm} u(\cdot,t)$ which are given by

$$\partial_k^{+} u(\cdot,t) \;:=\; k^{-1}(u(\cdot,t+k) - u(\cdot,t)) \quad , \quad t \in \bar{I}_k \setminus \{T\} \;, \tag{3.34a}$$

$$\partial_k^{-} u(\cdot,t) \;:=\; k^{-1}(u(\cdot,t) - u(\cdot,t-k)) \quad , \quad t \in \bar{I}_k \setminus \{0\} \;. \tag{3.34b}$$

We denote by $(u_n^{(m)}, p_n^{(m)}) \in X_n \times Q_n$ an approximation of $(u(\cdot,t_m), p(\cdot,t_m)) \in X \times L^2(\Omega)$ at time t_m. Using a convex combination of the discretizations by the forward and difference quotients in time results in the so-called Θ-scheme which at each time level amounts to the solution of the following nonlinear system of finite dimensional variational equations

$$\langle F_n^{(\Theta)}(u_n^{(m)}), w_n\rangle \;-\; \langle B_n^* p_n^{(m)}, w_n\rangle \;=\; \langle h_n^{(\Theta)}, w_n\rangle \;,\; w_n \in X_n \;, \tag{3.35a}$$

$$(B_n u_n^{(m)}, q_n\rangle \;=\; 0 \;,\; q_n \in Q_n \;, \tag{3.35b}$$

where the nonlinear operator $F_n^{(\Theta)} : X_n \to X_n^*$ and the right-hand side $h_n^{(\Theta)} \in X_n^*, \Theta \in [0,1]$, are given by

$$\langle F_n^{(\Theta)}(v_n), w_n\rangle := \rho\, k^{-1}\, \langle v_n, w_n\rangle + \Theta\, \Big(\langle (v_n\cdot\nabla)v_n, w_n\rangle + \langle S_n(v_n), w_n\rangle \Big) \;, \tag{3.36a}$$

$$h_n^{(\Theta)} := f_n + g_n + k^{-1} u_n^{(m)} - (1-\Theta)\, \Big((u_n^{(m)}\cdot\nabla)u_n^{(m)} + S_n(u_n^{(m)}) \Big) \;. \tag{3.36b}$$

For $\Theta = 0$ and $\Theta = 1$, we recover the standard explicit and implicit difference approximation, respectively. The difference approximation for $\Theta = 1/2$ is called the Crank-Nicolsen method. It is well-known that the Θ-scheme is consistent with the initial-boundary value problem of order $O(k)$ in time for $\Theta \neq 1/2$, whereas the Crank-Nicolsen method is consistent of order $O(k^2)$. Moreover, the Θ-scheme is only conditionally stable for $\Theta < 1/2$ and unconditionally stable for $\Theta \in [1/2, 1]$ (cf., e.g., STRIKWERDA [2004], THOMAS [1995]). Usually, the stability condition for $\Theta \in [0, 1/2)$ imposes a severe restriction on the choice of the step size k so that the corresponding schemes are not used in practice.

The nonlinear system (3.35a),(3.35b) can be solved using the same techniques as described in subsection 3.1. In particular, we may use the analogues of the inexact nonlinear Uzawa algorithm (3.9a),(3.9b) and the inexact augmented Lagrangian algorithm (3.13a),(3.13b) provided we have suitable approximate inverses $(\tilde{F}_n^{(\Theta)})^{-1}$ of $(F_n^{(\Theta)})^{-1}$, $\Theta \in (1/2, 1]$, at hand. For the construction of such inverses, the Picard iteration or fixed point iterations can be used as well. The only difference is that we are faced with the additional nonlinear convective term $(v_n\cdot\nabla)v_n$ which, however, can be treated in much the same way as the nonlinearity in the operator S_n. For instance, in case of the standard implicit scheme ($\Theta = 1$) we use

$$\langle \tilde{F}_n^{(1)}(v_n), w_n\rangle := \rho\, k^{-1}\, \langle v_n, w_n\rangle + \Big(\langle (u_n^{(m)}\cdot\nabla)v_n, w_n\rangle + \langle \tilde{S}_n(u_n^{(m)}), w_n\rangle \Big) \;, \tag{3.37}$$

with $\tilde{S}_n$ given by (3.14).
For the Crank-Nicolsen scheme, an appropriate modification has to be used in order to retain second order accuracy (cf., e.g., ELMAN [2002]).

3.3. **Non-isothermal incompressible electrorheological flow problems.** We use the notations from subsection 2.5 and assume $\{X_n\}_{\mathbb{N}}$, $\{Q_n\}_{\mathbb{N}}$ and $\{Y_n\}_{\mathbb{N}}$ to be limit dense nested sequences of finite dimensional subspaces of X, $L^2(\Omega)$ and $W_{0,\Gamma}^{1,2}(\Omega)$, respectively, and we consider the following sequence of approximating systems of finite dimensional variational equations: Find $(v_n, p_n, \theta_n) \in X_n \times Q_n \times Y_n$ such that

$$\langle N(v_n,\theta_n), w_n\rangle - \langle B_n^* p_n, w_n\rangle \;=\; \langle f+g, w_n\rangle \;, w_n \in X_n \tag{3.38a}$$

$$(B_n v_n, q_n)_{0,\Omega} \;=\; 0 \;,\; q_n \in Q_n \;, \tag{3.38b}$$

$$(\nabla\theta_n, \nabla\zeta_n)_{0,\Omega} - \langle A_\beta(v_n,\theta_n), \zeta_n\rangle \;=\; (f_3, \zeta_n)_{0,\Omega} \;,\; \zeta_n \in Y_n \;, \tag{3.38c}$$

where $B_n \in \mathcal{L}(X_n, Q_n)$ refers to the discrete divergence operator (cf. subsection 2.2.1).

THEOREM 3.5. *Let the assumptions of Theorem* 2.6 *be satisfied and let* $\{(v_n, p_n, \theta_n)\}_{\mathbb{N}}$ *be a sequence of solutions of* (3.38a)-(3.38c). *Then, there exist a subsequence* $\mathbb{N}' \subset \mathbb{N}$ *and a triple* $(v, p, \theta) \in X \times L^2(\Omega) \times W^{1,2}_{0,\Gamma}(\Omega)$ *that solves* (2.101a)-(2.101c) *such that for* $\mathbb{N}' \ni n \to \infty$

$$v_n \to v \quad \text{in } X \ , \tag{3.39a}$$

$$p_n \to p \quad \text{in } L^2(\Omega) \ , \tag{3.39b}$$

$$\theta_n \to \theta \quad \text{in } W^{1,2}_{0,\Gamma}(\Omega) \ . \tag{3.39c}$$

Proof. Setting $V_n = \text{Ker}(B_n)$, (3.38a)-(3.38c) can be equivalently stated as: Find $(v_n, \theta_n) \in V_n \times Y_n$ such that

$$\langle N(v_n, \theta_n), w_n \rangle \ = \ \langle f_1 + g, w_n \rangle \ , w_n \in X_n \tag{3.40a}$$

$$(\nabla \theta_n, \nabla \zeta_n)_{0,\Omega} - \langle A_\beta(v_n, \theta_n), \zeta_n \rangle \ = \ (f_3, \zeta_n)_{0,\Omega} \ , \ \zeta_n \in Y_n \ . \tag{3.40b}$$

It follows from Theorem 2.6 that for each $n \in \mathbb{N}$ problem (3.40a),(3.40b) admits a solution $(v_n, \theta_n) \in V_n \times Y_n$. Moreover, there are constants $C_i > 0, 1 \le i \le 2$, such that

$$\|v_n\|_X \ \le \ C_1 \quad , \quad \|\theta_n\|_{1,\Omega} \ \le \ C_2 \tag{3.41}$$

uniformly in $n \in \mathbb{N}$. We have $N(v_n, \theta_n) - (f_1 + g) \in V_n^0$, and hence, Lemma 2.2 implies that there is a unique $p_n \in Q_n$ such that

$$\langle N(v_n, \theta_n), w_n \rangle \ - \ \langle B_n^* p_n, w_n \rangle \ = \ \langle f_1 + g, w_n \rangle \ , \ w_n \in X_n \ , \tag{3.42}$$

i.e., (v_n, p_n, θ_n) solves (3.38a)-(3.38c). Lemma 2.2 and (3.41) yield

$$\|p_n\|_{0,\Omega} \ \le \ C_3 \quad , \quad n \in \mathbb{N} \tag{3.43}$$

for some constant $C_3 > 0$. Consequently, there exist a subsequence $\mathbb{N}' \subset \mathbb{N}$ and $(v, p, \theta) \in X \times L^2(\Omega) \times W^{1,2}_{0,\Gamma}(\Omega)$ such that for $\mathbb{N}' \ni n \to \infty$

$$v_n \rightharpoonup u \quad \text{in } X \ , \tag{3.44a}$$

$$v_n \to v \quad \text{in } L^4(\Omega)^d \ , \tag{3.44b}$$

$$v_n \to v \quad \text{a.e. in } \Omega \ , \tag{3.44c}$$

$$p_n \rightharpoonup p \quad \text{in } L^2(\Omega) \ , \tag{3.44d}$$

$$\theta_n \rightharpoonup \theta \quad \text{in } W^{1,2}_{0,\Gamma}(\Omega) \ , \tag{3.44e}$$

$$\theta_n \to \theta \quad \text{in } L^4(\Omega) \ , \tag{3.44f}$$

$$\theta_n \to \theta \quad \text{a.e. in } \Omega \ , \tag{3.44g}$$

$$N(v_n, \theta_n) \rightharpoonup \ell \quad \text{in } X^* \ . \tag{3.44h}$$

For a fixed integer $n_0 \in \mathbb{N}$ let $w_{n_0} \in X_{n_0}$ and $q_{n_0} \in Q_{n_0}$. Then, in view of (3.44a),(3.44d) and (3.44h), passing to the limit in (2.101a),(2.101b) yields

$$\begin{aligned} \langle \ell - B^* p, w \rangle \ &= \langle f_1 + g, w \rangle \quad , \quad w \in X_{n_0} \ , \\ (Bv, q)_{0,\Omega} \ &= 0 \quad , \quad q \in Q_{n_0} \ . \end{aligned}$$

