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ON THE CONTROL PROBLEM ASSOCIATED WITH THE HEATING PROCESS

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Abstract: In the papaer, we consider the initial-boundary problem for the heat conduction equation inside a bounded domain. On the part of the border of the considered domain, the value of the solution with control parameter is given. Restrictions on the control are given in such a way that the average value of the solution in some part of the considered domain gets a given value. It is supposed that on the boundary of this domain the heat exchange takes place according to Newton's law. The control parameter is equal to the magnitude of output of hot or cold air and is defined on a given part of the boundary, and the weight function is not assumed to be strictly positive in the given domain. Then, we found the dependence $T(\theta)$ on the parameters of the temperature process when θ is close to critical value.

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1. Introduction

Consider in the bounded domain $\Omega \subset \mathbb{R}^3$ with piecewise smooth boundary $\partial\Omega$ the heat conduction equation

$$u_t(x,t) = \Delta u(x,t), \quad x \in \Omega,\ t > 0, \tag{1}$$

with boundary conditions

$$\frac{\partial u}{\partial n} + h(x)u(x,t) \ = \ 0, \quad x \in \partial\Omega \setminus \Gamma,\ t > 0, \tag{2}$$

$$\frac{\partial u}{\partial n} = a(x)\,\mu(t), \quad x \in \Gamma,\ t > 0, \tag{3}$$

and initial condition

$$u(x,0) = 0. \tag{4}$$

Here Γ is some subset of $\partial\Omega$ (heater or air conditioner) with piecewise smooth boundary $\partial\Gamma$ and with $\operatorname{mes}\Gamma > 0$ (we denote by $\operatorname{mes}\Gamma$ the surface measure of Γ, distinct from Lebesgue measure $|\Gamma|$).

We suppose that $h(x)$ (thermal conductivity of the walls) and $a(x)$ (the density of the power of the heater or air conditioner) are given piecewise smooth non-negative functions, which are not identically zero. The condition (3) means that there is a blast of hot (or cold) air with magnitude of output given by a measurable real-valued function $\mu(t)$, and condition (2) means that on the surface $\partial\Omega$ a heat exchange takes place according to Newton's law (see, e.g. [1, , Sect. III.1.4]).

We may extend both functions $h(x)$ and $a(x)$ to the whole boundary $\partial\Omega$ by setting $h(x) = 0$ for $x \in \Gamma$, and $a(x) = 0$ for $x \notin \Gamma$. In this case we may write the conditions (2) and (3) in the following form

$$\frac{\partial u(x,t)}{\partial n} + h(x)u(x,t) = a(x)\,\mu(t), \quad x \in \partial\Omega,\ t > 0. \tag{5}$$

By the solution of the initial boundary value problem (1)–(5), we mean the generalized solution defined in [2] (see Chapter III, Sec. 5).

Let $M > 0$ be some given constant. We say that the function $\mu(t)$ is an *admissible control* if this function is measurable on the half line $t \geq 0$ and satisfies the following constraint

$$|\mu(t)| \leq M, \quad t \geq 0. \tag{6}$$

Let the function $\rho : \overline{\Omega} \to R$ satisfies conditions

$$\int\limits_{\Omega} \rho(x)\,dx = 1, \quad \rho(x) \geq 0.$$

For any $\theta > 0$ consider the condition

$$\int\limits_{\Omega} u(x,t)\rho(x)\,dx = \theta. \tag{7}$$

Note that the weight function $\rho(x)$ is not assumed to be strictly positive. In particular, the value (7) may be the average value over some subdomain of the main region Ω.

Denote by the symbol $T(\theta)$ the minimal time required to reach the given value θ by the average value of the temperature. This means that the equation (7) is fulfilled for $t = T(\theta)$ and is not valid for $t < T(\theta)$.

We present the critical value θ^* such that for any $\theta < \theta^*$ there exists the required admissible control $\mu(t)$ and corresponding value of $T(\theta) < +\infty$, and for $\theta \geq \theta^*$ the equality (7) is impossible.

The purpose of this work is to determine the dependence $T(\theta)$ on the parameters of the temperature process when θ is close to critical value.

The difference between the problem under consideration and paper [3] is that in this work the weight function $\rho(x)$ is not required to be exactly greater than zero in the given domain. Therefore, we considered the three-dimensional domain in this work. A special case of this problem is studied in [4, 5].

