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On the type and generators of monomial curves

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Abstract: Let $n_1, n_2, \ldots, n_d$ be positive integers and H be the numerical semigroup generated by $n_1, n_2, \ldots, n_d$. Let $A := k[H] := k[t^{n_1}, t^{n_2}, \ldots, t^{n_d}] \cong k[x_1, x_2, \ldots, x_d]/I$ be the numerical semigroup ring of H over k. In this paper we give a condition $(*)$ that implies that the minimal number of generators of the defining ideal I is bounded explicitly by its type. As a consequence for semigroups with $d = 4$ satisfying the condition $(*)$ we have $\mu(\mathrm{in}(I)) \leq 2(t(H)) + 1$.

Key words: Frobenius number, pseudo-Frobenius number, almost Gorenstein ring, semigroup rings, monomial curve

1. Introduction

Let $n_1 < n_2 < \ldots < n_d$ be positive integers such that $\gcd(n_1, n_2, \ldots, n_d) = 1$ and $H = \langle n_1, n_2, \ldots, n_d \rangle = \{\sum_{i=1}^{d} c_i n_i \mid c_i \in \mathbb{N} \text{ for all } 1 \leqslant i \leqslant n\}$ be the numerical semigroup minimally generated by $n_1, n_2, \ldots, n_d$, where $\mathbb{N}$ stands for the set of nonnegative integers. Let k be a field, $A := k[H] := k[t^{n_1}, t^{n_2}, \ldots, t^{n_d}]$ the numerical semigroup ring of H and $R := k[x_1, x_2, \ldots, x_d]$, the polynomial ring in d variables over k. We can regard R and A as graded rings by $R_0 = A_0 = k, \deg t = 1$ and $\deg x_i = n_i$ for all $1 \leqslant i \leqslant d$. If we set $I := I(H)$ the kernel of the graded ring homomorphism $\Phi : R \to A$ defined by $\Phi(x_i) = t^{n_i}$ for each $1 \leqslant i \leqslant d$, then the ring $A \subset k[t]$ has a presentation as a quotient R/I and I is called the *defining ideal* of H. Now let consider the *affine monomial curve* C in the affine d-space $\mathbb{A}^d(k)$ defined parametrically by

$$x_1 = t^{n_1}, \ldots, x_d = t^{n_d}.$$

The vanishing ideal $I(C)$ of C is the kernel of the k-algebra homomorphism $\Phi : R \to k[t]$ defined by $\Phi(x_i) = t^{n_i}$ for all $1 \leqslant i \leqslant d$. If k is infinite we have $I(C) = I(H)$, the defining ideal of the corresponding semigroup. In this paper we work from the algebraic point of view. From now on we do not need any hypothesis on the field k.

One of the important problems in commutative algebra is finding the minimal system of generators $\mu(I)$ and the minimal free resolution of I. When the embedding dimension $d = 3$, Herzog [11] proved that $\mu(I) \leqslant 3$. When $d = 4$, the problem is rather wild and there are some results for special cases. Namely, if H is symmetric, then Bresinsky [6] gave a complete description of the defining ideal I and he has proved that $\mu(I) \leqslant 5$. Komeda in [12] was the first person to give 5 binomials generating the toric ideal of a pseudo-symmetric semigroup.

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The full resolution is given by Barucci et al. in [5]. When $d = 4$ and $n_1 = 5$, the complete list of numerical semigroups $H = \langle 5, b, c, d\rangle$ which are almost symmetric is given by Nari et al. [16]. In that paper, the authors proved that $\mu(I) = 5$ if H is pseudo-symmetric and $\mu(I) = 6$ if H is almost symmetric with $t(H) = 3$. Moscariello in [15] proved that if H is pseudo-symmetric then its type is at most 3. More recently, Eto in [9] classified almost symmetric semigroups and gave minimal generators as well as free resolutions for the toric ideal.

The purpose of the present paper is to show the relation between the reduced Gröbner basis of the defining ideal I and the minimal number of generators of I. To state the main result, we need additional definitions and notations. Recall that for two natural numbers $d < r$, the *cyclic polytope* $C_d(r)$ is the convex hull of any r distinct points on the moment curve $t \mapsto (t, t^2, \ldots, t^d)$. The number of $i-$dimensional faces of $C_d(r)$ is denoted by $C_{i,d,r}$ (see [13]). Throughout this paper, we use the degree revlex lexicographical order $\prec_{degrevlex}$ on the monomials of the ring R with $x_1 \prec_{degrevlex} \cdots \prec_{degrevlex} x_d$ and $\deg x_i = n_i$ for all $1 \leqslant i \leqslant d$. Let $G(H)$ be the reduced Gröbner basis of the ideal I. We say that the semigroup H (or the ring $k[H]$) satisfies the *condition (*)* if for every binomial $M_i - N_i \in G(H)$, with $N_i \prec_{degrevlex} M_i$, the variable x_1 divides the monomial N_i for all i. With this notation, the main result is stated as follows.

Theorem 1.1 *If the semigroup H satisfies the condition (*) then we have $\mu(\mathrm{in}(I)) \leq C_{d-2,d-1,(t(H)+d-1)} - 1$. Moreover, $G(H)$ is a minimal set of generators of I.*

This paper is divided into 3 sections. In the next section, we recall some results about corner elements. In section 3, by using the condition $(*)$, we will prove that the minimal number of generators of the defining ideal I is bounded explicitly by its type (Theorem 1.1). As a consequence for semigroups satisfying the condition $(*)$ and the ring A being almost Gorenstein, we have $\mu(I) \leq 7$ (Corollary 3.7).

