On Hill's Worst-Case Guarantee for Indivisible Bads

BO LI, Department of Computing, The Hong Kong Polytechnic University, Hong Kong, Hong Kong

HERVE MOULIN, Adam Smith Business School, University of Glasgow, Glasgow, United Kingdom of Great Britain and Northern Ireland

ANKANG SUN, Department of Computing, The Hong Kong Polytechnic University, Hong Kong, Hong Kong

YU ZHOU, College of Information Sciences and Technology, The Pennsylvania State University, University Park, United States

We evaluate the fairness of a rule allocating among agents with equal rights by computing their utility for a hypothetical worst-case share, which only depends on their own valuation and the number of agents. For indivisible goods, Budish proposed the *maximin share*: the least utility of a bundle in the best partition of the objects; unfortunately maximin share is not always satisfiable. Earlier, Hill proposed the worst maximin share over all utilities with the same largest possible single-object value. More conservative than maximin share, it is guaranteed to be satisfiable for any possible profile of utilities and its computation is elementary, whereas it is NP-hard to compute maximin share. We apply Hill's approach to the allocation of indivisible bads (objects with disutilities) and compute in closed form the worst-case minimax share for a given value of the worst single bad. We show that the worst-case minimax share is close to the original minimax share and that its monotonic closure is the best guaranteed share for all allocation instances with a common upper bound on the value of the worst single bad.

1 Introduction

The task is to fairly allocate a given pile of indivisible objects among agents with equal rights but different preferences. Since the very beginning of the fair division literature [25], allocation rules

have been evaluated in part by the worst-case utility they guarantee to each participant over all possible utility profiles of the other agents. The higher the guaranteed utility, the safer it is for an agent clueless about the others' utilities and actions to participate in the allocation process defined by the rule. To evaluate the guaranteed utility, the literature normally introduces a well-designed *share* that is a function depending solely on each agent herself and the number of other agents (but not on the particulars of the agents), and checks if there are allocation rules that can guarantee every agent her share for any possible profile of utilities.

For example, when the objects are divisible and desirable (i.e., *goods*) and utilities are additive, the proportional share $\frac{1}{n} \cdot v_i(M)$, where M is the set of goods, n is the number of agents, and i is a generic agent with utility function v_i, can be guaranteed [12]. In other words, for any profile of utilities v_i's, there always exists an allocation such that every agent's utility is at least her proportional share. Moreover, proportional share is indeed the best guaranteed utility, considering the case when all agents have identical preferences. But in all the important practical contexts where the objects are indivisible while utilities remain additive, the search for maximal guarantees (those that cannot be improved over the entire domain of utilities) cannot be that simple. The difficulty is obvious when we consider the "one diamond and several worthless rocks" example: unless we throw away the diamond, all agents but one end up with a negligible fraction of $v_i(M)$.

To capture exactly the diamond effect when indivisible goods are distributed, the concept of maximin share [10] has been intensely studied over the past decade [3, 18, 19]. An agent's maximin share is motivated by an imaginary divide-and-choose experiment: the agent gets the chance to partition the objects into n bundles but is the last one to choose one bundle. Then, the agent's maximin share is the utility of her worst bundle in the best n-partition of the objects. Maximin share bears some disadvantages. On one hand, the definition is not trivial and computing its value involves solving an NP-hard problem [26]. On the other hand, in some rare cases, the maximin share cannot be guaranteed [24]—that is, there exist profiles of utilities that no matter how the objects are allocated to the agents, at least one agent's utility is smaller that her maximin share. So far, the best-known approximation is that a $(\frac{3}{4} + \frac{3}{3836})$ fraction of maximin share can be guaranteed [1] and the allocation guaranteeing $\frac{3}{4}$-maximin share can be computed in polynomial time [14].

Back in the 1980s, Hill [17] also investigated how the indivisibility of the objects affects the worst-case utility of agents by restricting attention to additive utility functions v such that $v(M) = 1$ (without loss of generality) and the most valuable object of v is worth α, $0 < \alpha < 1$; we write $\mathcal{V}(\alpha)$ for this subdomain of additive valuations. Hill proposed to study the worst-case maximin share among all valuations in $\mathcal{V}(\alpha)$, which is referred to as the *Hill's share* throughout this article. Hill [17] computed for every $n \geq 2$ a function $V_n : [0, 1] \rightarrow [0, \frac{1}{n}]$, which lower-bounds Hill's share. By the definition, for any $\alpha \in [0, 1]$, $V_n(\alpha)$ is also a lower bound on the maximin share of every utility in $\mathcal{V}(\alpha)$. Depending on α, $V_n(\alpha)$ may or may not improve upon the $(\frac{3}{4} + \frac{3}{3836})$-approximate maximin share, but its great advantage is that whether a given allocation meets Hill's share for a given utility profile is immediately verifiable. Furthermore, Hill proved that if every agent's utility is in $\mathcal{V}(\alpha)$, it is always possible to simultaneously give each agent a utility of at least $V_n(\alpha)$—that is, $V_n(\cdot)$ can always be guaranteed. Markakis and Psomas [22] proved a stronger result: the share $V_n(\alpha_i)$ where $\max_{e \in M} v_i(e) = \alpha_i$ is a bona fide guarantee over the full domain of additive and non-negative utilities. Moreover, an allocation ensuring the Hill's share for all agents can be computed in polynomial time. Gourvès et al. [16] found that $V_n(\alpha)$ is not the tight characterisation of Hill's share and proved a tighter function. An interesting fact is that the tight function is not monotone in α, and its exact computation is still open.

All the aforementioned work, as well as the majority of fair division literature, focuses on the allocation of goods, and the mirror problem of bads (undesirable objects like chores, liabilities
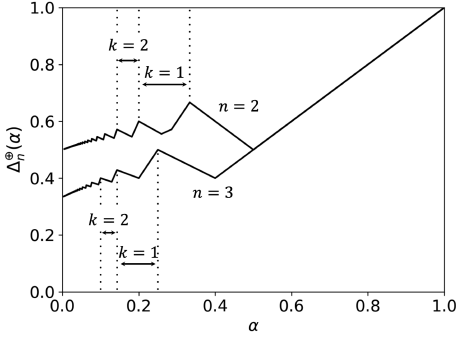
Fig. 1. Hill's share $\Delta_n^{\oplus}(\alpha)$ when $n = 2$ and 3 and m is not restricted. The parameter k categorises the interval to which α belongs.
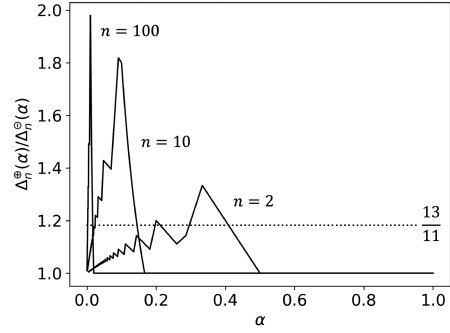
Fig. 2. The ratio between the upper and lower bounds of the minimax share of valuations in $\mathcal{V}(\alpha)$. Note that 13/11 is the state-of-the-art approximation guarantee known to be achievable for minimax share.

when a partnership is dissolved, etc.; see [6, 18, 20]) is not as well understood as that of goods, which motivates the current work.

1.1 Our Problem and Results

We apply Hill's approach to the allocation of indivisible bads and prove a set of results parallel to those just mentioned. The diamond effect now becomes the "chore from hell" effect where the *dis*utility is concentrated in a single bad, and now $\mathcal{V}(\alpha)$ collects all disutility functions where the disutility/cost of the worst bad equals α, maintaining the normalisation $v(M) = 1$.

Our first result for bads resembles those just mentioned for goods and, in addition, the connection between Hill's share and minimax share (*the largest disutility of a bundle in the best partition of the bads*). To be more precise, we compute first the tight characterisation of Hill's share, refined to problems with a given number m of bads—that is, the exact upper bound $\Delta_n^{\oplus}(\alpha; m)$ of the minimax share in the domain $\mathcal{V}(\alpha; m)$, where $\mathcal{V}(\alpha; m)$ contains the valuations over m objects with the largest disutility of an object being α. This result is stated in Theorem 3.1. If m is not restricted (i.e., $\mathcal{V}(\alpha) = \bigcup_m \mathcal{V}(\alpha; m)$ and $\Delta_n^{\oplus}(\alpha) = \max_m \Delta_n^{\oplus}(\alpha; m)$), we illustrate the function $\Delta_n^{\oplus}(\alpha)$ for $n = 2, 3$ in Figure 1 where k categorises the interval to which α belongs. Just like Gourvès et al. [16] observed for the problem of goods, this function is not monotone in α. In passing, we tighten the bounds proposed by Hill [17] and Gourvès et al. [16] for the worst-case maximin share in the two-agent problem of goods; see Remark 1.

Compared to the minimax share, Hill's share $\Delta_n^{\oplus}(\alpha; m)$ is immediately verifiable, whereas deciding whether (a multiple of) the minimax share is met at a given allocation involves solving an NP-hard problem. Moreover, the function $\alpha \rightarrow \Delta_n^{\oplus}(\alpha; m)$ relating the guaranteed share to the disutility of the worst bad (relative to total disutility) is a transparent hard design constraint of which all participants should be aware. Although $\Delta_n^{\oplus}(\alpha; m)$ seems more conservative than the minimax share of a specific disutility function, we argue that $\Delta_n^{\oplus}(\alpha; m)$ is approximately as effective as minimax share. First, $\Delta_n^{\oplus}(\alpha; m)$ is at most twice the minimax share of every disutility in $\mathcal{V}(\alpha; m)$. We plot the exact ratio of $\Delta_n^{\oplus}(\alpha)$ and the best minimax share of disutilities in $\mathcal{V}(\alpha)$ for every α in Figure 2 when $n = 2, 10$ and 100. As we can see, although the largest ratio may reach 2 (only happens when n is large), for most values of α, the ratio is not far from 1. Besides the preceding worst-case comparison, in Section 5 we conduct numerical experiments with synthetic and real-world data to illustrate the real distances between Hill's share and minimax share. The experiments show that Hill's share is actually very close to (e.g., within 1.1 fraction of) the minimax share for the majority of the instances.

Finally, we obtain the second main result of this work—a counterpart for bads of the worst-case guaranteed utility for goods improved by Markakis and Psomas [22]. Letting $V_n(\alpha; m)$ denote the monotonic closure of $\Delta_n^\oplus(\alpha; m)$ with respect to α, Theorem 4.1 shows that $V_n(\alpha_i; m)$ can be guaranteed over the full domain of additive disutilities with m bads. We also provide an algorithm to implement this guarantee in polynomial time. To the best of our knowledge, no other similarly simple guarantee for allocating bads has been identified.

1.2 More Relevant Literature

The properties known as proportionality up to one object (Prop1) and up to any object (PropX) offer different relaxations of the equal share $\frac{1}{n} v_i(M)$ when the objects (goods or bads) are indivisible [5, 23]. These relaxations require the equal share to be satisfiable if at most one object is added or removed. Like the Hill's share for goods or bads we discuss here, they are immediately verifiable, but unlike these, they are not always preserved by Pareto improvements, a serious limitation of their implementation. They also do not provide agents with any guaranteed utility or disutility. The same remark applies to the popular ex-post notions/criteria envy-free up to one item/object, or up to any item/object [10, 11, 21]. Another easily verifiable test is the truncated proportional share bound of Babaioff et al. [7], but unlike Hill's share, it improves upon the maximin share for goods, so it cannot be guaranteed either.

Intuitively, the allocation of undesirable bads is the mirror image of that of goods. However, adapting the results is not a simple matter of switching signs. For instance, when objects are indivisible, the approximations Prop1 and PropX behave quite differently for goods and bads [5, 23].[1] Our results confirm this general observation: in our case, the general allure of the critical functions $\Delta_n^\oplus$ is the same for goods and for bads, but the details and the proofs are quite different. We refer the readers to recent surveys by Moulin [23] and Amanatidis et al. [2] for a more detailed discussion on the fair division of indivisible goods and bads. In particular, Amanatidis et al. [2] and Aziz et al. [4] explicitly listed computing Hill's share for bads as an open problem.

2 Preliminaries

For any positive integer k, let $[k] = \{1, \ldots, k\}$. We consider allocating m indivisible objects, denoted by $M = [m]$, among n agents, and let $\mathcal{A}dd(m)$ be the domain containing the non-negative additive disutility functions v on object set $[m]$, normalised without loss of generality, as follows:

$$v(S) = \sum_{e \in S} v(\{e\}) \text{ for all } S \subseteq M \text{ and } v(M) = 1.$$

Following the convention of the literature, disutility functions are also called *valuations*. For simplicity, we write $v(e)$ to represent $v(\{e\})$ for each $e \in M$. For any $\alpha \in [0, 1]$, the subdomain $\mathcal{V}(\alpha; m) \subseteq \mathcal{A}dd(m)$ is defined by the property $\max_{e \in M} v(e) = \alpha$ and $\mathcal{U}(\alpha; m) \subseteq \mathcal{A}dd(m)$ by $v(e) \leq \alpha$ for all $e \in M$. According to the definitions, $\mathcal{V}(\alpha; m) \subseteq \mathcal{U}(\alpha; m)$ for any valid pair of α and m. Note that since functions are all normalised, $\mathcal{V}(\alpha; m)$ is only well defined if $\alpha \times m \geq 1$, equivalently for $m \geq m_* = \lceil \frac{1}{\alpha} \rceil$ (the upper integer part of $\frac{1}{\alpha}$).

An allocation, denoted by $\mathbf{A} = (A_1, \ldots, A_n)$, is a partition of M into n disjoint subsets of objects; note that some of these subsets can be empty. The set of all allocations is denoted by $\mathcal{X}_n(M)$. When there are n agents, MMS (minimax share) with respect to the disutility $v \in \mathcal{A}dd(m)$ is defined as

$$\text{MMS}_n(v) = \min_{\mathbf{A} \in \mathcal{X}_n(M)} \max_{1 \leq \ell \leq n} v(A_\ell).$$

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[1]See also the work of Bogomolnaia et al. [8] for the competitive equilibrium from equal incomes when objects are divisible.

Hereafter, we use minimax share and MMS interchangeably. We next define the upper and lower bounds of MMS among all disutilities in $\mathcal{V}(\alpha; m)$:

$$\Delta_n^{\oplus}(\alpha; m) = \max_{v \in \mathcal{V}(\alpha; m)} \text{MMS}_n(v) \text{ and}$$

$$\Delta_n^{\ominus}(\alpha; m) = \min_{v \in \mathcal{V}(\alpha; m)} \text{MMS}_n(v).$$

The upper bound $\Delta_n^{\oplus}(\alpha; m)$ (i.e., the worst-case MMS) is indeed Hill's share, and we use these terms interchangeably in this article. We remark that $\mathcal{V}(\alpha; m)$ is a compact set and all valuation functions are continuous, so $\Delta_n^{\oplus}(\alpha; m)$ can be achieved by some $v^* \in \mathcal{V}(\alpha; m)$.

It is not difficult to obtain the following formula of $\Delta_n^{\ominus}(\alpha; m)$.

LEMMA 2.1. *Given $0 < \alpha < 1$, $n \geq 2$, and $m \geq \lceil \frac{1}{\alpha} \rceil$, $\Delta_n^{\ominus}(\alpha; m)$ is as follows:*

$$\Delta_n^{\ominus}(\alpha; m) = \begin{cases} \alpha, & \text{if } \alpha > \frac{1}{n}, \\ \frac{1}{n}, & \text{if } \alpha = \frac{1}{kn}, \text{ or } \frac{1}{(k+1)n} < \alpha < \frac{1}{kn} \text{ and } m \geq kn + n \\ k\alpha + \frac{1-kn\alpha}{m-kn}, & \text{if } \frac{1}{(k+1)n} < \alpha < \frac{1}{kn} \text{ and } m \leq kn + n - 1 \end{cases}$$

for some integer $k \geq 1$.