Since $n_0 \in \mathbb{N}$ was arbitrarily chosen and the sequences $\{X_n\}_{\mathbb{N}}$ and $\{Q_n\}_{\mathbb{N}}$ are limit dense in X and $L^2(\Omega)$, it follows that

$$\ell \ - \ B^* p \ = \ f_1 + g \quad \text{in } X^* \ , \tag{3.45a}$$

$$\nabla \cdot v \ = \ 0 \quad \text{a.e. in } \Omega \ . \tag{3.45b}$$

We define $L_{z_1,z_2} : X \to X^*$ according to

$$\langle L_{z_1,z_2}(w_1), w_2\rangle \ := \\ 2 \int\limits_\Omega \varphi(I(\tilde{u}+w_1), |E|, \mu(\tilde{u}+z_1, E), \tilde{\theta}+z_2)\varepsilon(\tilde{u}+w_1) : \varepsilon(w_2)\, dx \ , \ w_1, w_2 \in X \ .$$

For $z_1 = v_n, z_2 = \theta_n$ Lemma 2.3 gives

$$\langle L_{(v_n,\theta_n)}(v_n) - L_{(v_n,\theta_n)}(v), v_n - w\rangle \ \geq \ 0 \quad , \quad w \in X \ , \ n \in \mathbb{N} \ . \tag{3.46}$$

Moreover, by (3.44b),(3.44c) and (3.44f),(3.44g) and the Lebesgue theorem

$$L_{(v_n,\theta_n)}(w) \to L_{(v,\theta)}(w) \quad \text{in } X^* \ , \ w \in X \ .$$

It follows that for $w \in X$ there holds

$$\lim_{\mathbb{N}' \ni n \to \infty} \langle L_{(v_n,\theta_n)}(w), v_n\rangle \ = \langle L_{(v,\theta)}(w), v\rangle \ , \tag{3.47a}$$

$$\lim_{\mathbb{N}' \ni n \to \infty} \langle L_{(v_n,\theta_n)}(w), w\rangle \ = \langle L_{(v,\theta)}(w), w\rangle \ . \tag{3.47b}$$

Observing (3.44h) and (3.45a), we obtain

$$\lim_{\mathbb{N}' \ni n \to \infty} \Big(\langle L_{(v_n,\theta_n)}(v_n), w\rangle - \langle B^* p, w\rangle \Big) \ = \ \langle f_1 + g, w\rangle \quad , \quad w \in X \ . \tag{3.48}$$

Taking into account that

$$\langle B^* p_n, v_n\rangle \ = \ (p_n, B_n v_n)_{0,\Omega} \ ,$$

(2.101a) and (3.44a) imply that for $\mathbb{N}' \ni n \to \infty$ there holds

$$\langle L_{(v_n,\theta_n)}(v_n), v_n\rangle \ = \ \langle f_1 + g, v_n\rangle \to \langle f_1 + g, v\rangle \ . \tag{3.49}$$

Due to (3.47a),(3.47b) and (3.48),(3.49), we pass to the limit in (3.46) and get

$$\langle f_1 + g - L_{(v,\theta)}(w) + B^* p, v - w\rangle \ \geq \ 0 \quad , \quad w \in X \ . \tag{3.50}$$

If we choose $w = v - \gamma z, z \in X, \gamma > 0$, in (3.50), for $\gamma \to 0$ it follows that

$$\langle f_1 + g - N(v,\theta) + B^* p, z\rangle \ \geq \ 0 \quad , \quad z \in X \ .$$

Since $z \in X$ can be arbitrarily chosen, we may replace z by $-z$ and thus obtain

$$\langle N(v,\theta), z\rangle \ - \ \langle B^* p, z\rangle \ = \langle f_1 + g, z\rangle \quad , \quad z \in X \ , \tag{3.51a}$$

$$\ell \ = N(v,\theta) \ . \tag{3.51b}$$

On the other hand, (3.44a)-(3.44c) and (3.44e)-(3.44g) as well as Lebesgue's theorem imply

$$\lim_{\mathbb{N}' \ni n \to \infty} \langle A_\beta(v_n,\theta_n), \xi\rangle \ = \ \langle A_\beta(v,\theta), \xi\rangle \quad , \quad \xi \in W^{1,2}_{0,\Gamma}(\Omega) \ .$$

Choosing $n_0 \in \mathbb{N}$ and $\xi_{n_0} \in Y_{n_0}$ arbitrarily, but fixed, and passing to the limit in (2.101c), we get

$$(\nabla\theta, \nabla\xi_{n_0})_{0,\Omega} \ - \ \langle A_\beta(v,\theta), \xi_{n_0}\rangle \ = \ \langle f_3, \xi_{n_0}\rangle \ .$$

Since the sequence $\{Y_n\}_{\mathbb{N}}$ is limit dense in $W^{1,2}_{0,\Gamma}(\Omega)$, we thus have

$$(\nabla\theta, \nabla\xi)_{0,\Omega} \ - \ \langle A_\beta(v,\theta), \xi\rangle \ = \ \langle f_3, \xi\rangle \quad , \quad \xi \in W^{1,2}_{0,\gamma}(\Omega) \ . \tag{3.52}$$

Now, (3.45b), (3.51a) and (3.52) show that the triple (v,p,θ) is a solution of (2.101a)-(2.101c).

What remains to be shown is the strong convergence (3.39a)-(3.39c). We first note that due to (3.44a), (3.48) (with $w = v$), and (3.49)

$$\Lambda_n \ := \ \langle L_{(v_n,\theta_n)}(v_n) - L_{v,\theta)}(v), v_n - v\rangle \to 0 \quad (\mathbb{N}' \ni n \to \infty) \ . \tag{3.53}$$

We split Λ_n according to

$$\Lambda_n \;=\; \langle L_{(v_n,\theta_n)}(v_n) - L_{v_n,\theta_n)}(v), v_n - v\rangle \;+\; \langle L_{(v_n,\theta_n)}(v) - L_{v,\theta)}(v), v_n - v\rangle \;. \tag{3.54}$$

In view of (3.44a) and (3.47a),(3.47b) we have

$$\langle L_{(v_n,\theta_n)}(v) - L_{v,\theta)}(v), v_n - v\rangle \to 0 \quad (\mathbb{N}' \ni n \to \infty)\;,$$

and hence, due to (3.53),(3.54)

$$\langle L_{(v_n,\theta_n)}(v_n) - L_{v_n,\theta_n)}(v), v_n - v\rangle \to 0 \quad (\mathbb{N}' \ni n \to \infty)\;. \tag{3.55}$$

Now, Lemma 2.3 implies

$$v_n \to v \quad \text{in } X \quad (\mathbb{N}' \ni n \to \infty)\;, \tag{3.56}$$

whence

$$I(\tilde{u} + v_n) \to I(\tilde{u} + v) \quad \text{a.e. in } \Omega \quad (\mathbb{N}' \ni n \to \infty)\;. \tag{3.57}$$

We choose $w = w_n \in X_n$ in (2.101a) and subtract (2.101a) from (3.38a) which shows that for $q_n \in Q_n$ there holds

$$\langle B^*(p_n - q_n), w_n\rangle \;=\; \langle N(v_n,\theta_n) - N(v,\theta), w_n\rangle \;+\; \langle B^*(p - q_n), w_n\rangle\;. \tag{3.58}$$

Applying Lemma 2.2 in (3.58) yields

$$\begin{aligned} \|p_n - q_n\|_{0,\Omega} \;&\le\; \sup_{w_n \in X_n} \frac{\langle B^*(p_n - q_n), w_n\rangle}{\beta\, |w_n\|_X} \;\le \\ &\le\; \beta^{-1}\, \|N(v_n,\theta_n) - N(v,\theta)\|_{X^*} + C\, \|p - q_n\|_{0,\Omega} \quad , \quad q_n \in Q_n\;, \end{aligned}$$

where $C \in \mathbb{R}$ is a positive constant. It follows that

$$\begin{aligned} \|p - p_n\|_{0,\Omega} \;&\le\; \inf_{q_n \in Q_n} \Big(\|p - q_n\|_{0,\Omega} \;+\; \|p_n - q_n\|_{0,\Omega} \Big) \;\le \\ &\le\; \beta^{-1}\, \|N(v_n,\theta_n) - N(v,\theta)\|_{X^*} \;+\; (C+1) \inf_{q_n \in Q_n} \|p - q_n\|_{0,\Omega}\;. \end{aligned} \tag{3.59}$$

