Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Francisco F. Lasheras & Michael Mihalik

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Article à paraître, mis en ligne le 3 juillet 2023, 17 p.

CENTRE MERSENNE
Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
Article à paraître
Mis en ligne le 3 juillet 2023.

LIFTING SEMISTABILITY IN FINITELY GENERATED ASCENDING HNN-EXTENSIONS

by Francisco F. LASHERAS & Michael MIHALIK

ABSTRACT. — If a finitely generated group G maps epimorphically onto a group H, we are interested in the question: When does the semistability of H imply G is semistable? In this paper, we give an answer within the class of ascending HNN-extensions. More precisely, our main theorem states: Suppose that the 1-ended finitely generated ascending HNN-extension $H = \langle S, t; R, t^{-1}st = \phi(s), s \in S\rangle$ is semistable at infinity. Let $\overline{R}$ be the kernel of the obvious homomorphism from the free group $F(\{t\} \cup S)$ onto H, then there is a finite subset $R_0 \subseteq \overline{R}$ such that those finitely generated ascending HNN-extensions $H_1 = \langle S, t; R_1, t^{-1}st = \phi(s), s \in S\rangle$, with $R_0 \subseteq R_1 \subset \overline{R}$, are all 1-ended and semistable at infinity as well. Furthermore H_1 has such a presentation with $R_1 \subset R$. Note that there is an obvious epimorphism from H_1 to H. It is unknown whether all finitely presented ascending HNN-extensions are semistable at infinity.

RÉSUMÉ. — La question fondamentale de cet article est de savoir sous quelles conditions la semistabilité d'un groupe H entraîne la semistabilité d'un groupe G qui admet une surjection sur H. Nous allons y répondre dans le cadre des extensions HNN ascendantes. Plus précisement, considérons une extension HNN de type fini ayant un seul bout $H = \langle S, t; R, t^{-1}st = \phi(s), s \in S\rangle$ qu'on suppose être semistable à l'infini. Soit $\overline{R}$ le noyau du morphisme tautologique du groupe libre $F(\{t\} \cup S)$ sur H. Alors il existe un sous-ensemble fini $R_0 \subseteq \overline{R}$ tel que toute extension HNN de type fini $H_1 = \langle S, t; R_1, t^{-1}st = \phi(s), s \in S\rangle$, ayant $R_0 \subseteq R_1 \subset \overline{R}$, n'a qu'un seul bout et est semistable à l'infini. De plus H_1 admet une telle présentation avec $R_1 \subset R$. Notons qu'il y a un épimorphisme de H_1 dans H. A l'heure actuelle, nous ne savons pas si toutes les extensions HNN ascendantes sont semistables à l'infini.

1. Introduction

We consider an asymptotic property of finitely presented groups that has been well studied for over 40 years called semistable fundamental group at ∞. A locally finite complex Y has semistable fundamental group at ∞ if

Keywords: Proper homotopy, semistability at infinity, ascending HNN-extension, group presentation.
2020 *Mathematics Subject Classification:* 20F69, 20F65, 20E22.

any two proper rays $r, s : [0, \infty) \to Y$ that converge to the same end of Y are properly homotopic in Y. A finitely presented group G has semistable fundamental group at ∞ (often shortened to say G is semistable) if for some (equivalently any) finite complex X with $\pi_1(X) = G$, the universal cover of X has semistable fundamental group at ∞. Suppose H is a finitely generated group, G is a finitely presented group and $h : G \to H$ is an epimorphism. If X is a finite complex with $\pi_1(X) = G$ and Y is the regular cover of X corresponding to the kernel of h, then we say H has semistable fundamental group at ∞ if Y does. The idea of extending semistability from finitely presented groups to finitely generated groups has been used successfully to show many finitely presented groups are semistable, see [8] and [10]. While semistability has a number of interesting connections to other asymptotic properties of groups, perhaps the primary application of semistability is the following: If a finitely presented 1-ended group G is semistable at ∞, then the fundamental group at infinity of G is well defined. It is unknown at this time, whether or not all finitely presented groups have semistable fundamental group at ∞, but in [9] the problem is reduced to considering 1-ended groups. The finitely presented group G satisfies a weaker geometric condition called semistable first homology at ∞ if and only if $H^2(G : \mathbb{Z}G)$ is free abelian (see [3]). The question of whether or not $H^2(G : \mathbb{Z}G)$ is free abelian for all finitely presented groups G goes back to H. Hopf.