PROOF. For each case, we show that $\text{MMS}_n(v) \geq \Delta_n^{\ominus}(\alpha; m)$ for any $v \in \mathcal{V}(\alpha; m)$, and design a disutility function such that the MMS is exactly $\Delta_n^{\ominus}(\alpha; m)$. By the definition of $\mathcal{V}(\alpha; m)$, there exists an object with disutility α, thus $\text{MMS}_n(v) \geq \alpha$ for any $v \in \mathcal{V}(\alpha; m)$. Moreover, when $\alpha > 1/n$, there exists a disutility function such that the MMS is exactly α. Specifically, among the m objects, there are $\lfloor \frac{1}{\alpha} \rfloor$ objects with disutility α and one object with disutility $(1 - \lfloor \frac{1}{\alpha} \rfloor \cdot \alpha) < \alpha$ (if 1 is indivisible by α). Each of the remaining objects (if any) has a disutility zero. $\text{MMS}_n(v_1) = \alpha$ follows from the fact that there are at most n objects with positive disutility.

By the definition of minimax share, $\text{MMS}_n(v) \geq \frac{1}{n}$, where the equality is achieved when the total disutility of M can be evenly distributed among the n-partition. When $1/n$ is divisible by α (i.e., $\alpha = \frac{1}{kn}$ for some positive integer k), or $1/n$ is not divisible by α (i.e., $\frac{1}{(k+1)n} < \alpha < \frac{1}{kn}$) and the number of objects m is at least $kn + n$, there exists a disutility function such that the MMS is exactly $1/n$. For the former, the disutility function v_2 contains $1/\alpha = kn$ objects with disutility α. Clearly, each bundle in the best n-partition contains k objects with disutility α and $\text{MMS}_n(v_2) = 1/n$. For the latter, intuitively, the total disutility of M can also be evenly distributed by letting each bundle contain $\lfloor \frac{1}{n\alpha} \rfloor = k$ objects with disutility α and one object with disutility $\frac{1}{n} - k\alpha < \alpha$. In total, $kn + n \leq m$ objects are needed. Hence, the disutility function v_3 contains kn objects with disutility α, n objects with disutility $\frac{1}{n} - k\alpha$ and $m - kn - n$ objects with disutility 0, and $\text{MMS}_n(v_3) = 1/n$.

However, when $1/n$ is indivisible by α but the number of objects m is limited to $kn + n - 1$, $1/n$ cannot be achieved since some bundles in any n-partition contain no more than k objects, and the disutilities of these bundles are at most $k\alpha < 1/n$. For this case, we show that $\text{MMS}_n(v) \geq k\alpha + \frac{1-kn\alpha}{m-kn}$ for any $v \in \mathcal{V}(\alpha; m)$. Let x be the number of bundles in the n-partition that contain no more than k objects, and it follows that $x \geq kn + n - m$. Since the disutility of each of these bundles is at most $k\alpha$, the average disutility of the other bundles is at least

$$\frac{1 - k\alpha x}{n - x} \geq \frac{1 - (kn + n - m) \cdot k\alpha}{m - kn} = k\alpha + \frac{1 - kn\alpha}{m - kn} > k\alpha$$

where the leftmost side is an increasing function of x since $k\alpha < 1/n$, and the last inequality is because $m \geq \lceil \frac{1}{\alpha} \rceil > kn$. Therefore, the largest disutility of any n-partition is at least $k\alpha + \frac{1-kn\alpha}{m-kn}$ — that is, $\text{MMS}_n(v) \geq k\alpha + \frac{1-kn\alpha}{m-kn}$ for any $v \in \mathcal{V}(\alpha; m)$. Let v_4 contain kn objects with disutility α and $m - kn$ objects with disutility $\frac{1-kn\alpha}{m-kn} < \alpha$. Clearly, the worst bundle of the best n-partition contains k objects with disutility α and one object with disutility $\frac{1-kn\alpha}{m-kn}$, thus $\text{MMS}_n(v_4) = k\alpha + \frac{1-kn\alpha}{m-kn}$. □

Computing $\Delta_n^\oplus(\alpha; m)$ is non-trivial, as shown in Section 3, but the following lemma presents two simple properties.

LEMMA 2.2. *(1) $\Delta_n^\oplus(\alpha; m)$ is weakly decreasing in n; (2) $\Delta_n^\oplus(\alpha; m)$ is weakly increasing in m from $\lceil \frac{1}{\alpha} \rceil$ to $\lceil \frac{2}{\alpha} \rceil - 1$ and constant thereafter.*

PROOF. That $\Delta_n^\oplus(\alpha; m)$ decreases in n is clear by comparing the minimax share of an arbitrary n-partition and the $(n + 1)$-partition obtained by adding one empty share. The monotonicity in m (i.e., $\Delta_n^\oplus(\alpha; m) \leq \Delta_n^\oplus(\alpha; m+1)$) is due to the fact that every disutility in $\mathcal{V}(\alpha; m)$ can be transformed to one in $\mathcal{V}(\alpha; m + 1)$ by adding an object with disutility 0, without changing the minimax share.

We then show when $m \geq \lceil \frac{2}{\alpha} \rceil - 1$, $\Delta_n^\oplus(\alpha; m) \geq \Delta_n^\oplus(\alpha; m + 1)$, thus $\Delta_n^\oplus(\alpha; m)$ remains constant. First, note that if $\Delta_n^\oplus(\alpha; m+1) = \alpha$, then $\Delta_n^\oplus(\alpha; m) = \alpha$ as the largest disutility of a single object is α. Then, we further assume $\Delta_n^\oplus(\alpha; m + 1) > \alpha$. Let $v \in \mathcal{V}(\alpha; m + 1)$ be a disutility with $\text{MMS}_n(v) > \alpha$. Note that the existence of such a v is guaranteed as $\Delta_n^\oplus(\alpha; m + 1) > \alpha$. Let $(A_1, \ldots, A_n)$ be the n-partition that gives MMS of v, and moreover, $(A_1, \ldots, A_n)$ minimises the number of bundles achieving $\text{MMS}_n(v)$ among all n-partition yielding $\text{MMS}_n(v)$. Without loss of generality, assume $v(A_1) \geq v(A_2) \geq \cdots \geq v(A_n)$.

We next claim that, among bundles A_i's, there exists one bundle such that the total disutility of two of its objects is no more than α. Denote by $\mathcal{J} = \{j \in [n] \mid |A_j| \geq 2\}$, and as $v(A_1) = \text{MMS}_n(v) > \alpha$ and the largest disutility of a single object is α, we have $\mathcal{J} \neq \emptyset$. For a contradiction, assume that for any A_t with $t \in \mathcal{J}$, the total disutility of any two objects is larger than α, which means that $v(A_t) > \frac{|A_t|}{2}\alpha$. If every bundle contains at least two objects (i.e., $\mathcal{J} = [n]$), then by summing up the lower bounds over all bundles, $1 = \sum_{j \in [n]} v(A_j) > \frac{\alpha}{2} \cdot \lceil \frac{2}{\alpha} \rceil \geq 1$, a contradiction. However, for the case of $\mathcal{J} \subsetneq [n]$, we claim that for any $j \in [n] \setminus \mathcal{J}$, it holds that $v(A_j) > \frac{\alpha}{2}$; otherwise, reallocating the object with the least disutility from A_1 to bundle A_{j^*} where $j^* \in \arg\min_{j \in [n] \setminus \mathcal{J}} v(A_j)$ violates the construction of $(A_1, \ldots, A_n)$. Then, again, upon summing up the lower bounds over all bundles, $1 = \sum_{j \in [n]} v(A_j) > \frac{\alpha}{2} \cdot \lceil \frac{2}{\alpha} \rceil \geq 1$, a contradiction.

By the claim in the previous paragraph, there exists a bundle A_i such that two objects $e_1, e_2 \in A_i$ satisfy $v(e_1) + v(e_2) \leq \alpha$. We derive a disutility $v' \in \mathcal{V}(\alpha; m)$ by merging e_1 and e_2 into one object e, and show that $\text{MMS}_n(v) = \text{MMS}_n(v')$. On one hand, let $A_i' = A_i \setminus \{e_1, e_2\} \cup \{e\}$, and since $(A_1, \ldots, A_{i-1}, A_i', A_{i+1}, \ldots, A_n)$ is an n-partition regarding v' with the largest disutility of a bundle being $\text{MMS}_n(v)$, it follows that $\text{MMS}_n(v) \geq \text{MMS}_n(v')$. On the other hand, by decomposing e into e_1 and e_2, we can convert any allocation regarding v' to an allocation regarding v without changing the largest disutility, thus $\text{MMS}_n(v) \leq \text{MMS}_n(v')$.

Therefore, when $m \geq \lceil \frac{2}{\alpha} \rceil - 1$, every disutility v with $\text{MMS}_n(v) > \alpha$ can be transformed to a disutility v' with $\text{MMS}_n(v) = \text{MMS}_n(v')$, which gives $\Delta_n^\oplus(\alpha; m) \geq \Delta_n^\oplus(\alpha; m+1)$. By combining the monotonicity in m, $\Delta_n^\oplus(\alpha; m)$ remains constant when $m \geq \lceil \frac{2}{\alpha} \rceil - 1$.   □

By the second property in Lemma 2.2, and also following other works [16, 17, 22], we also consider the case when m is not restricted. Let $\mathcal{V}(\alpha) = \bigcup_m \mathcal{V}(\alpha, m)$ and $\mathcal{U}(\alpha) = \bigcup_m \mathcal{U}(\alpha; m)$. Accordingly, we have

$$\Delta_n^\oplus(\alpha) = \max_{v \in \mathcal{V}(\alpha)} \text{MMS}_n(v) \text{ and}$$

$$\Delta_n^\ominus(\alpha) = \min_{v \in \mathcal{V}(\alpha)} \text{MMS}_n(v).$$

By Lemma 2.1, $\Delta_n^\ominus(\alpha) = \max\{\alpha, 1/n\}$.

Hill's share $\Delta_n^\oplus(\alpha; m)$ (and $\Delta_n^\oplus(\alpha)$) behave much like the MMS in the following senses. First, for any $v \in \mathcal{V}(\alpha; m)$, there is an allocation $(A_1, \ldots, A_n)$ such that $v(A_i) \leq \Delta_n^\oplus(\alpha; m)$ for all i. This follows from the definition of MMS plus that $\Delta_n^\oplus(\alpha; m)$ is an upper bound of MMS. Second, the *max*

in the definition of $\Delta_n^\oplus(\alpha; m)$ is achieved by some $v^* \in \mathcal{V}(\alpha; m)$—that is, $\Delta_n^\oplus(\alpha; m) = \text{MMS}_n(v^*)$. This is because $\mathcal{V}(\alpha; m)$ is a compact set and all the functions are continuous. Then we know that for any allocation $(B_1, \ldots, B_n)$, there is some i such that $v^*(B_i) \geq \Delta_n^\oplus(\alpha; m)$. Note that these two facts have nothing to do with what the function $\Delta_n^\oplus(\alpha; m)$ actually looks like, and they can be easily adapted to $\Delta_n^\oplus(\alpha)$.

3 Characterising Hill's Share

3.1 Main Result

We now characterise Hill's share—that is, the exact upper bound of the minimax share values, $\Delta_n^\oplus(\alpha; m)$ and $\Delta_n^\oplus(\alpha)$. For any integers $n \geq 2$ and $k \geq 0$, define the following real intervals:

$$D(n, k) = \left(\frac{1}{kn + n + 1}, \frac{k + 2}{n(k + 1)^2 + k + 2} \right] \tag{1}$$

$$I(n, k) = \left(\frac{k + 2}{n(k + 1)^2 + k + 2}, \frac{1}{kn + 1} \right]. \tag{2}$$

It is not hard to check that all the intervals are well defined, non-overlapping, and $\bigcup_{k \geq 0}(D(n, k) \cup I(n, k)) = (0, 1]$.

Our first main theorem gives the tight characterisation of Hill's share with the parameter m.

THEOREM 3.1. *(i) General case: For any $0 < \alpha < 1$, $n \geq 2$, and $m \geq \lceil \frac{1}{\alpha} \rceil$,*

$$\Delta_n^\oplus(\alpha; m) = \begin{cases} \frac{k+2}{k+1} \cdot \frac{1-\alpha}{n}, & \text{if } \alpha \in D(n, k) \text{ and } m \geq kn + n + 1, \\ (k + 1)\alpha, & \text{if } \alpha \in D(n, k) \text{ and } m \leq kn + n, \\ (k + 1)\alpha, & \text{if } \alpha \in I(n, k) \end{cases} \tag{3}$$

for any integers $n \geq 2$ and $k \geq 0$ except $n = 2$ and simultaneously $k = 1$.

(ii) Special case: If $n = 2$ and $k = 1$, $\Delta_2^\oplus(\frac{1}{3}; 3) = \frac{2}{3}$, $\Delta_2^\oplus(\alpha; 4) = 2\alpha$ for $\alpha \in [\frac{1}{4}, \frac{1}{3}]$, and $\Delta_2^\oplus(\alpha; 5)$ is as follows:

$$\Delta_2^\oplus(\alpha; 5) = \begin{cases} \frac{3-3\alpha}{4}, & \text{if } \alpha \in (\frac{1}{5}, \frac{3}{11}], \\ 2\alpha, & \text{if } \alpha \in (\frac{3}{11}, \frac{1}{3}], \end{cases} \tag{4}$$

and for $m \geq 6$,

$$\Delta_2^\oplus(\alpha; m) = \begin{cases} \frac{3-3\alpha}{4}, & \text{if } \alpha \in (\frac{1}{5}, \frac{7}{27}] \\ \alpha + \frac{2-2\alpha}{5}, & \text{if } \alpha \in (\frac{7}{27}, \frac{2}{7}] \\ 2\alpha, & \text{if } \alpha \in (\frac{2}{7}, \frac{1}{3}]. \end{cases} \tag{5}$$

Theorem 3.1 also implies the expression of Hill's share when the number of objects is not restricted, as shown in the following corollary.

COROLLARY 3.2. *For any $0 < \alpha < 1$, $n \geq 2$, $\Delta_n^\oplus(\alpha) = \max_{m \geq \lceil \frac{1}{\alpha} \rceil} \Delta_n^\oplus(\alpha; m)$. In other words,*

$$\Delta_n^\oplus(\alpha) = \begin{cases} \frac{k+2}{k+1} \cdot \frac{1-\alpha}{n}, & \text{if } \alpha \in D(n, k), \\ (k + 1)\alpha, & \text{if } \alpha \in I(n, k) \end{cases} \tag{6}$$

for any integers $n \geq 2$ and $k \geq 0$ except $n = 2$ and simultaneously $k = 1$. If $n = 2$ and $k = 1$,

$$\Delta_2^\oplus(\alpha) = \begin{cases} \frac{3-3\alpha}{4}, & \text{if } \alpha \in (\frac{1}{5}, \frac{7}{27}] \\ \alpha + \frac{2-2\alpha}{5}, & \text{if } \alpha \in (\frac{7}{27}, \frac{2}{7}] \\ 2\alpha, & \text{if } \alpha \in (\frac{2}{7}, \frac{1}{3}]. \end{cases} \tag{7}$$

Actually, Corollary 3.2 is a special case of Theorem 3.1 when m is sufficiently large (e.g., $m \geq \lceil \frac{2}{\alpha} \rceil - 1$ by Lemma 2.2). Recall that we illustrated $\Delta_2^{\oplus}(\alpha)$ and $\Delta_3^{\oplus}(\alpha)$ in Figure 1. We observe two interesting and somewhat unintuitive facts about Theorem 3.1. First, $\Delta_n^{\oplus}(\cdot)$ is not monotone in α, just like Gourvès et al. [16] observed for the problem with goods. To characterise $\Delta_n^{\oplus}(\alpha; m)$, we want to understand the worst-case disutility in $\mathcal{V}(\alpha; m)$, for which the objects can be hardly partitioned into bundles with similar disutilities. Intuitively, when the single-object disutility gets larger, it becomes harder to find such a balanced partition. However, this turns out to be imprecise. Second, the case of $n = 2$ makes a difference from $n \geq 3$. When $n = 2$ and $k = 1$, there are three steps in $\Delta_n^{\oplus}(\cdot)$: the worst-case MMS has two increasing intervals with different slops following a decreasing interval. For all the other values of n and k, there are two intervals with one decreasing and the other increasing.