2. Corner elements of monomial ideals

Let $\mathfrak{m} := (x_1, \ldots, x_d)$ be the maximal ideal of R, and $J \subset R$ be a monomial ideal. We denote by $[[R]]$ the set of all monomials of R and $\mu(J)$ the number of minimal generators of J. For any vector $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{N}^d$, set $\mathbf{a} + \mathbf{1} = (a_1 + 1, \ldots, a_d + 1) \in \mathbb{N}^d$, $\mathfrak{m}^{\mathbf{a}} := (x_1^{a_1}, \ldots, x_d^{a_d})$ and $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \ldots x_n^{a_d}$. Now we need some results from the book still not published and cited here as [MRS]: *Monomial ideals and their decompositions* by W. Frank Moore, Mark Rogers, Sean Sather-Wagstaff.

Definition 2.1 A monomial $z \in [[R]]$ is a *J-corner element* if $z \notin J$ but $x_1 z, \ldots, x_d z \in J$. The set of corner elements of J in $[[R]]$ is denoted by $C_R(J)$.

Fact 2.2 (i) It is clear that the J-corner elements are precisely the monomials in $(J :_R \mathfrak{m}) \setminus J$, or, in other words, $C_R(J) = [[(J :_R \mathfrak{m})]] \setminus [[J]]$.

(ii) The set $C_R(J)$ is finite.

(iii) If $\mathrm{rad}(J) = \mathfrak{m}$, it is well known that $t(R/J) = \mathrm{Card}(C_R(J))$ is the type of the ring R/J.

The following theorem gives us some methods for computing m-irreducible decompositions in general (see [MRS], Theorem 6.3.5, Theorem 7.5.3, and Theorem 7.5.5).

Theorem 2.3 *(i) Let $J \subset R$ be a monomial ideal such that* $\mathrm{rad}(J) = \mathfrak{m}$*. If the set of corner elements is given by* $C_R(J) = \{\mathbf{x}^{\mathbf{b_j}} \mid \mathbf{b_j} \in \mathbb{N}^d, j = 1, \ldots, t(R/J)\}$ *then* $J = \cap_{j=1}^{t(R/J)} \mathfrak{m}^{\mathbf{b_j}+\mathbf{1}}$ *is the unique irredundant irreducible decomposition of J.*

(ii) Assume that $\mathrm{rad}(J) \neq \mathfrak{m}$ *and* $J = (\mathbf{x}^{\mathbf{b_j}} \mid \mathbf{b_j} \in \mathbb{N}^d, j = 1, \ldots, \mu(J))R$*. Let m be an integer strictly bigger than any of the coordinates of the vectors $\mathbf{b_j}$. Set* $J' := J + (x_1^m, \ldots, x_d^m)R$ *and* $C_R(J') = \{\mathbf{x}^{\mathbf{c_j}} \mid \mathbf{c_j} \in \mathbb{N}^d, j = 1, \ldots, t(R/J')\}$*. Then* $J' = \cap_{j=1}^{t(R/J')} \mathfrak{m}^{\mathbf{c_j}+\mathbf{1}}$ *is the unique irredundant irreducible decomposition of J' and* $J = \cap_{j=1}^{t(R/J')} \widetilde{\mathfrak{m}^{\mathbf{c_j}+\mathbf{1}}}$ *is the unique irredundant irreducible decomposition of J, where* $\widetilde{\mathfrak{m}^{\mathbf{c_j}+\mathbf{1}}}$ *is obtained from* $\mathfrak{m}^{\mathbf{c_j}+\mathbf{1}}$ *by deleting all monomials of the type* $x_1^m, \ldots, x_d^m$ *from its generators.*

Example 2.4 (i) Let $R = k[x, y]$ be a polynomial ring of 2 variables over k and monomial ideal $J = (x^6, x^5y^2, x^2y^4, y^6)R$. We can find the set of corner elements $C_R(J) = \{xy^5, x^4y^3, x^5y\}$. Therefore, by Theorem 2.3, (i) the irredundant irreducible decomposition of J is

$$J = \cap_{j=1}^{t(R/J)} \mathfrak{m}^{\mathbf{b_j}+\mathbf{1}} = (x^2, y^6)R \cap (x^5, y^4)R \cap (x^6, y^2)R.$$

(ii) Let $R = k[x, y, z]$ be a polynomial ring of 3 variables over k and monomial ideal $J = (xy, xz, yz)R$. By Theorem 2.3, (ii) we may set $m = 2$ and

$$J' := (xy, xz, yz)R + (x^2, y^2, z^2)R = (xy, xz, yz, x^2, y^2, z^2)R = (x, y, z)^2R.$$

Since the set of corner elements is $C_R(J) = \{x, y, z\}$, we have that

$$J' = \cap_{j=1}^{t(R/J')} \mathfrak{m}^{\mathbf{c_j}+\mathbf{1}} = (x^2, y, z)R \cap (x, y^2, z)R \cap (x, y, z^2)R.$$

By removing x^2, y^2, z^2 from these ideals, we get the irredundant irreducible decomposition of J

$$J = (y, z)R \cap (x, z)R \cap (x, y)R.$$

Definition 2.5 Given two vectors $\mathbf{a} = (a_1, \ldots, a_d)$ and $\mathbf{b} = (b_1, \ldots, b_d) \in \mathbb{N}^d$ with $\mathbf{b} \preceq \mathbf{a}$ (that is, $b_i \leq a_i$ for all $i = 1, \ldots, d$). Let $\mathbf{a} \setminus \mathbf{b}$ denote the vector whose $i-$th coordinate is

$$a_i \setminus b_i = \begin{cases} a_i + 1 - b_i & \text{if } b_i \geqslant 1 \\ 0 & \text{if } b_i = 0. \end{cases}$$

If J is a monomial ideal whose minimal generators all divide $\mathbf{x}^{\mathbf{a}}$, then the *Alexander dual of J with respect to* $\mathbf{a}$ *is*

$$J^{[a]} = \bigcap \{\mathfrak{m}^{\mathbf{a}\setminus\mathbf{b}} \mid \mathbf{x}^{\mathbf{b}} \text{ is a minimal generator of J}\}.$$

The basic idea of the following theorem in Miller and Sturmfels [13] is making the irreducible components into generators.