The semistability of 1-ended finitely generated groups G was introduced in [8]. Given a finite generating set S for G, let R be the kernel of the map of the free group $F(S)$ to G and let $\Gamma_G(S)$ denote the Cayley graph of G with respect to S.

($*$). — The finitely generated group G is *semistable at infinity* if there is a finite subset $R_0 \subset R$ so that the (strongly) locally finite 2-dimensional CW-complex $\Gamma_G(S; R_0)$, obtained by attaching to every vertex of $\Gamma_G(S)$ 2-cells according to each element in R_0, is semistable at infinity. This notion is independent of finite generating set for G (see [8, Thm. 1]). The space $\Gamma_G(S; R_0)$ naturally identifies with a regular covering of the standard finite complex $X(S; R_0)$ (with $\pi_1(X(S; R_0)) \cong \langle S; R_0 \rangle$). This covering is obtained by taking the quotient of $\widetilde{X}(S; R_0)$, the universal cover of $X(S; R_0)$, by the kernel of the map $\langle S; R_0 \rangle \to G$ that takes $s \to s$ for each $s \in S$.

$$\begin{array}{ccc} \widetilde{X}(S;R_0) & & \\ \downarrow q & \searrow^{p} & \\ \Gamma_G(S;R_0) & \longrightarrow & X(S;R_0) \end{array}$$

In this situation, one may naturally ask whether for some (alternatively, for any) set R_1 such that $R_0 \subset R_1 \subset R$ the finitely generated group with presentation $A = \langle S; R_1 \rangle$ is semistable at infinity as well. In this paper, we give a partial answer to this question within the class of ascending HNN-extensions. In Section 2 we show that a finitely generated group is an ascending HNN-extension of another finitely generated group if and only if it has a presentation of the form $\langle S, t; R, t^{-1}st = \phi(s), s \in S \rangle$ where S is finite, R is a subset of the free group $F(S)$ and $\phi(s)$ is an element of $F(S)$ for each $s \in S$. The usual "HNN-base group" of such a presentation is the subgroup generated by S and we explain how ϕ determines a monomorphism of this base group.

Theorem 1.1. — *Suppose S is a finite set, $\phi : S \longrightarrow F(S)$ is a map to the free group generated by S, R is a subset of $F(S)$ and the finitely generated ascending HNN-extension H with presentation $\langle S, t; R, t^{-1}st = \phi(s), s \in S \rangle$ is 1-ended and semistable at infinity. Let $\overline{R}$ be the kernel of the obvious epimorphism of the free group $F(\{t\} \cup S)$ to H. Then there is a finite subset $R_0 \subseteq R$ such that those ascending HNN-extensions $H_1 = \langle S, t; R_1, t^{-1}st = \phi(s), s \in S \rangle$, with $R_0 \subseteq R_1 \subseteq \overline{R}$, are all 1-ended and semistable at infinity as well.*

Remark 1.2. — The subgroup of H generated by S is the base group of the ascending HNN-extension. The letter t is the stable letter. If the base group of H is infinite, then H is 1-ended (see [12, Thm. 3.3] or [13, Thm. 3.1]). Otherwise H is 2-ended. All 2-ended groups are semistable at infinity. Following the notation of the theorem, another presentation of H is $\langle S, t; \overline{R}, t^{-1}st = \phi(s), s \in S \rangle$. There is an obvious epimorphism of H_1 onto H. If R_0' is a finite subset of $\overline{R}$ such that R_0' along with the conjugation relations $t^{-1}st = \phi(s), s \in S$ satisfy the semistability condition for H (see $(*)$), then it is an elementary matter to show that R_0' is a consequence (in $F(\{t\} \cup S)$) of the conjugation relations and a finite set $R_0'' \subset R$. We show that R_0'' can play the role of R_0 in Theorem 1.1. Note that if the subset R_1 of $\overline{R}$ is finite then H_1 is a finitely presented ascending HNN-extensions that is semistable at infinity.