Remark 1. When $n = 2$ the problem of bads and that of goods are the same, since maximising the minimum bundle by partitioning the objects into two bundles is equivalent to minimising the maximum bundle. For $n = 2$, Gourvès et al. [16] provided a lower bound of the maximin share for goods which is not tight. It can be verified that $1 - \Delta_2^{\oplus}(\alpha)$ is strictly larger than their bound when $\alpha \in (\frac{1}{5}, \frac{3}{10})$ (Definition 2 in the work of Gourvès et al. [16]). Thus, as a byproduct, Corollary 3.2 improves the result in the work of Gourvès et al. [16] for goods with $n = 2$ by giving the tight worst-case bound—that is,

$$\min_{v \in \mathcal{V}(\alpha)} \max_{\mathbf{A} \in \mathcal{X}_2(M)} \min_{1 \leq \ell \leq 2} v(A_\ell) = 1 - \Delta_2^{\oplus}(\alpha).$$

In Remark 2, we show how to extend this result to two non-identical disutilities.

Before diving deep into proofs, we remark that the characterisation of Hill's share for goods can not be directly or trivially converted into the characterisation of Hill's share for chores. In particular, the worst-case scenario of chores differs from that of goods and precisely identifying the exact worst-case scenario for chores is quite challenging. Additionally, the up-to-date characterisation of goods by Gourvès et al. [16] is not tight, whereas we here aim to establish a tight characterisation for chores.

Roadmap for the Proof of Theorem 3.1. As we have discussed, after m reaches a certain value (e.g., $m \geq \lceil \frac{2}{\alpha} \rceil - 1$ by Lemma 2.2), Hill's share does not increase anymore, and thus Corollary 3.2 is a special case of Theorem 3.1 when m is sufficiently large. Therefore, we will first prove Corollary 3.2. The proof is by contradiction arguments, and we begin with assuming that there exists a disutility $v \in \mathcal{V}(\alpha)$ with MMS larger than the expression in Corollary 3.2. Then we consider the lexicographical MinMax allocation of v—that is, an allocation such that the largest disutility of bundles is minimised over all allocations, and among these allocations the second largest disutility is minimised, and so on. The lexicographical MinMax allocation ensures that starting from this allocation, swapping a subset of one bundle with a subset of another bundle should not improve the allocation. We then utilise such a property to derive the desired contradiction. After proving Corollary 3.2, we use similar proof ideas and carefully discuss Hill's share when m is not sufficiently large, which then completes the proof of Theorem 3.1. Our proofs involve careful and detailed case distinctions and are deferred to the appendix.

4 Hill's Share for Indivisible Bads

4.1 Main Result

In this section, we prove the counterpart result of the worst-case guaranteed disutility for indivisible bads. We know from the work of Aziz et al. [6] and Feige et al. [13] that the minimax share $\mathsf{MMS}_n(v_i)$ cannot be guaranteed because at some profiles no allocation ensures every
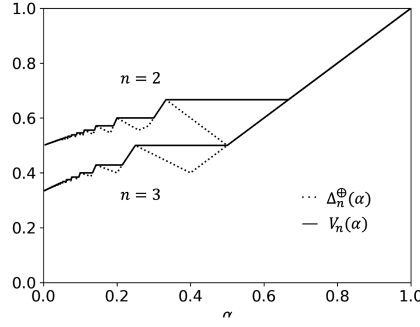
Fig. 3. The characterisation for heterogeneous agents.

agent's disutility to be $MMS_n(v_i)$ or below. Thus, the guaranteed disutility is lower bounded by the minimax share. In the following, we show that the monotonic closure of $\Delta_n^\oplus$ serves as the best guarantee in Hill's model. Recall that $\mathcal{U}(\alpha; m)$ contains all the disutility functions $v(\cdot)$ on objects M such that $\max_{e \in M} v(e) \leq \alpha$, and $\mathcal{U}(\alpha) = \bigcup_m \mathcal{U}(\alpha; m)$. For simplicity in the presentation and analysis, we ignore the restriction of the number of objects m in this section, and the result can be extended to the setting with parameter m using the same approach in Section 3. The definition of $\mathcal{U}(\alpha)$ is the same as in other works [16, 17, 22]. Note that $\mathcal{U}(\alpha') \subseteq \mathcal{U}(\alpha)$ if $\alpha' \leq \alpha$, and the difference between $\mathcal{V}(\alpha)$ and $\mathcal{U}(\alpha)$ is that the disutilities in $\mathcal{U}(\alpha)$ do not require that there must be one object with disutility α. It is straightforward that the tight guaranteed disutility regarding $\mathcal{U}(\cdot)$ must be monotonically non-decreasing since any worst-case disutility in $\mathcal{U}(\beta)$ is also a disutility in $\mathcal{U}(\alpha)$ for $\beta \leq \alpha$. We write V_n the monotonic closure of $\Delta_n^\oplus$

$$V_n(\alpha) = \max_{0 \leq \beta \leq \alpha} \Delta_n^\oplus(\beta),$$

as illustrated in Figure 3 when $n = 2, 3$. In more detail, we have the following formula of V_n:

$$V_n(\alpha) = \begin{cases} \frac{k+2}{(k+1)n+1}, & \text{if } \alpha \in NI(n, k) \\ (k+1)\alpha, & \text{if } \alpha \in I(n, k) \end{cases}$$

where for any integer $k \geq 0$,

$$NI(n, k) = \left(\frac{1}{(k+1)n+1}, \frac{k+2}{(k+1)((k+1)n+1)} \right)$$

and

$$I(n, k) = \left[\frac{k+2}{(k+1)((k+1)n+1)}, \frac{1}{kn+1} \right].$$

By Theorem 3.1 and the construction of $V_n(\cdot)$, $V_n(\cdot)$ provides the tight bound of the worst-case MMS regarding $\mathcal{U}(\cdot)$. We further prove that $V_n(\cdot)$ is a guaranteed disutility, and moreover, an allocation satisfying $V_n(\cdot)$ can be found in polynomial time.

THEOREM 4.1. *Given any* $0 \leq \alpha_i \leq 1$ *and* $v_i \in \mathcal{U}(\alpha_i)$ *for* $i = 1, \ldots, n$, *there exists an allocation* $(A_1, \ldots, A_n)$ *with*

$$v_i(A_i) \leq V_n(\alpha_i) \text{ for all } i = 1, \ldots, n$$

and such an allocation can be computed in polynomial time. Moreover, for any $0 \leq \alpha \leq 1$, *there exists* $\{v_i'\}_{i=1}^n$ *with* $v_i' \in \mathcal{U}(\alpha)$ *for any* $i \in [n]$ *such that* $V_n(\alpha)$ *is the best possible guarantee—that is, for any allocation* $(B_1, \ldots, B_n)$,

there exists $i \in N$ *such that* $v_i'(B_i) \geq V_n(\alpha)$.

The two key features of this guarantee are as follows: its computation is elementary, and it does not depend on the number of bads to allocate. As far as we know, no other similarly simple guarantee for the allocation of bads has been identified.

Remark 2. By Theorem 4.1, $V_n(\alpha)$ is the best guaranteed disutility for preferences in $\mathcal{U}(\alpha)$, and thus we get the tight counterpart result of Hill [17] for bads. However, it may not be the best in the model of Gourvès et al. [16]—that is, for preferences in $\mathcal{V}(\alpha)$. For example, when $n = 2$, we can show that $\Delta_2^\oplus(\max_{e \in M} v_i(e))$ is a better guaranteed disutility in the later model. Given two disutility functions v_1 and v_2, without loss of generality, suppose $\Delta_2^\oplus(\max_e v_1(e)) \leq \Delta_2^\oplus(\max_e v_2(e))$. Then we find the MinMax partition of v_1 so that the disutilities of both bundles are no greater than $\Delta_2^\oplus(\max_e v_1(e))$ to agent 1. We ask agent 2 to choose a better bundle whose disutility must be no greater than $\frac{1}{2}$ and thus no greater than $\Delta_2^\oplus(\max_e v_2(e))$ to agent 2. It is still open whether $\Delta_n^\oplus(\max_{e \in M} v_i(e))$ can be guaranteed or not when $n \geq 3$ in the model of Gourvès et al. [16], which is an interesting future research direction.

4.2 Proof of Theorem 4.1

To show that one can compute an allocation satisfying the required bound in Theorem 4.1, we derive a variation of the moving-knife algorithm. When the objects are goods and divisible, Dubins and Spanier [12] proved that such an algorithm (also known as the Dubins-Spanier moving knife algorithm) gives the optimal worst-case bound—that is, every agent gets value for at least $\frac{1}{n}$. Markakis and Psomas [22] further proved that a variation of this algorithm also guarantees the optimal worst-case bound for indivisible goods. In a nutshell, towards proving Theorem 4.1, we first use the reduction proved in other works [9, 18] to restrict our attention to the ordered instances when agents have the same ranking over all objects, which significantly simplifies our analysis. Then we show that using $V_n(\cdot)$ to set the parameters in the moving-knife algorithm always returns an allocation ensuring the bound in Theorem 4.1.

The following lemma says that it suffices to only focus on the ordered instances.

LEMMA 4.2 ([9, 18]). *Suppose there is an algorithm that takes any ordered instance as input, runs in $T(n, m)$ time and returns an allocation where each agent i's disutility is at most $V_n(\alpha_i)$. Then, we have an algorithm that takes any instance as input, runs in $T(n, m) + O(nm \log m)$ time and returns an allocation with the same disutility guarantee.*

Our approach is similar to that in the work of Markakis and Psomas [22], but the detailed proof differs. Our algorithm runs in recursions. In each recursion, the algorithm allocates a bundle of objects to one agent in a moving-knife fashion. Each time, each of the remaining agents moves her "knife" one object towards the objects with smaller disutilities, until for every agent i the total disutility of the objects before her "knife" is larger than $V_n(\alpha_i)$. After that, one of the last agents (denoted by agent k) for whom the total utility of the objects before her knife is larger than $V_n(\alpha_k)$ receives the objects except the one right before her knife. If there remains only one agent who has not received a bundle, she will get all the remaining objects. Otherwise, all the remaining agents enter the next recursion with their disutility functions being normalised such that for each of them the total disutility of the remaining objects is 1. The formal description of our algorithm is presented in Algorithm 1.

Now, we are going to prove Theorem 4.1. Without loss of generality, let $1, \ldots, n$ be the order in which agents receive bundles in Algorithm 1. Denote $C_i = v_i(A_1)$ for every $i \in N \setminus \{1\}$, and the following lemma gives a lower bound of C_i.

LEMMA 4.3. *For any agent $i \in N \setminus \{1\}$ with $\alpha_i \in NI(n, k) \cup I(n, k)$ for some $k \geq 0$, we have*

$$C_i \geq \frac{1 - V_n(\alpha_i)}{n - 1}.$$

ALGORITHM 1: Algorithm for heterogeneous disutilities

Require: An ordered instance with agents N, objects M and disutility functions $\{v_i\}_{i \in N}$.
Ensure: An allocation $\mathbf{A} = \{A_1, \ldots, A_n\}$ with $v_i(A_i) \leq V_n(\alpha_i)$ for every $i \in N$.
 1: Initialise $S_i = \emptyset$ for every $i \in N$.
 2: **while** there exists an agent j with $v_j(S_j) \leq V_n(\alpha_j)$ **do**
 3:   **for** every $i \in N$ **do**
 4:     $S_i \leftarrow S_i \cup \{$the object in $M \setminus S_i$ with the largest disutility for agent i (tie breaks arbitrarily)$\}$.
 5:   **end for**
 6: **end while**
 7: Pick the agent $t \in N$ with $v_t(S_t \setminus \{\widetilde{e}\}) \leq V_n(\alpha_t)$ where $\widetilde{e}$ is the last object that t added into S_t (tie breaks arbitrarily).
 8: $A_t \leftarrow S_t \setminus \{\widetilde{e}\}$.
 9: **if** $|N| = 2$ **then**
10:   Allocate $M \setminus A_t$ to the remaining agent.
11: **else**
12:   Construct a new disutility function v_i' for every $i \in N \setminus \{t\}$ by setting $v_i'(e) = \frac{v_i(e)}{1 - v_i(A_t)}$ for every $e \in M \setminus A_t$.
13:   Run Algorithm 1($N \setminus \{t\}, M \setminus A_t, \{v_i'\}_{i \in N \setminus \{t\}}$).
14: **end if**

PROOF. Denote by q the index such that $\sum_{e=1}^{q} v_i(e) \leq V_n(\alpha_i)$ and $\sum_{e=1}^{q+1} v_i(e) > V_n(\alpha_i)$, whose existence is guaranteed since $v_i(M) > V_n(\alpha_i)$. Since $V_n(\alpha_i) \geq (k+1)\alpha_i$ (this can be easily verified from the definition of $V_n(\alpha)$ and can also be seen from Figure 3) and $v_i(e) \leq \alpha_i$ for every $e \in M$, $q \geq k+1$. Otherwise, $\sum_{e=1}^{q+1} v_i(e) \leq (k+1)\alpha_i \leq V_n(\alpha_i)$, which contradicts the definition of q. According to Algorithm 1, $C_i \geq \sum_{e=1}^{q} v_i(e)$. Since only ordered instances are considered, $v_i(\{q+1\}) \leq v_i(\{q\}) \leq \frac{C_i}{k+1}$, which gives

$$C_i + \frac{C_i}{k+1} \geq \sum_{e=1}^{q+1} v_i(e) > V_n(\alpha_i).$$

Therefore, $C_i > \frac{k+1}{k+2} \cdot V_n(\alpha_i)$. We consider the following two cases regarding the ranges of α_i.

Case 1: $\alpha_i \in I(n, k)$. In this case, $\frac{k+2}{(k+1)((k+1)n+1)} \leq \alpha_i \leq \frac{1}{kn+1}$ and $V_n(\alpha_i) = (k+1)\alpha_i$. Then,

$$C_i > \frac{k+1}{k+2} \cdot V_n(\alpha) \geq \frac{1 - V_n(\alpha)}{n-1},$$

where the last inequality holds since $\alpha_i \geq \frac{k+2}{(k+1)((k+1)n+1)}$.

Case 2: $\alpha_i \in NI(n, k)$. In this case, $V_n(\alpha_i) = \frac{k+2}{(k+1)n+1}$, which gives

$$C_i > \frac{k+1}{k+2} \cdot V_n(\alpha) = \frac{1 - V_n(\alpha)}{n-1},$$

which completes the proof. □

Interestingly, the following lemma shows the connection between the ranges of α_i and $\frac{\alpha_i}{1 - \frac{1-V_n(\alpha_i)}{n-1}}$.

LEMMA 4.4. *For any $\alpha_i \in NI(n, k) \cup I(n, k)$ for some $k \geq 0$, we have*

$$\frac{\alpha_i}{1 - \frac{1-V_n(\alpha_i)}{n-1}} \in \begin{cases} I(n-1, k), & \text{if } \alpha_i \in I(n, k) \\ NI(n-1, k), & \text{if } \alpha_i \in NI(n, k). \end{cases}$$

PROOF. We consider the following two cases regarding the ranges of α_i.

Case 1: $\alpha_i \in I(n, k)$. In this case, $\frac{k+2}{(k+1)((k+1)n+1)} \leq \alpha_i \leq \frac{1}{kn+1}$ and $V_n(\alpha_i) = (k+1)\alpha_i$. Then, we have

$$\frac{\alpha_i}{1 - \frac{1-V_n(\alpha_i)}{n-1}} = \frac{(n-1)\alpha_i}{n-2+(k+1)\alpha_i} \leq \frac{n-1}{(n-2)(kn+1)+k+1} = \frac{1}{k(n-1)+1},$$

where the inequality is because $\alpha_i \leq \frac{1}{kn+1}$. Besides,

$$\frac{\alpha_i}{1 - \frac{1-V_n(\alpha_i)}{n-1}} = \frac{(n-1)\alpha_i}{n-2+(k+1)\alpha_i} \geq \frac{(k+2)(n-1)}{(k+1)((n-2)(kn+n+1)+k+2)}$$

$$= \frac{k+2}{(k+1)((k+1)(n-1)+1)},$$

where the inequality is because $\alpha_i \geq \frac{k+2}{(k+1)((k+1)n+1)}$.