Theorem 2.6 *([13, Theorems 5.24, 5.27]) If all minimal generators of J divide $\mathbf{x}^{\mathbf{a}}$, then* *(i) All minimal generators of* $J^{[a]}$ *divide* $\mathbf{x}^{\mathbf{a}}$, *and* $(J^{[a]})^{[a]} = J$.

(ii) J *has a unique irredundant irreducible decomposition, and it is given by*

$$J = \bigcap \{\mathfrak{m}^{\mathbf{a}\setminus\mathbf{b}} \mid \mathbf{x}^{\mathbf{b}} \text{ is a minimal generator of } J^{[a]}\}.$$

Equivalently, the Alexander dual of J *with respect to* $\mathbf{a}$ *is given by minimal generators as*

$$J^{[a]} = \left\langle \mathbf{x}^{\mathbf{a}\setminus\mathbf{b}}, \text{ where } \mathfrak{m}^{\mathbf{b}} \text{ is an irreducible component of } J \right\rangle.$$

Example 2.7 Let $R = k[x, y, z]$ be a polynomial ring over k, $J = (xz^2, y)R$ be the monomial ideal of R, and $\mathbf{a} = (3, 3, 3) \in \mathbb{N}^3$. Then we have by the above relation $\mathbf{b_1} = (1, 0, 2)$, $\mathbf{b_2} = (0, 1, 0)$ and $\mathbf{a} \setminus \mathbf{b_1} = (3, 0, 2)$, $\mathbf{a} \setminus \mathbf{b_2} = (0, 3, 0)$. Therefore we have by Theorem 2.6 that

$$J^{[(3,3,3)]} = \mathfrak{m}^{(3,0,2)} \cap \mathfrak{m}^{(0,3,0)} = (x^3, z^2)R \cap (y^3)R = (x^3y^3, y^3z^2)R.$$

Now we set $c_1 = (3, 3, 0)$ and $c_2 = (0, 3, 2)$ be the vector components of x^3y^3, y^3z^2. Then we have $\mathbf{a} \setminus \mathbf{c_1} = (1, 1, 0)$, $\mathbf{a} \setminus \mathbf{c_2} = (0, 1, 2)$. By Theorem 2.6 again we have

$$(J^{[(3,3,3)]})^{[(3,3,3)]} = \mathfrak{m}^{(1,1,0)} \cap \mathfrak{m}^{(0,1,2)} = (x, y)R \cap (y, z^2)R = (xy, y, yz^2, xz^2)R = (xz^2, y)R = J.$$

3. Initial ideals

In this part we study some properties of the initial ideal $\mathrm{in}(I)$. Recall that for two natural numbers $d < r$, the *cyclic polytope* $C_d(r)$ is the convex hull of any r distinct points on the moment curve $t \mapsto (t, t^2, \ldots, t^d)$. The number of $i-$dimensional faces of $C_d(r)$ is denoted by $C_{i,d,r}$. It is well known that

$$C_{d-1,d,r} = 2\binom{r - \frac{d+1}{2}}{r - d} \text{ if } d \text{ is odd,} \tag{1}$$

$$C_{d-1,d,r} = \binom{r - \frac{d}{2}}{r - d} + \binom{r - \frac{d}{2} - 1}{r - d} \text{ if } d \text{ is even.} \tag{2}$$

Recall that any graded ideal J of a polynomial ring R has a minimal finite free resolution

$$0 \longrightarrow R^{\beta_s(J)} \longrightarrow R^{\beta_{s-1}(J)} \longrightarrow \ldots \longrightarrow R^{\beta_0(J)} \longrightarrow J \longrightarrow 0.$$

The number $\beta_i(J)$ is called the *i-Betti number*. For $i = 0$, we have $\beta_0(J) = \mu(J)$ and if R/J is Cohen–Macaulay then $s = \mathrm{ht}(J) - 1$ and $\beta_s(J)$ is the type of R/J. The following theorem is an important result given by [13, Theorem 6.29].

Theorem 3.1 *The number* $\beta_i(J)$ *of minimal* $i^{th}-$*syzygies of the monomial ideal* $J \subset R$ *minimally generated by* $\mu(J) > d$ *monomials is bounded above by* $C_{i,d,\mu(J)}$ *the number of* i*-dimensional faces of the cyclic* d*-polytope with* $\mu(J)$ *vertices. For* $i = d - 1$ *this bound is strict.*

As a consequence we will prove the following theorem.

Theorem 3.2 *Suppose that the monomial ideal* $J \subset R$ *is minimally generated by* $\mu(J)$ *monomials and* $\mathrm{rad}(J) = \mathfrak{m}$. *Then* $\mu(J) \leq C_{d-1,d,(t(R/J)+d)} - 1$. *In particular for* $d = 3$ *we have* $\mu(J) \leq 2(t(R/J)) + 1$.

Proof By Theorem 2.6 we have that $J^{[a]}$ is minimally generated by $t(R/J)$ monomials since $\mathrm{rad}(J) = \mathfrak{m}$. Let m be an integer strictly bigger than the highest power of any variables appearing in the set of generators of $J^{[a]}$; hence $(J^{[a]})' := J^{[a]} + (x_1^m, \ldots, x_d^m)R$ is minimally generated by $t(R/J) + d$ monomials.

By Theorem 2.3 (ii), the number of irreducible components of $(J^{[a]})'$ is the number of irreducible components of $J^{[a]}$ and the number of irreducible components of $J^{[a]}$ coincides with the number of generators of $(J^{[a]})^{[a]} = J$ by Theorem 2.6. Now by applying Theorem 3.1 we get that $\beta_{d-1}((J^{[a]})') = t(R/(J^{[a]})') \leq C_{d-1,d,(t(R/J)+d)} - 1$. However, $t(R/(J^{[a]})')$ equals the number of irreducible components of $(J^{[a]})'$ and by Theorem 2.3 (i) it equals the number of irreducible components of $J^{[a]}$, which is $\mu(J)$. Hence we have that $\mu(J) \leq C_{d-1,d,(t(R/J)+d)} - 1$.