We recall that the time-optimal control problem for partial differential equations of parabolic type was first investigated in [6] and [7]. More recent results concerned with this problem were established in [3–5, 8–10]. Detailed information on the problems of optimal control for distributed parameter systems is given in [11] and in the monographs [12–14].

To formulate the main result we describe some spectral properties of the corresponding self-adjoint extension of Laplace operator.

Consider the following eigenvalue problem for the Laplace operator

$$-\Delta v_k(x) = \lambda_k \, v_k(x), \quad x \in \Omega, \tag{8}$$

with boundary condition

$$\frac{\partial v_k(x)}{\partial n} + h(x) v_k(x) = 0, \quad x \in \partial\Omega. \tag{9}$$

Under assumptions made above this problem is self-adjoint in $L_2(\Omega, dx)$ and there exists a sequence of eigenvalues $\{\lambda_k\}$ so that

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty, \quad k \to \infty.$$

The corresponding eigenfunctions form a complete orthonormal system $\{v_k\}_{k\in\mathbb{N}}$ in $L_2(\Omega, dx)$ and these functions belong to $C(\overline{\Omega})$, where $\overline{\Omega} = \Omega \cup \partial\Omega$.

It is well-known that the asymptotic behavior of the solution of the heat conduction equation mainly depends on the first (minimal) eigenvalue of the corresponding selfadjoint extension of Laplace operator (see, e.g. [1]).

We obtain an estimate of the minimal time of heating by the characteristics of the first eigenfunction v_1.

According to (8), we get

$$\lambda_k = -(\Delta v_k, v_k) = \int\limits_{\Omega} |\nabla v_k(x)|^2 \, dx + \int\limits_{\partial\Omega} |v_k(x)|^2 h(x) \, d\sigma(x) \geq 0.$$

If $h(x) \geq 0$ and $h(x) \not\equiv 0$ then $\lambda_1 > 0$. Indeed, assume that $\lambda_1 = 0$. Then the first eigenfunction is an harmonic function

$$\Delta v_1(x) = 0,$$

and, in accordance with the theorem of Giraud and Theorem I.5.II in the book [15], we may state that $v_1 \equiv 0$.

According to the non-negative of the first eigenfunction (see, e.g. [16]) and from the orthogonality of the eigenfunctions v_1 and v_2, we can write

$$\lambda_1 < \lambda_2.$$

Recall that we consider the behavior of the function

$$U(t) = \int\limits_{\Omega} u(x,t) \rho(x) \, dx, \tag{10}$$

where the solution $u(x,t)$ of the problem (1)–(4) depends on the control function $\mu(t)$.

Set

$$\theta^* = M \int\limits_{\Gamma} [(-\Delta)^{-1} \rho(x)] a(x) \, d\sigma(x), \tag{11}$$

and

$$b = \frac{M}{\lambda_1} \cdot (\rho, v_1) \int\limits_{\Gamma} v_1(y) a(y) \, d\sigma(y). \tag{12}$$

Theorem 1. *Let* $\theta^* > 0$ *be defined by equation* (11). *Then*
(1) for every θ *from the interval* $0 < \theta < \theta^*$ *there exist* $T(\theta)$ *such that*

$$U(t) < \theta, \quad 0 < t < T(\theta),$$

and

$$U(T(\theta)) = \theta;$$

(2) *for* $\theta \to \theta^*$ *the following estimate is valid:*

$$T(\theta) = \ln \frac{1}{\varepsilon(\theta)} + \frac{1}{\lambda_1} \ln b + O(\varepsilon^{\lambda_2 - \lambda_1}),$$

where $\varepsilon = |\theta^* - \theta|^{1/\lambda_1}$;

(3) *for every* $\theta \geq \theta^*$ *the* $T(\theta)$ *does not exists.*

The proof of theorem we give step by step as propositions.