Setting

$$\varphi_{nm} \;:=\; \varphi(I(\tilde{u} + v_n), |E|, \mu(\tilde{u} + v_m, E), \tilde{\theta} + \theta_m) \quad , \quad n,m \in \mathbb{N}_0\;,$$

straightforward estimation results in

$$\begin{aligned} \frac{1}{2}\, \|N(v_n,\theta_n) - N(v,\theta)\|_{X^*} \;&\le\; \Big(\int_\Omega (\varphi_{nn}\varepsilon(\tilde{u} + v_n) - \varphi_{00}\varepsilon(\tilde{u} + v)^2\, dx \Big)^{1/2} \\ &=\; \Big(\int_\Omega \Big((\varphi_{nn}(\varepsilon(\tilde{u} + v_n) - \varepsilon(\tilde{u} - v)) + (\varphi_{nn} - \varphi_{00})\varepsilon(\tilde{u} + v) \Big)^2 dx \Big)^{1/2} \;\le \\ &\le\; \Big(\int_\Omega \varphi_{nn}^2 I(v_n - v)\, dx \Big)^{1/2} + \Big(\int_\Omega (\varphi_{nn} - \varphi_{00})^2 I(\tilde{u} + v)\, dx \Big)^{1/2}\;. \end{aligned} \tag{3.60}$$

It follows from $(\mathbf{T_1})$, (3.44b),(3.44c), (3.44f),(3.44g) and (3.56),(3.57) as well as the Lebesgue theorem that the right-hand side in (3.60) converges to zero as $\mathbb{N}' \ni n \to \infty$. Consequently,

$$\|N(v_n,\theta_n) - N(v,\theta)\|_{X^*} \to 0 \quad (\mathbb{N}' \ni n \to \infty)\;. \tag{3.61}$$

Since the sequence $\{Q_n\}_{\mathbb{N}}$ is limit dense in $L^2(\Omega)$, (3.59) and (3.61) imply

$$p_n \to p \quad \text{in } L^2(\Omega) \quad (\mathbb{N}' \ni n \to \infty)\;. \tag{3.62}$$

Finally, from (3.44b),(3.44c), (3.44f),(3.44g) and (3.56),(3.57) we also get

$$A_\beta(v_n,\theta_n) \to A_\beta(v,\theta) \quad \text{in } W^{-1,2}(\Omega) \quad (\mathbb{N}' \ni n \to \infty)\ . \tag{3.63}$$

Choosing $\zeta_n = \theta_n$ in (3.38c), we have

$$\|\theta_n\|^2_{1,2,\Omega} \ = \ \langle A_\beta(v_n,\theta_n),\theta_n\rangle \ + \ \langle f_3,\theta_n\rangle\ ,$$

whence in view of (2.101c),(3.44f) and (3.63) for $\mathbb{N}' \ni n \to \infty$ we have

$$\lim_{\mathbb{N}' \ni n \to \infty} \Big(\langle A_\beta(v_n,\theta_n),\theta_n\rangle + \langle f_3,\theta_n\rangle \Big) = \langle A_\beta(v,\theta,\theta\rangle + \langle f_3,\theta\rangle = \|\theta\|^2_{1,2,\Omega}\ .$$

Consequently, $\|\theta_n\|^2_{1,2,\Omega} \to \|\theta\|^2_{1,2,\Omega}$ as $\mathbb{N}' \ni n \to \infty$, which together with (3.44f) results in

$$\theta_n \to \theta \quad \text{in } W^{1,2}_{0,\Gamma}(\Omega) \quad (\mathbb{N}' \ni n \to \infty)\ .$$

This concludes the proof of the theorem. □

4. NUMERICAL SIMULATION AND OPTIMIZATION OF ELECTRORHEOLOGICAL DEVICES

We shall consider the application of the algorithmic tools developed in the previous section 3 to the simulation and the optimal design of electrorheological devices and systems. The most elementary devices are rheometers used for the measurement of rheological properties which shall be discussed in 4.1. Examples for more advanced devices are given by electrorheological shock absorbers which feature a much wider spectrum of damper characteristics than absorbers based on conventional fluids. The simulation of the operational behavior of such electrorheological shock absorbers, in particular their compression and rebound states, shall be treated in 4.2. Finally, 4.3 is devoted to a brief presentation of a methodology for the shape optimization of the inlet and outlet boundaries of piston ducts in electrorheological shock absorbers. For general aspects of optimization problems related to fluid mechanical processes we refer to LITVINOV [2000] and MOHAMMADI and PIRONNEAU [2001].

4.1. **Electrorheological rheometers.** Electrorheological rheometers are devices for the measurement of the rheological properties of electrorheological fluids. Figure 4.5 displays a simple model consisting of two coaxial cylinders of lengths l_i, l_e and radii r_r, r_e, respectively. The inner cylinder features a high voltage lead to an external electric circuit which supplies the lateral surface. The inner cylinder thus serves as the electrode. The lateral surface of the outer cylinder represents the counter electrode. The gap between the cylinders is filled with an electrorheological fluid.
One of the cylinders may rotate, whereas the other one remains at rest. When one of the cylinders starts revolving, the other one experiences a torque due to the viscosity of the fluid. Applying a voltage through the external electric circuit, the electrorheological effect results in an enhanced viscosity and the strength of the torque felt by the other cylinder increases. Commercial rheometers operate within a frequency range of 10^{-7} - 100 Hz, a temperature range of -150 - 1000 ^{o}C and allow angular velocities of 0 - 320 rad/s. The normal force range is between 10^{-3} and 50 N.

The arrangement has full rotational symmetry so that the computational domain reduces to the domain Ω as shown in Figure 4.5 (right). Given a cylindrical coordinate system

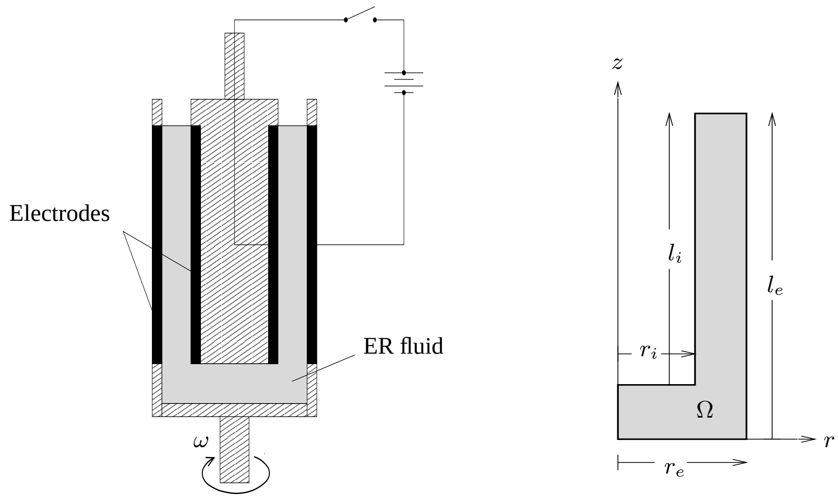

FIGURE 4.5. Electrorheological clutch (left) and computational domain (right).

(r, α, z) with basis vectors e_r, e_α and e_z, the velocity vector only features an angular component $u(r, z)e_\alpha$ which results in the following components of the strain tensor

$$\begin{aligned}\varepsilon_{12}(u) = \varepsilon_{21}(u) &= \frac{1}{2}(\frac{\partial u}{\partial r} - \frac{u}{r}) \; , \; \varepsilon_{23}(u) = \varepsilon_{32}(u) = \frac{1}{2}\frac{\partial u}{\partial z} \; , \\ \varepsilon_{11}(u) = \varepsilon_{22}(u) &= \varepsilon_{33}(u) = \varepsilon_{13}(u) = \varepsilon_{31}(u) = 0 \; .\end{aligned} \tag{4.1}$$

Hence, for the second invariant of the rate of strain tensor we obtain

$$I(u) \; = \; \frac{1}{2}(\frac{\partial u}{\partial r} - \frac{u}{r})^2 \; + \; \frac{1}{2}(\frac{\partial u}{\partial z})^2 \; . \tag{4.2}$$

In our case, $\mu(u, E) = 0$ and hence, the viscosity function φ is given by

$$\varphi(I(u), |E|, 0) \; := \; b(|E|, 0)(\kappa + I(u))^{-1/2} \; + \; c(I(u), |E|, 0) \; , \tag{4.3}$$

where κ is the regularization parameter. Note that $\kappa = 0$ refers to the extended Bingham fluid. Assuming no volume force acting on the fluid, the steady state equations take the form

$$\begin{aligned}&\frac{\partial}{\partial r}(\varphi(I(u), |E|, 0)(\frac{\partial u}{\partial r} - \frac{u}{r})) \; + \; \frac{\partial}{\partial z}(\varphi(I(u), |E|, 0)\frac{\partial u}{\partial z}) \; + \\ &\qquad + \; \frac{2}{r}\varphi(I(u), |E|, 0)(\frac{\partial u}{\partial r} - \frac{u}{r}) \; = \; 0 \; ,\end{aligned} \tag{4.4a}$$

$$\frac{\partial p}{\partial r} \; = \; \frac{\partial p}{\partial z} \; = \; 0 \; . \tag{4.4b}$$

The incompressibility condition is automatically satisfied.