For $d = 3$, by formula (1) we have $C_{2,3,r} = 2r - 4$. Hence $\mu(J) \leq 2(t(R/J)) + 1$. □

Example 3.3 Let consider the case of two variables. Let $R = k[x, y]$, $J \subset R$ be a monomial ideal such that $\mathrm{rad}(J) = \mathfrak{m}$. The generators of J can be represented in Figure 1 and so the monomials in J are in the up part of a stair. The corner elements can also be drawn. It follows that the number of corner elements is $\mu(J) - 1$, that is $t(R/J) = \mu(J) - 1$.

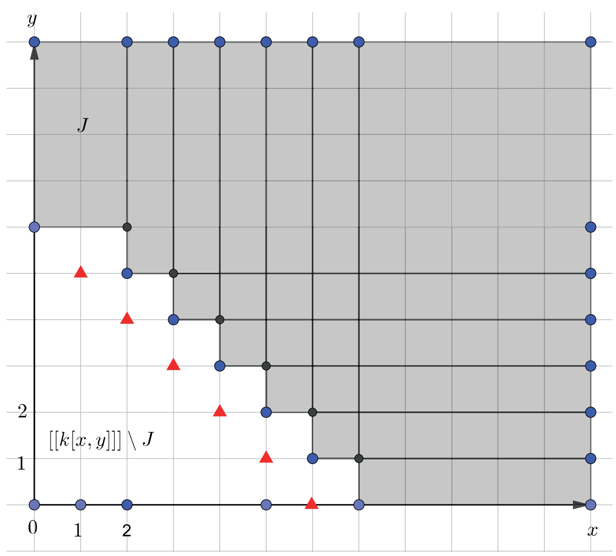

Figure 1. Two variables case.

On the other hand, the cyclic polytope $C_2(r)$ is a convex polygon with r vertices and r faces (see Figure 2).

In particular, we have $C_{1,2,(t(R/J)+2)} - 1 = t(R/J) + 1 = \mu(J)$. Hence the bound in the above theorem is tight.

Recall that the *Frobenius number*, denoted by $F(H)$, is the biggest integer not belonging to H and an integer x is called a *pseudo-Frobenius number* if $x \notin H$ and $x + h \in H$, for all $h \in H \setminus \{0\}$. We denote

Figure 2. Cyclic polytope.

by $PF(H)$ the set of pseudo-Frobenius numbers of H. The *type* of H, denoted by $t(H)$, is the cardinality of $PF(H)$. A numerical semigroup H is called *symmetric* if for any integer $x \in \mathbb{Z}$, either $x \in H$ or $F(H)-x \in H$, and H is called *pseudo-symmetric* if $F(H)$ is even and for any integer $x \in \mathbb{Z} \setminus \{F(H)/2\}$, either $x \in H$ or $F(H)-x \in H$. A larger class of semigroups than pseudo-symmetric class defined in [4] (see also [10]) is called *almost symmetric*, namely H is almost symmetric if for any $x \notin H$ such that $x \leq F(H)$ we have either $F(H)-x \in H$ or $x \in PF(H)$.

Now let us go back to the case of monomial curves. Let $\mathrm{in}(I)$ be the initial ideal of the reduced Gröbner basis $G(H)$ for $\prec_{degrevlex}$. Set $R' = k[x_2, \ldots, x_d]$ and denote by $[[R']]$ the set of all monomials of R'. Let $\varphi : [[R']] \to \mathbb{N}$ be the map defined by $\varphi(M) = k_2n_2 + \ldots + k_dn_d$ for every monomial $M = x_2^{k_2} \ldots x_d^{k_d} \in [[R']]$. Recall that the *Apéry set* as defined in [1] with respect to n_1 in H is

$$\mathrm{Ap}(H, n_1) = \{h \in H \mid h - n_1 \notin H\} = \{0 = \omega(0), \omega(1), \ldots, \omega(n_1 - 1)\},$$

where $\omega(i)$ is the least element of H congruent to i modulo n_1. We consider two sets

$$\mathrm{Ap}\widetilde{(H, n_1)} = \{M \in [[R']] \mid M \notin \mathrm{in}(I)\}$$

and

$$\widetilde{PF(H)} = \{M \in \mathrm{Ap}\widetilde{(H, n_1)} \mid \forall i \neq 1, \exists N_i \in R \text{ such that } Mx_i - x_1^{\alpha_i} N_i \in I, \alpha_i > 0\}.$$

Theorem 3.4 *([14]) (i) The polynomial ring $k[x_1] \subset A$ is a Noether normalization and*

$$A \simeq \oplus_{x_2^{k_2} \ldots x_d^{k_d} \notin \mathrm{in}(I)} k[t^{n_1}][t^{k_2n_2 + \ldots + k_dn_d}].$$

(ii) The restriction of φ to $\mathrm{Ap}\widetilde{(H, n_1)}$ is injective and $\varphi(\mathrm{Ap}\widetilde{(H, n_1)}) = \mathrm{Ap}(H, n_1)$. In particular $\sharp(\{M \in [[R']] \mid M \notin \mathrm{in}(I)\}) = n_1$.

(iii) The Frobenius number $F(H) = \max\{\varphi(M) \mid M \in \mathbb{A}p\widetilde{(H, n-1)}\} - n_1$.

Remark 3.5 From the definition of the pseudo-Frobenius set $PF(H)$ and the above theorem we have that every element $\omega \in PF(H)$ corresponds to exactly one monomial $M_\omega \in \widetilde{PF(H)}$ such that $\omega = \varphi(M_\omega) - n_1$.

