

CONSTRUCTING HÖLDER MAPS TO CARNOT GROUPS

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ABSTRACT. In this paper, we construct Hölder maps to Carnot groups equipped with a Carnot metric, especially the first Heisenberg group \mathbb{H} . Pansu and Gromov [7] observed that any surface embedded in \mathbb{H} has Hausdorff dimension at least 3, so there is no α -Hölder embedding of a surface into \mathbb{H} when $\alpha > \frac{2}{3}$. Züst [17] improved this result to show that when $\alpha > \frac{2}{3}$, any α -Hölder map from a simply-connected Riemannian manifold to \mathbb{H} factors through a metric tree. In the present paper, we show that Züst's result is sharp by constructing $(\frac{2}{3} - \epsilon)$ -Hölder maps from D^2 and D^3 to \mathbb{H} that do not factor through a tree. We use these to show that if $0 < \alpha < \frac{2}{3}$, then the set of α -Hölder maps from a compact metric space to \mathbb{H} is dense in the set of continuous maps and to construct proper degree-1 maps from \mathbb{R}^3 to \mathbb{H} with Hölder exponents arbitrarily close to $\frac{2}{3}$.

1. INTRODUCTION AND STATEMENT OF RESULTS

The first Heisenberg group \mathbb{H} , equipped with a Carnot metric, is a subriemannian manifold. The Hausdorff dimension of such a manifold is greater than its topological dimension; the Heisenberg group, for instance, has topological dimension 3 and Hausdorff dimension 4. It follows that there is no surjective Lipschitz map from \mathbb{R}^3 to \mathbb{H} , since Lipschitz maps cannot increase Hausdorff dimension. Indeed, the image of the 3-dimensional unit ball D^3 under an α -Hölder map has Hausdorff dimension at most $\frac{3}{\alpha}$, so when $\alpha > \frac{3}{4}$, there is no α -Hölder map from D^3 to \mathbb{H} whose image contains a metric ball.

When $\alpha < \frac{3}{4}$, a construction like that of Kaufman [10] can be used to construct an α -Hölder map from D^3 to \mathbb{H} whose image contains a ball, but when $\frac{2}{3} < \alpha < \frac{3}{4}$, the topology of such maps is very restricted. These conditions arise from the fact, proved by Gromov in [7, 0.6.C, 2.1] and therein also attributed to Pansu, that any surface embedded in \mathbb{H} has topological dimension 2 but Hausdorff dimension at least 3, so if $\alpha > \frac{2}{3}$, then the image of a surface under an α -Hölder map cannot be a surface. Indeed, Züst [17] showed that if M is a simply-connected Riemannian manifold and $f: M \rightarrow \mathbb{H}$ is α -Hölder with $\alpha > \frac{2}{3}$, then f factors through a metric tree. Moreover, Le Donne and Züst [11] proved that if

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$\alpha > \frac{1}{2}$ then any α -Hölder surface in \mathbb{H} (if it exists) must intersect many vertical lines in a topological Cantor set.

In [7, 0.5.C], Gromov asked:

Given two [Carnot–Carathéodory] spaces V and W and a real number $0 < \alpha \leq 1$, describe the space of C^α -maps $f: W \rightarrow V$. For example, when can each continuous map $W \rightarrow V$ be uniformly approximated by C^α -maps? When can W be C^α -embedded into V ? When are V and W C^α homeomorphic? etc.

The special case of finding the maximum α such that there is an α -Hölder homeomorphism from \mathbb{R}^3 to \mathbb{H} has become known as the Hölder equivalence problem. It follows from the results of Pansu and Gromov [7, 2.1] or Züst [17], both mentioned above, that if $\alpha > \frac{2}{3}$ then there is no locally α -Hölder homeomorphism from \mathbb{R}^3 to \mathbb{H} and that continuous maps from \mathbb{R}^3 to \mathbb{H} cannot be approximated by α -Hölder maps. On the other hand, a smooth or C^2 map from \mathbb{R}^3 to \mathbb{H} is locally $\frac{1}{2}$ -Hölder, so a continuous map from \mathbb{R}^3 to \mathbb{H} can be approximated by a locally $\frac{1}{2}$ -Hölder map, and there are many locally $\frac{1}{2}$ -Hölder homeomorphisms from \mathbb{R}^3 to \mathbb{H} .

In this paper, we will partially answer Gromov’s question by showing that there are many α -Hölder maps from \mathbb{R}^n to \mathbb{H} for α arbitrarily close to $\frac{2}{3}$, including maps that are topologically nontrivial (e.g., proper and degree-1) and maps that approximate arbitrary continuous functions. Our constructions build on techniques developed in [15] and [12]. Results like this were first suggested by unpublished work of Piotr Hajłasz, Jake Mirra, and Armin Schikorra, who explored constructing Hölder maps by numerical methods and found results pointing to the possible existence of nontrivial surfaces in \mathbb{H} with Hölder exponent larger than $\frac{1}{2}$ [13].

Our first result provides Hölder extensions of maps from subsets of \mathbb{R}^2 to general Carnot groups equipped with a Carnot metric. In order to state our theorem we recall the following definition. A pair (X, Y) of metric spaces X and Y is said to have the α -Hölder extension property, $0 < \alpha \leq 1$, if there exists $L \geq 1$ such that for every subset $Z \subset X$ every (λ, α) -Hölder map $\varphi: Z \rightarrow Y$ has an $(L\lambda, \alpha)$ -Hölder extension $\bar{\varphi}: X \rightarrow Y$.

Theorem 1.1. *Let G be a Carnot group of step k , endowed with a Carnot metric d_c . Then the pair $(\mathbb{R}^2, (G, d_c))$ has the α -Hölder extension property for every $\alpha < \frac{2}{k+1}$.*

In particular, given a closed Lipschitz curve $\gamma: S^1 \rightarrow \mathbb{H}$, we can extend γ to a α -Hölder map of a disc for any $\alpha < \frac{2}{3}$. Our construction produces a disc which is not even locally an embedding, even if γ is an embedding, and it is an open question (see [7, 0.5.D]) whether there are α -Hölder embeddings from \mathbb{R}^2 to \mathbb{H} for $\frac{1}{2} < \alpha < \frac{2}{3}$.

We can extend the construction used in the theorem above to produce Hölder maps from 3-dimensional Riemannian manifolds to \mathbb{H} . Let d_R be the distance

coming from a left-invariant Riemannian metric on \mathbb{H} and let d_c be the associated Carnot metric. Let $\mathbb{H}_{\mathbb{Z}}$ be the integer lattice in \mathbb{H} .

Theorem 1.2. *For any $\alpha < \frac{2}{3}$ there is a locally α -Hölder map $g: (\mathbb{H}, d_R) \rightarrow (\mathbb{H}, d_c)$ which is $\mathbb{H}_{\mathbb{Z}}$ -equivariant (that is, $g(h \cdot x) = h \cdot g(x)$ for all $h \in \mathbb{H}_{\mathbb{Z}}$ and $x \in \mathbb{H}$) and equivariantly homotopic to the identity.*

As a consequence, we obtain the following approximation result.

Theorem 1.3. *Let Y be a compact metric space and let $0 < \alpha < \frac{2}{3}$. Any continuous map $\varphi: Y \rightarrow \mathbb{H}$ can be approximated uniformly by α -Hölder maps.*

This exponent is sharp; when $\frac{2}{3} < \alpha \leq 1$, [17] implies that any α -Hölder map $\psi: D^2 \rightarrow \mathbb{H}$ factors through a metric tree, so $\psi(\partial D^2)$ has filling radius zero. A curve with nonzero filling radius cannot be uniformly approximated by curves with filling radius zero, so if $\varphi: D^2 \rightarrow \mathbb{H}$ is a continuous map such that $\varphi(\partial D^2)$ is a simple closed curve, then φ cannot be uniformly approximated by α -Hölder maps when $\alpha > \frac{2}{3}$.

The map constructed in Theorem 1.2 is self-similar, and by taking a tangent cone at a carefully-chosen point, we furthermore obtain the following result. Recall that a continuous map between metric spaces is called proper if preimages of compact sets are compact. A Euclidean similarity is a composition of a scaling, translation, and rotation/reflection. A Heisenberg similarity is a scaling composed with a left-translation.

Theorem 1.4. *For any $\epsilon > 0$, there is an $0 < \epsilon' < \epsilon$ such that there is a globally $(\frac{2}{3} - \epsilon')$ -Hölder map $F: \mathbb{R}^3 \rightarrow (\mathbb{H}, d_c)$ which is proper and of degree 1. This map is self-similar in the sense that there is a Euclidean similarity $h: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and a Heisenberg similarity $m: \mathbb{H} \rightarrow \mathbb{H}$, both with scaling factors bigger than 1, such that $m(F(h^{-1}(x))) = F(x)$ for all $x \in \mathbb{R}^3$.*

Gromov showed [7, 3.1.A] that α -Hölder maps from \mathbb{R}^3 to \mathbb{H} must have local degree zero when $\alpha > \frac{2}{3}$ and asked whether this exponent can be improved; this construction shows that Gromov's result is sharp.

We now give an outline of the proofs of our results. In order to prove Theorem 1.1 it is enough to show that there exists $L \geq 1$ such that every L -Lipschitz curve $\gamma: S^1 \rightarrow (G, d_c)$ admits an $(L\lambda, \alpha)$ -Hölder extension $f: D^2 \rightarrow (G, d_c)$ to the 2-dimensional unit ball D^2 (see [12, Theorem 6.4]).

We construct such an extension using methods based on the Hölder extension results in [12]. The main ingredient is the so-called coarse Dehn function, also known as Gromov's mesh function, which is roughly defined as follows. Let X be a geodesic metric space and $\epsilon > 0$. For $r > 0$ the coarse Dehn function $\text{Ar}_{X,\epsilon}(r)$ is the smallest number such that any closed curve in X of length at most r can be subdivided into $\text{Ar}_{X,\epsilon}(r)$ closed curves of length at most ϵ . We refer to Section 2 for a precise definition. When X is a Carnot group G of step s , equipped with a Carnot metric d_c , it can be shown that

$$\text{Ar}_{(G,d_c),\frac{\epsilon}{n}}(r) \leq C \cdot n^{s+1} \tag{1}$$

(see Lemma 3.3).

Fix a closed Lipschitz curve γ in (G, d_c) of length r . Using the bound (1) we can construct nested subdivisions of γ as follows. Let $n \in \mathbb{N}$ be sufficiently large, only depending on α and C , and set $N = Cn^{s+1}$. Then there exist closed curves $\gamma_1^1, \dots, \gamma_N^1$ of length $n^{-1}r$ that subdivide γ . Each curve γ_j^1 can then be subdivided into curves $\gamma_{(j-1)N+1}^2, \dots, \gamma_{jN}^2$ of length $n^{-2}r$, and so on. We will construct the Hölder extension f of γ so that its image is the closure of the union $\bigcup_{i,j} \gamma_j^i$.

We start by constructing a family of nested discs in D^2 . Let $B_1^0 = D^2$ and let $B_1^1, \dots, B_N^1 \subset B_1^0$ be disjoint discs of equal radius ρ , where $\rho \approx N^{-\frac{1}{2}}$. We repeat the process on each disc; for each $i \geq 0$ and $j = 1, \dots, N^i$, we choose N disjoint discs of radius ρ^{i+1} inside B_j^i and label them $B_{(j-1)N+1}^{i+1}, \dots, B_{jN}^{i+1}$. Let $M_i = D^2 \setminus \bigcup_j \text{int } B_j^i$, so that

$$S^1 = M_0 \subset M_1 \subset M_2 \subset \dots \subset D^2$$

is an increasing sequence of subsets and $\bigcup_i M_i$ is the complement of a Cantor set K .

Next, we define f on the boundaries of the discs so that $f(\partial D^2) = \gamma$ and $f(\partial B_j^i) = \gamma_j^i$ for every i and j . We extend f to the rest of D^2 by noting that the complement $D^2 \setminus \bigcup_{i,j} \partial B_j^i$ consists of the Cantor set $K = \bigcap (D^2 \setminus M_i)$ and infinitely many connected components that are each homeomorphic to a genus 0 surface with $N+1$ boundary components. Let

$$S_j^i = B_j^i \setminus \bigcup_{m=(j-1)N+1}^{jN} \text{int } B_m^{i+1}$$

be one such component. Then f sends the outer boundary ∂B_j^i to γ_j^i and the N inner boundary components to N curves γ_m^{i+1} that subdivide γ_j^i . Consequently, we can extend f over S_j^i so that

$$f(S_j^i) \subset \bigcup_{m=(j-1)N+1}^{jN} \gamma_m^{i+1}$$

and f is Lipschitz on S_j^i . This defines f on $D^2 \setminus K$. If we construct the extensions to the S_j^i 's carefully, we can ensure that f is Hölder on $D^2 \setminus K$ and extend f continuously to K to obtain the desired map.

Note that f is far from injective. In fact, for any neighborhood U of K , the image $f(D^2 \setminus U)$ has Hausdorff and topological dimension 1; actually, the restriction $f|_{D^2 \setminus U}$ factors through a graph.

This construction uses two main ideas: First, we can reduce the problem of constructing a Hölder extension of a closed curve γ to the subproblem of constructing Hölder extensions of each curve in a subdivision of γ . That is, we can extend γ by subdividing γ into $\gamma_1, \dots, \gamma_N$, constructing a map from an N -holed disc M to G that sends the outer boundary of M to γ and the boundaries of the holes to the γ_i 's, and constructing Hölder extensions of the γ_i 's. Second, we

don't need to actually solve any of the subproblems. As long as we can split the original problem into smaller and smaller subproblems, we can pass to a limit where all of them disappear.

We use a version of these ideas to construct the map from (\mathbb{H}, d_R) to (\mathbb{H}, d_c) in Theorem 1.2, but the extra dimension adds some complications. To simplify matters, consider constructing a map g from the Euclidean ball D^3 to (\mathbb{H}, d_c) . As in the two-dimensional case, one can construct a sequence of nested sets $M_1 \subset M_2 \subset \dots \subset D^3$, where M_i consists of D^3 with k_i balls $B_1^i, \dots, B_{k_i}^i$ removed. One could attempt to construct a Hölder map by following the outline of Theorem 1.1. That is, each ball B_j^i contains some collection of smaller balls C_1, \dots, C_m . If g is already defined on the outer sphere ∂B_j^i , one can extend it to $B_j^i \setminus \bigcup_k C_k$ so that it sends the inner spheres to a subdivision $g(\partial C_1), \dots, g(\partial C_m)$ of $g(\partial B_j^i)$. The images $g(\partial M_i)$ then form a sequence of finer and finer subdivisions of $g(\partial D^3)$, and we can extend the map to all of D^3 by passing to a limit.