2. The main integral equation

We consider the following Green function:

$$G(x, y, t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} v_k(x) v_k(y), \quad x \in \Omega,\ y \in \Omega,\ t > 0.$$

This function is the solution of the initial-boundary value problem for the equation

$$G_t(x, y, t) = \Delta G(x, y, t), \quad x \in \Omega,\ t > 0,$$

with boundary condition

$$\frac{\partial G(x, y, t)}{\partial n} + h(x) G(x, y, t) = 0, \quad x \in \partial\Omega,\ t > 0,$$

and initial condition

$$G(x, y, 0) = \delta(x - y).$$

Set

$$H(x, t) = \int_{\Omega} \rho(y) G(x, y, t)\, dy, \quad x \in \Omega,\ t > 0. \tag{13}$$

It is clear that the function (13) is a solution of the following initial-boundary value problem:

$$H_t(x, t) - \Delta H(x, t) = 0, \quad x \in \Omega,\ t > 0,$$

$$\frac{\partial H(x, t)}{\partial n} + h(x) H(x, t) = 0, \quad x \in \partial\Omega,\ t > 0,$$

and

$$H(x, 0) = \rho(x), \quad x \in \Omega.$$

In this using the spectral theorem in $L_2(\Omega, dx)$ we may write

$$H(x, t) = \int_0^{\infty} e^{-\lambda t}\, dE_\lambda \rho(x).$$

Obviously,

$$H(x,t) = (\rho, v_1)e^{-\lambda_1 t}v_1(x) \ + \ H_1(x,t), \quad t \geq 0, \tag{14}$$

where

$$H_1(x,t) = \int\limits_{\lambda_2}^{\infty} e^{-\lambda t}\, dE_\lambda \rho(x). \tag{15}$$

Set

$$A_k = \int\limits_{\Gamma} v_k(y)a(y)\, d\sigma(y).$$

Proposition 1. *The following estimate is true:*

$$A_1 = \int\limits_{\Gamma} v_1(y)a(y)d\sigma(y) > 0. \tag{16}$$

Proof. Assume that this integral is equal to 0. Then on some surface $\Gamma_1 \subset \Gamma$ v_1 equals 0:

$$v(s) = 0, \quad s \in \Gamma_1.$$

It follows from (9) that

$$\frac{\partial v(s)}{\partial n} = 0, \quad s \in \Gamma_1.$$

Hence, $v_1(x)$ is a solution to homogeneous Cauchy problem and from the uniqueness of the solution $v_1(x) \equiv 0$, and this contradicts the assumption that $v_1(x)$ is an eigenfunction.

Proposition 1 proved.

Set

$$G_2(x,y) = \sum_{k=2}^{\infty} \frac{v_k(x)v_k(y)}{\lambda_k^2}. \tag{17}$$

Proposition 2. *The function $H_1(x,t)$ satisfies the following estimate:*

$$\big|H_1(x,t)\big| \ \leq \|\Delta\rho\| \sqrt{G_2(x,x)}e^{-\lambda_2 t}, \quad t \geq 0,$$

uniformly in $x \in \overline{\Omega}$.

Proof. From (15) we can write

$$H_1(x,t) = \int\limits_{\lambda_2}^{\infty} e^{-\lambda t}\, dE_\lambda \rho(x) = \sum_{k=2}^{\infty} (\rho, v_k)e^{-\lambda_k t}v_k(x), \quad t \geq 0.$$

Then we have

$$|H_1(x,t)|^2 = \left| \sum_{k=2}^{\infty} (\rho, v_k)e^{-\lambda_k t}v_k(x) \right|^2$$

$$\leq \left(\sum_{k=2}^{\infty} |(\rho, v_k)|^2 \lambda_k^2 \right) \left(\sum_{k=2}^{\infty} e^{-2\lambda_k t}|v_k(x)|^2 \lambda_k^{-2} \right), \quad t \geq 0.$$

Then we get the following estimate:

$$|H_1(x,t)| \le \|\Delta\rho\| \sqrt{G_2(x,x)} e^{-\lambda_2 t}.$$

Proposition 2 proved.

Now we introduce the kernel of a main integral operator:

$$K(t) = \int\limits_{\Gamma} H(y,t) a(y) d\sigma(y). \tag{18}$$

According to (14), we may write

$$K(t) = (\rho, v_1) e^{-\lambda_1 t} \int\limits_{\Gamma} v_1(y) a(y)\, d\sigma(y) + \int\limits_{\Gamma} H_1(y,t) a(y)\, d\sigma(y)$$
$$= A_1 \cdot (\rho, v_1) e^{-\lambda_1 t} + \beta(t) e^{-\lambda_2 t}, \tag{19}$$

where

$$|\beta(t)| \le B = \|\Delta\rho\| \int\limits_{\Gamma} \sqrt{G_2(y,y)} a(y)\, d\sigma(y).$$

The proof of the following Proposition 3 and Proposition 4 can be seen [8].