Recall that the semigroup H (or the ring $k[H]$) satisfies the condition (*) if for every binomial $M_i - N_i$ in the reduced Gröbner basis $G(H)$ with $N_i \prec_{degrevlex} M_i$, the variable x_1 divides the monomial N_i, that is $N_i = x_1^{a_i} N_i'$ with $N_i' \in [[R']], N_i' \notin \mathrm{in}(I), a_i > 0$ for all $i = 1, \ldots, \mu(\mathrm{in}(I))$.

The following lemma is an improvement of Bresinsky [7, Corollary 1]. For the commodity of the reader we give here a short proof.

Lemma 3.6 *Let $R = k[x_1, \dots, x_d]$ be the polynomial ring with the order $\prec_{degrevlex}$, $I \subset R$ an homogeneous ideal. Let $G = \{F_1, \dots, F_s\}$ be a reduced Gröbner basis of I for a monomial order $\prec_{degrevlex}$. If no leading monomial in G is a multiple of a non-leading monomial of G, then G is a minimal set of generators for I.*

Proof We know that a Gröbner basis of I is a system of generators for I. Suppose that G is not minimal. We can assume that F_1 belongs to the ideal generated by $F_2, \dots, F_s$, hence we have

$$F_1 = H_2F_2 + \ldots + H_sF_s \quad (3)$$

for some polynomials $H_2, \dots, H_s$. Let M be the leading monomial of F_1, this monomial should appear on the right side of equality (3), and so is a multiple of a monomial of some of the $F_2, \dots, F_s$. The hypothesis implies that M is a multiple of a leading monomial of some $F_2, \dots, F_s$. This is a contradiction to the hypothesis that G is a reduced Gröbner basis of I. $\square$

Proof [**of Theorem 1.1**] By Theorem 3.4 the generators of $\mathrm{in}(I)$ are contained in R'. Let $J \subset R'$ be the ideal generated by the generators of $\mathrm{in}(I)$. In order to apply Theorem 3.2 for $J \subset R'$, we need to prove that $t(R'/J) = t(H)$. Indeed, it is clear from Remark 3.5 that $\widetilde{PF(H)} \subset C_{R'}(J)$, hence $t(H) \leq t(R'/J)$. Let $M \in C_{R'}(J)$. Then for any $i = 2, \dots, d$ we have $Mx_i \in J$, hence there exists a generator M_j of J such that M_j divides Mx_i, say $Mx_i = M_jL$ for some monomial $L \in [[R']]$. Since H satisfies the condition (*), for any $i = 2, \dots, d$ we have $Mx_i - x_1^{a_i}N_j'L \in I$, where $a_i > 0$. Hence $M \in \widetilde{PF(H)}$, which implies that $t(R'/J) \leq t(H)$.

Note that if H satisfies the condition (*) then no N_i divides any M_j. Hence by Lemma 3.6, $G(H)$ is a minimal set of generators for I. Our claim is done. $\square$

The following result is a consequence of Theorem 1.1.

Corollary 3.7 *Assume that H satisfies the condition (*) and $d = 4$. Then $\mu(\mathrm{in}(I)) \leq 2(t(H)) + 1$. In particular, if H is almost symmetric then $\mu(I) \leq \mu(\mathrm{in}(I)) \leq 7$.*

Proof Applying Theorem 1.1 and the formula (1) for $d = 4$, we have the first claim. For the second claim, since H is almost symmetric we have by [15] that $t(H) \leqslant 3$ and now applying the first claim we get the result. $\square$

Now we investigate the condition (*) in the cases of 4 variables.

Lemma 3.8 *Let $R = k[x, y, z, w]$. Then $z^\beta w^\alpha - y^\gamma \in G(H)$ if and only if γ is the smallest integer such that $\gamma n_2 \in \langle n_3, n_4 \rangle$ and $\gamma n_2 \in \mathrm{Ap}(H, n_1)$. In particular, there is at most one generator of the type $z^\beta w^\alpha - y^\gamma$ in $G(H)$.*

Proof Assume that $z^\beta w^\alpha - y^\gamma \in G(H)$. Since $y^\gamma \notin \mathrm{in}(I)$, we have $y^\gamma \in \widetilde{\mathrm{Ap}(H, n_1)}$ by the definition of $\widetilde{\mathrm{Ap}(H, n_1)}$ and then $\gamma n_2 \in \mathrm{Ap}(H, n_1)$ by Theorem 3.4. We need only to show that γ is the smallest integer such that $\gamma n_2 \in \langle n_3, n_4\rangle$. Let b be the smallest integer such that $bn_2 = cn_3 + dn_4$, then $b \leq \gamma$. If $b < \gamma$ then we have an equation $z^\beta w^\alpha - y^{\gamma-b} z^c w^d \in I$, it implies that $z^\beta w^\alpha$ is a generator of $\mathrm{in}(I)$. We have three cases:

Case 1. $z^{\beta-c} - y^{\gamma-b} w^{d-\alpha} \in I$. This implies $z^{\beta-c} \in \mathrm{in}(I)$ and hence divides $z^\beta w^\alpha$, which is a generator of $\mathrm{in}(I)$.

Case 2. $w^{\alpha-d} - y^{\gamma-b} z^{c-\beta} \in I$. It is impossible since this implies $w^{\alpha-d} \in \mathrm{in}(I)$ and divides $z^\beta w^\alpha$, which is a generator of $\mathrm{in}(I)$.

Case 3. $z^{\beta-c} w^{\alpha-d} - y^{\gamma-b} \in I$. It is also impossible since this implies $z^{\beta-c} w^{\alpha-d} \in \mathrm{in}(I)$ and divides $z^\beta w^\alpha$, which is a generator of $\mathrm{in}(I)$.

Conversely, since $\gamma n_2 \in \mathrm{Ap}(H, n_1)$ by hypothesis, we have by Theorem 3.4 that $y^\gamma \in \widetilde{\mathrm{Ap}(H, n_1)}$, that is $y^\gamma \notin \mathrm{in}(I)$.