Case 2: $\alpha_i \in NI(n, k)$. In this case, $\frac{1}{(k+1)n+1} < \alpha_i < \frac{k+2}{(k+1)((k+1)n+1)}$ and $V_n(\alpha_i) = \frac{k+2}{(k+1)n+1}$. Then, we have

$$\frac{\alpha_i}{1 - \frac{1-V_n(\alpha_i)}{n-1}} = \frac{((k+1)n+1)\alpha_i}{(k+1)(n-1)+1} < \frac{k+2}{(k+1)((k+1)(n-1)+1)},$$

where the inequality is because $\alpha_i < \frac{k+2}{(k+1)((k+1)n+1)}$. Besides,

$$\frac{\alpha_i}{1 - \frac{1-V_n(\alpha_i)}{n-1}} = \frac{((k+1)n+1)\alpha_i}{(k+1)(n-1)+1} > \frac{1}{(k+1)(n-1)+1},$$

where the inequality is because $\alpha_i > \frac{1}{(k+1)n+1}$. □

PROOF OF THEOREM 4.1. We prove Theorem 4.1 by mathematical induction. When $n = 2$, it is easy to see the correctness of Theorem 4.1 from Lemma 4.3 since $v_1(A_1) \leq V_2(\alpha_1)$ and $v_2(A_2) = 1 - v_2(A_1) \leq 1 - (1 - V_2(\alpha_2)) = V_2(\alpha_2)$, We assume as our induction hypothesis that Theorem 4.1 holds for $n - 1$. Then we aim to prove the correctness for n.

From Algorithm 1, $v_1(A_1) \leq V_n(\alpha_1)$ clearly holds for agent 1. For any other agent $i \in N \setminus \{1\}$, denote $\widetilde{\alpha}_i = \max_{e \in M \setminus A_1} v'_i(e)$. We know from Algorithm 1 that $\widetilde{\alpha}_i \leq \frac{\alpha_i}{1-C_i}$ and from the induction hypothesis that $v'_i(A_i) \leq V_{n-1}(\widetilde{\alpha}_i)$, which together give

$$v_i(A_i) = (1 - C_i)v'_i(A_i) \leq (1 - C_i)V_{n-1}(\widetilde{\alpha}_i) \leq (1 - C_i)V_{n-1}\left(\frac{\alpha_i}{1-C_i}\right),$$

where the last inequality holds by recalling that $V_{n-1}(\widetilde{\alpha}_i)$ is an non-decreasing function of $\widetilde{\alpha}_i$. Therefore, it remains to show

$$(1 - C_i)V_{n-1}\left(\frac{\alpha_i}{1-C_i}\right) \leq V_n(\alpha_i).$$

Note that $(1 - C_i)V_{n-1}(\frac{\alpha_i}{1-C_i})$ is a non-increasing function of C_i. This is because when $\frac{\alpha_i}{1-C_i} \in I(n-1, k)$ for some k, $(1 - C_i)V_{n-1}(\frac{\alpha_i}{1-C_i}) = (1 - C_i)(k+1)\frac{\alpha_i}{1-C_i} = (k+1)\alpha_i$, which is a constant with respect to C_i; when $\frac{\alpha_i}{1-C_i} \in NI(n-1, k)$ for some k, $(1 - C_i)V_{n-1}(\frac{\alpha_i}{1-C_i}) = (1 - C_i)\frac{k+2}{(k+1)(n-1)+1}$, a decreasing function of C_i. It follows that

$$(1 - C_i)V_{n-1}\left(\frac{\alpha_i}{1-C_i}\right) \leq \left(1 - \frac{1-V_n(\alpha_i)}{n-1}\right)V_{n-1}\left(\frac{\alpha_i}{1 - \frac{1-V_n(\alpha_i)}{n-1}}\right) = V_n(\alpha_i),$$

where the inequality is due to $C_i \geq \frac{1-V_n(\alpha_i)}{n-1}$ (according to Lemma 4.3), and the equality can be verified by considering the following two cases regarding the ranges of α_i.

Case 1: $\alpha_i \in I(n, k)$. In this case, Lemma 4.4 gives $\frac{\alpha_i}{1 - \frac{1 - V_n(\alpha_i)}{n-1}} \in I(n - 1, k)$. Thus, we have

$$\left(1 - \frac{1 - V_n(\alpha_i)}{n - 1}\right) V_{n-1}\left(\frac{\alpha_i}{1 - \frac{1 - V_n(\alpha_i)}{n-1}}\right) = \left(1 - \frac{1 - V_n(\alpha_i)}{n - 1}\right) \cdot (k + 1)\frac{\alpha_i}{1 - \frac{1 - V_n(\alpha_i)}{n-1}}$$

$$= (k + 1)\alpha_i = V_n(\alpha_i).$$

Case 2: $\alpha_i \in NI(n, k)$. In this case, $\frac{\alpha_i}{1 - \frac{1 - V_n(\alpha_i)}{n-1}} \in NI(n - 1, k)$. Thus, we have

$$\left(1 - \frac{1 - V_n(\alpha_i)}{n - 1}\right) V_{n-1}\left(\frac{\alpha_i}{1 - \frac{1 - V_n(\alpha_i)}{n-1}}\right) = \left(1 - \frac{\frac{k+2}{(k+1)n+1}}{n - 1}\right) \cdot \frac{k + 2}{(k + 1)(n - 1) + 1}$$

$$= \frac{k + 2}{(k + 1)n + 1} = V_n(\alpha_i).$$

Therefore, we complete the proof of Theorem 4.1. □

The instances provided in proofs (in the appendix) of Corollary 3.2 and Theorem 3.1 show the tightness of Theorem 4.1.

5 Numerical Experiments

To demonstrate that $\Delta_n^\oplus(\alpha; m)$ can serve as a good alternative of the minimax share, in this section we empirically estimate the difference between the two shares. We first evaluate the worst-case ratio of $\Delta_n^\oplus(\alpha; m)$ and $\Delta_n^\ominus(\alpha; m)$ (recall that $\Delta_n^\ominus(\alpha; m)$ is the best-case MMS over all disutilities in $\mathcal{V}(\alpha; m)$). Denote by $r_n(\alpha; m) = \frac{\Delta_n^\oplus(\alpha; m)}{\Delta_n^\ominus(\alpha; m)}$. It is clear that $r_n(\alpha; m)$ is no smaller than the ratio between $\Delta_n^\oplus(\alpha; m)$ and the real MMS, and we have illustrated $r_n(\alpha; \infty)$ in Figure 2 for $n = 2, 10, 100$. As we can see, although the worst-case ratio can be close to 2, it only happens for sufficiently large n and a small range of values of α. Actually, it is not hard to verify that $r_n(\alpha; m) \leq \frac{2n}{n+1} < 2$ for all α, and $r_n(\alpha; m) \leq \frac{4}{3}$ for all α outside of $(\frac{4}{9n}, \frac{3}{2n+3})$. Note that $\frac{3}{2n+3} - \frac{4}{9n} < \frac{7}{6n}$.

CLAIM 1. *For any $n \geq 2$, $\alpha \in (0, 1]$ and $m \geq \lceil \frac{1}{\alpha} \rceil$, $r_n(\alpha; m) \leq \frac{2n}{n+1}$.*

PROOF. Notice that by Lemmas 2.1 and 2.2, $r_n(\alpha; m)$ is weakly increasing in m. Therefore, it suffices to prove the claim for the setting when m is unrestricted—that is, $r_n(\alpha) \leq \frac{2n}{n+1}$. We first consider the case where $n = 2$ and $k = 1$. In this case, $\alpha \in (\frac{1}{5}, \frac{1}{3}]$. Since $\alpha < \frac{1}{n} = \frac{1}{2}$, $\Delta_2^\ominus(\alpha) = \frac{1}{2}$. When $\alpha \in (\frac{1}{5}, \frac{7}{27}]$, $\Delta_2^\oplus(\alpha) = \frac{3-3\alpha}{4}$, thus $r_2(\alpha) = \frac{3-3\alpha}{2} < \frac{6}{5} < \frac{4}{3}$; when $\alpha \in (\frac{7}{27}, \frac{2}{7}]$, $\Delta_2^\oplus(\alpha) = \frac{2+3\alpha}{5}$, thus $r_2(\alpha) = \frac{4+6\alpha}{5} \leq \frac{8}{7} < \frac{4}{3}$; when $\alpha \in (\frac{2}{7}, \frac{1}{3}]$, $\Delta_2^\oplus(\alpha) = 2\alpha$, thus $r_2(\alpha) = 4\alpha \leq \frac{4}{3}$.

We next consider the cases when $n \geq 3$ or $k \neq 1$. When $\alpha > \frac{1}{n}$, which means $\alpha \in I(n, 0)$ or $\alpha \in (\frac{1}{n}, \frac{2}{n+2}] \in D(n, 0)$, $\Delta_n^\ominus(\alpha) = \alpha$. Thus, when $\alpha \in I(n, 0)$, $\Delta_n^\oplus(\alpha) = \alpha$ and $r_n(\alpha) = 1 < \frac{4}{3} \leq \frac{2n}{n+1}$ since $n \geq 2$; when $\alpha \in (\frac{1}{n}, \frac{2}{n+2}]$, $\Delta_n^\oplus(\alpha) = \frac{2 \cdot (1-\alpha)}{n}$ and $r_n(\alpha) = \frac{2}{n} \cdot \frac{1-\alpha}{\alpha} < 2 \cdot (1 - \frac{1}{n}) < \frac{2n}{n+1}$. When $\alpha \leq \frac{1}{n}$, it follows that $\alpha \in (\frac{1}{n+1}, \frac{1}{n}] \in D(n, 0)$ or $\alpha \in I(n, k)$ with $k \geq 1$ or $\alpha \in D(n, k)$ with $k \geq 1$. In these cases, $\Delta_n^\ominus(\alpha) = \frac{1}{n}$. When $\alpha \in (\frac{1}{n+1}, \frac{1}{n}]$, $\Delta_n^\oplus(\alpha) = \frac{2 \cdot (1-\alpha)}{n}$ and $r_n(\alpha) = 2 \cdot (1 - \alpha) < \frac{2n}{n+1}$; when $\alpha \in I(n, k) = (\frac{k+2}{n(k+1)^2+k+2}, \frac{1}{k+1}]$ with $k \geq 1$, $\Delta_n^\oplus(\alpha) = (k + 1)\alpha$ and $r_n(\alpha) = n(k + 1) \cdot \alpha \leq \frac{kn+n}{kn+1} \leq \frac{2n}{n+1}$; when $\alpha \in D(n, k) = (\frac{1}{kn+n+1}, \frac{k+2}{n(k+1)^2+k+2}]$ with $k \geq 1$, $\Delta_n^\oplus(\alpha) = \frac{k+2}{k+1} \cdot \frac{1-\alpha}{n}$ and $r_n(\alpha) = \frac{k+2}{k+1} \cdot (1 - \alpha) < \frac{kn+2n}{kn+n+1} \leq \frac{3n}{2n+1} < \frac{2n}{n+1}$. □

CLAIM 2. $r_n(\alpha; m) > \frac{4}{3}$ only when $\alpha \in (\frac{2}{9}, \frac{1}{3})$ if $n = 3$, or $\alpha \in (\frac{1}{6}, \frac{3}{11})$ if $n = 4$, or $\alpha \in (\frac{4}{45}, \frac{1}{9}) \cup (\frac{2}{15}, \frac{3}{13})$ if $n = 5$, or $\alpha \in (\frac{4}{9n}, \frac{3}{2n+3})$ if $n \geq 6$.

Proof. Note that we actually derive the ranges of α that satisfy $r_n(\alpha; +\infty) > \frac{4}{3}$, which are necessary conditions for $r_n(\alpha; m) > \frac{4}{3}$ but may not be sufficient ones. We use the formulas of $r_n(\alpha)$ derived in the proof of Claim 1 and only consider the following cases when $r_n(\alpha)$ may be larger than $\frac{4}{3}$:

— When $\alpha \in (\frac{1}{n}, \frac{2}{n+2}]$, $r_n(\alpha) = \frac{2}{n} \cdot \frac{1-\alpha}{\alpha}$, which is larger than $\frac{4}{3}$ when $\alpha < \frac{3}{2n+3}$. Since $\frac{3}{2n+3} > \frac{1}{n}$ only when $n \geq 4$ and $\frac{3}{2n+3} < \frac{2}{n+2}$, the range is $\alpha \in (\frac{1}{n}, \frac{3}{2n+3})$ with $n \geq 4$.

— When $\alpha \in (\frac{1}{n+1}, \frac{1}{n}]$, $r_n(\alpha) = 2 \cdot (1 - \alpha)$, which is larger than $\frac{4}{3}$ when $\alpha < \frac{1}{3}$. Since $\frac{1}{n+1} < \frac{1}{3}$ only when $n \geq 3$ and $\frac{1}{n} \leq \frac{1}{3}$ when $n \geq 3$, the range is $\alpha \in (\frac{1}{n+1}, \frac{1}{n})$ with $n \geq 3$.

— When $\alpha \in I(n, k) = (\frac{k+2}{n(k+1)^2+k+2}, \frac{1}{kn+1}]$ with $k \geq 1$, $r_n(\alpha) = n(k+1) \cdot \alpha$, which is larger than $\frac{4}{3}$ when $\alpha > \frac{4}{3n(k+1)}$. Note that $\frac{4}{3n(k+1)} < \frac{1}{kn+1}$ is equivalent to $(3 - k)n > 4$, which can be satisfied only when $k = 1$ or $k = 2$. When $k = 1$, $(3 - k)n > 4$ gives $n \geq 3$, $\alpha > \frac{4}{3n(k+1)}$ is equivalent to $\alpha > \frac{2}{3n}$, and $\frac{k+2}{n(k+2)^2+k+2} = \frac{3}{4n+3}$. Since $\frac{3}{4n+3} \geq \frac{2}{3n}$ when $n \geq 6$, the ranges are $\alpha \in (\frac{2}{3n}, \frac{1}{n+1})$ with $3 \leq n \leq 5$, and $\alpha \in (\frac{3}{4n+3}, \frac{1}{n+1})$ with $n \geq 6$. When $k = 2$, $(3 - k)n > 4$ gives $n \geq 5$, $\alpha > \frac{4}{3n(k+1)}$ is equivalent to $\alpha > \frac{4}{9n}$, and $\frac{k+2}{n(k+2)^2+k+2} = \frac{1}{4n+1}$. Since $\frac{4}{9n} > \frac{1}{4n+1}$, the range is $\alpha \in (\frac{4}{9n}, \frac{1}{2n+1})$ with $n \geq 5$.

— When $\alpha \in D(n, k) = (\frac{1}{kn+n+1}, \frac{k+2}{n(k+1)^2+k+2}]$ with $k \geq 1$, $r_n(\alpha) = \frac{k+2}{k+1} \cdot (1 - \alpha)$, which is larger than $\frac{4}{3}$ when $\alpha < \frac{2-k}{3k+6}$. Note that $\frac{2-k}{3k+6} > 0$ only when $k = 1$. Then, $\alpha \leq \frac{2-k}{3k+6}$ is equivalent to $\alpha < \frac{1}{9}$, $\frac{1}{kn+n+1} = \frac{1}{2n+1}$ and $\frac{k+2}{n(k+1)^2+k+2} = \frac{3}{4n+3}$. Since $\frac{3}{4n+3} \leq \frac{1}{9}$ when $n \geq 6$ and $\frac{1}{9} > \frac{1}{2n+1}$ when $n \geq 5$, the ranges are $(\frac{1}{2n+1}, \frac{3}{4n+3})$ with $n \geq 6$, and $(\frac{1}{2n+1}, \frac{1}{9})$ with $n = 5$.