As far as the boundary conditions on $\Gamma = \partial\Omega$ are concerned, we prescribe velocities on the left boundary of Ω

$$\Gamma_\ell \; := \; \{(r, z) \mid r = 0 \; , \; z \in (0, l_e - l_i)\}$$

and on the surface of the internal and external cylinder

$$\Gamma_s \; := \; \bigcup_{i=1}^{4} \Gamma_{s,i} \; ,$$

where the subsurfaces $\Gamma_{s,i}, 1 \leq i \leq 4$, are given by

$$\begin{aligned}
\Gamma_{s,1} &:= \{(r,z) \mid z = 0 \ , \ r \in ((0, r_e)\} \ , \\
\Gamma_{s,2} &:= \{(r,z) \mid r = r_e \ , \ z \in (0, l_e)\} \ , \\
\Gamma_{s,3} &:= \{(r,z) \mid z = l_e - l_i \ , \ r \in (0, r_i)\} \ , \\
\Gamma_{s,4} &:= \{(r,z) \mid r = r_i \ , \ z \in ((l_e - l_i), l_e)\} \ .
\end{aligned}$$

Moreover, surface forces are specified on

$$\Gamma_t \ := \ \Gamma \setminus (\bar{\Gamma}_\ell \cup \bar{\Gamma}_s) \ .$$

If the inner cylinder is rotating, the boundary conditions are chosen according to

$$u(r,z) \ = \ \begin{cases} 0 & \text{on } \Gamma_\ell \cup \Gamma_{s,1} \cup \Gamma_{s,2} \\ r\omega & \text{on } \Gamma_{s,3} \\ r_i\omega & \text{on } \Gamma_{s,4} \end{cases} \ , \tag{4.5a}$$

$$\lim_{r \to 0} (\frac{\partial u}{\partial r} - \frac{u}{r})(r,z) \ = \ 0 \ , \ z \in (0, l_e - l_i) \ , \tag{4.5b}$$

$$\varphi(I(u), |E|, 0) \frac{\partial u}{\partial z} \ = \ 0 \ , \ p \ = \ \text{const. on } \Gamma_t \ . \tag{4.5c}$$

On the other hand, if the outer cylinder is revolving, we have

$$u(r,z) \ = \ \begin{cases} 0 & \text{on } \Gamma_\ell \cup \Gamma_{s,3} \cup \Gamma_{s,4} \\ r\omega & \text{on } \Gamma_{s,1} \\ r_e\omega & \text{on } \Gamma_{s,2} \end{cases} \ , \tag{4.6a}$$

$$\lim_{r \to 0} (\frac{\partial u}{\partial r} - \frac{u}{r})(r,z) \ = \ 0 \ , \ z \in (0, l_e - l_i) \ , \tag{4.6b}$$

$$\varphi(I(u), |E|, 0) \frac{\partial u}{\partial z} \ = \ 0 \ , \ p \ = \ \text{const. on } \Gamma_t \ . \tag{4.6c}$$

Due to the rotational symmetry, the electric field

$$E(r,z) \ = \ E_r(r,z) e_r \ + \ E_z(r,z) e_z$$

has two components E_r and E_z which can be computed according to $E = -\nabla\psi = -(\partial\psi/\partial r, \partial\psi/\partial z)^T$ as the gradient of an electric potential $\psi = \psi(r,z)$. Denoting by

$$\begin{aligned}
\Gamma_i &:= \{(r,z) \mid r = r_i \ , \ z \in (l_e - l_i, l_e)\} \ , \\
\Gamma_e &:= \{(r,z) \mid r = r_e \ , \ z \in (l_e - l_i, l_e)\} \ ,
\end{aligned}$$

the lateral surfaces of the inner and outer cylinder, the electric potential ψ satisfies the boundary value problem

$$\frac{\partial}{\partial r}(\epsilon \frac{\partial \psi}{\partial r}) \ + \ \frac{\epsilon}{r}\frac{\partial \psi}{\partial r} \ + \ \frac{\partial}{\partial z}(\epsilon \frac{\partial \psi}{\partial z}) \ = \ 0 \quad \text{in } \Omega \ , \tag{4.7a}$$

$$\psi \ = \ U \quad \text{on } \Gamma_i \quad , \quad \psi \ = \ 0 \quad \text{on } \Gamma_e \ , \tag{4.7b}$$

$$\frac{\partial \psi}{\partial r} \ = \ 0 \quad \text{on } \Gamma_0 \quad , \quad \nu_r \epsilon \frac{\partial \psi}{\partial r} + \nu_z \epsilon \frac{\partial \psi}{\partial z} \ = \ 0 \quad \text{on } \Gamma_t \ ,$$

where U is the applied voltage, ϵ stands for the dielectric permittivity and $\nu = (\nu_r, \nu_z)^T$ is the exterior normal unit vector.

Given a simplicial triangulation of the computational domain Ω, we have discretized (4.4a) by conforming P1 finite elements in case of a regularized viscosity function, i.e., $\kappa > 0$, whereas for the extended Bingham fluid model, i.e., $\kappa = 0$, we have chosen the mixed formulation from subsection 3.1.4 and used conforming P1 elements for the primal variable and elementwise constants for the dual variables. The resulting algebraic systems have been solved by the augmented Lagrangian algorithm as described in section 3. In both cases, the boundary value problem (4.7a),(4.7b) has been discretized by conforming P1 elements, and the resulting algebraic system has been solved by the preconditioned conjugate gradient method.

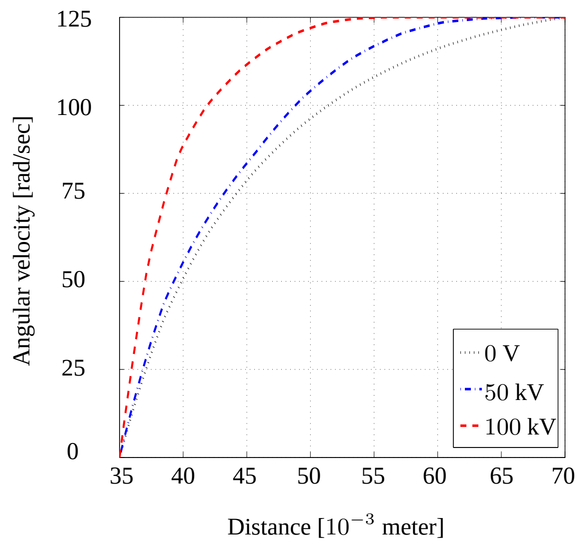

FIGURE 4.6. Wide-gap configuration: angular velocity profiles (revolving outer cylinder); from HOPPE, LITVINOV and RAHMAN [2005]

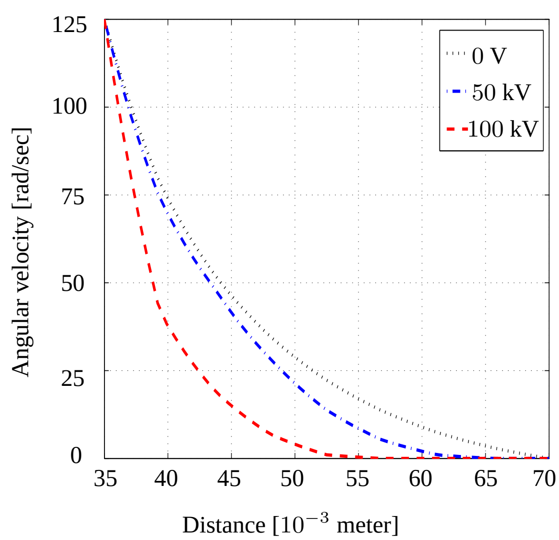

FIGURE 4.7. Wide-gap configuration: angular velocity profiles (revolving inner cylinder); from HOPPE, LITVINOV and RAHMAN [2005]

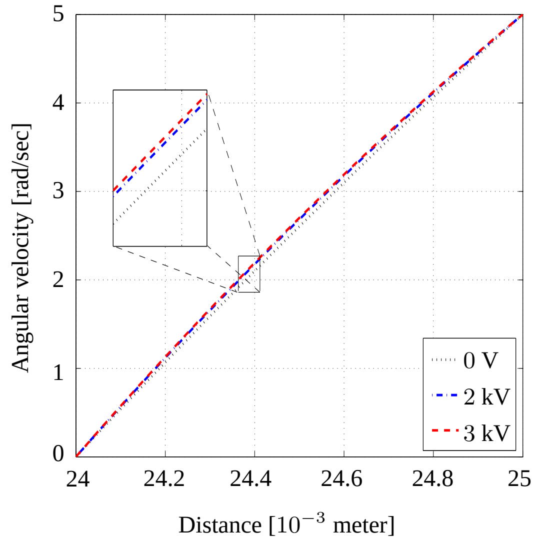

FIGURE 4.8. Narrow-gap configuration: angular velocity profiles (rotating outer cylinder); from HOPPE, LITVINOV and RAHMAN [2005]