The hypothesis $\gamma n_2 \in \langle n_3, n_4\rangle$ implies that an equation $z^\alpha w^\beta - y^\gamma \in I$. If this equation is not in $G(H)$, then there is an equation $z^c w^d - x^a y^b \in G(H)$, with $c \leq \alpha, d \leq \beta$. We have to consider two cases.

Case 1. If $a > 0$ then $b < \gamma$, and multiplying by $y^{\gamma-b}$ we have

$$z^c w^d y^{\gamma-b} - x^a y^\gamma = z^c w^d y^{\gamma-b} - x^a z^\alpha w^\beta.$$

Therefore $y^{\gamma-b} - x^a z^{\alpha-c} w^{\beta-d} \in G(H)$, hence $y^{\gamma-b} \in \mathrm{in}(I)$, a contradiction.

Case 2. If $a = 0$ then $b \geq \gamma$. By similar arguments, we have $y^{b-\gamma} - z^{\alpha-c} w^{\beta-d} \in G(H)$, hence $y^{\gamma-b} \in \mathrm{in}(I)$, a contradiction except $b = \gamma, \alpha = c, \beta = d$. □

Lemma 3.9 *Let n_1, n_3 be positive integers such that $\gcd(n_1, n_3) = 1$ and $a, b \in \mathbb{N}^*$ such that $an_1 + n_3 \in 2\mathbb{Z}, (2b + a)n_1 + 3n_3 \in 4\mathbb{Z}$ and $3a + 2b \notin n_3\mathbb{Z}$. Let $n_2 := (an_1 + n_3)/2, n_4 := ((2b + a)n_1 + 3n_3)/4$ and H the semigroup generated by n_1, n_2, n_3, n_4. Assume that H cannot be generated by less than 4 elements. Then the reduced Gröbner basis $G(H)$ satisfies the condition (*) with respect to the order $\prec_{degrevlex}$ with w, z, y, x and $\deg x = n_1, \deg y = n_2, \deg z = n_3, \deg w = n_4$.*

Proof By the definitions of n_2, n_4, we have the equations $y^2 - x^a z, w^2 - x^b yz \in I$, which implies $y^2, w^2 \in \mathrm{in}(I)$ and they are generators of $\mathrm{in}(I)$. The elements of $G(H)$ such that x eventually does not appear are of the form $z^\alpha w^\gamma - y^\beta$ or $z^\alpha - y^\beta w^\gamma$ for some integers α, β, γ. Let us study each case. If $z^\alpha w^\gamma - y^\beta \in G(H)$ then since $y^2 \in \mathrm{in}(I)$, we have $\beta = 1$, which implies that H is generated by less than 4 elements, a contradiction. If $z^\alpha - y^\beta w^\gamma \in G(H)$ then since $y^2, w^2 \in \mathrm{in}(I)$ we have $\beta \leqslant 1$ and $\gamma \leqslant 1$. The case $\beta = 0$ or $\gamma = 0$ implies that H is generated by less than 4 elements, a contradiction. Thus $\beta = 1$ and $\gamma = 1$, which implies $\alpha n_3 = n_2 + n_1$. After some computations we have $(4(\alpha - 1) - 1)n_3 = (3a + 2b)n_1$, it cannot be possible since $\gcd(n_1, n_3) = 1$ and $3a + 2b \notin n_3\mathbb{Z}$. Hence $G(H)$ satisfies the condition (*). □

Note that Eto [9] has recently given the classification of almost Gorenstein numerical semigroup rings generated by 4 elements into families named UF1, UF2, nUF1, and nUF2. He also gave a minimal system of generators for their defining ideals. By inspecting the generators of the defining ideal in each case, we can see that the cases UF1, UF2 satisfy the condition (*). Moreover, it can be proved that in all cases the generators of the defining ideal form a Gröbner basis for the order $\prec_{degrevlex}$ and a suitable order of the variables.

In the following Lemma 3.10, we will give a family of rings that satisfies the condition (*) and the rings R/I are almost Gorenstein. This family corresponds to the case UF1 by Eto and was found by the author independently.

Lemma 3.10 *Let n_1, n_2 be positive integers such that $\gcd(n_1, n_2) = 1$, n_1 is even and $a \in \mathbb{N}^*$ an odd number. Set $n_3 := (an_1)/2, n_4 := ((a-2)n_1 + 2n_2)/2$ and H the semigroup generated by n_1, n_2, n_3, n_4. Assume that H cannot be generated by less than 4 elements. Then the reduced Gröbner basis $G(H)$ satisfies the condition (*) with respect to the order $\prec_{degrevlex}$ with w, z, y, x and $\deg x = n_1, \deg y = n_2, \deg z = n_3, \deg w = n_4$. Moreover, the ring R/I is almost Gorenstein.*

Proof We have the equations $z^2 - x^a, w^2 - x^{a-2}y^2 \in I$ by the definitions of n_3 and n_4. It implies $z^2, w^2 \in \operatorname{in}(I)$ and hence they are generators of $\operatorname{in}(I)$. From these equations we get $zw - x^{a-1}y, yz - xw \in G(H)$. Since $z^2, w^2 \in \operatorname{in}(I)$, there is an equation $y^c - x^d z^e w^f \in G(H)$ such that $d \geq 1$ and $e, f \leq 1$, hence y^c is a generator of $\operatorname{in}(I)$. Combining the last equation with $yz - xw$ we get the equation $xy^{c-1}w - x^d z^{e+1} w^f \in I$. Suppose that $f = 1$, then we have $y^{c-1} - x^{d-1}z^{e+1} \in I$, but since $y^{c-1} \notin \operatorname{in}(I)$ we must have $d = 1$, that is $z^{e+1} - y^{c-1} \in I$. If $e = 0$ we get $z - y^{c-1} \in I$, a contradiction, and if $e = 1$ we get $z^2 - y^{c-1} \in I$, that is $y^{c-1} - x^a \in I$, a contradiction since y^c is a generator of $\operatorname{in}(I)$.