By summarising the preceding ranges, we complete the proof.   □

From the formula of $r_n(\alpha; m)$, as well as Figure 2, we have the following observations:

Observation 1: As n increases, the worst-case ratio of $r_n(\alpha; m)$ (i.e., $\max_\alpha r_n(\alpha; m)$) increases.
Observation 2: As n increases, large values of $r_n(\alpha; m)$ happen increasingly more rarely if α is randomly generated from $[0, 1]$.

Next, we conduct numerical experiments with synthetic and real-world data to illustrate the real distances between $\Delta_n^\oplus(\alpha; m)$ and the MMS of specific disutility functions, which also validate the preceding two observations.

5.1 Experiments with Synthetic Data

In this section, we randomly generate a number of disutility functions, and for each of them, we compute the ratio between the corresponding Hill's share and the minimax share. In particular, for each given n and m, we randomly generate 100 instances; for each instance, we randomly generate $m-1$ numbers in $[0, 1]$. These $m-1$ numbers separate the interval $[0, 1]$ into m contiguous segments, and the lengths of these segments are used as the disutilities of the m objects. Then we compute α as the maximum of these value and compute $\Delta_n^\oplus(\alpha; m)$ by substituting α and m to the expression in Theorem 3.1. The MMS is calculated by enumerating all possible allocations of the m items. For each instance, we record the ratio of these two quantities.

The results are summarised in Figure 4. We slice the ratios into small intervals, each of which has a length of 0.1, and count the number of instances falling into each interval for each setting. The figure validates the previous two observations: when $n = 2$ and 3, the largest ratio can only reach interval $[1.3, 1.4)$ and $[1.4, 1.5)$, but when $n \geq 4$, it reaches $[1.5, 1.6)$; however, looking at the number of instances, for larger n, fewer and fewer instances fall into these large intervals, and instead, the number of instances in $[1.0, 1.1)$ significantly dominates the other intervals. Specifically,
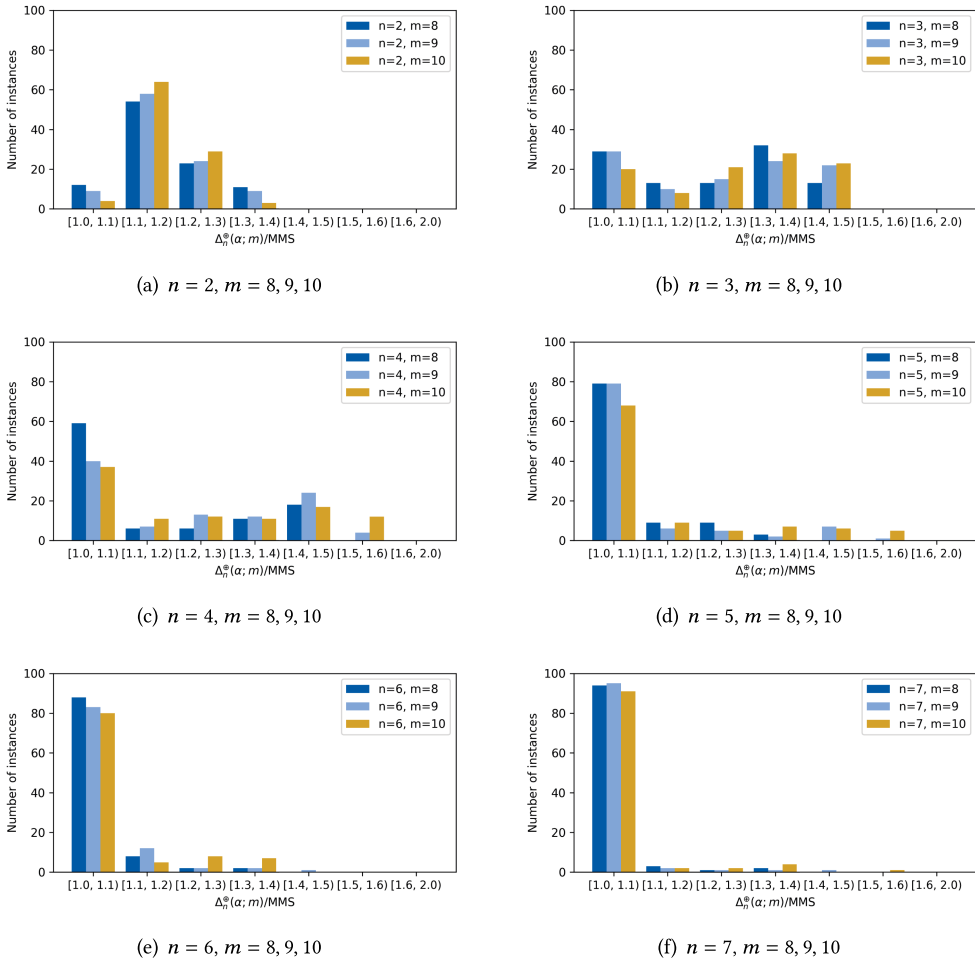
(a) $n = 2$, $m = 8, 9, 10$

(b) $n = 3$, $m = 8, 9, 10$

(c) $n = 4$, $m = 8, 9, 10$

(d) $n = 5$, $m = 8, 9, 10$

(e) $n = 6$, $m = 8, 9, 10$

(f) $n = 7$, $m = 8, 9, 10$

Fig. 4. Ratios in random data.

when $n = 6$ and 7, $[1.0, 1.1)$ contains more than 80% of all random instances, and none of them reaches a ratio beyond 1.6, whereas the worst-case ratio can be greater than 1.7.

5.2 Experiments with Real-World Data

The real-world dataset is collected from the Spliddit platform (Spliddit.org)—a well-known platform that provides implementations of fair allocation algorithms for various practical problems [15]. The dataset contains 8,409 instances created between October 2014 and May 2020, involving 22,530 agents and 42,469 objects. We randomly select 10,000 disutility functions from the data, where the largest value of n is 14. After normalising all the disutility functions, for each of them, we record the ratio of the corresponding Hill's share and MMS. The results are shown in Figure 5. As we can see, very few instances have ratios higher than 1.4, and more than 65% of the instances have ratios within $[1.0, 1.1)$. Actually, there are only 173 (= 1.73%) and 26 (= 0.26%) instances falling into $[1.6, 1.7)$ and $[1.7, 1.8)$, respectively, and none is beyond 1.8. Note that in the 10,000 disutility functions, there are only 14 instances with $n \geq 9$, which further amplifies the rare happening of large ratios.
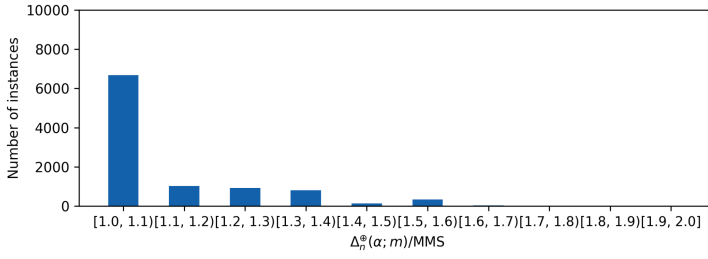
Fig. 5. Ratios in Spliddit data.

6 Conclusion

In this work, we gave the tight characterisation of Hill's share for allocating indivisible bads—that is, the exact upper bound of the minimax share of disutility functions with the same largest single-object value. Hill's share exhibits several advantages including elementary computation, being close to the minimax share, and displaying the effect of an agent's disutility in her share of all objects. More importantly, the monotonic closure of Hill's share is guaranteed to be satisfiable; as far as we know, no other similarly simple guarantee for the allocation of bads has been identified. There are some open problems. Hill's share is tight for the domain of disutility functions whose largest single-object disutility is *no greater than* a given parameter, but we do not know whether it is tight when the domain only contains the disutility functions whose largest single-object disutility *equals* this parameter. The same problem is also open for the mirror problem of allocating goods, for which the tight characterisation of Hill's share is also unknown (when $n \geq 3$). Our work also uncovers some other related research problems, such as the algorithmic problem of finding a Pareto optimal allocation satisfying Hill's share and the game-theoretic problem of designing truthful mechanisms to incentivise the agents to report their disutility functions honestly while achieving (approximations of) Hill's share.

Appendix

A Missing Proofs in Section 3

A.1 Proof of Corollary 3.2

We prove Corollary 3.2 by contradiction, and assume that there exists a disutility $v \in \mathcal{V}(\alpha)$ whose MMS is larger than the expressions in Corollary 3.2. With a slight abuse of notations, in the following we also use $\Delta_n^{\oplus}(\alpha)$ to denote those expressions in Corollary 3.2. Let $\mathbf{A} = (A_1, \ldots, A_n)$ be a lexicographical MinMax allocation of v—that is, the largest disutility of bundles in $\mathbf{A}$ is minimised over all allocations, and among these allocations the second largest disutility is minimised, and so on. Without loss of generality, assume $v(A_1) \geq \cdots \geq v(A_n)$ and $v(A_1) = \mathrm{MMS}_n(v) > \Delta_n^{\oplus}(\alpha)$. Let E_α denote the subset of objects whose disutilities are exactly α—that is, $E_\alpha = \{e \in M \mid v(e) = \alpha\}$. It can be verified that $\Delta_n^{\oplus}(\alpha) \geq (k+1)\alpha$ (this is also illustrated in Figure 1), which gives $v(A_1) > (k+1)\alpha$. Moreover, since $v(e) \leq \alpha$ for any $e \in M$, $|A_1| \geq k + 2$. We have the following property.

CLAIM 3. *Letting j be an agent in $N \setminus \{1\}$, for any $S_1 \subseteq A_1$ and $S_j \subseteq A_j$ such that $v(S_1) > v(S_j)$, $v(S_1) - v(S_j) \geq v(A_1) - v(A_j)$.*

PROOF OF CLAIM 3. For the sake of contradiction, we assume that there exists $S_1' \subseteq A_1$ and $S_{j'}' \subseteq A_{j'}$ such that $v(S_1') > v(S_{j'}')$ and $v(S_1') - v(S_{j'}') < v(A_1) - v(A_{j'})$. Then we construct another allocation $\mathbf{B} = (B_1, \ldots, B_n)$ by exchanging S_1' and $S_{j'}'$—that is, $B_1 = A_1 \setminus S_1' \cup S_{j'}'$, $B_{j'} = A_{j'} \setminus S_{j'}' \cup S_1'$ and $B_j = A_j$ for any $j \in N \setminus \{1, j'\}$. It follows that $v(B_1) < v(A_1)$, $v(B_{j'}) < v(A_1)$ and $v(B_j) = v(A_j)$

Table 1. Disutility Function for
Subcases 1.1 and 1.2

Object Disutility	Quantity
α	1
$\frac{1-\alpha}{n}$	n

for any $j \in N \setminus \{1, j'\}$, which contradicts the assumption that **A** is a lexicographical MinMax allocation of v. □

The contraposition of Claim 3 gives the following.

CLAIM 4. *Letting j be an agent in $N \setminus \{1\}$, for any $S_1 \subseteq A_1$ and $S_j \subseteq A_j$ such that $v(A_j \setminus S_j \cup S_1) < v(A_1)$, $v(S_j) \geq v(S_1)$.*

As a warm-up, we start from the case with large α, where $k = 0$, and distinguish two subcases depending on the domain of α.
Case 1: $n \geq 2$ and $k = 0$

Subcase 1.1: $\alpha \in D(n, 0)$

When $\alpha \in D(n, 0)$, $\frac{1}{n+1} < \alpha \leq \frac{2}{n+2}$ and $v(A_1) > \Delta_n^{\oplus}(\alpha) = \frac{2-2\alpha}{n}$. If $E_\alpha \cap A_1 \neq \emptyset$, there exists $e^* \in A_1$ such that $v(e^*) = \alpha < v(A_1)$. Then Claim 4 gives a lower bound of $v(A_j)$ for any $j \in N \setminus \{1\}$—that is, $v(A_j) \geq v(e^*) = \alpha$. Summing up these lower bounds leads to the following contradiction:

$$1 = \sum_{j \in N} v(A_j) > \frac{2 - 2\alpha}{n} + (n-1) \cdot \alpha = \frac{(n+1)(n-2)\alpha + 2}{n} \geq 1,$$

where the last inequality is because $\alpha > \frac{1}{n+1}$ and $n \geq 2$.

Therefore, $E_\alpha \cap A_1 = \emptyset$. Then by the definition of $\mathcal{V}(\alpha)$, there must exist $j' \in N \setminus \{1\}$ such that $E_\alpha \cap A_{j'} \neq \emptyset$, and thus $v(A_{j'}) \geq \alpha$. Recall that $|A_1| \geq k + 2 = 2$, which implies that there exists $S \subseteq A_1$ such that $v(A_1) > v(S) \geq \frac{1}{2}v(A_1) > \frac{1-\alpha}{n}$. According to Claim 4, $v(A_j) \geq v(S) > \frac{1-\alpha}{n}$ holds for any $j \in N \setminus \{1, j'\}$. As a result,

$$1 = \sum_{j \in N} v(A_j) > \frac{2 - 2\alpha}{n} + \alpha + (n-2) \cdot \frac{1-\alpha}{n} = 1,$$

which is also a contradiction. Therefore, $v(A_1) > \Delta_n^{\oplus}(\alpha)$ never holds when $\alpha \in D(n, 0)$.

For the other direction, the disutility function for this subcase (Table 1) contains one object with disutility α and n objects with disutility $\frac{1-\alpha}{n}$. Since $\frac{1}{n+1} < \alpha \leq \frac{2}{n+2}$, it follows that $\frac{1-\alpha}{n} < \alpha \leq 2 \cdot \frac{1-\alpha}{n}$. Clearly, the MMS of this disutility function is $2 \cdot \frac{1-\alpha}{n} = \Delta_n^{\oplus}(\alpha)$.

Subcase 1.2: $\alpha \in I(n, 0)$

When $\alpha \in I(n, 0)$, by similar reasonings, we can show that $v(A_1) > \Delta_n^{\oplus}(\alpha)$ does not hold either. In this subcase, $\frac{2}{n+2} < \alpha \leq 1$ and $\Delta_n^{\oplus}(\alpha) = \alpha$. If $E_\alpha \cap A_1 \neq \emptyset$, then there exists $e^* \in A_1$ such that $v(e^*) = \alpha < v(A_1)$ and Claim 4 gives a lower bound of $v(A_j)$ for any $j \in N \setminus \{1\}$—that is, $v(A_j) \geq v(e^*) = \alpha$. Summing up these lower bounds leads to the following contradiction:

$$1 = \sum_{j \in N} v(A_j) > n\alpha > \frac{2n}{n+2} \geq 1,$$

where the last inequality is because $n \geq 2$.

Therefore, it must hold that $E_\alpha \cap A_1 = \emptyset$, and moreover, there exists $j' \in N \setminus \{1\}$ with $E_\alpha \cap A_{j'} \neq \emptyset$. Thus, $v(A_{j'}) \geq \alpha$. Since $|A_1| \geq k + 2 = 2$, there exists $S \subseteq A_1$ such that $v(A_1) > v(S) \geq \frac{1}{2}v(A_1) > \frac{\alpha}{2}$.

According to Claim 4, $v(A_j) \geq v(S) > \frac{\alpha}{2}$ holds for any $j \in N \setminus \{1, j'\}$. As a result,

$$1 = \sum_{j \in N} v(A_j) > \alpha + \alpha + (n-2) \cdot \frac{\alpha}{2} = \frac{n+2}{2} \alpha > 1,$$

which is also a contradiction.

For the other direction, the disutility function for this subcase also contains one object with disutility α and n objects with disutility $\frac{1-\alpha}{n}$ (see Table 1). Since $\frac{2}{n+2} < \alpha \leq 1$, it follows that $2 \cdot \frac{1-\alpha}{n} < \alpha \leq 1$. Clearly, the MMS of this disutility function is $\alpha = \Delta_n^\oplus(\alpha)$.