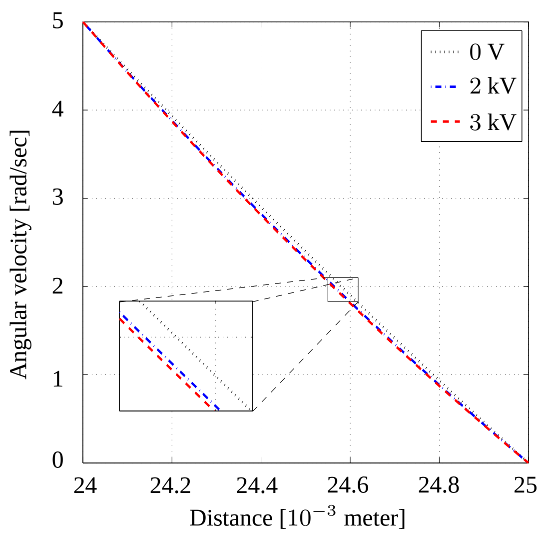

FIGURE 4.9. Narrow-gap configuration: angular velocity profiles (rotating inner cylinder); from HOPPE, LITVINOV and RAHMAN [2005]

The computations have been performed for the commercially available polyurethane-based electrorheological fluid Rheobay TP AI 3565 (cf. BAYER [1997a]). Using experimental measurements for various electric field strengths, the viscosity function φ has been specified by cubic spline approximations of the $\tau(\gamma)$-flow curves (cf. section 2).
We have considered two different geometrical configurations of the rheometer, namely a

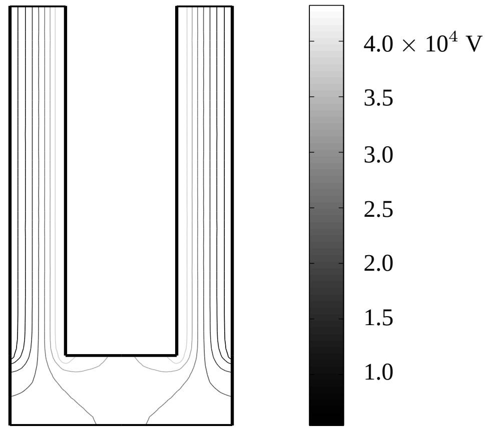

FIGURE 4.10. Isolines of the electric potential (wide-gap configuration)

wide-gap configuration with the specifications

Wide-gap: $r_i = 35$ mm , $r_e = 70$ mm , $l_i = 250$ mm , $l_e = 300$ mm ,
$\omega = 125$ rad/s , $U = 0, 2, 3$ kV

and a narrow-gap configuration with

Narrow-gap: $r_i = 24$ mm , $r_e = 25$ mm , $l_i = 25$ mm , $l_e = 30$ mm ,
$\omega = 5$ rad/s , $U = 0, 50, 100$ kV .

The following results have been obtained based on the regularized viscosity function φ with $\kappa = 10^{-11}$ (for related results based on the extended Bingham fluid model, i.e., $\kappa = 0$ we refer to ENGELMANN et al. [2000]).
Figures 4.6 and 4.7 display the angular velocity profiles for the wide-gap configuration with revolving outer cylinder (Figure 4.6) and revolving inner cylinder (Figure 4.7) at applied voltages of $U = 0\ V$, $U = 50\ kV$, and $U = 100\ kV$, respectively. In both cases a zone with a constant angular velocity occurs close to the outer cylinder which increases for increasing voltage. This is the typical velocity profile for electrorheological Couette-type flows.
On the other hand, Figures 4.8 and 4.9 show the angular velocity profiles for the narrow-gap configuration with revolving outer cylinder (Figure 4.8) and revolving inner cylinder (Figure 4.9) at applied voltages of $U = 0\ V$, $U = 2\ kV$, and $U = 3\ kV$. We observe that in both cases there is no zone with a constant angular velocity. Indeed, independent of the applied voltage, the velocity profile is almost linear.

Finally, Figure 4.10 contains the isolines of the electric potential ψ with respect to the wide-gap configuration. In fact, for both the wide-gap and the narrow-gap configuration the electric field $E = (E_r, E_z)^T$ in the gap between the inner and outer cylinder is close to the constant vector $(U/(r_i - r_e), 0)^T$ and thus perpendicular to the velocity. The electric field decays rapidly with increasing distance to the electrodes.

4.2. **Electrorheological shock absorbers.** Due to their fast response to outer electrical fields, electrorheological fluids are much better suited for automotive shock absorbers than conventional oils. In fact, electrorheological shock absorbers feature a much wider

characteristics than conventional ones and thus allow for an ideal adaptation to different road conditions and driving styles (cf., e.g., BAYER [1997b, 1998], BÖSE, HOPPE and MAZURKEVICH [2001], FILISKO [1995], GAVIN et al. [1996a,b], HOPPE, LITVINOV and RAHMAN [2003, 2007], HOPPE et al. [2000]).
Figure 4.11 (left) displays the longitudinal section of an electrorheological shock absorber. The absorber consists of two chambers filled with an electrorheological fluid, a piston featuring two transfer ducts that connect the chambers, and a third gas-filled chamber separated from the others by a floating piston. The inner walls of the transfer ducts act as electrodes and counter electrodes, respectively. They are connected with an outer electric circuit by a high voltage lead within the piston rod. We distinguish between the compression mode and the rebound mode. In the compression mode, the piston moves down and the fluid passes from the lower chamber through the ducts into the upper chamber, whereas in the rebound mode the piston moves up and the fluid flow is in the opposite direction. The variation of the applied voltage almost instantaneously changes the viscosity of the fluid and thus allows to control the damper characteristics.

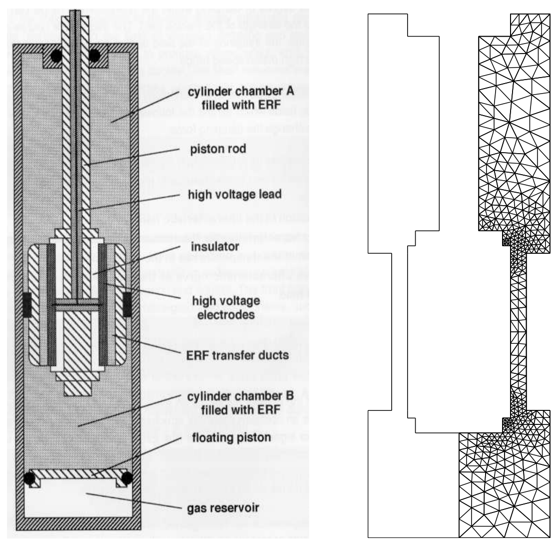

FIGURE 4.11. Schematic representation of an electrorheological shock absorber (left) and simplicial triangulation of the computational domain (right)

The fluid flow is assumed to be axially symmetric so that the computational domain can be restricted to the right half of the fluid chamber and displayed in cylindrical coordinates r, z. Figure 4.12 illustrates the computational domain in the situation where the piston is at an upper position (left) and at a lower position (right). Due to the displacement $a(t)$ of the piston, the computational domain changes in time and will thus be denoted by $\Omega_{a(t)}$. If the piston is displaced by $a(t) = l_1(t) - l_1(0)$, the floating piston is displaced from its initial position by $b(t) = a(t)(R_1/R)^2$, where R and R_1 are the radii of the floating piston and the piston rod. For a proper specification of the boundary conditions, we refer to $\Gamma_{a(t)} = \partial\Omega_{a(t)}$ as the boundary of the right half of the fluid chamber. In particular, $\Gamma^{(p)}_{a(t)}$ and $\Gamma^{(f)}_{a(t)}$ stand for the boundary of the piston and the upper boundary of the floating piston. We further denote by $\Gamma^{(e)}_{a(t)}$ and $\Gamma^{(c)}_{a(t)}$ the inner wall (CD in Figure 4.12) and the

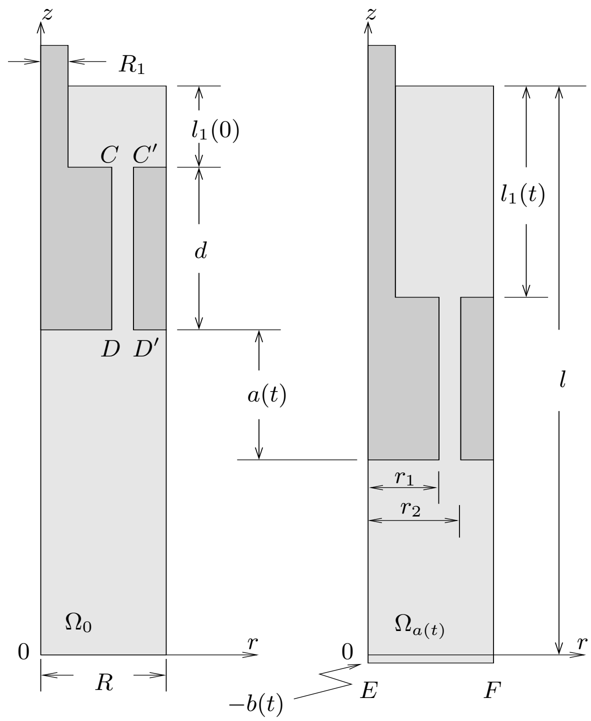

FIGURE 4.12. Domain of flow of the electrorheological fluid at time instants $t = 0$ (left) and $t > 0$ (right).

outer wall (C'D' in Figure 4.12) of the transfer duct which serve as the electrode and counter electrode, respectively. Finally, $\Gamma^{(\ell)}_{a(t)} := \{(r,z) \in \overline{\Omega}_{a(t)} \mid r = 0\}$ stands for the left boundary of the computational domain which coincides with the symmetry axis. We set $Q := \Omega_{a(t)} \times (0,T), \Sigma_{a(t)} := \Gamma_{a(t)} \times (0,T)$ and use analogous notations for the other space-time domains involving the specific parts of the boundary of the computational domain.