Hence we have $f = 0$, and thus $y^c - x^d z^e \in G(H)$ and $y^{c-1}w - x^{d-1}z^{e+1} \in I$. If $e = 0$ then since $\gcd(n_1, n_2) = 1$ we have $c = n_1, d = n_2$, but if y^{n_1} is a generator of $\operatorname{in}(I)$ then the set $\widetilde{\operatorname{Ap}(H, n_1)}$ would contain the monomials $1, z, w, y, \ldots, y^{n_1-1}$ and hence has more than $n_1 + 1$ elements, which is a contradiction. Therefore, we can conclude that $y^c - x^d z \in G(H)$ and $y^{c-1}w - x^{d-1+a} \in I$.

We will prove that the set of six equations

$$z^2 - x^a, w^2 - x^{a-2}y^2, zw - x^{a-1}y, yz - xw, y^c - x^d z, y^{c-1}w - x^{d-1+a}$$

is a Gröbner basis of I. Since z^2, w^2, yz, zw, y^c are a part of a system of generators of $\operatorname{in}(I)$, and $y^{c-1}w - x^{d-1+a} \in I$ we have to look for a generator $y^g w$ of $\operatorname{in}(I)$, where $g \leq c-1$. If $g = c-1$, then $y^{c-1}w - x^{d-1+a} \in G(H)$ and the proof is done. Assume that $g < c-1$. Then we have an element $y^g w - x^h z^i \in G(H)$, with $h \geq 1, i \leq 1$. By combining this element with $zw - x^{a-1}y$ we get the equation $x^h z^{i+1} - x^{a-1}y^{g+1} \in I$. We have to consider two cases:

(1) If $h > a-1$, then we have $y^{g+1} - x^{h-(a-1)}z^{i+1} \in I$. Hence $y^{g+1} \in \operatorname{in}(I)$, a contradiction.

(2) If $h \leq a-1$, then in this case $i = 1$ and we have $y^{g+1} - x^{h+1} \in I$. Hence $y^{g+1} \in \operatorname{in}(I)$, a contradiction.

Both cases lead to a contradiction, hence the set of these 6 equations is a Gröbner basis of I. It then follows that the elements of $\widetilde{PF(H)}$ are $z, y^{c-1}, y^{c-2}w$ and they satisfy the relation $zy^{c-1} - y^{c-2}wx \in I$. Thus R/I is almost Gorenstein. □

Thanks are due to the referee for suggesting some interesting examples related to our result. Namely, Bresinsky [8] and Arslan [2] gave a family of examples in 4-space with a toric ideal generated by an arbitrarily large number of binomials. Arslan [2] proved that their tangent cones are Cohen–Macaulay. We will show that both families of examples satisfy the condition (*). As we will see, in both families, the Gröbner basis obtained by Arslan in order to compute the tangent cone coincides with the Gröbner basis with respect to the order $\prec_{degrevlex}$.

Firstly, we need the following lemma, which will appear in [14]. We give here a short proof for the convenience of the reader.

Lemma 3.11 *Let $R = k[x_1, \ldots, x_d]$ be the ring with respect to the order $\prec_{degrevlex}$, $R' = k[x_2, \ldots, x_d]$ and $[[R']]$ the set of all monomials of R'. Let G be a finite family of reduced binomials in $I := I(H)$ and J the monomial ideal generated by the leading monomials of the elements in G. If $\sharp([[R']] \setminus J) = n_1$ then G is a Gröbner basis of I.*

Proof Since $J \subseteq \operatorname{in}(I)$ we have the exact sequence

$$0 \to \operatorname{in}(I)R'/JR' \to R'/JR' \to R'/\operatorname{in}(I)R' \to 0.$$

We have by Theorem 3.4, (ii) that $\dim_k R'/\operatorname{in}(I)R' = n_1$ and by the hypothesis that $\dim_k R'/JR' = n_1$. Hence we have $J = \operatorname{in}(I)$, which implies that G is a Gröbner basis of I. □

From now on, let $R = k[x, y, z, w]$ be the polynomial ring with respect to the order $\prec_{degrevlex}$ such that $\deg x = n_1, \deg y = n_2, \deg z = n_3, \deg w = n_4$, $R' = k[y, z, w]$ and $[[R']]$ the set of all monomials of R'.

The following family is based on Bresinsky [8]. Note that his aim was to show that when $d \geq 4$ the first Betti number of the resolution of I is unbounded while we show that not only the first Betti number but all the Betti numbers of the resolution of I can be unbounded.

Example 3.12 Let $m \geqslant 2$ be any integer and $H = \langle n_1, n_2, n_3, n_4 \rangle$ be a semigroup, where $n_1 = (2m)(2m-1), n_2 = (2m+1)(2m-1), n_3 = (2m)(2m+1), n_4 = (2m)(2m+1) + 2m - 1$. Then we have the following results.

1. The following family of binomials is the Gröbner basis of I with respect to the order $\prec_{degrevlex}$

$$\begin{array}{l|l}
z^{2m-1} - x^{2m+1} & y^{2m} - x^{2m+1} \\
z^{2m-2}w - x^{2m}y & y^{2m-1}w - x^{2m}z \\
z^{2m-3}w^2 - x^{2m-1}y^2 & y^{2m-2}w^2 - x^{2m-1}z^2 \\
\ldots & \ldots \\
zw^{2m-2} - x^3y^{2m-2} & y^2w^{2m-2} - x^3z^{2m-2} \\
w^{2m-1} - x^2y^{2m-1} & yz - xw.
\end{array}$$

   Indeed, let J be the monomial ideal generated by the leading monomials of the above binomials

$$J = (z^{2m-1}, z^{2m-2}w, \ldots, zw^{2m-2}, w^{2m-1}, y^{2m}, y^{2m-1}w, \ldots, y^2w^{2m-2}, yz).$$

   From Figure 3, it is very simple to count the monomials $\sharp([[R']] \setminus J) = n_1$. Hence by Lemma 3.11, the above families are a Gröbner basis of I with respect to the order $\prec_{degrevlex}$.