It is unknown whether all finitely presented ascending HNN-extensions are semistable at infinity, see [11] for some recent results. Notice that, even when R is finite (and so H is finitely presented), the base group for the ascending HNN-extension H need not be finitely presented and hence might not admit $\langle S; R \rangle$ as a presentation. For an explicit presentation of such base groups see [11, §4]. It is worth mentioning that ascending

HNN-extensions with finitely presented base group are all known to be semistable at infinity [6]; moreover, their fundamental pro-group is known to be semistable and pro-(finitely generated free), see [4].

Remark 1.3. — As an interesting example, Theorem 1.1 applies to any ascending HNN-extension of the (first) Grigorchuk group Γ, which is an infinite finitely generated (non finitely presentable) 2-group. Notice that such a finitely generated ascending HNN-extension is always 1-ended and semistable at infinity by [8, Prop. 2] and [11, Thm. 1.3] (see also [8, Thm. 4]) as Γ and $\Gamma \times \Gamma$ are commensurable. In fact, in [10] it is shown that Γ as well as any of its ascending HNN-extensions is simply connected at infinity. See [1, Chap. VIII] for a complete description of the group Γ.

In Section 2 we give basic facts about ascending HNN-extensions. In particular, we show that a finitely generated group H is an ascending HNN-extension of a finitely generated group if and only if H has a presentation matching the one in our Theorem 1.1. Section 3 contains basic definitions. We prove two elementary lemmas about certain proper rays in a space that we must show is semistable at infinity. The lemmas reduce the proof of our theorem to the more interesting rays described in Proposition 3.3. The Main Lemma (4.3) of the paper is proved in Section 4. It states that any proper ray (in the space we are trying to show is semistable at infinity) is properly homotopic to one that projects (by a covering map) to a proper ray in a space we already know is semistable at infinity. A lifting argument then concludes the proof of our main theorem.

2. Ascending HNN-extension combinatorics

Suppose S is a finite set and B is a group with presentation $\langle S; R\rangle$ (so that R is a subset of the free group $F(S)$). Let $q : F(S) \to B$ be the quotient homomorphism with kernel N equal to the normal closure of R. Suppose $\bar{\phi} : B \to B$ is a monomorphism then there is a homomorphism $\phi : F(S) \to F(S)$ such that $\bar{\phi} q = q\phi$.

$$\begin{array}{ccc} F(S) & \xrightarrow{\ \phi\ } & F(S) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{\ \bar{\phi}\ } & B \end{array}$$

A *standard presentation* for G, the ascending HNN-extension of B and $\bar{\phi}$ is $\langle S, t; R, t^{-1}st = \phi(s), s \in S\rangle$. Let $q_1 : F(S \cup \{t\}) \to G$ be the quotient map with kernel equal to the normal closure of the set $R \cup \{t^{-1}st(\phi(s))^{-1}, s \in S\}$. Let $i : F(S) \to F(S \cup \{t\})$ be the inclusion homomorphism. There is a homomorphism $\bar{i} : B \to G$ so that $q_1 i = \bar{i} q$ and q_1 restricted to $F(S)$ is equal to q.

$$\begin{CD} F(S) @>{i}>> F(S \cup \{t\}) \\ @V{q}VV @VV{q_1}V \\ B @>{\bar{i}}>> G \end{CD}$$

By Britton's Lemma, the map $\bar{i} : B \to G$ is a monomorphism. The image of B under $\bar{i}$ is called the *base group* of the ascending HNN-extension. The letter t is called the *stable letter* of the standard presentation.