The main difficulty with this outline is that the spheres and their subdivisions need to be parametrized by Hölder maps. These are more difficult to construct than the horizontal subdivisions of curves that we used in the two-dimensional case. That is, in the two-dimensional case, we reduced from a horizontal curve γ to its horizontal subdivision $\gamma_1, \dots, \gamma_N$ by constructing a map from a N -holed disc M to the horizontal graph $\bigcup_i \gamma_i$ that sends the outer boundary to γ and the inner boundaries to the γ_i . Since the edges of the graph are horizontal, the map on M can be taken to be Lipschitz and thus Hölder. Suppose instead that S is a sphere in \mathbb{H} , subdivided into spheres S_1, \dots, S_N . Then there are continuous maps from the N -holed ball M to $\bigcup S_i$ that send the outer boundary to S and the inner boundaries to the S_i 's, but those maps may not be Hölder, even if all of the S_i 's are images of Hölder spheres.

This is difficult to solve directly, so in Sections 4–5, we develop a different approach to constructing Hölder maps. We give a brief sketch. We start by choosing a $0 < s < 1$ and constructing a sequence of simplicial complexes X_i . We scale the metric on X_i so that each simplex is a regular simplex with sides of length s^i and ask that the X_i approximate (\mathbb{H}, d_c) more and more closely in the sense that there are $C > 1$ and bilipschitz homeomorphisms $\iota_i: X_i \rightarrow (\mathbb{H}, d_R)$ that satisfy

$$C^{-1}d_{X_i}(x, y) - Cs^i \leq d_c(\iota_i(x), \iota_i(y)) \leq Cd_{X_i}(x, y) + Cs^i$$

for all $x, y \in X_i$.

We construct a map $P: X_0 \rightarrow \mathbb{H}$ by composing a sequence of Lipschitz cellular maps $Q_i: X_i \rightarrow X_{i+1}$ that are *admissible maps*. An admissible map is a cellular map such that for each $d > 0$ and each d -cell $\delta \in \mathcal{F}^d(X_i)$, there is a collection of closed balls $B_1^\delta, \dots, B_b^\delta \subset \delta$ with disjoint interiors such that Q_i sends each B_j^δ homeomorphically to a d -cell of X_{i+1} and collapses $\delta \setminus \bigcup_j B_j^\delta$ to a lower-dimensional set, i.e.,

$$Q_i(\delta \setminus \bigcup_j B_j^\delta) \subset X_{i+1}^{(d-1)},$$

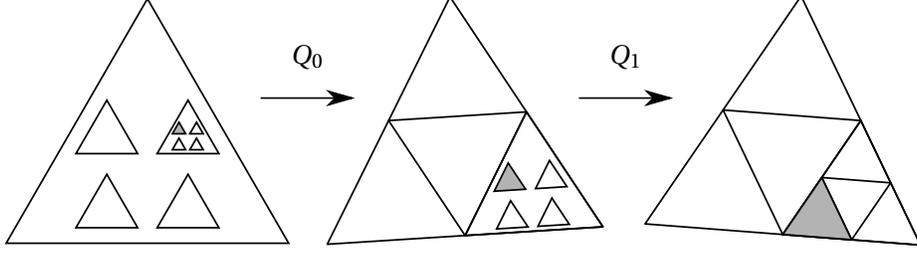


FIGURE 1. The maps P_i and Q_i send uncollapsed balls (small triangles) to cells of X_{i+1} by similarities and collapse the region outside the uncollapsed balls to lower-dimensional skeleta. The four mid-sized triangles on the left are the uncollapsed balls of $P_1 = Q_0$ and the small triangles are uncollapsed balls of $P_2 = Q_1 \circ Q_0$. Note that P_i collapses more and more of X_0 to the 1-skeleton of X_i as i increases.

where $X_{i+1}^{(d-1)}$ denotes the $(d-1)$ -skeleton of X_{i+1} . We call the B_i^δ 's *uncollapsed balls*, and we construct Q_i so that the B_i^δ 's are pairwise disjoint, as in Figure 1.

Let

$$P_i = Q_{i-1} \circ \cdots \circ Q_0: X_0 \rightarrow X_i.$$

If the Q_i 's have small displacement in the sense that $d_c(\iota_{i+1}(Q_i(x)), \iota_i(x)) \lesssim s^i$, then $\iota_i \circ P_i$ converges uniformly to a locally Hölder map $P: X_0 \rightarrow \mathbb{H}$; indeed, $d_c(\iota_i \circ P_i, P) \lesssim s^i$. Since ι_0 is a bilipschitz homeomorphism from $X_0 \rightarrow (\mathbb{H}, d_R)$, the map $g = P \circ \iota_0^{-1}$ will also be locally Hölder.

In fact, this construction results in a map based on nested subdivisions of curves and spheres, like the one outlined above. As i increases, P_i sends larger and larger pieces of X_0 to the 1-skeleton of X_i . That is, for any d , let $M_i^{(d)} = P_i^{-1}(X_i^{(d)})$ be the preimage of the d -skeleton of X_i . The cellularity of Q_i implies that $M_i^{(1)} \subset M_{i+1}^{(1)}$ for all i . Let δ be a 2-cell of X_0 and let $M_{\delta,i} := \delta \cap M_i^{(1)}$. This is the complement of the uncollapsed discs of P_i . Each uncollapsed disc of P_{i+1} is contained in an uncollapsed disc of P_i (see Figure 1), so $\bigcup_i M_i^{(1)}$ contains all of δ except for a Cantor set.

Let $\gamma := P(\partial\delta)$; when s is sufficiently small, γ will be a $(1-\epsilon)$ -Hölder curve in \mathbb{H} with diameter roughly 1. Such curves are generally not differentiable, so they do not have horizontal velocities, but several authors have noted that they satisfy an integral version of the horizontality condition, see for instance [11, Lemma 3.1]. Let $M_{\delta,i} := \delta \cap M_i^{(1)}$ be the complement of the uncollapsed discs of P_i . Then $P_i(M_{\delta,i}) \subset X_i^{(1)}$, and there is an N such that $M_{\delta,i}$ is an N -holed disc. The outer boundary of $M_{\delta,i}$ is $\partial\delta$, so the images of the inner boundaries of $M_{\delta,i}$ subdivide γ into closed curves $\gamma_1, \dots, \gamma_N$, each of diameter roughly s^i . As i increases, these subdivisions grow finer and finer, and as in the proof of Theorem 1.1, the restriction $P|_\delta$ is Hölder.

We see the same picture in the 3-cells. Let Δ be a 3-cell of X_0 and let $S := P(\partial\Delta)$ be a Hölder sphere. Let $M_{\Delta,i} := \Delta \cap M_i^{(2)}$; as before, this is Δ minus some uncollapsed balls. Since $P_i(M_{\Delta,i})$ is 2-dimensional, P sends the outer boundary of $M_{\Delta,i}$ to S and sends the inner boundaries of $M_{\Delta,i}$ to a subdivision of S into Hölder spheres of diameter roughly s^i . As i increases, the subdivisions grow finer and finer, so that ultimately $P|_{\Delta}$ is Hölder.

Though we can write P in terms of subdivisions, our bound on the Hölder exponent takes a different approach. We sketch the bound here; for full details, see Section 5. The main idea of the Hölder bound is that because $d_c(t_i \circ P_i, P) \lesssim s^i$, the Hölder exponent of P is bounded by the growth rate of $\text{Lip}(P_i)$, and we control $\text{Lip}(P_i)$ by placing geometric conditions on the Q_i 's.

First, we require that Q_i is Lipschitz on $X_i^{(1)}$ with Lipschitz constant independent of s . This ensures that $\text{Lip}(P_i|_{X_0^{(1)}})$ does not grow too quickly. Second, we require that there is an $0 < a < s$ such that for all i , the uncollapsed balls of Q_i are disjoint regular simplices of radius as^i and that Q_i sends each uncollapsed ball to a cell of X_{i+1} by a similarity with scaling factor $\frac{s}{a}$.

The Dehn function of the Heisenberg group puts an upper bound on the size of a . That is, if δ is a 2-cell of X_i , then $Q_i(\partial\delta)$ is a curve of length roughly s^i in X_{i+1} , which is to say that $Q_i(\partial\delta)$ is a loop of roughly s^{-1} edges. Filling such a loop with a disc typically requires on the order of s^{-3} triangles, so $Q_i|_{\delta}$ might have roughly s^{-3} uncollapsed balls. If these all have radius as^i and are contained in a ball of radius s^i , then $a \lesssim s^{\frac{3}{2}}$. In fact, we may choose Q_i so that $a = cs^{\frac{3}{2}}$ for some c independent of s .

We claim that there is a C such that

$$\left(\frac{s}{a}\right)^i \leq \text{Lip}(P_i) \leq C \left(\frac{s}{a}\right)^i.$$

The lower bound follows from considering uncollapsed balls. If B is an uncollapsed ball of P_i , then for each $0 \leq j < i$, $P_j(B)$ is contained in an uncollapsed ball of Q_j . Since Q_j scales $P_j(B)$ by $\frac{s}{a}$, the composition P_i scales B by a factor of $\left(\frac{s}{a}\right)^i$.

The upper bound is more difficult, but the key idea is that for any $x \in X_0$, we may consider the sequence $D_i(x)$ of integers such that $D_i(x)$ is the dimension of the cell of X_i whose interior contains $P_i(x)$. This sequence is non-increasing, and if $D_i(x) > 1$ and $P_i(x)$ is not contained in an uncollapsed ball of Q_i , then $D_{i+1}(x) < D_i(x)$. Since $D_i(x) \leq 3$, this can happen at most three times. That is, for all but at most three values of i , either $P_i(x)$ lies in an uncollapsed ball of Q_i or $P_i(x) \in X_i^{(1)}$. But Q_i is $\frac{s}{a}$ -Lipschitz on uncollapsed balls and uniformly Lipschitz on $X_i^{(1)}$, so

$$\text{Lip}(P_i) \leq C \left(\frac{s}{a}\right)^i$$

for some C depending on $\max_i \text{Lip}(Q_i)$.

We use this bound and the bound $d_c(t_i \circ P_i, P) \lesssim s^i$ to show that P is locally $\frac{\log s}{\log a}$ -Hölder, see Section 5. Since $a \approx s^{\frac{3}{2}}$, $\frac{\log s}{\log a}$ approaches $\frac{2}{3}$ as $s \rightarrow 0$.

1.1. Structure of paper. In Section 2 we establish basic notation used throughout the text. We furthermore recall the precise definition of the coarse Dehn function $\text{Ar}_{X,\varepsilon}$ and the definition of an admissible map between complexes. Finally, we collect some background on Carnot groups needed for the rest of the paper. In Section 3 we deduce Theorem 1.1 from the extension results proved in [12].

In Section 4 we construct the triangulations X_i of \mathbb{H} and the maps $Q_i: X_i \rightarrow X_{i+1}$ used in the sketch. In practice, the X_i 's are all scalings of one triangulation X_0 and the Q_i 's are all scalings of one map $X_0 \rightarrow X_1$. We compose scalings of Q to produce a map P from X_0 to (\mathbb{H}, d_c) . We give two constructions of Q ; to prove Theorem 1.2, we mainly need bounds on the Lipschitz constant of Q and the volume of its cells, but to prove Theorem 1.4, we need some additional conditions, which are provided by Lemma 4.3 and Lemma 4.4. These lemmas deal with approximating continuous maps to arbitrary simplicial complexes by admissible maps, and we defer their proofs to Appendix A.

In Section 5 we show that the map P satisfies the properties of Theorem 1.2 and prove Theorem 1.3. In Section 6 we analyze the tangent cone of P and prove Theorem 1.4.

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2. PRELIMINARIES

2.1. Basic definitions and notation. The Euclidean metric on \mathbb{R}^d will be denoted by $|\cdot|$. For $x \in \mathbb{R}^d$ and $r > 0$, we let $B(x, r) := \{x \in \mathbb{R}^d : |x| \leq r\}$ and let $D^d := B(0, 1)$ be the closed unit ball.

Let (X, d_X) be a metric space. The length of a curve c in X is denoted by $\ell_{d_X}(c)$ or simply by $\ell(c)$. Let (Z, d_Z) be another metric space. A map $\varphi: Z \rightarrow X$ is called (λ, α) -Hölder if

$$d_X(\varphi(z), \varphi(z')) \leq \lambda \cdot d_Z(z, z')^\alpha$$

for all $z, z' \in Z$. We say φ is α -Hölder if it is (λ, α) -Hölder for some $\lambda > 0$.

Let $U \subset \mathbb{R}^d$ be open. The (parametrized) volume of a Lipschitz map $\varphi: U \rightarrow X$ is defined by

$$\text{vol}^d(\varphi) = \text{vol}^d(\varphi; X) = \int_X \#\{z \in U : \varphi(z) = x\} d\mathcal{H}^d(x),$$

where \mathcal{H}^d denotes the d -dimensional Hausdorff measure on X . If X is a Riemannian manifold or a CW complex with piecewise Riemannian metric then, by the area formula, $\text{vol}^d(\varphi)$ agrees with the volume defined by integrating the jacobian of the derivative of φ .

2.2. Admissible maps. Let X be a simplicial complex. For $d \geq 0$ we denote by $\mathcal{F}^d(X)$ the set of closed d -simplices in X and by $X^{(d)}$ the d -skeleton of X . Let $\mathcal{F}(X)$ be the set of cells in X of every dimension.

We say that a continuous map $f: D^n \rightarrow X^{(n)}$ is an *admissible map* if there is a collection of closed subsets $B_1, \dots, B_k \subset D^n$, called *uncollapsed balls*, such that the interiors of the B_i are pairwise disjoint, f sends each B_i homeomorphically to an n -cell of X , and $f(D^n \setminus \bigcup B_i) \subset X^{(n-1)}$. We call k the *admissible volume* of f , denoted $\text{avol}(f; X)$ or simply $\text{avol}(f)$ when it is clear what the target complex is. This differs slightly from the definition given in [2], where the target space is a CW complex and the uncollapsed balls are open balls.