Proposition 3. *The derivative of the kernel* (18) *satisfies the following estimates:*

$$K'(t) = \frac{O(1)}{\sqrt{t}}, \quad 0 < t < 1,$$

and

$$K'(t) = -\lambda_1 A_1 e^{-\lambda_1 t} + O(1) e^{-\lambda_2 t}, \quad t \ge 1.$$

where A_1 *is defined by the equality* (16).

Proposition 4. *Let* $u(x,t)$ *be the solution of the initial-boundary value problem* (1)–(4). *Then the following equality*

$$u(x,t) = \int\limits_0^t \mu(s)\, ds \int\limits_{\Gamma} G(x,y,t-s) a(y)\, d\sigma(y),$$

is valid.

According to condition (10) we can write

$$\int\limits_{\Omega} \rho(x) u(x,t)\, dx = \int\limits_0^t \mu(s)\, ds \int\limits_{\partial\Omega} a(y)\, d\sigma(y) \int\limits_{\Omega} \rho(x) G(x,y,t-s)\, dx = U(t)$$

Then, from (13) and (18), we get the following integral equation:

$$\int\limits_{\Omega} \rho(x) u(x,t)\, dx = \int\limits_0^t K(t-s) \mu(s)\, ds = U(t). \tag{20}$$

3. Proof of Theorem 1

Set

$$L(x,t) = \int\limits_0^t H(x,s)\,ds. \tag{21}$$

Then we can write

$$L(x,t) = \sum_{k=1}^{\infty}(\rho, v_k)v_k(x)\int\limits_0^t e^{-\lambda_k s}\,ds = \sum_{k=1}^{\infty}\frac{1-e^{-\lambda_k t}}{\lambda_k}(\rho, v_k)v_k(x)$$

$$= (-\Delta)^{-1}\rho(x) - \frac{e^{-\lambda_1 t}}{\lambda_1}(\rho, v_1)v_1(x) - L_1(x,t),$$

where

$$L_1(x,t) = \sum_{k=2}^{\infty}\frac{e^{-\lambda_k t}}{\lambda_k}(\rho, v_k)v_k(x).$$

We have the following estimate:

$$|L_1(x,t)| \le e^{-\lambda_2 t}\left(\sum_{k=2}^{\infty}|(\rho, v_k)|^2\right)^{1/2}\left(\sum_{k=2}^{\infty}\frac{|v_k(x)|^2}{\lambda_k^2}\right)^{1/2}.$$

Hence,

$$|L_1(x,t)| \;\le\; e^{-\lambda_2 t}\sqrt{G_2(x,x)}\|\rho\|. \tag{22}$$

Further,

$$\int\limits_{\Gamma} L(x,t)a(x)\,d\sigma(x) = \int\limits_{\Gamma}[(-\Delta)^{-1}\rho(x)]a(x)\,d\sigma(x)$$

$$- \frac{A_1}{\lambda_1}(\rho, v_1)e^{-\lambda_1 t} - \int\limits_{\Gamma} L_1(x,t)a(x)\,d\sigma(x). \tag{23}$$

We introduce a specific heating as

$$Q(t) = \int\limits_0^t K(t-s)ds = \int\limits_0^t K(s)\,ds. \tag{24}$$

The physical meaning of this function is evident: $Q(t)$ equals the average temperature of Ω in case where the heater is acting unit load (see, e.g. [3, 8]).

It is clear that $Q(0) = 0$ and $Q'(t) = K(t) \ge 0$.

According to (18), we have

$$\int\limits_{\Gamma} L(x,t)a(x)\,d\sigma(x) = \int\limits_0^t ds\int\limits_{\Gamma} H(x,s)a(x)\,d\sigma(x) = \int\limits_0^t K(s)\,ds = Q(t). \tag{25}$$

Set

$$Q^* = \lim_{t\to\infty} Q(t) = \int\limits_0^{\infty} K(s)\,ds. \tag{26}$$

Obviously, the average temperature of Ω in the case where the heater is acting with unit load cannot exceed Q^*.