Taking advantage of the axial symmetry, the velocity u is given by

$$u(r,z) \; = \; u_1(r,z)e_r \; + \; u_2(r,z)e_z \; ,$$

which gives rise to the following components of the strain tensor

$$\begin{aligned}
&\varepsilon_{11}(u) \; = \; \frac{\partial u_1}{\partial r} \quad , \quad \varepsilon_{22}(u) \; = \; \frac{u_1}{r} \quad , \quad \varepsilon_{33}(u) \; = \; \frac{\partial u_2}{\partial z} \; , \\
&\varepsilon_{13}(u) \; = \; \varepsilon_{31}(u) \; = \; \frac{1}{2}(\frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial z}) \; , \\
&\varepsilon_{12}(u) \; = \; \varepsilon_{21}(u) \; = \; \varepsilon_{23}(u) \; = \; \varepsilon_{32}(u) \; = \; 0 \; .
\end{aligned}$$

The second invariant of the rate of strain tensor turns out to be

$$I(u) \; = \; (\frac{\partial u_1}{\partial r})^2 \; + \; (\frac{u_1}{r})^2 \; + \; (\frac{\partial u_2}{\partial z})^2 \; + \; \frac{1}{2}(\frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial r})^2 \; .$$

Denoting by ρ the density of the fluid, by φ the viscosity function according to (2.19), and by $f = (f_1, f_2)^T$ the volume force with the radial and axial components f_1 and f_2, the equations of motion take the form

$$\rho\Big(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial r} + u_2\frac{\partial u_1}{\partial z}\Big) + \frac{\partial p}{\partial r} - \tag{4.8a}$$
$$- 2\frac{\partial}{\partial r}(\varphi\varepsilon_{11}(u)) - 2\frac{\partial}{\partial z}(\varphi\varepsilon_{13}(u)) - \frac{2}{r}\varphi(\varepsilon_{11}(u) - \varepsilon_{22}(u)) \;=\; f_1 \quad \text{in } Q\ ,$$
$$\rho\Big(\frac{\partial u_2}{\partial t} + u_1\frac{\partial u_2}{\partial r} + u_2\frac{\partial u_2}{\partial z}\Big) + \frac{\partial p}{\partial r} - \tag{4.8b}$$
$$- 2\frac{\partial}{\partial r}(\varphi\varepsilon_{13}(u)) - 2\frac{\partial}{\partial z}(\varphi\varepsilon_{33}(u)) - \frac{2}{r}\varphi\varepsilon_{13}(u) \;=\; f_2 \quad \text{in } Q\ ,$$
$$\nabla\cdot u \;=\; \frac{\partial u_1}{\partial r} + \frac{\partial u_2}{\partial z} + \frac{u_1}{r} \;=\; 0 \quad \text{in } Q\ . \tag{4.8c}$$

Moreover, referring to $v^{(p)}$ as the piston velocity and to $u^{(0)}$ as some given initial velocity, the boundary conditions and the initial condition are given by

$$u_1 \;=\; 0 \qquad \text{on } \Sigma_{a(t)}\ , \tag{4.9a}$$
$$u_2 \;=\; v^{(p)} \quad \text{on } \Sigma_{a(t)}^{(p)}\ , \tag{4.9b}$$
$$u_2 \;=\; v^{(p)}(R_1/R)^2 \quad \text{on } \Sigma_{a(t)}^{(f)}\ , \tag{4.9c}$$
$$u_2 \;=\; 0 \qquad \text{on } \Sigma_{a(t)} \setminus (\overline{\Sigma}_{a(t)}^{(f)} \cup \overline{\Sigma}_{a(t)}^{(\ell)} \cup \overline{\Sigma}_{a(t)}^{(p)})\ , \tag{4.9d}$$
$$\frac{\partial u_2}{\partial r} \;=\; 0 \qquad \text{on } \Sigma_{a(t)}^{(\ell)}\ , \tag{4.9e}$$
$$u(\cdot,0) \;=\; u^{(0)} \quad \text{in } \Omega_{a(t)}\ . \tag{4.9f}$$

The motion of the piston satisfies the initial-value problem

$$m\,\frac{dv^{(p)}}{dt}(t) \;=\; g(t, v^{(p)}(t), U(t)) \quad , \quad t \in (0,T)\ , \tag{4.10a}$$
$$v^{(p)}(0) \;=\; v_0^{(p)} < 0\ , \tag{4.10b}$$

where m is the sum of the mass of the piston and the mass of the body that strikes the piston at $t = 0$, $U(t)$ stands for the applied voltage, and the drag force $g(t, v^{(p)}(t), U(t))$ is given by

$$g(t, v^{(p)}(t), U(t)) \;:=\; -\int\limits_{\Sigma_{a(t)}^{(p)}} \Big(2\varphi\varepsilon_{31}(u)\nu_r + (2\varphi\varepsilon_{33}(u) - p)\nu_z\Big)\, ds\ . \tag{4.11}$$

The electric field E has the form

$$E(r,z) \;=\; E_1(r,z)e_r \;+\; E_2(r,z)e_z\ .$$

As in the previous example (cf. subsection 4.1), it can be computed by means of an electric potential $\psi(t)$ which at each time instant $t \in [0,T]$ satisfies the following elliptic boundary value problem

$$\nabla \cdot (\epsilon \nabla \psi(t)) = 0 \quad \text{in } \Omega_{a(t)} , \tag{4.12a}$$

$$\psi(t) = U(t) \quad \text{on } \Gamma^{(e)}_{a(t)} , \tag{4.12b}$$

$$\psi(t) = 0 \quad \text{on } \Gamma^{(c)}_{a(t)} , \tag{4.12c}$$

$$\frac{\partial \psi}{\partial r}(t) = 0 \quad \text{on } \Gamma^{(\ell)}_{a(t)} , \tag{4.12d}$$

$$\nu_r \epsilon \frac{\partial \psi}{\partial r}(t) + \nu_z \epsilon \frac{\partial \psi}{\partial z}(t) = 0 \quad \text{elsewhere} . \tag{4.12e}$$

For the numerical simulation of the operational behavior of the electrorheological shock absorber we have used a discretization in time with respect to a uniform partition of the time interval $[0, T]$ of step size $k := T/M, M \in \mathbb{N}$, using the explicit Euler scheme for the equation of motion (4.10) of the piston and the backward Euler scheme for the equations of motion (4.8a)-(4.8c) of the fluid with $\rho = 0$. Knowing the computation domain at time level $t_m, 0 \leq m \leq M-1$, the discretization in space has been taken care of by P_2/P_1 Taylor-Hood elements for the fluid variables and conforming P_1 elements for the electric potential with respect to a simplicial triangulation of $\Omega_{a(t_m)}$. The discretized fluid equations have been solved by the augmented Lagrangian algorithm as described in subsection 3.1, whereas the preconditioned conjugate gradient method has been used for the discretized potential equation. For details we refer to HOPPE, LITVINOV and RAHMAN [2007].

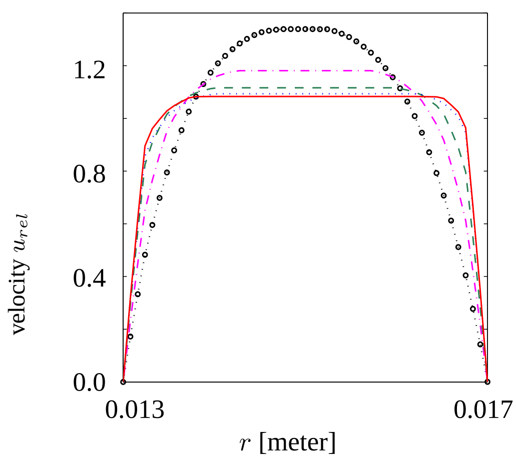

FIGURE 4.13. Profiles of the relative velocity of the fluid in the piston duct for various applied voltages: $U = 0$ Volt (dotted-circled line), 1 kV (dashed-dotted line), 3 kV (dashed line), 6 kV (dotted line) and 9 kV (solid line).