Up to here, the proof regarding the case of $k = 0$ is completed.

Next, we consider the general case of $k \geq 1$ excluding $n = 2$ and $k = 1$.

Case 2: $n \geq 3$ **and** $k \geq 1$ **or** $n \geq 2$ **and** $k \geq 2$

For this case, we again start with the subcases when $\alpha \in D(n, k)$. Recall that when $\alpha \in D(n, k)$, $\alpha \in (\frac{1}{(k+1)n+1}, \frac{k+2}{n(k+1)^2+k+2}]$ and $v(A_1) > \Delta_n^\oplus(\alpha) = \frac{k+2}{k+1} \cdot \frac{1-\alpha}{n}$.

Subcase 2.1: $\alpha \in D(n, k)$ **and** $E_\alpha \cap A_j = \emptyset$ **for any** $j \in N \setminus \{1\}$

In this subcase, all the objects with disutility α are in A_1, and thus $v(e) < \alpha$ for any $e \in A_j$ and $j \in N \setminus \{1\}$. Due to the normalization, there exists an agent j_0 who receives disutility at most $\frac{1-v(A_1)}{n-1}$, which gives the following lower bound of the difference between the disutilities that agents 1 and j_0 receive:

$$v(A_1) - v(A_{j_0}) \geq \frac{n}{n-1} v(A_1) - \frac{1}{n-1} > \frac{1 - (k+2)\alpha}{(n-1)(k+1)}.$$

It can be shown that the rightmost side of the preceding inequality is no less than $\frac{\alpha}{2}$, which is equivalent to $\alpha \leq \frac{2}{(k+1)n+k+3}$. Since $\alpha \leq \frac{k+2}{n(k+1)^2+k+2}$, it suffices to show $\frac{2}{(k+1)n+k+3} \geq \frac{k+2}{n(k+1)^2+k+2}$, which holds since

$$\frac{2}{(k+1)n+k+3} - \frac{k+2}{n(k+1)^2+k+2} = \frac{(k+1)(nk-k-2)}{((k+1)n+k+3)(n(k+1)^2+k+2)} \geq 0,$$

where the last inequality is because $n \geq 3$ and $k \geq 1$, or $n \geq 2$ and $k \geq 2$.

Therefore, $v(A_1) - v(A_{j_0}) > \frac{\alpha}{2}$. Let e^* be an object in A_1 with disutility α. Since $v(A_1) > (k+1)\alpha > \alpha$, for any $S \subseteq A_{j_0}$ with disutility smaller than α, Claim 3 actually gives a tighter bound of its disutility—that is, $v(S) \leq v(e^*) - (v(A_1) - v(A_{j_0})) < \frac{\alpha}{2}$. Thus, $v(e) < \frac{\alpha}{2}$ for any $e \in A_{j_0}$. Besides, according to Claim 4, $v(A_{j_0}) \geq v(e^*) = \alpha$. These two facts together imply that there exists $S' \subseteq A_{j_0}$ such that $v(S') \in [\frac{\alpha}{2}, \alpha)$, which is a contradiction to Claim 3.

Subcase 2.2: $\alpha \in D(n, k)$ **and** $E_\alpha \cap A_{j'} \neq \emptyset$ **for some** $j' \in N \setminus \{1\}$

In this subcase, some objects with disutility α are in $A_{j'}$. Before diving into the proof for this subcase, we present the following claim, which shows the existence of a subset of A_1 whose disutility is within a specific range.

CLAIM 5. *There exists a subset $S \subseteq A_1$ such that $\frac{k}{k+2} v(A_1) \leq v(S) < v(A_1) - \alpha$.*

PROOF OF CLAIM 5. When $k = 1$, if there exists $e \in A_1$ such that $v(e) \geq \frac{1}{3} v(A_1)$, recall that $v(A_1) > (k+1)\alpha = 2\alpha$, Claim 5 holds since $v(e) \leq \alpha < v(A_1) - \alpha$. If $v(e) < \frac{1}{3} v(A_1)$ for any $e \in A_1$, denote by (A_1^1, A_1^2) one 2-partition of A_1 that minimises the disutility difference between the two bundles among all the 2-partitions. Without loss of generality, we assume $v(A_1^1) \leq v(A_1^2)$, then $v(A_1^1) \leq \frac{1}{2} v(A_1) < v(A_1) - \alpha$. Besides, $v(A_1^1) \geq \frac{1}{3} v(A_1)$ holds. Otherwise, $v(A_1^2) - v(A_1^1) = v(A_1) - 2v(A_1^1) > \frac{1}{3} v(A_1)$, implying that moving an object from A_1^2 to A_1^1 returns another 2-partition of A_1 that has a smaller disutility difference, which contradicts the definition of (A_1^1, A_1^2).

When $k \geq 2$, we first show that $v(e) > \frac{1}{k+2}\alpha$ for any $e \in A_1$. If not, $v(A_1) > v(A_1 \setminus \{e\}) \geq v(A_1) - \frac{1}{k+2}\alpha$. Then Claim 4 gives $v(A_j) \geq v(A_1) - \frac{1}{k+2}\alpha$ for any $j \in N \setminus \{1\}$. Summing up these lower bounds gives the following inequality:

$$1 = \sum_{j \in N} v(A_j) \geq v(A_1) + (n-1)v(A_1) - \frac{n-1}{k+2}\alpha > \frac{k+2}{k+1} - \frac{(k+2)^2 + (k+1)(n-1)}{(k+1)(k+2)}\alpha.$$

It can be shown that the rightmost side is at least 1, which constitutes a contradiction. This is equivalent to show that $\alpha \leq \frac{k+2}{(k+2)^2 + (k+1)(n-1)}$. Since $\alpha \leq \frac{k+2}{n(k+1)^2 + k+2}$, it suffices to show that $\frac{k+2}{(k+1)^2 + (k+1)(n-1)} \geq \frac{k+2}{n(k+1)^2 + k+2}$, which holds since

$$n(k+1)^2 + k + 2 - ((k+2)^2 + (k+1)(n-1)) = (k+1)(nk - k - 1) \geq 0,$$

where the last inequality is because $n \geq 2$ and $k \geq 1$.

We then let $S^* = \arg\min_{S \subseteq A_1, v(S) > \alpha} v(S)$, which is guaranteed to exist since $v(A_1) > (k+1)\alpha > \alpha$, and show by contradiction that $v(S^*) \leq \frac{2}{k+2}v(A_1)$. This gives $\frac{k}{k+2}v(A_1) \leq v(A_1 \setminus S^*) < v(A_1) - \alpha$. We assume for the sake of contradiction that $v(S^*) > \frac{2}{k+2}v(A_1)$. Then the definition of S^* gives the following lower bound of $v(e)$ for any $e \in S^*$:

$$v(e) > v(S^*) - \alpha > \frac{2}{k+2}v(A_1) - \alpha > \frac{k}{k+2}\alpha \geq \frac{1}{2}\alpha,$$

where the second last inequality is because $v(A_1) > (k+1)\alpha$ and the last inequality is because $k \geq 2$. This lower bound implies that S^* contains exactly two objects. Otherwise (i.e., $|S^*| \geq 3$), for any subset $S' \subseteq S^*$ that contains exactly two objects, $\alpha < v(S') < v(S^*)$ holds, which contradicts the definition of S^*.

Therefore, we can denote $S^* = \{e^l, e^s\}$ and assume without loss of generality that $v(e^l) \geq v(e^s)$. Accordingly, $v(e^l) \geq \frac{1}{2}v(S^*) > \frac{1}{k+2}v(A_1) > \frac{k+1}{k+2}\alpha$. Recall that $v(e) > \frac{1}{k+2}\alpha$ holds for any $e \in A_1$. These two facts together imply that the total disutility of e^l and any other object in A_1 is larger than α. From the definition of S^*, we know that $e^s \in \arg\min_{e \in A_1} v(e)$, which gives $v(e) \geq v(e^s) > \frac{k}{k+2}\alpha$ for any $e \in A_1$. Let S' be the subset of A_1 that contains the two objects with the smallest disutilities, and the following inequality leads to a contradiction to the definition of S^*:

$$\alpha \leq \frac{2k}{k+2}\alpha < v(S') \leq \frac{2}{k+2}v(A_1) < v(S^*),$$

where the first inequality is because $k \geq 2$ and the second last inequality is due to $|A_1| \geq k+2$. □

We are now ready to reveal the contradiction in the subcase. Denote by e^* one object in $A_{j'}$ that has disutility α and by S a subset of A_1 that satisfies Claim 5, and Claim 4 gives $v(A_{j'} \setminus \{e^*\}) \geq v(S) \geq \frac{k}{k+2}v(A_1)$—that is, $v(A_{j'}) \geq \frac{k}{k+2}v(A_1) + \alpha$. For any $j \in N \setminus \{1, j'\}$, recall that $|A_1| \geq k + 2$, which implies that there exists $S' \subseteq A_1$ such that $v(A_1) > v(S') \geq \frac{k+1}{k+2}v(A_1)$, and Claim 4 gives $v(A_j) \geq v(S') \geq \frac{k+1}{k+2}v(A_1)$. Summing up these lower bounds leads to the following contradiction:

$$1 = \sum_{j \in N} v(A_j) \geq v(A_1) + \frac{k}{k+2}v(A_1) + \alpha + (n-2) \cdot \frac{k+1}{k+2}v(A_1) = \frac{n(k+1)}{k+2}v(A_1) + \alpha > 1 - \alpha + \alpha = 1.$$

For the other direction, the disutility function for the subcases when $\alpha \in D(n, k)$ (Table 2) containing one object with disutility α and $n(k+1)$ objects with disutility $\frac{1-\alpha}{n(k+1)}$. Since $\alpha > \frac{1}{kn+n+1}$, it follows that $\alpha > \frac{1-\alpha}{n(k+1)}$. Besides, it can be verified that $\alpha < \frac{2-2\alpha}{n(k+1)}$, which is equivalent to

Table 2. Disutility Function for
Subcases $\alpha \in D(n, k)$ with $n \geq 3$
and $k \geq 1$, or $n \geq 2$ and $k \geq 2$

Object Disutility	Quantity
α	1
$\frac{1-\alpha}{n(k+1)}$	$n(k+1)$

$\alpha < \frac{2}{nk+n+2}$. Since $\alpha \leq \frac{k+2}{n(k+1)^2+k+2}$, it suffices to show $\frac{k+2}{n(k+1)^2+k+2} < \frac{2}{n(k+1)+2}$, which holds since

$$\frac{2}{n(k+1)+2} - \frac{k+2}{n(k+1)^2+k+2} = \frac{nk(k+1)}{(n(k+1)+2)(n(k+1)^2+k+2)} > 0.$$

By the pigeonhole principle, there exists a bundle that contains at least $k+2$ objects in any allocation. This implies that MMS of this disutility function is $(k+2) \cdot \frac{1-\alpha}{n(k+1)}$, which happens in the allocation where one bundle contains $k+2$ objects with disutility $\frac{1-\alpha}{n(k+1)}$, one bundle contains k objects with disutility $\frac{1-\alpha}{n(k+1)}$ and one object with disutility α, and each of the other bundles contains $k+1$ objects with disutility $\frac{1-\alpha}{n(k+1)}$.

Next we consider the subcases when $\alpha \in I(n,k)$. Recall that when $\alpha \in I(n,k)$, $\alpha \in (\frac{k+2}{n(k+1)^2+k+2}, \frac{1}{kn+1}]$ and $v(A_1) > \Delta_n^{\oplus}(\alpha) = (k+1)\alpha$.

Subcase 2.3: $\alpha \in I(n,k)$ and $E_\alpha \cap A_j = \emptyset$ for any $j \in N \setminus \{1\}$

For this subcase, we first derive a lower bound of $v(A_j)$ for any $j \in N \setminus \{1\}$—that is, $v(A_j) \geq (\frac{(k+1)^2}{k+2} + \frac{1}{(n-1)(k+2)})\alpha$. Letting $D = \frac{(k+1)^2}{k+2} + \frac{1}{(n-1)(k+2)}$, we assume for the sake of contradiction that $v(A_{j'}) < D\alpha$ for some $j' \in N \setminus \{1\}$. It can be verified that $k < D < k+1$, where the first inequality is equivalent to $n > 0$, and the second inequality is equivalent to $(n-1)(k+1) > 1$. Denote by e^* one object in A_1 with disutility α and by e' any object in A_j, and we have

$$v(A_{j'} \setminus (A_{j'} \setminus \{e'\}) \cup (A_1 \setminus \{e^*\})) = v(A_1 \setminus \{e^*\} \cup \{e'\}) < v(A_1).$$

Then from Claim 4, $v(A_{j'} \setminus \{e'\}) \geq v(A_1 \setminus \{e^*\})$, which gives

$$v(e') \leq v(A_{j'}) - v(A_1) + v(e^*) < D\alpha - (k+1)\alpha + \alpha = (D-k)\alpha.$$

However, we next show that the disutility of some object in $A_{j'}$ must be larger than $(D-k)\alpha$, which leads to a contradiction. To achieve this, we denote $S^* \in \arg\min_{S \subseteq A_{j'}, v(S) > (D-1)\alpha} v(S)$, whose existence is guaranteed since Claim 4 gives $v(A_{j'}) \geq v(A_1 \setminus \{e^*\}) > k\alpha > (D-1)\alpha$. Notice that

$$v(A_{j'} \setminus S^* \cup (A_1 \setminus \{e^*\})) < D\alpha - (D-1)\alpha + v(A_1) - \alpha = v(A_1),$$

and from Claim 4, $v(S^*) \geq v(A_1 \setminus \{e^*\}) > k\alpha$. Then the definition of S^* implies that the disutility of any object in S^* is at least

$$v(S^*) - (D-1)\alpha > (k-D+1)\alpha \geq (D-k)\alpha,$$

where the last inequality is equivalent to $D - k - \frac{1}{2} = \frac{k+2-kn}{2(n-1)(k+2)} \leq 0$, which holds when $n \geq 3$ and $k \geq 1$, or $n \geq 2$ and $k \geq 2$.

Therefore, $v(A_j) \geq (\frac{(k+1)^2}{k+2} + \frac{1}{(n-1)(k+2)})\alpha$ holds for any $j \in N \setminus \{1\}$. Summing up these lower bounds leads to the following contradiction:

$$1 = \sum_{j \in N} v(A_j) > (k+1)\alpha + (n-1) \cdot \left(\frac{(k+1)^2}{k+2} + \frac{1}{(n-1)(k+2)} \right) \alpha = \frac{n(k+1)^2+k+2}{k+2}\alpha > 1.$$

Subcase 2.4: $\alpha \in I(n,k)$ and $E_\alpha \cap A_{j'} \neq \emptyset$ for some $j' \in N \setminus \{1\}$

Table 3. Disutility Function for
Subcases $\alpha \in I(n, k)$ with $n \geq 3$
and $k \geq 1$, or $n \geq 2$ and $k \geq 2$

Object Disutility	Quantity
α	$kn + 1$
$\frac{1-(nk+1)\alpha}{n-1}$	$n - n1$

The proof is similar to that of Subcase 2.2. First, it can be verified that Claim 5 still holds here.

PROOF OF CLAIM 5 FOR $\alpha \in I(n, k)$. Notice that Claim 5 holds as long as $k = 1$, thus we can focus on $k \geq 2$. We first show that $v(e) > \frac{1}{k+2}\alpha$ for any $e \in A_1$. If not, $v(A_1 \setminus \{e\}) \geq v(A_1) - \frac{1}{k+2}\alpha$. Then Claim 4 gives $v(A_j) \geq v(A_1) - \frac{1}{k+2}\alpha$ for any $j \in N \setminus \{1\}$. Summing up these lower bounds gives the following formula:

$$1 = \sum_{j \in N} v(A_j) \geq v(A_1) + (n-1)v(A_1) - \frac{n-1}{k+2}\alpha > \frac{n(k+1)(k+2) - n + 1}{k+2}\alpha.$$

It can be shown that the rightmost side of the preceding inequality is at least 1, which is a contradiction. This is equivalent to show that $\alpha \geq \frac{k+2}{n(k+1)(k+2)-n+1}$. Since $\alpha \geq \frac{k+2}{n(k+1)^2+k+2}$, it suffices to show that $\frac{k+2}{n(k+1)(k+2)-n+1} \leq \frac{k+2}{n(k+1)^2+k+2}$, which holds since

$$n(k+1)(k+2) - n + 1 - (n(k+1)^2 + k + 2) = nk - k - 1 \geq 0$$

where the last inequality is because $n \geq 2$ and $k \geq 1$.