2. It is clear that the condition (*) is satisfied. Moreover, we have by the definition of the set $\widetilde{PF(H)}$ (see the red triangles in Figure 3)

$$\widetilde{PF(H)} = \{z^{2m-2}, z^{2m-3}w, \ldots, zw^{2m-3}, y^{2m-1}, y^{2m-2}w, \ldots, yw^{2m-2}\},$$

   and so $\sharp(\widetilde{PF(H)}) = 4m - 3$ and we have the free resolution of R/I

$$0 \to R^{4m-3} \to R^{8m-4} \to R^{4m} \to R \to R/I \to 0.$$

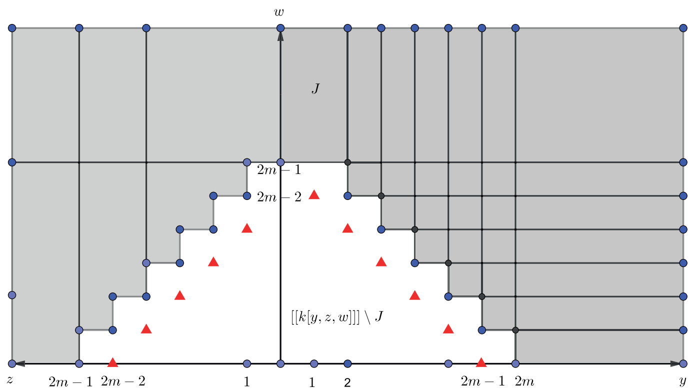

Figure 3. Bresinsky's example.

With the same method as in Example 3.12, we can study the following family, which is based on Arslan [2].

Example 3.13 Let $m \geqslant 2$ be any integer and $H = \langle n_1, n_2, n_3, n_4 \rangle$ be a semigroup, where $n_1 = m(m+1), n_2 = m(m+1)+1, n_3 = (m+1)^2, n_4 = (m+1)^2+1$. Then

1. It follows by Lemma 3.11 that the following family of binomials is the Gröbner basis of I with respect to the order $\prec_{degrevlex}$

$$\begin{array}{l|l}
z^m - x^{m+1} & y^{m+1} - x^m z \\
z^{m-1}w - x^m y & y^m w - x^{m-1}z^2 \\
z^{m-2}w^2 - x^{m-1}y^2 & y^{m-1}w^2 - x^{m-2}z^3 \\
\dots & \dots \\
z^2 w^{m-2} - x^3 y^{m-2} & y^3 w^{m-2} - x^2 z^{m-1} \\
z w^{m-1} - x^2 y^{m-1} & y^2 w^{m-1} - x^{m+2} \\
w^m - x y^m & yz - xw.
\end{array}$$

(In Figure 4, we present the ideal J and all monomials in $[[R']] \setminus J$, where J is generated by the leading monomials of the above binomials).

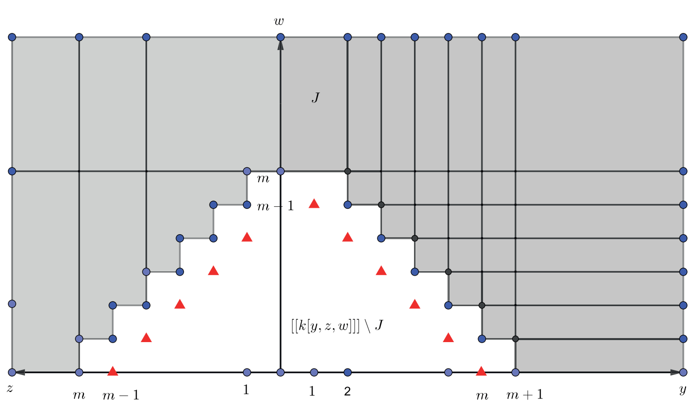

Figure 4. Arslan's example.

2. We have the set $\widetilde{PF(H)}$ (see the red triangles in Figure 3)

$$\widetilde{PF(H)} = \{z^{m-1}, z^{m-2}w, \ldots, zw^{m-2}, y^m, y^{m-1}w, \ldots, yw^{m-1}\},$$

and hence $\sharp(\widetilde{PF(H)}) = 2m-1$ and we have the free resolution of R/I

$$0 \to R^{2m-1} \to R^{4m} \to R^{2(m+1)} \to R \to R/I \to 0.$$

In the rest of this section, we give the following example of a numerical semigroup satisfying the condition (*) but the tangent cone of R/I is not a Cohen–Macaulay ring.

Example 3.14 Let $n_1 = 23, n_2 = 35, n_3 = 56, n_4 = 78$. Then the following binomials are a Gröbner basis of I with respect to the order $\prec_{degrevlex}$

$$\begin{array}{l|l|l} -z^2y + wx^3, & -wy^2 + zx^4, & -zy^3 + x^7, \\ -w^2y + z^3x, & w^2z^2 - y^7x, & z^5 - w^3x^2, \\ -y^8 + w^3x^2, & w^3z - y^5x^5, & wz^4 - y^6x^4, \\ w^4 - y^3x^9. & & \end{array}$$

It is clear that the condition (*) is satisfied. By a computation with the software Cocoa, we get the ideal generated by

$$wy^2, w^2y, z^2y, w^3z, w^4, w^2z^2, zy^3, wz^4, z^5, w^3x^2, wz^3x^3, w^2zx^6, -z^4x^7 + y^{11}$$

that defines the tangent cone. Since x divides several leading monomials, we have by Aslan et al. [3] [Lemma 2.7] that the tangent cone of R/I is not a Cohen–Macaulay ring.

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