Given simplicial complexes X and Y , we call a continuous map $f: X \rightarrow Y$ *admissible* if f is cellular (i.e., $f(X^{(i)}) \subset Y^{(i)}$ for all i) and for every d -cell $\sigma \in \mathcal{F}^d(X)$ with $d > 0$, the map $f|_\sigma$ is admissible in the sense above. Note that if X , Y , and Z are simplicial complexes and $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are admissible, then $g \circ f$ is admissible.

2.3. The coarse Dehn function. The coarse Dehn function which we introduce here is a slight variant of Gromov's definition of mesh function given in [6], see also [3, III.H.2.1] or [4]. A triangulation of the closed unit disc D^2 is a homeomorphism from D^2 to a combinatorial 2-complex τ in which every 2-cell is a triangle. We endow D^2 with the cell-structure of τ . Let X be a geodesic metric space and $c: S^1 \rightarrow X$ a Lipschitz curve. Let $\epsilon > 0$. An ϵ -filling of c is a pair (P, τ) consisting of a triangulation τ of D^2 and a continuous map $P: \tau^{(1)} \rightarrow X$ such that $P|_{S^1} = c$ and such that $\ell(P|_{\partial F}) \leq \epsilon$ for every triangle F in τ . The ϵ -area of c is

$$\text{Ar}_\epsilon(c) := \min \{ |\tau| \mid (P, \tau) \text{ is an } \epsilon\text{-filling of } c \}.$$

Here, $|\tau|$ denotes the number of triangles in τ . If no ϵ -filling exists then we set $\text{Ar}_\epsilon(c) := \infty$. The ϵ -coarse Dehn function of X is defined by

$$\text{Ar}_{X,\epsilon}(r) := \sup \{ \text{Ar}_\epsilon(c) \mid c: S^1 \rightarrow X \text{ Lipschitz, } \ell(c) \leq r \}$$

for $r > 0$.

It is not difficult to show that the asymptotic growth of $\text{Ar}_{X,\epsilon}(r)$ is a quasi-isometry invariant. Moreover, under mild conditions on the underlying space the function $\text{Ar}_{X,\epsilon}(r)$ has the same growth as the so-called Lipschitz Dehn function $\delta_X^{\text{Lip}}(r)$ (see [12]). We will however not need this. Recall here that the Lipschitz Dehn function $\delta_X^{\text{Lip}}(r)$ measures how much (parametrized) area is needed to fill a curve of length r by a Lipschitz disc in X .

2.4. Carnot groups and Carnot-Carathéodory distance. A connected, simply-connected nilpotent Lie group G is called a *Carnot group* if its Lie algebra \mathfrak{g} admits a stratification into subspaces

$$\mathfrak{g} = V_1 \oplus \dots \oplus V_k$$

such that $[V_1, V_i] = V_{i+1}$ for all $i = 1, \dots, k-1$ and $[V_1, V_k] = 0$. Here, $[V_1, V_i]$ is the subspace spanned by the elements $[v, v']$ with $v \in V_1$ and $v' \in V_i$. The number k is called the *step* or *nilpotency class* of G .

As G is a simply-connected nilpotent Lie group, the exponential map $\exp: \mathfrak{g} \rightarrow G$ is a diffeomorphism. Since it is Carnot, G comes with a family of scaling homomorphisms $\delta_r: G \rightarrow G$ for $r \geq 0$. They are given by

$$\delta_r(\exp(v)) = \exp\left(\sum_{i=1}^k r^i v_i\right)$$

for $v = v_1 + \dots + v_k$ with $v_i \in V_i$. The derivative $D_x \delta_r$ of δ_r at $x \in G$ is given by

$$D_x \delta_r(D_0 L_x(v)) = r^i \cdot D_0 L_{\delta_r(x)}(v) \quad (2)$$

for all $v \in V_i$, where L_x denotes left-translation by x and 0 is the identity element of G .

The *horizontal bundle* TH is the subbundle of TG obtained by left-translation of the subspace V_1 . Let g_0 be a left-invariant Riemannian metric on G such that at 0 the subspaces V_i are pairwise orthogonal with respect to g_0 . Let d_R be the distance coming from g_0 . A curve $c: [0, 1] \rightarrow G$ which is absolutely continuous with respect to d_R is called *horizontal* if $c'(t) \in T_{c(t)}H$ for almost every $t \in [0, 1]$. The *Carnot-Carathéodory distance* (or *Carnot metric* for short) associated with d_R is defined by

$$d_c(x, y) := \inf\{\ell_R(c) \mid c: [0, 1] \rightarrow G \text{ horizontal curve from } x \text{ to } y\}.$$

Here, $\ell_R(c)$ denotes the length of c with respect to the metric d_R . It can be shown that the Carnot-Carathéodory distance is always finite and thus defines a metric. It is moreover left-invariant and 1-homogeneous with respect to the scaling automorphisms; that is,

$$d_c(\delta_r(x), \delta_r(x')) = r d_c(x, x')$$

for all $x, x' \in G$ and all $r > 0$. The topologies induced on G by d_R and d_c agree, however it is well-known that these metrics are not even locally bilipschitz equivalent (except when G is abelian). Throughout this article, when we talk about a Carnot metric on G we always mean one which is associated with a distance coming from a left-invariant Riemannian metric such that at 0 the subspaces V_i are orthogonal.

We will need the following simple facts.

Lemma 2.1. *Let G be a Carnot group of step k . Let d_R be the distance coming from a left-invariant Riemannian metric on G and let d_c be the associated Carnot metric. Then there exists $L \geq 1$ such that*

- (i) $d_R(x, x') \leq d_c(x, x') \leq L \cdot d_R(x, x') + L$ for all $x, x' \in G$.
- (ii) for every $x \in G$ the curve $\gamma(r) := \delta_r(x)$ satisfies

$$\ell_R(\gamma|_{[s,t]}) \leq L \cdot (d_c(x, 0) + 1)^k \cdot |t - s|$$

for all $0 \leq s \leq t \leq 1$.

We provide a short proof for the convenience of the reader, compare with [14].

Proof. The first inequality in (i) is clear. To prove the second inequality in (i), notice that

$$L := \sup\{d_c(y, z) : d_R(y, z) \leq 1\} = \sup\{d_c(0, x) : d_R(0, x) \leq 1\} < \infty.$$

Let $x, x' \in G$ be distinct and set $l := d_R(x, x')$. Let $\gamma: [0, l] \rightarrow (G, d_R)$ be a geodesic from x to x' , parametrized by arc-length. Let m be the largest integer smaller than l . Then

$$d_c(x, x') \leq \sum_{i=1}^m d_c(\gamma(i-1), \gamma(i)) + d_c(\gamma(m), \gamma(l)) \leq L \cdot d_R(x, x') + L,$$

which proves the second inequality in (i).

We now prove statement (ii). For $x \in G$ define the curve $\gamma_x: [0, 1] \rightarrow G$ by $\gamma_x(r) := \delta_r(x)$. Notice that

$$L := \sup\{\|\gamma'_x(r)\| : d_R(0, x) \leq 1, r \in [0, 1]\} < \infty.$$

Let $x \in G$ and $0 \leq s < t \leq 1$. Suppose that $d_R(0, x) \leq 1$. Then

$$\ell_R(\gamma_x|_{[s,t]}) = \int_s^t \|\gamma'_x(r)\| dr \leq L \cdot |t-s| \leq L \cdot (d_c(x, 0) + 1)^k \cdot |t-s|.$$

Now suppose that $d_R(0, x) > 1$. Let β be a geodesic with respect to d_R from 0 to x . By (2), if $\xi \in (0, 1)$, then

$$\ell_R(\delta_\xi \circ \beta) \leq \xi \ell_R(\beta) = \xi d_R(0, x).$$

Let $\xi = (d_R(0, x))^{-1}$, so that $d_R(0, \delta_\xi(x)) \leq \ell_R(\delta_\xi \circ \beta) \leq 1$.

By (2), $\delta_{\xi^{-1}}$ distorts length by at most a factor of ξ^{-k} , so

$$\ell_R(\gamma_x|_{[s,t]}) \leq \xi^{-k} \cdot \ell_R(\delta_\xi \circ \gamma_x|_{[s,t]}) = \xi^{-k} \cdot \ell_R(\gamma_{\delta_\xi(x)}|_{[s,t]}) \leq L \cdot \xi^{-k} \cdot |t-s|$$

and thus

$$\ell_R(\gamma_x|_{[s,t]}) \leq L \cdot (d_R(0, x))^k \cdot |t-s| \leq L \cdot (d_c(0, x))^k \cdot |t-s|.$$

This proves statement (ii). \square

Finally, we consider the first Heisenberg group which is the Carnot group of step 2 given by $\mathbb{H} := \mathbb{R}^3$ with the multiplication

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy'). \quad (3)$$

A basis of left-invariant vector fields on \mathbb{H} is given by

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z},$$

and the Lie algebra \mathfrak{h} of \mathbb{H} has the stratification $\mathfrak{h} = \text{span}\{X, Y\} \oplus \text{span}\{Z\}$. We denote by $\mathbb{H}_{\mathbb{Z}} := \mathbb{Z}^3$ the integer lattice in \mathbb{H} .

Let g_0 be a left-invariant Riemannian metric on \mathbb{H} such that at 0 the subspaces V_1 and V_2 are orthogonal. It follows from (2) that for $r \geq 1$ the scaling automorphism δ_r on \mathbb{H} distorts 2-dimensional area in (\mathbb{H}, g_0) at most by a factor r^3 and 3-dimensional volume by a factor of exactly r^4 .

3. FROM TRIANGULATIONS TO HÖLDER EXTENSIONS

In this section we prove Theorem 1.1. The main ingredient is the next theorem, which follows from the results proved in [12].

Theorem 3.1. *Let X be a complete, geodesic metric space. Suppose there exist $K \geq 1$ and $\beta \geq 2$ such that for all $r > 0$ and $n \in \mathbb{N}$ we have*

$$\text{Ar}_{X, \frac{r}{n}}(r) \leq K \cdot n^\beta.$$

Then for every $\alpha < \frac{2}{\beta}$ the pair (\mathbb{R}^2, X) has the α -Hölder extension property.

Proof. Let $K \geq 1$ and $\beta \geq 2$ be as in the statement of the theorem and let $\alpha < \frac{2}{\beta}$. Choose n so large that

$$\eta := \frac{\log n}{\log(2\sqrt{Kn^\beta})} = \frac{\log n}{\log 2 + \frac{1}{2} \cdot \log K + \frac{\beta}{2} \cdot \log n} > \alpha.$$

It follows from Proposition 7.4 in [12] that the space X is η -Hölder 1-connected and, in particular, also α -Hölder 1-connected. Thus, there exists $C \geq 1$ such that every λ -Lipschitz curve $c: S^1 \rightarrow X$ admits a $(C\lambda, \alpha)$ -Hölder extension to D^2 . Now, Theorem 6.4 in [12] implies that the pair (\mathbb{R}^2, X) has the α -Hölder extension property, which completes the proof. \square

We now apply the theorem to Carnot groups:

Corollary 3.2. *Let G be a Carnot group and let d_c be a Carnot metric on G . Let $K \geq 1$ and $\beta \geq 2$ be such that for all $r \geq 1$,*

$$\text{Ar}_{(G, d_c), 1}(r) \leq K \cdot r^\beta.$$

Then the pair $(\mathbb{R}^2, (G, d_c))$ has the α -Hölder extension property for all $\alpha < \frac{2}{\beta}$.

Proof. Let $n \in \mathbb{N}$ and $r > 0$. Since scaling automorphisms are 1-homogeneous with respect to d_c we have

$$\text{Ar}_{(G, d_c), \frac{r}{n}}(r) = \text{Ar}_{(G, d_c), 1}(n) \leq K \cdot n^\beta$$

and hence the corollary follows from Theorem 3.1. \square

Now, Theorem 1.1 follows from the corollary above together with the next lemma.

Lemma 3.3. *Let G be a Carnot group of step k and let d_c be a Carnot metric on G . Then there exists $K \geq 1$ such that*

$$\text{Ar}_{(G, d_c), 1}(r) \leq K \cdot r^{k+1}$$

for all $r \geq 1$.

The lemma could easily be deduced from the upper bound on the growth of the Lipschitz Dehn function proved in [14] and the fact that the coarse Dehn function has the same growth as the Lipschitz Dehn function (see e.g. [12]). We prefer to give a direct proof here.

Proof. Let d_R be a left-invariant Riemannian metric on G such that d_c is the Carnot metric associated with d_R . Let $L \geq 1$ be as in Lemma 2.1.

Using the scaling homomorphisms it suffices to show that there exists K such that

$$\text{Ar}_{(G,d_c),6L}(r) \leq K \cdot r^{k+1} \quad (4)$$

for all $r \geq 6L$. In order to prove (4) let $c: S^1 \rightarrow (G, d_c)$ be a Lipschitz curve of length $r \geq 6L$. We may assume that c has constant speed and that c passes through $0 \in G$.