The simulations have been based on the commercial electrorheological fluid Rheobay TP AI 3565 (see BAYER [1997a]) by computing the viscosity function φ using experimentally available $\tau(\gamma)$-flow curves (cf. subsection 4.1). As far as the geometry of the

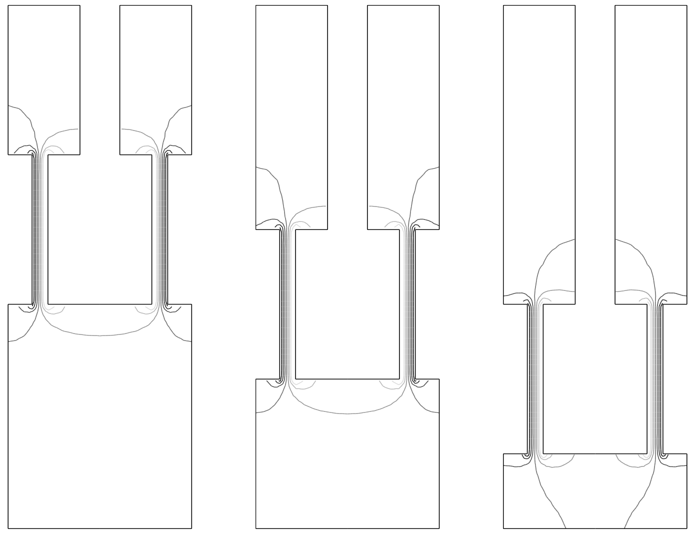

FIGURE 4.14. Isolines of the electric potential at three different piston positions

shock absorber is concerned, we have used the following data (cf. Figure 4.12):

$$R \,:=\, 0.023\, m \quad , \quad R_1 \,:=\, 0.005\, m \quad , \quad r_1 \,:=\, 0.013\, m \quad , \quad r_2 \,:=\, 0.017\, m \,,$$
$$l \,:=\, 0.14\, m \quad , \quad l_1(0) \,:=\, 0.02\, m \quad , \quad d \,:=\, 0.04\, m \,.$$

Figure 4.13 shows the relative velocity of the fluid $u_{rel} = (u - v)/\gamma$ in the piston duct for various electric field strengths, where $\gamma = (\int_{r_1}^{r_2} r dr)^{-1} \int_{r_1}^{r_2} r(u-v)(r, z_1) dr$ is the flow rate relative to the electrodes. In case of a vanishing electric field, we clearly observe a parabolic flow profile typical for flows of Newtonian fluids between two parallel plates. For increasing electric field strength the profile flattens in the center of the duct with an increasing zone of constant relative velocity. This is the typical flow pattern of electrorheological fluids.

Figure 4.14 displays the isolines of the electric potential ψ for various positions of the piston assuming an applied voltage of $U = 9\ kV$. Again, we see that the electric field is essentially concentrated within the transfer ducts in the direction of the r-axis and rapidly decays off the ducts.

Figures 4.15 and 4.16 contain visualizations of the velocity vector u at various stages of the compression mode (Figure 4.15) and the rebound mode (Figure 4.16). As has to be expected, in the transfer ducts the direction of the velocity vector essentially coincides with the direction of the z-axis and is thus orthogonal to the electric field E.
We note that the pressure in the gas reservoir should be sufficiently large, since otherwise the fluid chamber can not be fully filled with the fluid and cavitation may occur. For further details concerning the simulation results we refer to HOPPE, LITVINOV and RAHMAN [2007].

4.3. **Shape optimization of electrorheological devices.** An important issue in the design of electrorheological shock absorbers is to find a suitable geometry of the inflow and outflow boundaries of the piston ducts such that both in the compression mode and in the rebound mode pressure peaks are avoided which may cause inappropriate damping profiles. This amounts to the solution of a shape optimization problem which for simplicity

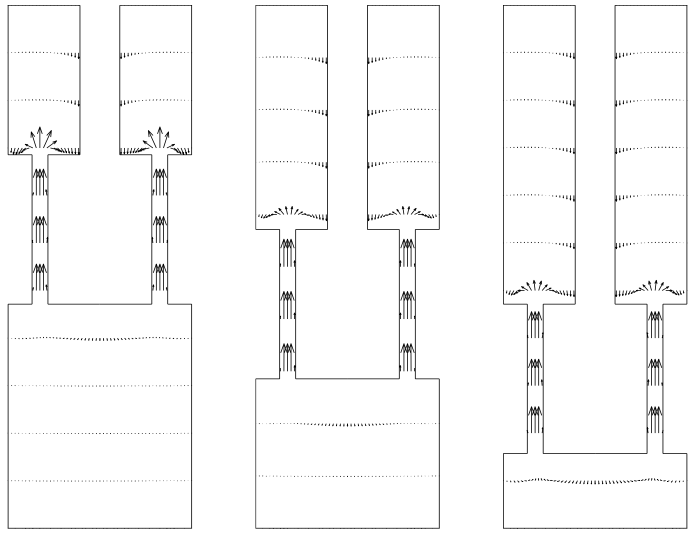

FIGURE 4.15. Velocity vectors during compression

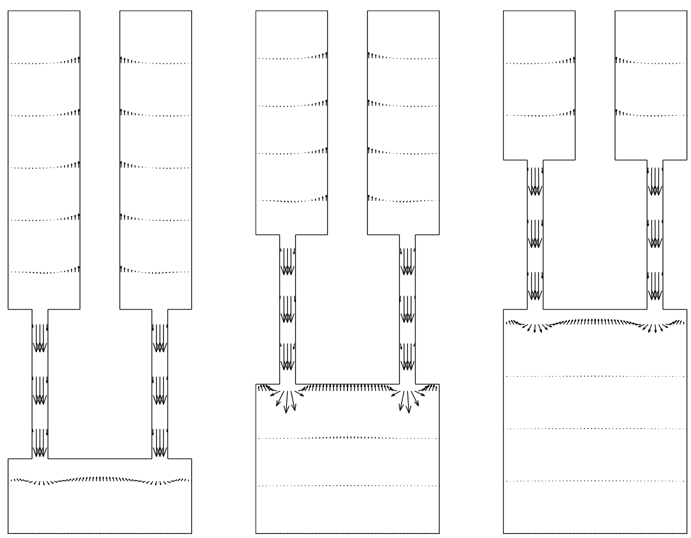

FIGURE 4.16. Velocity vectors during rebound

will be stated as a velocity and pressure tracking problem where the objective functional is given by

$$\text{(4.13)} \qquad \text{minimize} \quad J(u,p,d) \;:=\; \frac{\alpha_1}{2}\|u-u^d\|^2_{0,\Omega(d)} \;+\; \frac{\alpha_2}{2}\|p-p^d\|^2_{0,\Omega(d)}\;.$$

Here, $u^d \in H(\text{div}^0;\Omega(d))$ and $p^d \in L^2(\Omega(d))$ stand for a desired velocity profile and pressure distribution, respectively, $\alpha_i \in (0,1], 1 \le i \le 2$, and $\Omega(d)$ is the domain occupied by the fluid which depends on the design variables $d = (d_1,\cdots,d_m)^T \in \mathbb{R}^m$. The design variables are chosen as the Bézier control points of a Bézier curve representation (cf. FARIN [2002]) of the inlet and outlet boundaries (cf. Figure 4.17 (left)).

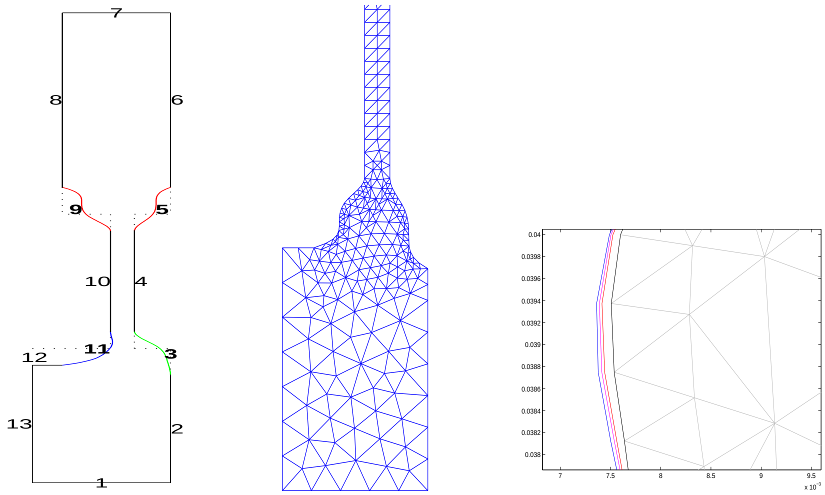

FIGURE 4.17. Bézier curve representation of the inlet and outlet boundaries of a piston duct (left), optimized outlet boundary (middle) and details of the optimal design for various electric field strengths (right)

The PDE constraints are given by

$$- \nabla \cdot \sigma(u) = f \quad \text{in } \Omega(d) \,, \tag{4.14a}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega(d) \,, \tag{4.14b}$$

along with appropriate boundary conditions (cf. subsection 4.2). The constitutive law is assumed to be given by

$$\sigma = -pI + 2\,\varphi(I(u), |E|, \mu(u,E))\,\varepsilon(u) \tag{4.15}$$

with a regularized viscosity function φ of the form (2.19), where the electric field E is computed via the gradient of an electric potential satisfying an elliptic boundary value problem (cf. (4.12a)-(4.12e)). We further assume bilateral constraints on the design variables according to

$$d \in K := \{ d \in \mathbb{R}^m \mid d_i^{min} \le d_i \le d_i^{max} \,,\, 1 \le i \le m \} \,. \tag{4.16}$$

Choosing $X \subset H^1(\Omega(d))^2$ and $Q := L_0^2(\Omega(d))$, we refer to $Y := X \times Q$ as the state space and denote by $S(\cdot, d), d \in K$, the nonlinear Stokes operator associated with (4.14a),(4.14b). Then, the state equations can be written in operator form according to

$$S(y,d) = g \,. \tag{4.17}$$

where $y := (u.p)^T$ and $g := (f,0)^T$. We choose $\hat{d} \in K$ as a reference design and refer to $\hat{\Omega} := \Omega(\hat{d})$ as the associated reference domain. Then, the actual domain $\Omega(d)$ can be obtained from the reference domain $\hat{\Omega}$ by means of an isomorphism

$$\begin{aligned} \Omega(d) &= \Phi(\hat{\Omega}; d) \,, \\ \Phi(\hat{x}; d) &= (\Phi_1(\hat{x}; d), \Phi_2(\hat{x}; d))^T \,,\ \hat{x} = (\hat{x}_1, \hat{x}_2)^T \end{aligned} \tag{4.18}$$