In the remainder of this section our goal is show that a finitely generated group G with a presentation of the form $\mathcal{P} = \langle S, t; R, t^{-1}st = \phi(s), s \in S\rangle$ (where S is a finite set, R is a subset of the free group $F(S)$ and $\phi(s)$ is an element of $F(S)$) is an ascending HNN-extension of the subgroup of G generated by S. Note that ϕ extends to a homomorphism $\phi : F(S) \to F(S)$. Then (up to isomorphism) the group G is the image of the quotient homomorphism $q : F(S \cup \{t\}) \to G$ with kernel equal to N, the normal closure of the set $R \cup \{t^{-1}st(\phi(s))^{-1}, s \in S\}$ in $F(S \cup \{t\})$. To simplify notation let $q(a) = \bar{a} = aN \in G$ for any $a \in F(S \cup \{t\})$. If B is the subgroup of G generated by $\overline{S}$ then any element of B is equal to $\bar{b}$ for some $b \in F(S)$. Furthermore,

$$\bar{t}^{-1}\bar{b}\bar{t} = \overline{\phi(b)}.$$

We want to see that $\mathcal{P}$ presents the ascending HNN-extension with base group B and some monomorphism of B. Consider the function $\bar{\phi} : B \to B$ defined by $\bar{b} \to \overline{\phi(b)}$ for all $b \in F(S)$. In order to see that $\bar{\phi}$ is a homomorphism, it is enough to see that ϕ takes relators of B (elements of the kernel of q restricted to $F(S)$) to relators of B. If r is a relator for B then $\bar{r} = q(r) = 1$ and $\overline{\phi(r)} = \bar{t}^{-1}\bar{r}\bar{t} = 1$. So, the function $\bar{\phi}$ is a homomorphism of B. Note that $\bar{\phi} q = q\phi$.

$$\begin{CD} F(S) @>{\phi}>> F(S) \\ @V{q}VV @VV{q}V \\ B @>{\bar{\phi}}>> B \end{CD}$$

Finally we want to see that $\bar{\phi}$ is a monomorphism. Let $\bar{b} \in B$ be in the kernel of $\bar{\phi}$. Then $1 = \bar{\phi}(\bar{b}) = \overline{\phi(b)} = \bar{t}^{-1}\bar{b}\bar{t}$. This implies that $\bar{b} = 1$ and $\bar{\phi}$ is a monomorphism. A presentation for B is $\langle S; \widetilde{R}\rangle$ where $\widetilde{R}$ is the kernel of q restricted to $F(S)$. A standard presentation for the HNN-extension of B

and $\bar{\phi}$ is $\langle S, t; \widetilde{R}, t^{-1}st = \phi(s), s \in S\rangle$. If $r \in \widetilde{R}$ then since r is in the kernel of q it belongs to N, the normal closure of the set $R \cup \{t^{-1}st(\phi(s))^{-1}, s \in S\}$ in $F(S \cup \{t\})$. Then $\widetilde{R}$ can be added to the relation set of the presentation $\langle S, t; R, t^{-1}st = \phi(s)\rangle$ of G. Since $R \subset \widetilde{R}$, $\langle S, t; \widetilde{R}, t^{-1}st = \phi(s), s \in S\rangle$ presents both G and the ascending HNN-extension of B and $\bar{\phi}$.

A final comment on presentations of the form $\mathcal{P} = \langle S, t; R, t^{-1}st = \phi(s), s \in S\rangle$ (where S is a finite set, R is a subset of the free group $F(S)$ and $\phi(s)$ is an element of $F(S)$). Extend ϕ to $\phi : F(S) \to F(S)$. The relation set (up to normal closure) of the subgroup B of this group generated by S is worked out in Theorem 4.1 of [11]. A presentation of B is

$$\left\langle S : \bigcup_{i=0}^{\infty} \phi^{-i}(N(\cup_{j=0}^{\infty} \phi^{j}(R))) \right\rangle.$$

Here if $A \subset F(S)$ then $N(A)$ is the normal closure of A in $F(S)$.