We construct a $6L$ -filling (P, τ) of c as follows. Let $M \in \mathbb{N}$ be the smallest integer larger than $\frac{r}{L}$ and set $v_l := e^{\frac{2\pi l i}{M}}$ for $l = 0, \dots, M-1$. Furthermore, let m be the smallest integer larger than $L \cdot (r+1)^k$. We define a triangulation τ of D^2 as follows. The set of vertices of τ is the subset of D^2 given by the points $v_{0,0} = 0$ and $v_{j,l} := \frac{j}{m} \cdot v_l$ for $j = 1, \dots, m$ and $l = 0, \dots, M-1$. For $l = 0, \dots, M-1$ we add edges between the points $v_{j,l}$ and $v_{j+1,l}$ for $j = 0, \dots, m-1$, where we have set $v_{0,l} = v_{0,0}$, between $v_{j,l}$ and $v_{j,l+1}$ for $j = 1, \dots, m$, where $v_{j,M} = v_{j,0}$, and between $v_{j,l}$ and $v_{j+1,l+1}$ for $j = 1, \dots, m-1$. This gives a triangulation τ of D^2 consisting of less than $2mM$ triangles and thus $|\tau| \leq Kr^{k+1}$, where K only depends on L and k . We now define a map $P: \tau^{(1)} \rightarrow G$ by setting $P|_{S^1} := c$ and by defining $P(v_{j,l}) := \delta_{\frac{j}{m}}(c(v_l))$. We extend P to $\tau^{(1)}$ by mapping the edges in $\tau^{(1)} \setminus S^1$ to geodesics with respect to the d_c -distance. It remains to show that $\ell_c(P|_{\partial F}) \leq 6L$ for every triangle F in τ , where ℓ_c denotes the length with respect to the Carnot metric d_c . For this we use the estimates in Lemma 2.1. Firstly, we have

$$d_c(P(v_{j,l}), P(v_{j,l+1})) = \frac{j}{m} \cdot d_c(c(v_l), c(v_{l+1})) \leq \frac{j}{m} \cdot \ell_c(c|_{[v_l, v_{l+1}]}) \leq L.$$

Secondly, writing $\gamma_l(t) := \delta_t(c(v_l))$ we obtain that

$$d_c(P(v_{j,l}), P(v_{j+1,l})) \leq L \cdot \ell_R(\gamma_l|_{[\frac{j}{m}, \frac{j+1}{m}]}) + L \leq L^2 \cdot (r+1)^k \cdot \frac{1}{m} + L \leq 2L.$$

The two inequalities together finally yield

$$d_c(P(v_{j,l}), P(v_{j+1,l+1})) \leq 3L,$$

from which it follows that $\ell_c(P|_{\partial F}) \leq 6L$ for every triangle F in τ . This proves (4). \square

4. CONSTRUCTING HÖLDER MAPS FROM ADMISSIBLE MAPS

While one can construct Hölder maps from discs to Carnot groups by constructing a sequence of nested subdivisions, it is difficult to generalize this construction to higher-dimensional domains. Instead, we will construct Hölder maps using admissible maps.

As in the introduction, we will construct a sequence of simplicial complexes X_0, X_1, \dots , equipped with path metrics, a sequence of maps $\iota_i: X_i \rightarrow \mathbb{H}$, and a sequence of admissible maps $Q_i: X_i \rightarrow X_{i+1}$. We then define $P_i: X_0 \rightarrow X_i$,

$$P_i := Q_{i-1} \circ \dots \circ Q_0,$$

and let $P: X_0 \rightarrow \mathbb{H}$, $P := \lim_i \iota_i \circ P_i$. The map P is the desired Hölder map. In this section, we will give the details of the construction by defining X_i , ι_i , and Q_i , and in the next section, we will show that P is Hölder.

We will define the X_i 's to be scalings of a single triangulation X of \mathbb{H} . Recall that $\mathbb{H}_{\mathbb{Z}} \subset \mathbb{H}$ is the lattice of integer points. Let X be an $\mathbb{H}_{\mathbb{Z}}$ -equivariant piecewise-linear (PL) triangulation of \mathbb{H} . That is, X is a tuple $X = (Y, \iota, \zeta)$ of a simplicial complex Y , a PL homeomorphism $\iota: Y \rightarrow \mathbb{H}$, and a left $\mathbb{H}_{\mathbb{Z}}$ -action $\zeta[g]: Y \rightarrow Y$, $g \in \mathbb{H}_{\mathbb{Z}}$, such that $\iota(\zeta[g](x)) = g\iota(x)$ for all $g \in \mathbb{H}_{\mathbb{Z}}$, $x \in Y$. The existence of such a ι relies on the fact that by (3), if $\alpha: \mathbb{R}^n \rightarrow \mathbb{H}$ is affine, then for any $g \in \mathbb{H}$, the translate $v \mapsto g\alpha(v)$ is also affine. This induces a subdivision of \mathbb{H} into simplices $(\iota(\sigma))_{\sigma \in \mathcal{F}(Y)}$. Since ι is equivariant, the action of $\mathbb{H}_{\mathbb{Z}}$ on \mathbb{H} permutes the simplices in this subdivision. We write $x \in X$ to denote a point in the underlying simplicial complex Y of X and define $gx = \zeta[g](x)$ for $g \in \mathbb{H}_{\mathbb{Z}}$, $x \in X$. Let d_{Δ} be the standard metric on X , i.e., the path metric such that every simplex of X is isometric to the unit regular simplex. Since ι is PL, it is a bilipschitz equivalence from (X, d_{Δ}) to (\mathbb{H}, d_R) , where d_R is the Riemannian metric on \mathbb{H} .

Let $n > 1$ be a large integer to be chosen later, and let $s = n^{-1}$. For each i and for each $i \geq 0$, let X_i be X scaled by s^i . That is, $X_i = (Y, \iota_i, \zeta_i)$, where $\iota_i = \delta_{n^{-i}} \circ \iota$ and $\zeta_i[g](x) = \zeta[\delta_{n^i}(g)](x) = \delta_{n^i}(g)x$ for all $g \in \mathbb{H}_{\mathbb{Z}}$, $x \in X$. Then X_i is a triangulation of \mathbb{H} , and for each $\sigma \in \mathcal{F}(X_i)$ with $\dim \sigma > 0$, the simplex $\iota_i(\sigma)$ has d_c -diameter roughly s^i . Furthermore, ι_i is $\mathbb{H}_{\mathbb{Z}}$ -equivariant, i.e., for all $x \in X_i$,

$$\iota_i(\zeta_i[g](x)) = \delta_{n^{-i}}(\iota(\zeta[\delta_{n^i}(g)](x))) = \delta_{n^{-i}}(\delta_{n^i}(g)\iota(x)) = g\delta_{n^{-i}}(\iota(x)) = g\iota_i(x).$$

As before, for $x \in X_i$, we let gx denote $\zeta_i[g](x)$.

We equip X_i with the metric $d_i = s^i d_{\Delta}$. Since ι is a bilipschitz equivalence from d_{Δ} to d_c and d_c is quasi-isometric to d_R (Lemma 2.1), ι is a quasi-isometry from (X, d_{Δ}) to (\mathbb{H}, d_c) . That is, there is a $k > 1$ such that for any $x, y \in X_0$,

$$k^{-1}d_{\Delta}(x, y) - k \leq d_c(\iota(x), \iota(y)) \leq kd_{\Delta}(x, y) + k. \quad (5)$$

By the scale-invariance of d_c , $d_c(\iota_i(x), \iota_i(y)) = s^i d_c(\iota(x), \iota(y))$, so multiplying (5) by s^i gives

$$k^{-1}d_i(x, y) - ks^i \leq d_c(\iota_i(x), \iota_i(y)) \leq kd_i(x, y) + ks^i. \quad (6)$$

That is, (X_i, d_i) approximates (\mathbb{H}, d_c) more and more closely as i increases.

We will construct $P: X_0 \rightarrow \mathbb{H}$ from a sequence of $\mathbb{H}_{\mathbb{Z}}$ -equivariant admissible maps $Q_i: X_i \rightarrow X_{i+1}$. Since all of the X_i 's have the same underlying simplicial complex, we can construct the Q_i 's as a scaling of a map $Q: X \rightarrow X$ which is equivariant as a map from X_0 to X_1 , that is, $Q(\zeta_0[g](x)) = \zeta_1[g](Q(x))$ for all $g \in \mathbb{H}_{\mathbb{Z}}$, $x \in X$. Since $\zeta_i[g](x) = \zeta[\delta_{n^i}(g)](x)$ for all $g \in \mathbb{H}_{\mathbb{Z}}$, $x \in X$, we have $Q(\zeta_i[g](x)) = \zeta[\delta_{n^i}(g)](Q(x))$ and

$$Q(\zeta_i[g](x)) = Q(\zeta[\delta_{n^i}(g)](x)) = \zeta[\delta_{n^{i+1}}(g)](Q(x)) = \zeta_{i+1}[g](Q(x))$$

for any $i \geq 0$. That is, if Q is $\mathbb{H}_{\mathbb{Z}}$ -equivariant as a map from X_0 to X_1 , then it is $\mathbb{H}_{\mathbb{Z}}$ -equivariant as a map from X_i to X_{i+1} for any $i \geq 0$.

We construct Q so that it satisfies the following properties. A *similarity* from one simplex to another is a map that scales the metric by a constant factor, i.e., a rotation or reflection followed by a scaling.

Lemma 4.1. *There are $c, n_0 > 1$ such that for every $n > n_0$ there is an admissible, surjective, degree-1, \mathbb{H}_Z -equivariant Lipschitz map $Q: X_0 \rightarrow X_1$ that has the following properties. Let $Q_i: (X_i, d_i) \rightarrow (X_{i+1}, d_{i+1})$ denote Q viewed as a map from X_i to X_{i+1} .*

- (1) For all i , $\text{Lip}(Q_i) = \text{Lip}(Q)$ and $\text{Lip}(Q_i|_{X_i^{(1)}}) < c$.
- (2) Q_i has small displacement. That is, for every $x \in X_i$,

$$d_c(\iota_i(x), \iota_{i+1}(Q_i(x))) < cs^i.$$

- (3) Let $a = c^{-1}n^{-\frac{3}{2}}$; note that $0 < a < s < 1$. For $\sigma \in \mathcal{F}^d(X_i)$ with $d = 2, 3$, every uncollapsed ball B of $Q_i|_\sigma$ is a regular simplex with edges of length as^i , i.e., B is a copy of σ scaled by a .
- (4) For any uncollapsed ball B , the restriction $Q_i|_B$ is a similarity from B to a simplex of X_{i+1} , with scale factor $\frac{s}{a}$.
- (5) There is an $r > 0$ independent of n such that for any $\sigma \in \mathcal{F}^3(X_0)$, there is a $q \in \mathbb{H}$ such that

$$d_c(q, \iota_1(Q(X_0 \setminus \sigma))) > r.$$

Before we prove the lemma, we will sketch how to use the lemma to construct Hölder maps. Indeed, the map in Theorem 1.2 can be constructed from a map Q_i that satisfies conditions (1), (3), (4), and the condition that there is some $D(n) > 0$ such that

$$d_c(\iota_i(x), \iota_{i+1}(Q_i(x))) < D(n)s^i \quad \text{for all } x \in X. \quad (7)$$

This is a version of (2) in which the bound on displacement is allowed to depend on n .

Given Q and Q_i satisfying these conditions, we let $P_i: X_0 \rightarrow X_i$, $P_i = Q_{i-1} \circ \dots \circ Q_0$. We claim that $(\iota_i \circ P_i)_i$ converges uniformly as $i \rightarrow \infty$. By equation (7), for any $x \in X_0$, we have

$$d_c(\iota_i(P_i(x)), \iota_{i+1}(P_{i+1}(x))) = d_c(\iota_i(P_i(x)), \iota_{i+1}(Q_i(P_i(x)))) < D(n)s^i,$$

so $(\iota_i \circ P_i)_i$ is uniformly Cauchy. Let $P: X_0 \rightarrow \mathbb{H}$, $P = \lim_i \iota_i \circ P_i$. Since $s \leq \frac{1}{2}$,

$$d_c(\iota_i(P_i(x)), P(x)) < \frac{D(n)s^i}{1-s} \leq 2D(n)s^i. \quad (8)$$

Combining (6) and (8) with a bound on the growth of the Lipschitz constants of the P_i 's leads to Hölder bounds on P . For example, condition (1) of Lemma 4.1 implies the following bound.

Lemma 4.2. *Let Q_i , P_i , and P be as above and let $\epsilon > 0$. If n is sufficiently large, then $P|_{X_0^{(1)}}$ is locally $(1 - \epsilon)$ -Hölder.*

Proof. Let $x, y \in X_0^{(1)}$ such that $0 < d_0(x, y) < 1$. Let $\rho = \frac{s}{c} < 1$ and let $i \geq 0$ be the integer such that $\rho^{i+1} \leq d_0(x, y) < \rho^i$. By condition (1) of Lemma 4.1 and the fact that Q_i is cellular, we have $\text{Lip}(P_i|_{X_0^{(1)}}) \leq c^i$. Let k be as in (6). Then by (8),

$$\begin{aligned} d_c(P(x), P(y)) &\leq 4D(n)s^i + d_c(\iota_i(P_i(x)), \iota_i(P_i(y))) \\ &\leq 4D(n)s^i + kc^i\rho^i + ks^i = \frac{4D(n) + 2k}{s}s^{i+1} \leq K' d_0(x, y)^{\frac{\log s}{\log \rho}}, \end{aligned}$$

where $K' = K'(n) = \frac{4D(n) + 2k}{s}$. As $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \frac{\log s}{\log \rho} = \lim_{n \rightarrow \infty} \frac{-\log n}{-\log n - \log c} = 1.$$

When n is large enough that $\frac{\log s}{\log \rho} > 1 - \epsilon$,

$$d_c(P(x), P(y)) \leq K' d_0(x, y)^{\frac{\log s}{\log \rho}} \leq K' d_0(x, y)^{1-\epsilon},$$

as desired. \square

That is, a bound of the form $\text{Lip}(P_i) \lesssim K(n)L^i$ implies that P is locally $\frac{\log s}{\log \frac{s}{L}}$ -Hölder. To prove Theorem 1.2, it suffices to show that $\text{Lip}(P_i) \lesssim K(n) \cdot (\frac{s}{a})^i$ for some $K(n)$, and we will use conditions (3) and (4) to prove this bound in Section 5.

When n is sufficiently large, condition (5) lets us show that for any $\sigma \in \mathcal{F}^3(X_0)$, there is a $y \in \sigma$ such that

$$d_c(P(y), P(X_0 \setminus \sigma)) > r - 2cs > \frac{r}{2};$$

we will use this in Section 6 to construct a proper tangent cone of P and prove Theorem 1.4.

Since Theorem 1.2 relies only on the existence of a map satisfying equation (7) and conditions (1), (3), and (4), we will sketch the construction of such a map for readers who are primarily interested in Theorem 1.2. For each vertex $v \in \mathcal{F}^0(X_0)$, choose $Q(v)$ to be a vertex of X_1 that minimizes the distance $d_c(\iota_0(v), \iota_1(Q(v)))$. By (6), for each edge $e = [v, w]$, we have $d_1(Q(v), Q(w)) \lesssim 1$. Define Q on e to be a shortest path in $X_1^{(1)}$ from $Q(v)$ to $Q(w)$, parametrized with constant speed. There is a $c_0 > 0$ independent of n such that this path has at most $c_0 n$ edges, so $\text{Lip}(Q|_{X_0^{(1)}}) \leq c_0$, i.e., Q satisfies condition (1).