Set

$$\theta^* = MQ^*. \tag{27}$$

Then, according to (22) and (23)

$$\theta(t) = MQ(t) = \theta^* - be^{-\lambda_1 t} + O(e^{-\lambda_2 t}), \tag{28}$$

where b defined by (12).

According to (26)–(28), for every θ from the interval $0 < \theta < \theta^*$ there exist $T(\theta)$ such that

$$U(t) \ < \ \theta, \quad 0 < t < T(\theta), \tag{29}$$

and we may write

$$U(T(\theta)) = \theta. \tag{30}$$

Proposition 5. *There exist $T(\theta) > 0$ and a real-valued measurable function $\mu(t)$ so that $|\mu(t)| \leq M$ and the following equality*

$$\int_0^T K(T-s)\mu(s)\,ds = U(T), \tag{31}$$

is valid.

Proof. This follows from the properties of the function Q. Indeed, if we set $\mu(t) = M$, then

$$\int_0^t K(t-s)\mu(s)\,ds = M\int_0^t K(t-s)\,ds = MQ(t),$$

and because of (31) there exists $T(\theta) > 0$ so that $MQ(T) = U(T)$.

Proposition 5 proved.

Remark. It is clear that the value $T(\theta)$, which was found in Proposition 5, gives a solution to the problem. Namely, $T(\theta)$ is the root of the equation

$$Q(T) = \frac{U(T)}{M} = \frac{\theta}{M}. \tag{32}$$

Proposition 6. *Let $f(r)$ be increasing on the interval $(0,1]$ and for some $\beta > 0$*

$$f(r) = br + O(r^{1+\beta}). \tag{33}$$

Then for inverse function $r = f^{-1}(s)$ the following estimate is valid:

$$\ln\frac{1}{r} = \ln\frac{1}{s} + \ln b + O(s^\beta), \tag{34}$$

where b defined by (12).

Proof. According to (33),

$$s = br\big[1 + \alpha(r)\big], \tag{35}$$

where

$$\alpha(r) = O(r^{\beta}). \tag{36}$$

Note that $f(r) > 0$ on the interval $0 < r \le 1$. Hence,

$$s \ge Cr, \quad 0 < r \le 1. \tag{37}$$

Then

$$r(s) = f^{-1}(s) \le \frac{1}{C} \cdot s,$$

and

$$r(s) = O(s). \tag{38}$$

Hence,

$$\alpha(r(s)) = O(s^{\beta}).$$

Then, according to (35),

$$\ln\frac{1}{s} = \ln\frac{1}{br} + \ln\frac{1}{1+\alpha(r)} = \ln\frac{1}{br} - \ln[1+\alpha(r)]$$
$$= \ln\frac{1}{r} + \ln\frac{1}{b} + O(|\alpha(r)|) = \ln\frac{1}{r} - \ln b + O(s^{\beta}). \tag{39}$$

Proposition 6 proved.

Corollary. *The following equality is true:*

$$t = \ln\frac{1}{|\theta^* - \theta(t)|^{1/\lambda_1}} + \frac{1}{\lambda_1}\ln b + O(|\theta^* - \theta(t)|^{(\lambda_2-\lambda_1)/\lambda_1}). \tag{40}$$

Indeed, according to (28),

$$\theta^* - \theta(t) = be^{-\lambda_1 t} + O(e^{-\lambda_2 t}). \tag{41}$$

Set

$$r = e^{-\lambda_1 t}, \quad s = \theta^* - \theta(t), \quad \beta = \frac{\lambda_2}{\lambda_1} - 1. \tag{42}$$

Then

$$e^{-\lambda_2 t} = e^{-\lambda_1 t(1+\beta)} = r^{1+\beta}.$$

We can apply Proposition 6 and get

$$t = \frac{1}{\lambda_1}\ln\frac{1}{\theta^* - \theta(t)} + \frac{1}{\lambda_1}\ln b + O(|\theta^* - \theta(t)|^{\beta}).$$

Then, for $\theta \to \theta^*$, we have the following estimate:

$$T(\theta) = \ln\frac{1}{\varepsilon(\theta)} + \frac{1}{\lambda_1}\ln b + O(\varepsilon^{\lambda_2-\lambda_1}),$$

where

$$\varepsilon = |\theta^* - \theta|^{1/\lambda_1}.$$

The proof of Theorem 1 follows from Propositions 5 and 6.

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