We then let $S^* = \arg\min_{S \subseteq A_1, v(S) > \alpha} v(S)$, which is guaranteed to exist since $v(A_1) > (k+1)\alpha > \alpha$. By the same proof as the counterpart in the proof of Claim 5 for $\alpha \in D(n, k)$, we can show that $v(S^*) \leq \frac{2}{k+2}v(A_1)$, which gives $\frac{k}{k+2}v(A_1) \leq v(A_1 \setminus S^*) < v(A_1) - \alpha$.   □

We are now ready to reveal the contradiction in this subcase. Denote by e^* one object in $A_{j'}$ that has disutility α and by S a subset of A_1 that satisfies Claim 5, and Claim 4 gives $v(A_{j'} \setminus \{e^*\}) \geq v(S) \geq \frac{k}{k+2}v(A_1)$—that is, $v(A_{j'}) \geq \frac{k}{k+2}v(A_1) + \alpha$. For any $j \in N \setminus \{1, j'\}$, recall that $|A_1| \geq k + 2$, which implies that there exists $S' \subseteq A_1$ such that $v(A_1) > v(S') \geq \frac{k+1}{k+2}v(A_1)$, Claim 4 gives $v(A_j) \geq v(S') \geq \frac{k+1}{k+2}v(A_1)$. Summing up these lower bounds leads to the following contradiction:

$$1 = \sum_{j \in N} v(A_j) \geq v(A_1) + \frac{k}{k+2}v(A_1) + \alpha + (n-2) \cdot \frac{k+1}{k+2}v(A_1) = \frac{n(k+1)}{k+2}v(A_1) + \alpha > \frac{n(k+1)^2 + k + 2}{k+2}\alpha > 1.$$

For the other direction, the disutility function for the subcases when $\alpha \in I(n, k)$ (Table 3) containing $kn + 1$ objects with disutility α and $n - 1$ objects with disutility $\frac{1-(nk+1)\alpha}{n-1}$. It can be verified that $\alpha > \frac{1-(kn+1)\alpha}{n-1}$, which is equivalent to $\alpha > \frac{1}{(k+1)n}$. Since $\alpha > \frac{k+2}{n(k+1)^2+k+2}$, it suffices to show $\frac{k+2}{n(k+1)^2+k+2} \geq \frac{1}{(k+1)n}$, which holds since

$$\frac{k+2}{n(k+1)^2 + k + 2} - \frac{1}{(k+1)n} = \frac{(k+1)n - k - 2}{(n(k+1)^2 + k + 2)(k+1)n} \geq 0,$$

where the inequality is because $n \geq 3$ and $k \geq 1$, or $n \geq 2$ and $k \geq 2$. By the pigeonhole principle, there exists a bundle that contains at least $k+1$ objects with disutility α. This implies that the MMS of this disutility function is $(k + 1)\alpha$, which happens in the allocation where one bundle contains $k + 1$ objects with disutility α, and each of the other bundles contains k objects with disutility α and one object with disutility $\frac{1-(nk+1)\alpha}{n-1}$.

We now prove Corollary 3.2 for the case of $n = 2$ and $k = 1$—that is, $\alpha \in D(2, 1) \cup I(2, 1)$.

Case 3: $n = 2$ **and** $k = 1$

Subcase 3.1: $\alpha \in (\frac{1}{5}, \frac{7}{27}]$

When $\alpha \in (\frac{1}{5}, \frac{7}{27}]$, $v(A_1) > \Delta_2^{\oplus}(\alpha) = \frac{3-3\alpha}{4}$. If $E_\alpha \cap A_2 \neq \emptyset$, then A_2 contains some objects with disutility α. Recall that Claim 5 holds as long as $k = 1$, thus there exists $S \subseteq A_1$ such that $\frac{1}{3}v(A_1) \leq v(S) < v(A_1) - \alpha$. Denote by e^* one object in A_2 with disutility α, and Claim 4 gives $v(A_2 \setminus \{e^*\}) \geq v(S) \geq \frac{1}{3}v(A_1)$. As a result,

$$1 = v(A_1) + v(A_2) \geq v(A_1) + \frac{1}{3}v(A_1) + \alpha,$$

which gives $v(A_1) \leq \frac{3-3\alpha}{4}$, thus contradicting the assumption that $v(A_1) > \Delta_2^{\oplus}(\alpha)$.

Therefore, $E_\alpha \cap A_2 = \emptyset$, which means that all the objects with disutility α are in A_1 and for any $e \in A_2$, $v(e) < \alpha$. We first derive an upper bound and a lower bound of the maximum disutility of the objects in A_2. Denote by e^* one object in A_1 with $v(e^*) = \alpha < v(A_1)$, and since $v(A_1) - v(A_2) = 2v(A_1) - 1 > \frac{1-3\alpha}{2}$, Claim 3 gives

$$\max_{e \in A_2} v(e) \leq v(e^*) - (v(A_1) - v(A_2)) < \frac{5\alpha - 1}{2}.$$

Notice that $\frac{1-3\alpha}{2} > \frac{\alpha}{3}$ since $\alpha \leq \frac{7}{27} < \frac{3}{11}$, $v(A_1) - v(A_2) > \frac{\alpha}{3}$. Then for every $S \subseteq A_2$ with $v(S) < \alpha$, Claim 3 actually gives a tighter bound of $v(S)$—that is, $v(S) \leq v(e^*) - (v(A_1) - v(A_2)) < \frac{2}{3}\alpha$. This also implies that for every $S' \subseteq A_2$ with $v(S') \geq \frac{2}{3}\alpha$, $v(S') \geq \alpha$ actually holds. Let $S^* = \arg\min_{S \subseteq A_2, v(S) \geq \frac{2}{3}\alpha} v(S)$ whose existence is guaranteed since Claim 4 gives $v(A_2) \geq v(e^*) = \alpha$, thus $v(S^*) \geq \alpha$. Then from the definition of S^*, $v(e) \geq v(S^*) - \frac{2}{3}\alpha \geq \frac{1}{3}\alpha$ holds for any $e \in A_2$, which implies

$$\max_{e \in A_2} v(e) \geq \frac{\alpha}{2}.$$

Otherwise (i.e., $\max_{e \in A_2} v(e) < \frac{\alpha}{2}$), the total disutility of any two objects in A_2 is at least $\frac{2}{3}\alpha$ and smaller than α, which is a contradiction to Claim 3.

We then show that $|A_1|$ is exactly 3. Otherwise (i.e., $|A_1| \geq 4$), there exists $S \subseteq A_1$ such that $v(A_1) > v(S) \geq \alpha + \frac{2}{3}(v(A_1) - \alpha)$. Then Claim 4 gives $v(A_2) \geq v(S) \geq \alpha + \frac{2}{3}(v(A_1) - \alpha)$. Summing up the lower bounds of $v(A_1)$ and $v(A_2)$ leads to a contradiction as follows:

$$1 = v(A_1) + v(A_2) \geq \frac{5}{3}v(A_1) + \frac{1}{3}\alpha > \frac{15 - 11\alpha}{12} > 1,$$

where the last inequality is because $\alpha \leq \frac{7}{27} < \frac{3}{11}$. Therefore, we can denote $A_1 = \{e_1^1, e_2^1, e_3^1\}$ and assume without loss of generality that $v(e_1^1) = \alpha \geq v(e_2^1) = x \geq v(e_3^1) = y$. We then derive the lower bounds of x and y, and reveal the contradiction in this subcase. Since $x \geq y$, the following formula holds:

$$x \geq \frac{x+y}{2} = \frac{v(A_1) - \alpha}{2} > \frac{3 - 7\alpha}{8} \geq \frac{5\alpha - 1}{2} > \max_{e \in A_2} v(e),$$

where the second last inequality is because $\alpha \leq \frac{7}{27}$. Then Claim 3 gives the following lower bound of x:

$$x \geq \max_{e \in A_2} v(e) + (v(A_1) - v(A_2)) > \frac{\alpha}{2} + \frac{1 - 3\alpha}{2} = \frac{1 - 2\alpha}{2}.$$

Claim 3 also gives $y \geq v(A_1) - v(A_2)$. Notice that

$$2 \cdot (v(A_1) - v(A_2)) > \frac{2 - 6\alpha}{2} > \alpha - \frac{1 - 3\alpha}{2} > x - (v(A_1) - v(A_2)),$$

where the second inequality is because $\alpha \le \frac{7}{27} < \frac{3}{11}$, and we have the following lower bound of y :

$$y > \frac{1}{2} \cdot (x - (v(A_1) - v(A_2))) \ge \frac{1}{2} \cdot \max_{e \in A_2} v(e) \ge \frac{\alpha}{4}.$$

Therefore, $v(A_1) = \alpha + x + y > \alpha + \frac{1-2\alpha}{2} + \frac{\alpha}{4} = \frac{2+\alpha}{4}$, which gives $v(A_1) - v(A_2) = 2v(A_1) - 1 > \frac{\alpha}{2}$. However, according to Claim 3, $v(A_1) - v(A_2) \le \alpha - \max_{e \in A_2} v(e) \le \frac{\alpha}{2}$, thus constituting a contradiction.

For the other direction, the disutility function for this subcase contains one object with disutility α and four objects with disutility $\frac{1-\alpha}{4}$. Since $\frac{1}{5} < \alpha \le \frac{7}{27}$, it follows that $\frac{1-\alpha}{4} < \alpha < 2 \cdot \frac{1-\alpha}{4}$, where the last inequality is because $\alpha \le \frac{7}{27} < \frac{1}{3}$. Clearly, the MMS of this disutility function is $3 \cdot \frac{1-\alpha}{4}$.

Subcase 3.2: $\alpha \in (\frac{7}{27}, \frac{2}{7}]$

When $\alpha \in (\frac{7}{27}, \frac{2}{7}]$, $v(A_1) > \Delta_2^{\oplus}(\alpha) = \frac{2+3\alpha}{5}$. If $E_\alpha \cap A_2 \ne \emptyset$, the proof is similar to that for the counterpart of Subcase 3.1. In other words, we also have $v(A_1) \le \frac{3-3\alpha}{4}$, which contradicts $v(A_1) > \Delta_n^{\oplus}(\alpha)$ since $\frac{3-3\alpha}{4} < \frac{2+3\alpha}{5}$ when $\alpha > \frac{7}{27}$.

Therefore, we can focus on $E_\alpha \cap A_2 = \emptyset$. We first derive an upper bound and a lower bound of the maximum disutility of the objects in A_2, which is similar to the counterpart of Subcase 3.1. Denote by e^* one object in A_1 with $v(e^*) = \alpha < v(A_1)$, and since $v(A_1) - v(A_2) = 2v(A_1) - 1 > \frac{6\alpha-1}{5}$, Claim 3 gives

$$\max_{e \in A_2} v(e) \le v(e^*) - (v(A_1) - v(A_2)) < \frac{1-\alpha}{5}.$$

Notice that $\frac{6\alpha-1}{5} > \frac{\alpha}{3}$ since $\alpha > \frac{7}{27} > \frac{3}{13}$, $v(A_1) - v(A_2) > \frac{\alpha}{3}$. Then for every $S \subseteq A_2$ with $v(S) < \alpha$, Claim 3 actually gives a tighter bound of $v(S)$—that is, $v(S) \le v(e^*) - v(A_1) - v(A_2) < \frac{2}{3}\alpha$. This also implies that for every $S' \subseteq A_2$ with $v(S') \ge \frac{2}{3}\alpha$, $v(S') \ge \alpha$ actually holds. Let $S^* = \arg\min_{S \subseteq A_2, v(S) \ge \frac{2}{3}\alpha} v(S)$ whose existence is guaranteed since Claim 4 gives $v(A_2) \ge v(e^*) = \alpha$, thus $v(S^*) \ge \alpha$. Then from the definition of S^*, $v(e) \ge v(S^*) - \frac{2}{3}\alpha \ge \frac{1}{3}\alpha$ holds for any $e \in A_2$, which implies

$$\max_{e \in A_2} v(e) \ge \frac{\alpha}{2}.$$

Otherwise (i.e., $\max_{e \in A_2} v(e) < \frac{\alpha}{2}$), the total disutility of any two objects in A_2 is at least $\frac{2}{3}\alpha$ and smaller than α, which is a contradiction to Claim 3.

Observe that A_1 contains exactly one object with disutility α. Otherwise (i.e., A_1 contains at least two objects with disutility α), Claim 4 gives $v(A_2) \ge 2\alpha$, which leads to the following contradiction:

$$1 = v(A_1) + v(A_2) > \frac{2+3\alpha}{5} + 2\alpha > 1,$$

where the last inequality is because $\alpha > \frac{7}{27} > \frac{3}{13}$. Recall that $|A_1| \ge 3$, A_1 contains at least two objects with disutility smaller than α. For each of such objects, we call it a *medium object* if its disutility is larger than $\max_{e \in A_2} v(e)$. Otherwise, we call it a *small object*. Then Claim 3 gives the following lower bound of the disutility of any medium object e :

$$v(e) \ge \max_{e \in A_2} v(e) + (v(A_1) - v(A_2)) = \max_{e \in A_2} v(e) + (2v(A_1) - 1) > \frac{\alpha}{2} + \frac{6\alpha-1}{5} = \frac{17\alpha-2}{10},$$

as well as the following lower bound of the disutility of any small object e' :

$$v(e') \ge v(A_1) - v(A_2) = 2v(A_1) - 1 > \frac{6\alpha-1}{5}.$$

We then reveal the contradiction by considering possible combinations of objects in A_1 and showing that no possible combination exists.

Combination 1: Besides the object with disutility α, A_1 also contains at least three small objects. Thus, $v(A_1) > \alpha + 3 \cdot \frac{6\alpha-1}{5} = \frac{23\alpha-3}{5}$. Then a lower bound of the difference between $v(A_1)$ and $v(A_2)$ is

$$v(A_1) - v(A_2) = 2v(A_1) - 1 > \frac{46\alpha - 11}{5} > \frac{\alpha}{2},$$

where the last inequality is because $\alpha > \frac{7}{27} > \frac{22}{87}$. However, according to Claim 3, $v(A_1) - v(A_2) \leq \alpha - \max_{e \in A_2} v(e) \leq \frac{\alpha}{2}$, which is a contradiction. Note that this also implies that except the object with disutility α, the total disutility of the other objects must be smaller than $3 \cdot \frac{6\alpha-1}{5}$. Since the total disutility of one medium object and one small object is larger than

$$\frac{17\alpha - 2}{10} + \frac{6\alpha - 1}{5} = \frac{29\alpha - 4}{10} \geq \frac{18\alpha - 3}{5} = 3 \cdot \frac{6\alpha - 1}{5},$$

where the inequality is because $\alpha \leq \frac{2}{7}$, the only combination that remains to consider is that A_1 contains two small objects besides the object with disutility α.

Combination 2: Besides the object with disutility α, A_1 contains two small objects. From the definition of small object, $v(e') \leq \max_{e \in A_2} v(e) < \frac{1-\alpha}{5}$ holds for any small object $e' \in A_1$. Thus, $v(A_1) < \alpha + 2 \cdot \frac{1-\alpha}{5} = \frac{2+3\alpha}{5}$, which is a contradiction to the assumption that $v(A_1) > \Delta_2^{\oplus}(\alpha)$.