For every triangle $\sigma \in \mathcal{F}^2(X_0)$, the image $Q(\partial\sigma)$ is a closed curve in $X_1^{(1)}$ with at most $3c_0 n$ edges. Since the Heisenberg group has a cubic Dehn function, results of [2] imply that there is a $c_1 > 0$ such that we can extend Q to an admissible map on σ with $\text{avol}(Q|_\sigma) \leq c_1 n^3$. We can take the uncollapsed discs of $Q|_\sigma$ to be a collection of disjoint balls. There is a homeomorphism from σ to itself that sends these disjoint balls to a collection of disjoint regular simplices of edge length roughly

$$\left(\frac{1}{c_1 n^3} \right)^{\frac{1}{2}} \approx n^{-\frac{3}{2}}.$$

We compose Q with this homeomorphism so that it satisfies condition (3), and by applying a further homotopy, we can ensure that it satisfies condition (4), see Lemma 4.4 and Appendix A.

Now, for every tetrahedron $\Delta \in \mathcal{F}^3(X_0)$, the image $Q(\partial\Delta)$ is a 2-sphere in X_1 consisting of at most $4c_1 n^3$ triangles. The filling volume function of the Heisenberg group grows like $\text{FV}^3(t) \approx t^{\frac{4}{3}}$, so, using results of [2] again, we can extend Q to an admissible map on Δ with disjoint uncollapsed balls and $\text{avol}(Q|_\Delta) \leq \text{FV}^3(4c_1 n^3) \leq c_2 n^4$. There is a homeomorphism that sends the uncollapsed balls of $Q|_\Delta$ to disjoint simplices, so we can adjust Q to satisfy conditions (3) and (4).

All of these constructions can be done equivariantly, so Q can be chosen to be $\mathbb{H}_{\mathbb{Z}}$ -equivariant. Since $\mathbb{H}_{\mathbb{Z}}$ acts cocompactly on \mathbb{H} , the map $\iota_1 \circ Q \circ \iota_0^{-1}$ has bounded displacement; say that there is some $D(n)$ such that

$$d_c(\iota_0(x), \iota_1(Q(x))) \leq D(n)$$

for all $x \in X$. Then

$$d_c(\iota_i(x), \iota_{i+1}(Q_i(x))) = d_c(\delta_{s^i}(\iota_0(x)), \delta_{s^i}(\iota_1(Q(x)))) \leq s^i D(n),$$

so equation (7) holds. Then Q and the Q_i satisfy the desired conditions, so Q can be used in the proof of Theorem 1.2. Readers who are primarily interested in Theorem 1.2 may turn to Section 5 for the proof.

To achieve conditions (2) and (5) as well, we need stronger control over the geometry of Q . We thus construct Q by first constructing a map $\phi: X_0 \rightarrow \mathbb{H}$ which is close to ι , then approximating $\iota_1^{-1} \circ \phi$ by an admissible map $A: X_0 \rightarrow X_1$, and finally adjusting A to satisfy conditions (3) and (4). Let $\kappa > 0$ be small enough that for every 3-simplex $\Delta \in \mathcal{F}^3(X_0)$, the image $\iota_0(\Delta)$ contains a d_c -ball of radius κ . Any curve in \mathbb{H} can be approximated arbitrarily closely by a horizontal curve, so there is an $\mathbb{H}_{\mathbb{Z}}$ -equivariant PL map $\phi: X_0 \rightarrow \mathbb{H}$ such that ϕ is horizontal on every edge of X_0 and

$$d_c(\phi(x), \iota_0(x)) < \frac{\kappa}{2} \tag{9}$$

for all $x \in X_0$. Then $\iota_1^{-1} \circ \phi$ is an $\mathbb{H}_{\mathbb{Z}}$ -equivariant PL map from X_0 to X_1 . By the following lemma, which is proved in Appendix A, we can approximate $\iota_1^{-1} \circ \phi$ by an admissible map $A: X_0 \rightarrow X_1$.

Lemma 4.3. *Let Y, Z be finite dimensional simplicial complexes equipped with the standard metric, and let $\psi: Y \rightarrow Z$ be a Lipschitz map. Then there is an admissible Lipschitz map $\bar{\psi}: Y \rightarrow Z$ such that:*

- (1) *For a subset $S \subset Z$, let $\text{supp } S$ be the smallest subcomplex of Z containing S . For any simplex $\sigma \in \mathcal{F}(Y)$, we have $\bar{\psi}(\sigma) \subset \text{supp } \psi(\sigma)$.*
- (2) *For any edge $e \in \mathcal{F}^1(Y)$, $\bar{\psi}(e)$ is an edge path parametrized with constant speed.*
- (3) *For any simplex $\sigma \in \mathcal{F}(Y)$,*

$$\text{avol}(\bar{\psi}|_\sigma) \leq b \cdot \sum_{\sigma' \in \mathcal{F}(\sigma)} \text{vol}^{\dim(\sigma')}(\psi|_{\sigma'}) \tag{10}$$

for some constant b depending only on the dimension of Z .

If ψ is equivariant then $\bar{\psi}$ can be taken to be equivariant as well.

Let $A = \overline{\iota_1^{-1} \circ \phi}: X_0 \rightarrow X_1$ be the map obtained by applying Lemma 4.3 to the map $\psi = \iota_1^{-1} \circ \phi$. We claim that A is Lipschitz on $X_0^{(1)}$ with a Lipschitz constant independent of n and that $\text{avol}(A|_\sigma) \lesssim Cn^{d+1}$ for all $d \geq 2$ and $\sigma \in \mathcal{F}^d(X_0)$. The bound on the Lipschitz constant will help us achieve condition (1) in Lemma 4.1, and the bound on $\text{avol}(A|_\sigma)$ will help us achieve conditions (3) and (4).

Let

$$M = \max_{\sigma \in \mathcal{F}^d(X_0), d \geq 1} \text{vol}_R^d(\phi|_\sigma),$$

where vol_R^d is d -volume with respect to the Riemannian metric d_R on \mathbb{H} . This maximum exists because there are only finitely many cells of $\mathbb{H}_{\mathbb{Z}} \setminus X_0$. We can bound the number of uncollapsed discs of A in terms of M and n . Let ℓ_0 and $\ell_1 = n^{-1}\ell_0$ be length with respect to d_0 (the standard metric) and d_1 , and let vol_0^d and vol_1^d denote d -volume with respect to d_0 and d_1 .

Let e be an edge of X_0 . Then $A|_e$ is an edge path in X_1 . By (10),

$$\ell_1(A|_e) = n^{-1} \text{avol}(A|_e) \leq n^{-1} b(\ell_0(\iota_1^{-1} \circ \phi|_e) + 2).$$

Since ϕ is horizontal on each edge e of X_0 and ι is bilipschitz,

$$\ell_0(\iota_1^{-1} \circ \phi|_e) = \ell_0(\iota^{-1} \circ \delta_n \circ \phi|_e) \approx \ell_R(\delta_n \circ \phi|_e) = n \ell_R(\phi|_e) \leq nM.$$

Therefore,

$$\ell_1(A|_e) \leq n^{-1} b(nM + 2) \leq bM + 2bn^{-1},$$

so $A|_{X_0^{(1)}}$ is $((M+2)b)$ -Lipschitz.

Recall from Section 2.4 that δ_n distorts 2-dimensional area by a factor of at most n^3 and distorts 3-dimensional volume by a factor of n^4 . Thus if $d \geq 2$,

$$\text{vol}_0^d(\iota_1^{-1} \circ \phi|_\sigma) \approx \text{vol}_R^d(\delta_n \circ \phi|_\sigma) \leq n^{d+1} M.$$

It follows that there is a $C = C(X, \phi) > 1$ such that $\text{Lip}(A|_{X_0^{(1)}}) < C$ and

$$\text{avol}(A|_\sigma) \leq Cn^{d+1} \tag{11}$$

for any $d \geq 2$ and any $\sigma \in \mathcal{F}^d(X_0)$.

We construct Q by adjusting A to make the uncollapsed balls into regular simplices. We use the following lemma, which will be proved in Appendix A. A *colored ball* in a d -dimensional manifold M is the image of the radius- $\frac{1}{2}$ Euclidean d -ball $B(0, \frac{1}{2}) \subset \mathbb{R}^d$ under an embedding of the unit ball $B(0, 1)$.

Lemma 4.4. *Let Y, Z be finite dimensional simplicial complexes equipped with a multiple of the standard metric, and let $f: Y \rightarrow Z$ be an admissible map. For each simplex $\sigma \in \mathcal{F}^d(Y)$ with $d \geq 2$, let $B_1^\sigma, \dots, B_{\text{avol}(f|_\sigma)}^\sigma \subset \sigma$ be the uncollapsed balls of $f|_\sigma$ and let $\tau_i^\sigma = f(B_i^\sigma) \in \mathcal{F}^d(Z)$ be the corresponding cells of Z .*

Let $C_1^\sigma, \dots, C_{\text{avol}(f|_\sigma)}^\sigma \subset \sigma$ be disjoint colored balls and let $g_i^\sigma: C_i^\sigma \rightarrow \tau_i^\sigma$ be homeomorphisms with the same orientation as $f|_{B_i^\sigma}$. Then f is homotopic to an admissible map $h: Y \rightarrow Z$ such that:

- (1) h agrees with f on $Y^{(1)}$.

- (2) For each simplex $\sigma \in \mathcal{F}^d(Y)$ with $d \geq 2$, the uncollapsed balls of $h|_\sigma$ are the C_i^σ 's, and for each i , h agrees with g_i^σ on C_i^σ .
- (3) For every simplex $\sigma \in \mathcal{F}^d(Y)$, $h(\sigma) \subset \text{supp } f(\sigma)$.

If f is Lipschitz and the g_i^σ 's are Lipschitz, we can take h to be Lipschitz on each cell of Y ; if f and the g_i^σ 's are equivariant, we can take h to be equivariant as well.

We use these lemmas to prove Lemma 4.1.

Proof of Lemma 4.1. We first construct Q . Let $c > 0$ be a large number to be chosen later. For each simplex $\sigma \in \mathcal{F}^d(X_0)$ with $d \geq 2$, let $B_1^\sigma, \dots, B_{\text{avol}(A|_\sigma)}^\sigma$ be the uncollapsed balls of $A|_\sigma$ and let $\tau_i^\sigma = A(B_i^\sigma) \in \mathcal{F}^d(X_1)$.

Let $C > 1$ be as in the remarks after Lemma 4.3, so that $\text{Lip}(A|_{X_0^{(1)}}) < C$ and $\text{avol}(A|_\sigma) \leq Cn^{d+1}$ for each $d = 2, 3$ and each simplex $\sigma \in \mathcal{F}^d(X_0)$. Let $\beta > 0$ be small enough that for $d = 2, 3$ and for any $N > 0$, one can construct N disjoint regular simplices of edge length $\beta N^{-\frac{1}{d}}$ in the interior of the unit regular simplex Δ^d (for instance, by inscribing a cube in Δ^d , subdividing the cube into a grid with $\lceil N^{-\frac{1}{d}} \rceil$ grid cells on an edge, then inscribing a regular simplex in each grid cell).

Let $c_0 = \beta^{-1}C^{\frac{1}{2}}$. For any $c > c_0$ and $d = 2, 3$,

$$a = c^{-1}n^{-\frac{3}{2}} < \beta C^{-\frac{1}{2}}n^{-\frac{3}{2}} \leq \beta(Cn^{d+1})^{-\frac{1}{d}}.$$

Thus for each $d = 2, 3$ and each cell $\sigma \in \mathcal{F}^d(X_0)$, we can construct

$$C_1^\sigma, \dots, C_{\text{avol}(A|_\sigma)}^\sigma \subset \text{int } \sigma$$

which are disjoint regular simplices of edge length a . For each σ and i , let $g_i^\sigma: C_i^\sigma \rightarrow \tau_i^\sigma$ be a similarity with the same orientation as $A|_{B_i^\sigma}$ and scale factor $\frac{\text{diam } \tau_i^\sigma}{\text{diam } C_i^\sigma} = \frac{s}{a}$.

Let $Q: X_0 \rightarrow X_1$ be a Lipschitz, $\mathbb{H}_\mathbb{Z}$ -equivariant, admissible map satisfying Lemma 4.4 for this choice of C_i^σ and g_i^σ . Then $Q_0 = Q$ satisfies conditions (3) and (4) of Lemma 4.1. Since Q_i is the same map as Q , with the metric on the domain and range scaled by s^i , Q_i satisfies conditions (3) and (4) for any i .

We claim that Q_i satisfies the rest of the desired conditions. Since $d_i = s^i d_0$ for all i , we have $\text{Lip}(Q) = \text{Lip}(Q_i)$ for all i . By Lemma 4.4, $Q|_{X_0^{(1)}} = A|_{X_0^{(1)}}$, so $\text{Lip}(Q_i|_{X_i^{(1)}}) < C$ by our choice of C . Thus, (1) holds when $c > C$.

Condition (2) claims that $d_c(t_0(x), t_1(Q(x))) \lesssim 1$ for all $x \in X_0$. Let $x \in X_0$ and let σ be a simplex of X_0 that contains x . Then

$$d_c(t_0(x), t_1(Q(x))) \leq d_c(t_0(x), \phi(x)) + d_c(\phi(\sigma), t_1(Q(x))) + \text{diam } \phi(\sigma).$$

By (9), $d_c(t_0(x), \phi(x)) < \frac{\kappa}{2} \lesssim 1$. Likewise,

$$\text{diam } \phi(\sigma) \leq \max_{\delta \in \mathcal{F}(X_0)} \text{diam } \phi(\delta) \lesssim 1.$$

Finally, we have

$$Q(x) \in Q(\sigma) \subset \text{supp } A(\sigma) \subset \text{supp } \text{supp}(t_1^{-1}(\phi(\sigma))) = \text{supp}(t_1^{-1}(\phi(\sigma))).$$

Since simplices of X_1 have side length s ,

$$d_1(Q(x), \iota_1^{-1}(\phi(\sigma))) \leq s.$$

By (6),

$$d_c(\iota_1(Q(x)), \phi(\sigma)) \leq 2ks \lesssim 1.$$

Summing these inequalities, we find that there is a universal constant $c_2 > 0$ such that $d_c(\iota_0(x), \iota_1(Q(x))) \leq c_2$. Then condition (2) holds for all $c > c_2$.