3. Semistability preliminaries

We recall some basic notions. A map of topological spaces $f : X \to Y$ is proper if for each compact set $C \subset Y$, $f^{-1}(C)$ is compact in X. We are only interested in 1-ended groups in this paper. A 1-ended space Y is semistable at ∞ if any two proper rays $r, s : [0, \infty) \to Y$ are properly homotopic. Recall that $pro - \pi_1(Y, \omega)$ is determined by the inverse sequence (tower) of groups

$$\pi_1(Y, \omega(0)) \xleftarrow{\phi_1} \pi_1(Y - C_1, \omega(t_1)) \xleftarrow{\phi_2} \pi_1(Y - C_2, \omega(t_2)) \longleftarrow \cdots$$

where $C_1 \subset C_2 \subset \cdots \subset Y$ is a filtration of Y by compact subspaces (C_i is a subset of the interior of C_{i+1} and $\bigcup_{i=1}^{\infty} C_i = Y$), $\omega([t_i, \infty)) \subset Y - C_i$ and the bonding homomorphisms ϕ_i are induced by the inclusions and basepoint-change isomorphisms. If Y is 1-ended and semistable, then changing base rays ω gives pro-isomorphic inverse sequences and the inverse limit of $pro - \pi_1(Y)$ is called the fundamental group at ∞ of Y. The space Y is semistable at ∞ if and only if $pro - \pi_1(Y)$ is pro-isomorphic to an inverse sequence of groups with epimorphic bonding maps. We refer to [2, 5] for more details.

Let G be a group with finite generating set S. Let R be the kernel of the natural map from the free group $F(S)$ to G. The Cayley graph $\Gamma_G(S)$ of G with respect to the set S has G as vertex set and one edge labeled s joining g to gs for each pair $(g, s) \in G \times S$ (the inverse of this edge is labeled s^{-1}). If $R_0 \subset R$ is a finite set then we denote by $\Gamma_G(S; R_0)$ the (strongly) locally finite 2-complex obtained by attaching to every vertex of $\Gamma_G(S)$, a 2-cell according to each $r \in R_0$, so that G acts as a group of covering transformations on $\Gamma_G(S; R_0)$. Any proper edge path ray in $\Gamma_G(S; R_0)$ can be represented as an infinite sequence $(s_1^{\pm 1}, s_2^{\pm 1}, \dots)$, where $s_i \in S$, once the initial vertex is given. Observe that if G is finitely presented and $\langle S; R\rangle$ is a finite presentation of G, then the corresponding 2-complex $\Gamma_G(S; R)$ is the so called Cayley 2-complex for this presentation. It is simply connected and in this case the quotient $X = G\backslash\Gamma_G(S; R)$ is the standard 2-complex associated to the presentation. The space X is a finite 2-complex with $\pi_1(X) \cong G$ and the quotient map $\Gamma_G(S; R) \longrightarrow X$ is the universal covering projection. If G is 1-ended and semistable at ∞ then $pro - \pi_1(\Gamma_G(S; R))$ (up to pro-isomorphism) is the fundamental pro-group at ∞ of G. The inverse limit of $pro - \pi_1(\Gamma_G(S; R))$ is the fundamental group at ∞ of G.