The advantage of using the reference domain $\hat{\Omega}$ is that finite element approximations of (4.17) can be performed with respect to that fixed domain without being forced to remesh for each update of the design variables.
We denote by $(\mathcal{T}_h(\hat{\Omega}))_{\mathbb{N}}$ a shape regular family of simplicial triangulations of $\hat{\Omega}$. By means of (4.18), these triangulations induce an associated family $(\mathcal{T}_h(\Omega(d)))_{\mathbb{N}}$ of simplicial triangulations of the actual physical domains $\Omega(d)$.
We use Taylor-Hood $P2/P1$ elements for the discretization of the velocity $u \in X$ and the pressure $p \in Q$ denoting the associated trial spaces by X_h and Q_h with dim $X_h = n_1$ and dim $Q_h = n_2$, respectively. This gives rise to an objective functional $J_h : \mathbb{R}^n \times \mathbb{R}^m, n := n_1 + n_2$, by means of

$$J_h(u_h, p_h, d) \;:=\; \frac{\alpha_1}{2}\,(u_h - u_h^d)^T I_{1,h}(d)(u_h - u_h^d) \;+\; \frac{\alpha_2}{2}\,p_h^T I_{2,h}(d) p_h \;, \tag{4.19}$$

where $I_{\nu,h}(d), 1 \le \nu \le 2$, are the associated mass matrices and $u_h^d \in \mathbb{R}^{n_1}, p_h^d \in \mathbb{R}^{n_2}$ result from the L^2-projections of u^d, p^d onto $X_h \cap H(div^0;\Omega)$ and Q_h, respectively. The discretized shape optimization problem can be stated as

$$\inf_{u_h, p_h, d} J_h(u_h, p_h, d) \tag{4.20}$$

subject to the discrete nonlinear Stokes system

$$S_h(y_h, d) \;=\; g_h \;. \tag{4.21}$$

and the constraints

$$d \in K \;. \tag{4.22}$$

For notational convenience, in the sequel we will drop the discretization index h.
Due to the dependence of the domain on the design parameters $d_i, 1 \le i \le m$, the objective functional is nonconvex. Therefore, there may exist a multitude of local minima. Throughout the following, we assume that $(y^*, d^*) \in \mathbb{R}^n \times K$ is a strict local solution of (4.20)-(4.22).
We solve the discrete minimization problem by an adaptive path-following primal-dual interior-point method. To this end, we couple the inequality constraints (4.22) by logarithmic barrier functions with a barrier parameter $\beta = 1/\mu > 0$, $\mu \to \infty$, resulting in the following parameterized family of minimization subproblems

$$\inf_{y,d} B^{(\mu)}(y, d) \tag{4.23}$$

subject to (4.21), where

$$B^{(\mu)}(y, d) \;:=\; J(y, d) \;-\; \frac{1}{\mu} \sum_{i=1}^{m} [\ln(d_i - d_i^{\min}) + \ln(d_i^{\max} - d_i)] \;. \tag{4.24}$$

The dual aspect is to couple the constraint (4.21) by a Lagrange multiplier $\lambda = (\lambda_u, \lambda_p)^T$ which leads to the saddle point problem

$$\inf_{y,d} \sup_{\lambda} L^{(\mu)}(y, \lambda, d) \;, \tag{4.25}$$

where the Lagrangian $L^{(\mu)}$ is given by

$$L^{(\mu)}(y, \lambda, d) \;=\; B^{(\mu)}(y, d) \;+\; \lambda^T (S(y, d) - g) \;. \tag{4.26}$$

The central path $\mu \longmapsto x(\mu) := (y(\mu),\lambda(\mu),d(\mu))^T$ is defined as the solution of the nonlinear system

$$F(x(\mu),\mu) \;=\; \begin{pmatrix} L_y^{(\mu)}(y,\lambda,d) \\ L_\lambda^{(\mu)}(y,\lambda,d) \\ L_d^{(\mu)}(y,\lambda,d) \end{pmatrix} \;=\; 0 \;, \tag{4.27}$$

which represents the first order necessary optimality conditions for (4.25).
For the solution of the parameter-dependent nonlinear system (4.27) we use an adaptive path-following predictor-corrector strategy along the lines of DEUFLHARD [2004].

Predictor Step: The predictor step relies on tangent continuation along the trajectory of the Davidenko equation

$$F_x(x(\mu),\mu)\; x'(\mu) \;=\; -F_\mu(x(\mu),\mu)\;. \tag{4.28}$$

Given some approximation $\tilde{x}(\mu_k)$ at $\mu_k > 0$, compute $\tilde{x}^{(0)}(\mu_{k+1})$, where $\mu_{k+1} = \mu_k + \Delta\mu_k^{(0)}$, according to

$$F_x(\tilde{x}(\mu_k),\mu_k)\;\delta x(\mu_k) = \;-\;F_\mu(\tilde{x}(\mu_k),\mu_k)\;, \tag{4.29a}$$

$$\tilde{x}^{(0)}(\mu_{k+1}) = \;\tilde{x}(\mu_k) \;+\; \Delta\mu_k^{(0)}\;\delta x(\mu_k)\;. \tag{4.29b}$$

We use $\Delta\mu_0^{(0)} = \Delta\mu_0$ for some given initial step size $\Delta\mu_0$, whereas for $k \geq 1$ the predicted step size $\Delta\mu_k^{(0)}$ is chosen by

$$\Delta\mu_k^{(0)} \;:=\; \Big(\frac{\|\Delta x^{(0)}(\mu_k)\|}{\|\tilde{x}(\mu_k) - \tilde{x}^{(0)}(\mu_k)\|}\;\frac{\sqrt{2}-1}{2\Theta(\mu_k)}\Big)^{1/2}\;\Delta\mu_{k-1}\;, \tag{4.30}$$

where $\Delta\mu_{k-1}$ is the computed continuation step size, $\Delta x^{(0)}(\mu_k)$ is the first Newton correction (see below), and $\Theta(\mu_k) < 1$ is the contraction factor associated with a successful previous continuation step.

Corrector step: As a corrector, we use Newton's method applied to $F(x(\mu_{k+1}),\mu_{k+1}) = 0$ with $\tilde{x}^{(0)}(\mu_{k+1})$ as a start vector. In particular, for $\ell \geq 0$ and $j_\ell \geq 0$ we compute $\Delta x^{(j_\ell)}(\mu_{k+1})$ according to

$$F'(\tilde{x}^{(j_\ell)}(\mu_{k+1}),\mu_{k+1})\;\Delta x^{(j_\ell)}(\mu_{k+1}) \;=\; -\;F(\tilde{x}^{(j_\ell)}(\mu_{k+1}),\mu_{k+1}) \tag{4.31}$$

and $\overline{\Delta x}^{(j_\ell)}(\mu_{k+1})$ as the associated simplified Newton correction

$$F'(\tilde{x}^{(j_\ell)}(\mu_{k+1}),\mu_{k+1})\;\overline{\Delta x}^{(j_\ell)}(\mu_{k+1}) \;=\; -\;F(\tilde{x}^{(j_\ell)}(\mu_{k+1}) + \Delta x^{(j_\ell)}(\mu_{k+1}),\mu_{k+1})\;. \tag{4.32}$$

We monitor convergence of Newton's method by means of

$$\Theta^{(j_\ell)}(\mu_{k+1}) \;:=\; \|\overline{\Delta x}^{(j_\ell)}(\mu_{k+1})\|/\|\Delta x^{(j_\ell)}(\mu_{k+1})\|\;.$$

In case of successful convergence, we accept the current step size and proceed with the next continuation step. However, if the monotonicity test

$$\Theta^{(j_\ell)}(\mu_{k+1}) \;<\; 1 \tag{4.33}$$

fails for some $j_\ell \geq 0$, the continuation step has to be repeated with the reduced step size

$$\Delta\mu_k^{(\ell+1)} \;:=\; \Big(\frac{\sqrt{2}-1}{g(\Theta^{(j_\ell)})}\Big)^{1/2}\;\Delta\mu_k^{(\ell)} \quad , \quad g(\Theta) := \sqrt{\Theta+1}-1 \tag{4.34}$$

until we either achieve convergence or for some prespecified lower bound $\Delta\mu_{min}$ observe

$$\Delta\mu_k^{(\ell+1)} \; < \; \Delta\mu_{min} \; .$$

In the latter case, we stop the algorithm and report convergence failure.

Actually, we perform the correction step by an inexact Newton method featuring right-transforming iterations. The derivatives have been computed by automatic differentiation. For details we refer to ANTIL et al. [2007], HOPPE, PETROVA and SCHULZ [2002], HOPPE and PETROVA [2004], HOPPE, LINSENMANN and PETROVA [2006], WITTUM [1989].

Figure 4.17 (middle) shows the optimized design of the outlet boundary of a piston duct in the rebound stage (cf. subsection 4.2) and details of the optimized outlet boundary for various electric field strengths (the lines show the different designs for increasing electric field strengths from right to left). Although the designs do not differ that much, the specification of a best compromise is the subject of a further optimization routine.

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