In fact, ι_0 and $\iota_1 \circ Q: X_0 \rightarrow \mathbb{H}$ are homotopic by a straight-line homotopy that moves each point distance at most c_2 . Since ι_0 is a degree-1 map, so is $\iota_1 \circ Q$. Thus Q is a surjective, degree-1 map.

Finally, recall that by the construction of ι_0 , there is a $\kappa > 0$ such that for every $\sigma \in \mathcal{F}^3(X_0)$, there is a point $q \in \mathbb{H}$ such that if $y \in X_0 \setminus \text{int } \sigma$, then $d_c(q, \iota_0(y)) > \kappa$. By (9),

$$d_c(q, \phi(y)) > \frac{\kappa}{2}. \quad (12)$$

Let $r = \frac{\kappa}{4}$, let

$$D = \max_{\alpha \in \mathcal{F}(X_0)} \text{diam } \iota_0(\alpha)$$

and let $n_0 = \frac{4D}{\kappa}$. Suppose that $n > n_0$.

Suppose that $\tau \in \mathcal{F}^3(X_0)$ and $\tau \neq \sigma$. Since $\tau \subset X_0 \setminus \text{int } \sigma$, we have $d_c(q, \phi(\tau)) > \frac{\kappa}{2}$, and we claim that $d_c(q, \iota_1(Q(\tau))) > r$. By Lemma 4.3 and Lemma 4.4,

$$Q(\tau) \subset \text{supp } A(\tau) = \text{supp } (\overline{\iota_1^{-1} \circ \phi(\tau)}) \subset \text{supp } (\iota_1^{-1}(\phi(\tau))).$$

That is, for any $w \in Q(\tau)$, there are $z \in \tau$ and $\alpha \in \mathcal{F}(X_1)$ such that $\{w, \iota_1^{-1}(\phi(z))\} \subset \alpha$ and thus

$$d_c(\iota_1(w), \phi(z)) \leq \text{diam } \iota_1(\alpha) \leq Dn^{-1} \leq \frac{\kappa}{4}.$$

By the triangle inequality,

$$d_c(q, \iota_1(w)) \geq d_c(q, \phi(z)) - d_c(\phi(z), \iota_1(w)) > \frac{\kappa}{4} = r.$$

Therefore, $d_c(q, \iota_1(Q(\tau))) > r$. Since this holds for every $\tau \neq \sigma$, we have

$$d_c(q, \iota_1(Q(X_0 \setminus \sigma))) > r.$$

This proves condition (5). □

5. HÖLDER BOUNDS FOR P

Let c be as in Lemma 4.1. By condition (2) of Lemma 4.1, for any $x \in X_0$, we have $d_c(\iota_i(P_i(x)), \iota_{i+1}(P_{i+1}(x))) < cs^i$, so the P_i 's converge uniformly. Let $P = \lim_i \iota_i \circ P_i$. Then

$$d_c(\iota_i(P_i(x)), P(x)) < \frac{cs^i}{1-s} \leq 2cs^i. \quad (13)$$

In this section, we will prove an exponential bound on $\text{Lip}(P_i)$ and combine it with (13) to prove Hölder bounds on P . We bound $\text{Lip}(P_i)$ as follows.

Lemma 5.1. *Let $c, n, a = c^{-1}n^{-\frac{3}{2}}$, and $Q_i: X_i \rightarrow X_{i+1}$ be as in Lemma 4.1. Let $P_i = Q_{i-1} \circ \dots \circ Q_0$. There is a $C > 0$ depending on X and n such that $\text{Lip}(P_i) \leq C \cdot \left(\frac{s}{a}\right)^i$.*

Proof. Let $\lambda_i: X_0 \rightarrow \mathbb{R}$ be the local Lipschitz constant

$$\lambda_i(x) = \lim_{r \rightarrow 0^+} \text{Lip}(P_i|_{B(x,r)}),$$

where $B(x, r)$ denotes the ball with respect to d_0 . Let $L = \text{Lip}(Q_1)$; since the Q_i 's are scalings of Q_1 , we have $L = \text{Lip}(Q_i)$ for all i . Since Q_i scales uncollapsed balls by $\frac{s}{a}$, we have $\frac{s}{a} \leq L$, but $\frac{s}{a}$ will typically be much smaller than L . In general, $\lambda_{i+1}(x) \leq L\lambda_i(x)$, and we will prove the lemma by showing that $\lambda_{i+1}(x) \leq \frac{s}{a}\lambda_i(x)$ for all but six values of i .

For each i , define $D_i: X_0 \rightarrow \{0, 1, 2, 3\}$ so that $D_i(x)$ is the dimension of the smallest cell of X_i containing $P_i(x)$. Let

$$\widehat{D}_i(x) = \lim_{r \rightarrow 0^+} \max D_i(B(x, r)).$$

Then $\widehat{D}_0(x) = 3$ and $\widehat{D}_i(x) \geq D_i(x)$ for all x and i . Since Q_i is a cellular map, we have $D_{i+1}(x) \leq D_i(x)$ and $\widehat{D}_{i+1}(x) \leq \widehat{D}_i(x)$ for all x and i .

Let $x \in X_0$. We claim that $\widehat{D}_i(x) = D_i(x) = D_{i+1}(x)$ for all but at most six values of i and that if $\widehat{D}_i(x) = D_i(x) = D_{i+1}(x)$, then $\lambda_{i+1}(x) \leq \frac{s}{a}\lambda_i(x)$.

First, since $(D_i(x))_i$ is non-increasing, there are at most three values of i such that $D_i(x) \neq D_{i+1}(x)$. Suppose that $\widehat{D}_i(x) \neq D_i(x)$; we must have $\widehat{D}_i(x) > D_i(x)$. Let $d = D_i(x)$, so that $P_i(x) \in X_i^{(d)}$. By condition (3) of Lemma 4.1, there is an $r > 0$ such that $P_i(B(x, r))$ does not intersect any uncollapsed $(d+1)$ -ball and thus $P_{i+1}(B(x, r)) \subset X_{i+1}^{(d)}$, i.e., $\widehat{D}_{i+1}(x) \leq D_i(x) < \widehat{D}_i(x)$. Since $(\widehat{D}_i(x))_i$ is non-increasing, this can happen for at most three values of i . There are thus at most six values of i such that $D_i(x) \neq D_{i+1}(x)$ or $\widehat{D}_i(x) \neq D_i(x)$.

For every other value of i , we have $\widehat{D}_i(x) = D_i(x) = D_{i+1}(x)$. If $\widehat{D}_i(x) \leq 1$, then there is some $r > 0$ such that $P_i(B(x, r)) \subset X_i^{(1)}$. By condition (1) of Lemma 4.1, $\text{Lip}(Q_i|_{P_i(B(x, r))}) < c \leq \frac{s}{a}$, so $\lambda_{i+1}(x) \leq \frac{s}{a}\lambda_i(x)$.

Otherwise, if $\widehat{D}_i(x) > 1$, let $d = D_i(x)$. Since $D_i(x) = D_{i+1}(x)$, $P_i(x)$ lies in the interior of some uncollapsed d -ball B of $Q_i(x)$. Since $\widehat{D}_i(x) = d$, there is some $r > 0$ such that $P_i(B(x, r)) \subset X_i^{(d)}$. Thus, if $r > 0$ is sufficiently small, we have $P_i(B(x, r)) \subset B$. Since Q_i rescales B by a factor of $\frac{s}{a}$, we have $\lambda_{i+1}(x) = \lambda_i(x) \frac{s}{a}$.

Thus, for up to six values of i , we have $\lambda_{i+1}(x) \leq L\lambda_i(x)$, and for all the rest, $\lambda_{i+1}(x) \leq \frac{s}{a}\lambda_i(x)$. Since $\lambda_0(x) = 1$, this implies

$$\lambda_i(x) \leq L^6 \left(\frac{s}{a}\right)^i$$

for all x . Since X_0 is a path metric space,

$$\text{Lip}(P_i) \leq \sup_{x \in X_0} \lambda_i(x) \leq L^6 \left(\frac{s}{a}\right)^i,$$

as desired. \square

Given the lemma, one can use (13) to show that P is locally Hölder and prove Theorem 1.2. This is the same argument as in Lemma 4.2.

Proof of Theorem 1.2. Let $\alpha < \frac{2}{3}$ and let $c > 1$ be as in Lemma 4.1. Since $a = c^{-1}n^{-\frac{3}{2}}$ and $s = n^{-1}$,

$$\lim_{n \rightarrow \infty} \frac{\log s}{\log a} = \lim_{n \rightarrow \infty} \frac{\log n^{-1}}{\log c^{-1}n^{-\frac{3}{2}}} = \frac{2}{3}.$$

Let n be large enough that $\frac{\log s}{\log a} \geq \alpha$. Let $k > 1$ be the quasi-isometry constant of ι as in (5) and (6), and let C be as in Lemma 5.1 so that $\text{Lip}(P_i) \leq C \cdot (\frac{s}{a})^i$.

We claim that P is locally (K, α) -Hölder, where $K = \frac{4c+kC+k}{s}$. Let $x, y \in X_0$ such that $0 < d_0(x, y) < 1$. Let $i \geq 0$ be the integer such that $a^{i+1} \leq d_0(x, y) < a^i$. Then, by (13) and (6),

$$\begin{aligned} d_c(P(x), P(y)) &\leq 4cs^i + d_c(\iota_i(P_i(x)), \iota_i(P_i(y))) \\ &\leq 4cs^i + k\text{Lip}(P_i)a^i + ks^i \leq Ks^{i+1} \leq Kd_0(x, y)^\alpha, \end{aligned}$$

as desired.

Since ι_0 is a bilipschitz equivalence from (\mathbb{H}, d_R) to X_0 , the composition $g = P \circ \iota_0^{-1}: (\mathbb{H}, d_R) \rightarrow (\mathbb{H}, d_c)$ is an $\mathbb{H}_{\mathbb{Z}}$ -equivariant, locally α -Hölder map. Since \mathbb{H} is contractible, any such map is equivariantly homotopic to the identity. \square

We prove Theorem 1.3 by using g to construct a sequence of locally Hölder maps that converge to the identity map.

Proof of Theorem 1.3. Let Y be a compact metric space and let $\varphi: Y \rightarrow \mathbb{H}$ be a continuous map. Let $\epsilon > 0$ and let $0 < \alpha < \frac{2}{3}$.

We first approximate φ by a Lipschitz map. Let $\delta > 0$ be such that if $x, y \in \mathbb{H}$ and $d_R(x, y) < \delta$, then $d_c(x, y) < \frac{\epsilon}{2}$. Let $B \subset \mathbb{H}$ be a ball containing $\varphi(Y)$ and note that the Riemannian metric on B is bilipschitz equivalent to the Euclidean metric on B (with a constant depending on the radius of B). Lipschitz maps from Y to \mathbb{R}^3 are dense in the space of continuous maps by [8, 6.8] or by the Stone-Weierstrass theorem, so there is a Lipschitz map $\lambda: Y \rightarrow (\mathbb{H}, d_R)$ such that $d_R(\varphi(y), \lambda(y)) < \delta$ for every $y \in Y$ and hence $d_c(\varphi(y), \lambda(y)) < \frac{\epsilon}{2}$.

By Theorem 1.2, there is an $\mathbb{H}_{\mathbb{Z}}$ -equivariant map $g: (\mathbb{H}, d_R) \rightarrow (\mathbb{H}, d_c)$ that is locally α -Hölder. The $\mathbb{H}_{\mathbb{Z}}$ -equivariance of g implies that there is an $m > 0$ such that $m \geq d_c(x, g(x))$ for all $x \in \mathbb{H}$. Let $r = \frac{\epsilon}{2m}$. For any $x \in \mathbb{H}$,

$$d_c(x, (\delta_r \circ g \circ \delta_r^{-1})(x)) = r \cdot d_c(\delta_r^{-1}(x), g(\delta_r^{-1}(x))) \leq rm = \frac{\epsilon}{2}.$$

Let $\psi = \delta_r \circ g \circ \delta_r^{-1} \circ \lambda$. Then ψ is locally α -Hölder with respect to d_c , and since Y is compact, a locally α -Hölder map from Y to \mathbb{H} is globally α -Hölder. For any $y \in Y$,

$$\begin{aligned} d_c(\varphi(y), \psi(y)) &\leq d_c(\varphi(y), \lambda(y)) + d_c(\lambda(y), \psi(y)) \\ &\leq \frac{\epsilon}{2} + d_c(\lambda(y), (\delta_r \circ g \circ \delta_r^{-1})(\lambda(y))) \leq \epsilon, \end{aligned}$$

so ψ is an α -Hölder map that is ϵ -close to φ . \square

6. A GLOBALLY HÖLDER MAP FROM \mathbb{R}^3 TO (\mathbb{H}, d_c)

In this section, we prove Theorem 1.4 by constructing a self-similar globally Hölder map from \mathbb{R}^3 to \mathbb{H} .

First, we show that the construction in Section 4 produces a self-similar map. Recall that all the complexes X_i have the same underlying simplicial complex X and that all of the maps $Q: X_0 \rightarrow X_1$ and $Q_i: X_i \rightarrow X_{i+1}$ are the same as maps from X to X . The main difference between the X_i 's is that they are equipped with different metrics $d_i = s^i d_\Delta$, different actions $\zeta_i[g] = \zeta[\delta_{n^i}(g)]$ (where $\zeta[g]: X \rightarrow X$ is the action of $\mathbb{H}_\mathbb{Z}$ on X), and different equivariant maps $\iota_i: X \rightarrow \mathbb{H}$, $\iota_i = \delta_{n^{-i}} \circ \iota$ (where $\iota: X \rightarrow \mathbb{H}$ is an equivariant homeomorphism).