For the other direction, the disutility function for this subcase contains one object with disutility α and five objects with disutility $\frac{1-\alpha}{5}$. Since $\frac{1}{6} < \frac{7}{27} < \alpha \leq \frac{2}{7}$, it follows that $\frac{1-\alpha}{5} < \alpha \leq 2 \cdot \frac{1-\alpha}{5}$. Clearly, the MMS of this disutility function is $\alpha + 2 \cdot \frac{1-\alpha}{5}$.

Subcase 3.3: $\alpha \in (\frac{2}{7}, \frac{1}{3}]$

When $\alpha \in (\frac{2}{7}, \frac{1}{3}]$, $v(A_1) > \Delta_2^{\oplus}(\alpha) = 2\alpha$. If $E_\alpha \cap A_2 \neq \emptyset$, then the proof is similar to those for the counterparts of Subcases 3.1 and 3.2. In other words, we also have $v(A_1) \leq \frac{3-3\alpha}{4}$, which contradicts $v(A_1) > \Delta_2^{\oplus}(\alpha)$ since $\frac{3-3\alpha}{4} < 2\alpha$ when $\alpha > \frac{2}{7} > \frac{3}{11}$.

Then we focus on $E_\alpha \cap A_2 = \emptyset$. Since $|A_1| \geq 3$, there exists $S \subseteq A_1$ such that $\alpha + \frac{1}{2}(v(A_1) - \alpha) \leq v(S) < v(A_1)$. From Claim 4, we have a lower bound of $v(A_2)$—that is, $v(A_2) \geq \alpha + \frac{1}{2}(v(A_1) - \alpha)$. Summing up the lower bounds of $v(A_1)$ and $v(A_2)$ leads to a contradiction:

$$1 = v(A_1) + v(A_2) \geq \frac{3}{2}v(A_1) + \frac{\alpha}{2} > \frac{7\alpha}{2} > 1,$$

where the last inequality is because $\alpha > \frac{2}{7}$.

For the other direction, the disutility function for this subcase contains three objects with disutility α and one object with disutility $1 - 3\alpha$ (if $\alpha < \frac{1}{3}$). Since $\alpha > \frac{2}{7} > \frac{1}{4}$, it follows that $1 - 3\alpha < \alpha$. Clearly, the MMS is 2α.

Up to here, we computed Hill's share for unrestricted m. Moving to the setting of restricted m, Hill's share can be computed by similar approaches with a more involved discussion.

A.2 Proof of Theorem 3.1

Moving to the setting when m is not sufficiently large, Hill's share can be computed by similar approaches with a more involved discussion. For the sake of contradiction, we assume that there exists a disutility $v \in \mathcal{V}(\alpha; m)$ such that $\mathrm{MMS}_n(v)$ is larger than whose MMS is larger than the expressions in Theorem 3.1. With a slight abuse of notations, in the following we use $\Delta_n^{\oplus}(\alpha; m)$ to denote those expressions in Theorem 3.1. Let $\mathbf{A} = (A_1, \ldots, A_n)$ be a lexicographical MinMax allocation of v, and without loss of generality, assume $v(A_1) \geq \cdots \geq v(A_n)$. We now split the proof into several cases based on the values of n and k, and it suffices to compute the share for the case where m is smaller than the number of objects in the worst-case disutility function in the unrestricted setting.

Table 4. Disutility Function for
Subcase $\alpha \in D(n,k)$ with $n \neq 2$ or
$k \neq 1$, and $m \leq kn + n$

Object Disutility	Quantity
α	$\lceil \frac{1}{\alpha} \rceil - 1$
$1 - (\lceil \frac{1}{\alpha} \rceil - 1)\alpha$	1
0	$m - \lceil \frac{1}{\alpha} \rceil$

Case 1: $n \neq 2$ or $k \neq 1$

We consider the subcases $\alpha \in D(n,k)$ and $\alpha \in I(n,k)$, separately.

Subcase 1.1: $\alpha \in D(n,k)$

Recall that when $\alpha \in D(n,k)$ with $n \neq 2$ or $k \neq 1$, the disutility function constructed in the setting when m is not restricted contains $kn + n + 1$ objects (see Tables 1 and 2). Therefore, if $m \geq kn + n + 1$, the tight bound remains unchanged.

Thus, we can focus on $m \leq kn + n$. Since $v(A_1) > \Delta_n^{\oplus}(\alpha; m) = (k+1)\alpha$, by Claim 4, $v(A_j) \geq v(A_1) - \alpha > k\alpha$ for any $j \in N \setminus \{1\}$. Moreover, since the disutility of any object is at most α, A_1 contains at least $k+2$ objects and A_j contains at least $k+1$ ones—that is, $|A_1| \geq k+2$ and $|A_j| \geq k+1$. Accordingly, the total number of objects is at least $k + 2 + (n-1)(k+1) = kn + n + 1 > m$, a contradiction. The disutility function that shows tightness (Table 4) contains $\lceil \frac{1}{\alpha} \rceil - 1$ objects with disutility α, one object with disutility $1 - (\lceil \frac{1}{\alpha} \rceil - 1)\alpha$, and $m - \lceil \frac{1}{\alpha} \rceil$ objects with disutility 0. This disutility function is valid since $m \geq \lceil \frac{1}{\alpha} \rceil$. Since $\alpha \in D(n,k)$, $\frac{1}{\alpha} \geq \frac{n(k+1)^2 + k + 2}{k+2} \geq kn + 1$, where the last inequality is because $n \geq 0$. Therefore, the disutility function contains at least $kn + 1$ objects with disutility α. By the pigeonhole principle, the MMS is at least $(k+1)\alpha$.

Subcase 1.2: $\alpha \in I(n,k)$

The bound for $\alpha \in I(n,k)$ remains unchanged regardless of the value of m, since there always exists a disutility function whose MMS is at least $\Delta_n^{\oplus}(\alpha; m) = (k+1)\alpha$. Specifically, the disutility function (see Table 4) also contains $\lceil \frac{1}{\alpha} \rceil - 1$ objects with disutility α, one object with disutility $1 - (\lceil \frac{1}{\alpha} \rceil - 1)\alpha$, and $m - \lceil \frac{1}{\alpha} \rceil$ objects with disutility 0. Since $\alpha \in I(n,k)$, $\frac{1}{\alpha} \geq kn + 1$, which means that there are at least $kn + 1$ objects with disutility α. By the pigeonhole principle, the MMS is at least $(k+1)\alpha$.

Case 2: $n = 2$ and $k = 1$

Recall that when $n = 2$ and $k = 1$, $\alpha \in (\frac{1}{5}, \frac{1}{3}]$, thus $m \geq \lceil \frac{1}{\alpha} \rceil \geq 3$. When $m = 3$, α can only be $\frac{1}{3}$. The tight bound remains unchanged (i.e., $\Delta_2^{\oplus}(\frac{1}{3}; 3) = \Delta_2^{\oplus}(\frac{1}{3})$), since the disutility function constructed in the unrestricted setting (i.e., Subcase 3.3 in Section A.1) contains three objects when $\alpha = \frac{1}{3}$.

When $m = 4$, $\alpha \in [\frac{1}{4}, \frac{1}{3})$. Since $v(A_1) > \Delta_2^{\oplus}(\alpha; 4) = 2\alpha$, by Claim 4, $v(A_2) > \alpha$. Therefore, A_1 contains at least three objects and A_2 contains at least two objects, a contradiction to $m = 4$. For the tightness, the disutility function contains $\lceil \frac{1}{\alpha} \rceil - 1$ objects with disutility α, and one object with disutility $1 - (\lceil \frac{1}{\alpha} \rceil - 1)\alpha$. Sine $\frac{1}{\alpha} > 3$, and by the pigeonhole principle, the MMS is at least 2α.

When $m = 5$, $\alpha \in (\frac{1}{5}, \frac{1}{3}]$. If $\alpha \in (\frac{1}{5}, \frac{7}{27}]$ or $(\frac{2}{7}, \frac{1}{3}]$, the disutility functions constructed in the unrestricted setting (i.e., Subcases 3.1 and 3.3 in Section A.1) contain five and four objects, respectively, thus the tight bounds do not change. If $\alpha \in (\frac{7}{27}, \frac{2}{7}]$, since $v(A_1) > \Delta_2^{\oplus}(\alpha; 5) \geq 2\alpha$, by Claim 4, $v(A_2) > \alpha$, thus A_1 contains at least three objects and A_2 contains at least two objects. More accurately, since $m = 5$, $|A_1|$ is exactly 3 and $|A_2|$ is exactly 2. Moreover, it can be verified that the largest disutility in A_1 is at most the smallest disutility in A_2. Since otherwise, by exchanging one object

in A_1 with a strictly larger disutility and one object in A_2 with a strictly smaller disutility, one can get another allocation $\mathbf{A}' = (A_1', A_2')$ such that $v(A_1') < v(A_1)$ and $v(A_2') \leq 2\alpha < v(A_1)$, which contradicts the assumption that $\mathbf{A}$ is a lexicographical MinMax allocation of v. Let $A_2 = \{e_1, e_2\}$ with $v(e_1) \geq v(e_2)$, and it follows that $v(e_1) = \alpha$ and $v(e_2) \geq \frac{1}{3} \cdot v(A_1)$. Therefore,

$$v(A_1 \cup A_2) \geq v(A_1) + \alpha + \frac{1}{3}v(A_1) = \frac{4}{3} \cdot v(A_1) + \alpha.$$

If $\alpha \in (\frac{7}{27}, \frac{3}{11}]$, $v(A_1) > \Delta_2^\oplus(\alpha; 5) = \frac{3-3\alpha}{4}$, thus

$$v(A_1 \cup A_2) > \frac{4}{3} \cdot \frac{3 - 3\alpha}{4} + \alpha = 1,$$

a contradiction. If $\alpha \in (\frac{3}{11}, \frac{2}{7}]$, $v(A_1) > \Delta_2^\oplus(\alpha; 5) = 2\alpha$, also a contradiction since

$$v(A_1 \cup A_2) > \frac{11}{3}\alpha > 1.$$

The disutility function that shows tightness for $\alpha \in (\frac{7}{27}, \frac{3}{11}]$ is the same as that in Subcase 3.1 in Section A.1—that is, one object with disutility α and four objects with disutility $\frac{1-\alpha}{4}$. Again, since $\frac{1}{5} < \frac{7}{27} < \alpha \leq \frac{3}{11} < \frac{1}{3}$, $\frac{1-\alpha}{4} < \alpha < 2 \cdot \frac{1-\alpha}{4}$, which gives that the MMS is $\frac{3-3\alpha}{4}$. For $\alpha \in (\frac{3}{11}, \frac{2}{7}]$, the disutility function is the same as that in Subcase 3.3 in Section A.1—that is, three objects with disutility α and one object with disutility $1 - 3\alpha$. Since $\alpha > \frac{3}{11} > \frac{1}{4}$, $1 - 3\alpha < \alpha$, thus the MMS is 2α.

When $m \geq 6$, $\alpha \in (\frac{1}{5}, \frac{1}{3}]$. Since the disutility functions constructed in the subcases of the unrestricted setting contain no more than six objects, the tight bounds remain unchanged.

Acknowledgments

References

[1] Hannaneh Akrami and Jugal Garg. 2023. Breaking the 3/4 barrier for approximate maximin share. *CoRR abs/2307.07304* (2023).

[2] Georgios Amanatidis, Haris Aziz, Georgios Birmpas, Aris Filos-Ratsikas, Bo Li, Hervé Moulin, Alexandros A. Voudouris, and Xiaowei Wu. 2023. Fair division of indivisible goods: Recent progress and open questions. *Artif. Intell.* 322 (2023), 103965.

[3] Georgios Amanatidis, Evangelos Markakis, Afshin Nikzad, and Amin Saberi. 2017. Approximation algorithms for computing maximin share allocations. *ACM Trans. Algorithms* 13, 4 (2017), Article 52, 28 pages.

[4] Haris Aziz, Bo Li, Hervé Moulin, and Xiaowei Wu. 2022. Algorithmic fair allocation of indivisible items: A survey and new questions. *ACM SIGecom Exch.* 20, 1 (2022), 24–40.

[5] Haris Aziz, Hervé Moulin, and Fedor Sandomirskiy. 2020. A polynomial-time algorithm for computing a Pareto optimal and almost proportional allocation. *Oper. Res. Lett.* 48, 5 (2020), 573–578.

[6] Haris Aziz, Gerhard Rauchecker, Guido Schryen, and Toby Walsh. 2017. Algorithms for max-min share fair allocation of indivisible chores. In *Proceedings of AAAI*. 335–341.

[7] Moshe Babaioff, Tomer Ezra, and Uriel Feige. 2022. On best-of-both-worlds fair-share allocations. In *Web and Internet Economics*. Lecture Notes in Computer Science, Vol. 13778. Springer, 237–255.

[8] Anna Bogomolnaia, Hervé Moulin, Fedor Sandomirskiy, and Elena Yanovskaia. 2019. Dividing bads under additive utilities. *Soc. Choice Welf.* 52, 3 (2019), 395–417.

[9] Sylvain Bouveret and Michel Lemaître. 2016. Characterizing conflicts in fair division of indivisible goods using a scale of criteria. *Auton. Agents Multi Agent Syst.* 30, 2 (2016), 259–290.

[10] Eric Budish. 2011. The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *J. Political Econ.* 119, 6 (2011), 1061–1103.

[11] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D. Procaccia, Nisarg Shah, and Junxing Wang. 2019. The unreasonable fairness of maximum Nash welfare. *ACM Trans. Econ. Comput.* 7, 3 (2019), Article 12, 32 pages.

[12] Lester E. Dubins and Edwin H. Spanier. 1961. How to cut a cake fairly. *Am. Math. Mon.* 68, 1P1 (1961), 1–17.

[13] Uriel Feige, Ariel Sapir, and Laliv Tauber. 2021. A tight negative example for MMS fair allocations. In *Web and Internet Economics*. Lecture Notes in Computer Science, Vol. 13112. Springer, 355–372.

[14] Jugal Garg and Setareh Taki. 2021. An improved approximation algorithm for maximin shares. *Artif. Intell.* 300 (2021), 103547.
[15] Jonathan R. Goldman and Ariel D. Procaccia. 2014. Spliddit: Unleashing fair division algorithms. *SIGecom Exch.* 13, 2 (2014), 41–46.
[16] Laurent Gourvès, Jérôme Monnot, and Lydia Tlilane. 2015. Worst case compromises in matroids with applications to the allocation of indivisible goods. *Theor. Comput. Sci.* 589 (2015), 121–140.
[17] Theodore P. Hill. 1987. Partitioning general probability measures. *Ann. Probab.* 15, 2 (1987), 804–813.
[18] Xin Huang and Pinyan Lu. 2021. An algorithmic framework for approximating maximin share allocation of chores. In *Proceedings of EC*. ACM, 630–631.
[19] David Kurokawa, Ariel D. Procaccia, and Junxing Wang. 2018. Fair enough: Guaranteeing approximate maximin shares. *J. ACM* 65, 2 (2018), Article 8, 27 pages.
[20] Jan Karel Lenstra, David B. Shmoys, and Éva Tardos. 1990. Approximation algorithms for scheduling unrelated parallel machines. *Math. Program.* 46 (1990), 259–271.
[21] Richard J. Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. 2004. On approximately fair allocations of indivisible goods. In *Proceedings of EC*. ACM, 125–131.
[22] Evangelos Markakis and Christos-Alexandros Psomas. 2011. On worst-case allocations in the presence of indivisible goods. In *Internet and Network Economics*. Lecture Notes in Computer Science, Vol. 7090. Springer, 278–289.
[23] Hervé Moulin. 2019. Fair division in the internet age. *Annu. Rev. Econ.* 11 (2019), 407–441.
[24] Ariel D. Procaccia and Junxing Wang. 2014. Fair enough: Guaranteeing approximate maximin shares. In *Proceedings of EC*. ACM, 675–692.
[25] Hugo Steinhaus. 1949. Sur la division pragmatique. *Econometrica* 17, Suppl. (1949), 315–319.
[26] Gerhard J. Woeginger. 1997. A polynomial-time approximation scheme for maximizing the minimum machine completion time. *Oper. Res. Lett.* 20, 4 (1997), 149–154.