In this section, we will identify all of the X_i 's with X . Under this identification,

$$P_i = Q_{i-1} \circ \cdots \circ Q_0 = Q^i.$$

Lemma 6.1. *With notation as in Section 4, let $\sigma \in \mathcal{F}^3(X)$. Let $D \subset X$ be an uncollapsed ball of Q and suppose that there is a $g \in \mathbb{H}_\mathbb{Z}$ such that $Q(D) = \zeta[g^{-1}](\sigma)$. Let $\tau = Q(D)$.*

With respect to the standard metric d_Δ , D is a regular simplex with side length a and σ is a regular simplex with side length 1, and there is a similarity $h: (D, d_\Delta) \rightarrow (\sigma, d_\Delta)$ with scaling factor $a^{-1} > 1$ and a Heisenberg similarity $m: \mathbb{H} \rightarrow \mathbb{H}$, $m(x) = g\delta_n(x)$ such that:

$$P(x) = m(P(h^{-1}(x))) \quad \text{for all } x \in \sigma, \quad (14)$$

and

$$P(y) = m^{-1}(P(\zeta[g](Q(y)))) \quad \text{for all } y \in X. \quad (15)$$

Proof. Let $m(x) = g\delta_n(x)$. Since $P = \lim_{i \rightarrow \infty} \iota_i \circ Q^i$ and P is equivariant,

$$\begin{aligned} m(P(y)) &= \lim_{i \rightarrow \infty} g\delta_n(\iota_i(Q^i(y))) = \\ &= \lim_{i \rightarrow \infty} g\iota_{i-1}(Q^{i-1}(Q(y))) = gP(Q(y)) = P(\zeta[g](Q(y))). \end{aligned}$$

Applying m^{-1} to both sides proves (15).

Equation (14) follows from (15). By condition (3) of Lemma 4.1, D is a regular simplex with sides of d_Δ -length a and Q sends D to τ by a similarity that scales the d_Δ -metric by a^{-1} . Let $h: D \rightarrow \sigma$ be the similarity $h = \zeta[g] \circ Q|_D: D \rightarrow \sigma$. Let $x \in \sigma$ and let $y = h^{-1}(x)$. By (15),

$$P(x) = P(\zeta[g](Q(y))) = m(P(y)) = m(P(h^{-1}(x))),$$

which proves (14). \square

If we can find simplices σ satisfying Lemma 6.1, we can use them to construct self-similar Hölder maps $F: \mathbb{R}^3 \rightarrow \mathbb{H}$, i.e., maps such that there are expanding similarities $m: \mathbb{H} \rightarrow \mathbb{H}$ and $h: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that satisfy $m(F(h^{-1}(x))) = F(x)$ for all

$x \in \mathbb{R}^3$. In order to prove Theorem 1.4, however, we want F to be proper, i.e., $F^{-1}(K)$ is compact for every compact $K \subset \mathbb{H}$.

Since m is expanding, it has a unique fixed point $q \in \mathbb{H}$. If F is proper and $F(p) = q$, then $F(h(p)) = m(F(p)) = q$, so $h(p) \in F^{-1}(q)$, i.e., $F^{-1}(q)$ is invariant under h . But the only compact sets that are invariant under h are the empty set and the set $\{p\}$ where p is the unique fixed point of h . Thus, in order to use Lemma 6.1 to construct a proper self-similar map $F: \mathbb{R}^3 \rightarrow \mathbb{H}$, we will need to adjust the construction so that there is a point $q \in \mathbb{H}$ such that $P^{-1}(q)$ is a single point.

We need the following lemma (see also [16]).

Lemma 6.2. *Let $d \geq 3$, let Y be a $(d-2)$ -connected simplicial complex, and let $f: D^d \rightarrow Y$ be a continuous map such that $f(\partial D^d) \subset Y^{(d-1)}$ and $f(D^d) \subset Y^{(d)}$. For each d -cell $\delta \in \mathcal{F}^d(Y)$, the degree of f is the same at every point in the interior of δ ; let $\deg_\delta(f)$ be this degree. Then there is an admissible map $g: D^d \rightarrow Y$ which agrees with f on ∂D^d and such that for every d -cell $\delta \in \mathcal{F}^d(Y)$, the number of uncollapsed balls in $g^{-1}(\delta)$ is $|\deg_\delta(f)|$.*

Proof. The image $f([D^d])$ of the fundamental class of D^d is a cellular d -chain in Y that can be written

$$f([D^d]) = \sum_{\delta \in \mathcal{F}^d(Y)} \deg_\delta(f) [\delta],$$

where $[\delta]$ represents the fundamental class of δ , and $\partial f([D^d]) = f([\partial D^d])$.

Let $k = \sum_\delta |\deg_\delta(f)|$ and let $B_1, \dots, B_k \subset \text{int } D^d$ be k disjoint smoothly embedded balls. For each ball, choose a cell $\delta_i \in \mathcal{F}^d(Y)$ and a homeomorphism $g_i: B_i \rightarrow \delta_i$ so that each cell δ is chosen $|\deg_\delta(f)|$ times and so that the orientation of g_i corresponds to the sign of $\deg_\delta(f)$.

We will extend the maps g_i to the desired map g . We proceed as in [15]. Consider the complement $E = D^d \setminus \bigcup_i \text{int}(B_i)$. Choose $v \in \partial D^d$ and for each i , let $v_i \in \partial B_i$. For each i , let γ_i be a simple smooth curve connecting v to v_i , and suppose that the γ_i 's are disjoint. The interior of the complement $E \setminus \bigcup_i \gamma_i$ is homeomorphic to an open d -ball, and we can give E the structure of a CW complex with vertices v, v_1, \dots, v_k ; edges $\gamma_1, \dots, \gamma_k$; $(d-1)$ -cells $\partial D^d, \partial B_1, \dots, \partial B_k$; and a single d -cell, which we call σ . Define g on $E^{(d-1)}$ so that it agrees with f on ∂D^d and with g_i on each ∂B_i ; since Y is connected, we can extend g on each edge. It remains to extend g on σ .

Since Y is $(d-2)$ -connected, so is $Y^{(d-1)}$, and Hurewicz's Theorem implies that $\pi_{d-1}(Y^{(d-1)}) \cong H_{d-1}(Y^{(d-1)})$. Let $\alpha: S^{d-1} \rightarrow E^{(d-1)}$ be the attaching map of σ . Then

$$\begin{aligned} g(\alpha([S^{d-1}])) &= f([\partial D^d]) - \sum_i g([\partial B_i]) \\ &= \sum_\delta \deg_\delta(f) [\partial \delta] - \sum_\delta \deg_\delta(f) [\partial \delta] \\ &= 0. \end{aligned}$$

Therefore, $g \circ \alpha$ is null-homotopic, and the obstruction to extending g to a map from D^d to $Y^{(d-1)}$ vanishes. \square

Applying this lemma to Q produces a map that satisfies Lemma 4.1 and also has a simplex with a small preimage.

Lemma 6.3. *For any sufficiently large n , there is an admissible Lipschitz map $Q: X \rightarrow X$ which is \mathbb{H}_Z -equivariant as a map from X_0 to X_1 , satisfies Lemma 4.1, and satisfies the following conditions.*

- For every $\sigma \in \mathcal{F}^3(X)$, there is a $\tau \in \mathcal{F}^3(X)$ such that $\sigma = \zeta[g](\tau)$ for some $g \in \mathbb{H}_Z$ and $Q^{-1}(\text{int } \tau)$ is the interior of a single uncollapsed ball in σ .
- Let $r > 0$ be as in Lemma 4.1. For every $y \in X \setminus \text{int } \sigma$,

$$d_c(P(y), \iota_1(\tau)) \geq \frac{r}{2}.$$

Proof. By Lemma 6.2, for any sufficiently large n , there is a map Q satisfying Lemma 4.1 with the property that for every $\sigma \in \mathcal{F}^3(X)$ and every $\delta \in \mathcal{F}^3(X)$, exactly $|\deg_\delta(Q|_\sigma)|$ uncollapsed balls of $Q|_\sigma$ map to δ .

Let $\sigma \in \mathcal{F}^3(X)$. By Lemma 4.1, there are $r > 0$ and $q \in \mathbb{H}$ such that if $y \in X \setminus \sigma$, then $d_c(q, \iota_1(Q(y))) > r$. Suppose that $n > \frac{8c}{r}$ and that n is large enough that for any d_c -ball B of radius $\frac{rn}{4}$ in \mathbb{H} , there is a $g \in \mathbb{H}_Z$ such that $g\iota(\sigma) \subset B$. Then there is a $g \in \mathbb{H}_Z$ such that

$$g\iota(\sigma) \subset B\left(\delta_n(q), \frac{rn}{4}; d_c\right),$$

where $B(q, r; d_c)$ is the metric ball with respect to d_c . Let $\tau = \zeta[g](\sigma)$. Then

$$\iota_1(\tau) = \delta_{n^{-1}}(g\iota(\sigma)) \subset B\left(q, \frac{r}{4}; d_c\right),$$

and by the triangle inequality, for every $y \in X \setminus \sigma$,

$$d_c(\iota_1(Q(y)), \iota_1(\tau)) > \frac{3r}{4},$$

i.e., $Q(y) \notin \tau$. Thus $Q^{-1}(\tau) \subset \sigma$. Since Q is degree-1, we have $\deg_r(Q|_\sigma) = 1$, so there is exactly one uncollapsed ball that maps to τ . Finally, by (13) and our choice of n , for every $y \in X \setminus \sigma$,

$$d_c(P(y), \iota_1(\tau)) > \frac{3r}{4} - 2cs > \frac{r}{2}.$$

By continuity, if $y \in X \setminus \text{int } \sigma$, then $d_c(P(y), \iota_1(\tau)) \geq \frac{r}{2}$. \square

Now we prove Theorem 1.4.

Proof of Theorem 1.4. Let $\sigma \in \mathcal{F}^3(X)$ and let τ satisfy Lemma 6.3 for some large n to be chosen later. Let $D \subset \sigma$ be the uncollapsed ball of Q such that $Q(D) = \tau$. Let $g \in \mathbb{H}_Z$ be such that $\tau = \zeta[g^{-1}]\sigma$.

By Lemma 6.1, there are a similarity $h: D \rightarrow \sigma$ with scale factor a^{-1} and a similarity $m: \mathbb{H} \rightarrow \mathbb{H}$, $m(x) = g\delta_n(x)$ such that $P(x) = m(P(h^{-1}(x)))$ for all $x \in \sigma$. We identify σ with a unit simplex in \mathbb{R}^3 and $D \subset \sigma$ with a subset of that simplex. Then

we can extend h to a similarity $h: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that sends D to σ and has scaling factor n . Since $n > 1$, h has a unique fixed point p such that $\lim_{i \rightarrow \infty} h^{-i}(x) = p$ for every $x \in \mathbb{R}^3$; since $h^{-1}(\sigma) = D \subseteq \sigma$, we have $p \in \text{int} D \subset \text{int} \sigma$.

We define $F: \mathbb{R}^3 \rightarrow \mathbb{H}$ as follows. Let $x \in \mathbb{R}^3$. Since $\lim_{i \rightarrow \infty} h^{-i}(x) = p$, there is an $i \in \mathbb{Z}$ such that $h^{-i}(x) \in \sigma$. Let

$$F(x) = m^i(P(h^{-i}(x))).$$

Then F is well-defined; if $i < j$ and $h^{-i}(x) \in \sigma$, then Lemma 6.1 implies

$$m^i(P(h^{-i}(x))) = m^i(m^{j-i} \circ P \circ h^{i-j}(h^{-i}(x))) = m^j(P(h^{-j}(x))).$$

Furthermore, for all $x \in h^i(\sigma)$, we have $h^{-1}(x) \in h^{i-1}(\sigma)$, so

$$m(F(h^{-1}(x))) = m(m^{i-1}(P(h^{-i+1}(h^{-1}(x))))) = m^{-i}(P(h^i(x))) = F(x).$$

That is, F is self-similar.

It remains to show that F is globally Hölder, proper, and degree 1.

First, we show that F is η -Hölder for $\eta := \frac{\log s}{\log a}$. Since P is locally η -Hölder, the restriction $P|_\sigma$ is (C, η) -Hölder for some constant C . Let $x, y \in \mathbb{R}^3$. Then there exists $i \in \mathbb{Z}$ such that $x, y \in h^i(\sigma)$ and

$$\begin{aligned} d_c(F(x), F(y)) &= d_c(m^i(P(h^{-i}(x))), m^i(P(h^{-i}(y)))) \\ &= n^i d_c(P(h^{-i}(x)), P(h^{-i}(y))) \\ &\leq C n^i |h^{-i}(x) - h^{-i}(y)|^\eta \\ &\leq C n^i (a^i |x - y|)^\eta \\ &= C |x - y|^\eta. \end{aligned}$$

Thus F is globally η -Hölder.

Next, we show F is proper. Let $q := F(p) = P(p) \in \mathbb{H}$. Then q is the unique fixed point of m , because $m(q) = m(F(p)) = F(h(p)) = F(p)$. Let r be as in Lemma 6.3, so that $d_c(P(y), \iota_1(\tau)) > \frac{r}{2}$ for every $y \in X \setminus \sigma$. We will show that if n is sufficiently large, and $z \in \mathbb{R}^3 \setminus \sigma$, then $d_c(P(z), q) > \frac{r}{4}$. We will then use self-similarity to conclude that F is proper.

First, we claim that when n is sufficiently large

$$d_c(P(y), q) > \frac{r}{4} \quad \text{for all } y \in X \setminus \text{int} \sigma. \quad (16)$$

Let c be as in Lemma 4.1 and Lemma 6.3 and suppose that $n > \frac{8c}{r}$. Since $p \in D$, we have $Q(p) \in \tau$. On one hand, by Lemma 6.3, for any $y \in X \setminus \text{int} \sigma$, we have $d_c(P(y), \iota_1(Q(p))) \geq \frac{r}{2}$; on the other hand, by (13),

$$d_c(q, \iota_1(Q(p))) = d_c(P(p), \iota_1(Q(p))) < 2cs < \frac{r}{4},$$

so the triangle inequality implies (16).

Now suppose that $z \in \mathbb{R}^3 \setminus \sigma$. Let $i > 0$ be such that $h^{-i}(z) \in \sigma \setminus D$ and let $y = h^{-i}(z)$ so that $F(z) = m^i(P(h^{-i}(z))) = m^i(P(y))$. Since $y \in \sigma \setminus D$, we have $Q(y) \notin \text{int} \tau$ and $\zeta[g](Q(y)) \notin \text{int} \sigma$. By (15),

$$P(y) = m^{-1}(P(\zeta[g](Q(y)))),$$

so by (16),

$$d_c(P(y), q) = n^{-1} d_c(m(P(y)), q) = n^{-1} d_c(P(\zeta[g](Q(y))), q) > \frac{r}{4n},$$

and

$$d_c(F(z), q) = n^i d_c(P(y), q) > \frac{r}{4},$$

as desired.

By the scale-invariance of F , if $z \in \mathbb{R}^3 \setminus h^i(\sigma)$, then $d_c(F(z), q) > \frac{r}{4} n^i$. Let $K \subset \mathbb{H}$ be a compact set. Then there is some $i \in \mathbb{Z}$ such that $K \subset B(q, \frac{r}{4} n^i; d_c)$ and thus $F^{-1}(K) \subset h^i(\sigma)$. Since $F^{-1}(K)$ is closed and bounded, it is compact. Therefore, the preimage of any compact set is compact, and F is proper.

Finally, we claim that F has degree 1. It suffices to show that $\deg_q(F) = 1$. Let $B = B(q, \frac{r}{4}; d_c)$. Then $F^{-1}(B) \subset \sigma$, so

$$\deg_q(F) = \deg_q(F|_\sigma) = \deg_q(P|_\sigma) = \deg_q(P) = 1.$$

Since F is proper, the degree is constant on \mathbb{H} . \square

APPENDIX A. PROOFS OF LEMMAS 4.3 AND 4.4

The aim of the appendix is to prove Lemmas 4.3 and 4.4. We first show:

Lemma A.1. *Let Z be a finite dimensional simplicial complex equipped with the standard metric and let $\rho: D^d \rightarrow Z$ be a Lipschitz map such that $\rho(\partial D^d) \subset Z^{(d-1)}$. Then ρ is homotopic relative to ∂D^d to an admissible map $\bar{\rho}: D^d \rightarrow Z$ with*

$$\text{avol}(\bar{\rho}) \leq c \cdot \text{vol}^d(\rho),$$

where c only depends on the dimension of Z . The homotopy between ρ and $\bar{\rho}$ is Lipschitz with image in $\text{supp}(\rho(D^d))$ and its $(d+1)$ -volume is bounded from above by $c \cdot \text{vol}^d(\rho)$.

If $d = 1$ then we may assume that $\bar{\rho}$ is an edge path parametrized with constant speed.

Proof. By the (proof of the) Federer-Fleming deformation theorem, see for example [5, Theorem 10.3.3], ρ is Lipschitz homotopic relative to ∂D^d to a map $\hat{\rho}: D^d \rightarrow Z^{(d)}$ such that the d -volume of $\hat{\rho}$ and the $(d+1)$ -volume of the homotopy are both bounded from above by $c \cdot \text{vol}^d(\rho)$ for some constant c only depending on the dimension of Z . Moreover, the image of the homotopy is contained in $\text{supp}(\rho(D^d))$. Arguing almost as in the proof of Lemma 2.3 of [2], one shows that $\hat{\rho}$ is homotopic relative to ∂D^d to an admissible map $\bar{\rho}$ with $\text{avol}(\bar{\rho}) \leq c \cdot \text{vol}^d(\rho)$ through a Lipschitz homotopy of zero volume and with image in $\text{supp}(\hat{\rho}(D^d))$. This proves the lemma. \square

Proof of Lemma 4.3. Denote by N the dimension of Y . Let $\psi: Y \rightarrow Z$ be as in the statement of the lemma. For $0 \leq d \leq N$, let Y_d be the mapping cylinder of the inclusion $Y^{(d)} \subset Y$, i.e.

$$Y_d = (Y \times \{0\}) \cup (Y^{(d)} \times [0, 1]) \subset Y \times [0, 1].$$

We will construct $\bar{\psi}$ by constructing a sequence of auxiliary maps $\psi_d: Y_d \rightarrow Z$ such that for all d and all $y \in Y$, $\psi_d(y, 0) = \psi(y)$, ψ_{d+1} extends ψ_d , and $\bar{\psi}|_{Y^{(d)}} := \psi_d(\cdot, 1)$ satisfies the conditions of the lemma. Then $\bar{\psi} = \psi_N(\cdot, 1)$ is the desired map.

Let ψ_0 be the Lipschitz map that coincides with ψ on $Y \times \{0\}$ and such that for $v \in \mathcal{F}^0(Y)$ the following holds: $\psi_0(v, 1)$ is a point in $Z^{(0)}$ which is closest to $\psi_0(v, 0) = \psi(v)$ and $\psi_0(v, \cdot)$ is the constant speed parametrization of the segment between its endpoints. Since $\psi_0(v, 1)$ and $\psi(v)$ lie in the same simplex of Z , $\bar{\psi}|_{Y^{(0)}} := \psi_0(\cdot, 1)$ satisfies the lemma and each segment $\psi_0(v \times [0, 1])$ has length at most 1.

We use Lemma A.1 to extend ψ_0 to a map ψ_1 as follows. Let $e \in \mathcal{F}^1(Y)$ be an edge. Then $(e \times \{0\}) \cup (\partial e \times [0, 1])$ is a path in Y_0 , and

$$\alpha_e = \psi_0|_{(e \times \{0\}) \cup (\partial e \times [0, 1])}$$

is a path in Z . By Lemma A.1, we can define ψ_1 on $e \times [0, 1]$ so that $\psi_1|_{e \times \{1\}}$ is a constant-speed edge path approximating α_e and so that $\psi_1|_{e \times [0, 1]}$ is a homotopy between α_e and its approximation. We obtain ψ_1 by repeating this process for every edge of Y .

Then ψ_1 is a Lipschitz map extending ψ_0 and it has the following properties. For each edge $e \in \mathcal{F}^1(Y)$, $\psi_1|_{e \times \{1\}}$ is an edge path parametrized with constant speed. Moreover, the length of $\psi_1|_{e \times \{1\}}$ and the 2-volume of $\psi_1|_{e \times [0, 1]}$ are both bounded from above by $c \cdot \ell(\alpha_e) \leq c \cdot \ell(\psi|_e) + 2c$ for some $c = c(\dim Z)$. Finally, $\psi_1(e \times [0, 1]) \subset \text{supp}(\psi(e))$, so $\bar{\psi}|_{Y^{(1)}} := \psi_1(\cdot, 1)$ satisfies the lemma.

We repeat the process to obtain a Lipschitz map $\psi_d: Y_d \rightarrow Z$ for each $d = 2, 3, \dots, N$. That is, we suppose that ψ_{d-1} is already defined and for each $\sigma \in \mathcal{F}^d(Y)$, we define

$$\alpha_\sigma = \psi_{d-1}|_{(\sigma \times \{0\}) \cup (\partial \sigma \times [0, 1])}.$$

This is a Lipschitz map from a d -disc to Z . By Lemma A.1, we can define ψ_d on $\sigma \times [0, 1]$ so that $\psi_d|_{\sigma \times \{1\}}$ is an admissible map approximating α_σ and so that $\psi_d|_{\sigma \times [0, 1]}$ is the homotopy between α_σ and its approximation.

Then ψ_d extends ψ_{d-1} and has the following properties. For every $\sigma \in \mathcal{F}^d(Y)$ the restriction $\psi_d|_{\sigma \times \{1\}}$ is admissible and $\psi_d(\sigma \times [0, 1]) \subset \text{supp}(\psi(\sigma))$. Moreover, both $\text{avol}(\psi_d|_{\sigma \times \{1\}})$ and $\text{vol}^{d+1}(\psi_d|_{\sigma \times [0, 1]})$ are bounded from above by

$$c \cdot \text{vol}^d(\alpha_\sigma) \leq c \cdot \text{vol}^d(\psi|_\sigma) + c \cdot \text{vol}^d(\psi_{d-1}|_{\partial \sigma \times [0, 1]}) \leq b_d \sum_{\sigma' \in \mathcal{F}(\sigma)} \text{vol}^{\dim(\sigma')}(\psi|_{\sigma'})$$

for some $b_d > 0$ depending on d and the dimension of Z . That is, $\bar{\psi}|_{Y^{(d)}} := \psi_d(\cdot, 1)$ satisfies the lemma for some constant b_N depending on the dimensions of Y and Z , and we define $\bar{\psi} := \psi_N(\cdot, 1)$. In fact, if $\sigma \in \mathcal{F}^d(Y)$ and $d > \dim Z$, then $\text{avol}(\bar{\psi}|_\sigma) = 0$, so $\bar{\psi}$ satisfies the lemma for $b = b_{\dim Z}$.

If ψ is equivariant then we can take all the maps ψ_d to be equivariant as well, and hence also $\bar{\psi}$. \square

In preparation for the proof of Lemma 4.4 we first establish the following results.

Lemma A.2. *Let Z be a finite dimensional simplicial complex. If $f: D^d \rightarrow Z$ is an admissible map with uncollapsed balls B_1, \dots, B_k , it is homotopic to an admissible map $f': D^d \rightarrow Z$ such that f and f' agree on $D^d \setminus \bigcup B_i$, $\text{avol}(f) = \text{avol}(f')$, and each uncollapsed ball B'_i of f' is a collared ball contained in B_i .*

Proof. Each B_i is homeomorphic to a closed ball. Let $B'_i \subset B_i$ be the image of the ball $B(0, \frac{1}{2}) \subset \mathbb{R}^d$ under a homeomorphism from $B(0, 1) \subset \mathbb{R}^d$ to B_i . Then there is a map $\zeta_i: B_i \rightarrow B_i$ that sends B'_i homeomorphically to B_i , such that $\zeta_i(B_i \setminus B'_i) \subset \partial B_i$ and ζ_i is the identity on ∂B_i . Let $f': D^d \rightarrow Z$ be the map such that f' agrees with f outside the uncollapsed balls B_i and such that $f'(x) = f(\zeta_i(x))$ for all $x \in B_i$. This is an admissible map whose uncollapsed balls are the collared balls B'_1, \dots, B'_k . \square

Lemma A.3. *Let Z be a finite dimensional simplicial complex and let $f: D^d \rightarrow Z$ be an admissible map with uncollapsed balls B_1, \dots, B_k . Let $\sigma_i = f(B_i) \in \mathcal{F}^d(Z)$ for all i . Let $C_1, \dots, C_k \subset D^d$ be a set of disjoint collared balls, and for $i = 1, \dots, k$, let $\varphi_i: C_i \rightarrow \sigma_i$ be a homeomorphism such that $(f|_{B_i})^{-1} \circ \varphi_i: C_i \rightarrow B_i$ preserves orientation. Then there is an admissible map $g: D^d \rightarrow Z$ with uncollapsed balls C_1, \dots, C_k such that $g|_{C_i} = \varphi_i$ and $g|_{\partial D^d} = f|_{\partial D^d}$. Furthermore, there is a homotopy between f and g that fixes ∂D^d pointwise and has image in $\text{supp } f(D^d)$. If $f|_{\partial D^d}$ and φ_i are Lipschitz then we can take g to be Lipschitz as well.*

Proof. By Lemma A.2, we may suppose that the B_i are disjoint collared balls. By the uniqueness of connected sums, the complements $J = D^d \setminus \bigcup_i \text{int}(B_i)$ and $K = D^d \setminus \bigcup_i \text{int}(C_i)$ are homeomorphic; indeed, any collection of orientation-preserving homeomorphisms $h_i: \partial C_i \rightarrow \partial B_i$ can be extended to a homeomorphism $h: K \rightarrow J$ that fixes ∂D^d pointwise. Thus, there exists a homeomorphism $h: D^d \rightarrow D^d$ which fixes ∂D^d pointwise and satisfies $h|_{C_i} = (f|_{B_i})^{-1} \circ \varphi_i$ for all $i = 1, \dots, k$. Let $g = f \circ h$. This map has the desired restrictions. Since h is homotopic to id_{D^d} , g is homotopic to f .

Suppose now that $f|_{\partial D^d}$ and the φ_i are Lipschitz. Then $g(D^d) \subset f(D^d) \subset \text{supp } f(D^d)$ and $g(K) \subset Z^{(d-1)}$. Since $\text{supp } g(K) \subset Z^{(d-1)}$ is locally Lipschitz n -connected for every n we can use the Lipschitz extension theorems in [1] or [9] to approximate g arbitrarily closely by an admissible Lipschitz map g' which still satisfies $g'|_{C_i} = \varphi_i$ and $g'|_{\partial D^d} = f|_{\partial D^d}$ and is still homotopic to f via a homotopy that fixes ∂D^d pointwise. \square

We are ready to prove Lemma 4.4.

Proof of Lemma 4.4. We construct a homotopy between f and a suitable map h satisfying the properties of the lemma by repeatedly applying Lemma A.3 as follows. Let N be the dimension of Y . For $1 \leq d \leq N$ set

$$Y_d = (Y \times \{0\}) \cup (Y^{(d)} \times [0, 1]) \subset Y \times [0, 1].$$

We will define maps $H_d: Y_d \rightarrow Z$ such that H_{d+1} extends H_d and such that H_N is the desired homotopy. Let H_1 be the map such that $H_1(y, 0) = f(y)$ for all $y \in Y$ and $H_1(y, t) = f(y)$ for all $(y, t) \in Y^{(1)} \times [0, 1]$.

Let $2 \leq d \leq N$ and suppose that we have already defined H_{d-1} with $H_{d-1}|_{\sigma \times \{1\}}$ admissible and $H_{d-1}(\sigma \times [0, 1]) \subset \text{supp } f(\sigma)$ for all $\sigma \in \mathcal{F}^{d-1}(Y)$. Let $\sigma \in \mathcal{F}^d(Y)$. Then $H_{d-1}(\partial\sigma \times [0, 1]) \subset \text{supp } f(\partial\sigma) \subset Z^{(d-1)}$ and thus H_{d-1} is admissible on the d -disc $(\sigma \times \{0\}) \cup (\partial\sigma \times [0, 1])$. By Lemma A.3 there exists a continuous map $H_d|_{\sigma \times [0, 1]}$ which coincides with H_{d-1} on $(\sigma \times \{0\}) \cup (\partial\sigma \times [0, 1])$ and such that $H_d|_{\sigma \times \{1\}}$ satisfies Lemma 4.4. That is, $H_d|_{\sigma \times \{1\}}$ is admissible and its uncollapsed balls are the C_i^σ 's and for each i it agrees with g_i^σ on C_i^σ . Moreover, $H_d(\sigma \times [0, 1]) \subset \text{supp } f(\sigma)$. Since σ was arbitrary this defines a map H_d on all of Y_d .

Applying this construction repeatedly we obtain maps $H_d: Y_d \rightarrow Z$ for $d = 2, \dots, N$. The map $H = H_N: Y \times [0, 1] \rightarrow Z$ is a homotopy from f to the map $h: Y \rightarrow Z$ given by $h = H_N(\cdot, 1)$ and it follows from the construction that h has the properties asserted in the statement of the lemma. \square

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