Suppose S is a finite set together with a function $\phi : S \longrightarrow F(S)$ and $R \subset F(S)$. Let R_0 be a finite subset of R. Consider the finitely generated ascending HNN-extension H given by the presentation:

$$H = \langle S, t; R, t^{-1}st = \phi(s) \text{ (for s} \in S)\rangle$$

and the locally finite 2-complex:

$$\widehat{X} = \Gamma_H(S, t; R_0, t^{-1}st = \phi(s) \text{ (for } s \in S)).$$

The homomorphism $L : H(= \widehat{X}^0) \longrightarrow \mathbb{Z}$ that kills the normal closure of S in H extends to the "level" map $L : \widehat{X} \longrightarrow \mathbb{R}$ which sends each 2-cell corresponding to a relator in R to $L(v)$ for any vertex v of it, and each 2-cell corresponding to a "conjugation relation" to the corresponding interval $[N, N+1] \subset \mathbb{R}$ in the obvious way (i.e., sending the corresponding edge labeled s to N, the edges in $\phi(s)$ to $N+1$ and the edges labeled t, t^{-1} to the interval $[N, N+1]$). Consider a vertex $v \in \widehat{X}$ and the proper edge path ray $r_v \equiv (t, t, \dots)$ based at v each of whose edges is labeled t. It is clear that the translate of r_v by t or t^{-1} is another ray in $\widehat{X}$ properly homotopic to r_v. On the other hand, [11, Lem. 3.1] shows that if an edge labeled $s \in S$ in $\widehat{X}$ has v and w as initial and final vertices, then the corresponding proper edge path rays r_v and r_w are properly homotopic (and the same is true for an edge labeled s^{-1}). See Figure 3.1 for a pictorial description of the proper homotopy.

Therefore, if we denote by $*$ the vertex of $\Gamma_H(S, t; R_0, t^{-1}st = \phi(s)$ (for $s \in S$)) corresponding to the identity element in H, one can easily conclude the following:

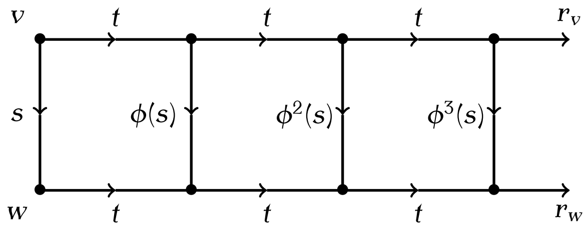

Figure 3.1. Conjugation Homotopies

LEMMA 3.1. — *Any two translates in $\Gamma_H(S,t;R_0,t^{-1}st = \phi(s)$ (for $s \in S$)) of the proper edge path ray $(t,t,\dots)$ based at $*$ are properly homotopic.*

Again, let $\widehat{X} = \Gamma_H(S,t;R_0,t^{-1}st = \phi(s)$ (for $s \in S$)). Suppose $r \equiv (\sigma_1,\sigma_2,\dots)$ is a proper edge path ray based at v in $\widehat{X}$, where each $\sigma_i \in S^{\pm 1} \cup \{t^{\pm 1}\}$. We may always assume that no two consecutive edges are labeled σ,σ^{-1} or σ^{-1},σ in r, with $\sigma \in S \cup \{t\}$. Of course, if there is an edge in r labeled σ_i after which all edges occurring in r are labeled by elements in $S^{\pm 1}$, then r is clearly properly homotopic to the proper edge path ray $(\sigma_{i+1},\sigma_{i+2},\dots)$ based at the final vertex w of σ_i. This ray is properly homotopic to the proper edge path ray r_w (and hence to r_*) all of whose edges are labeled t, by [11, Lem. 3.3]. One simply combines proper homotopies of the type shown in Figure 3.1 (one for each σ_j for $j > i$). In fact, the following result may be thought of as an extension of [11, Lem. 3.3].

LEMMA 3.2. — *Let $r \equiv (\sigma_1,\sigma_2,\dots)$ be a proper edge path ray based at v in $\widehat{X} = \Gamma_H(S,t;R_0,t^{-1}st = \phi(s)$ (for $s \in S$)) (with R_0 finite). If no edge in r is labeled t^{-1} then r is properly homotopic to the proper edge path ray $r_v \equiv (t,t,\dots)$ based at v in $\widehat{X}$.*

Proof. — By the above considerations, we may assume that for each $i \geqslant 1$ there are $j,k \geqslant i$ so that $\sigma_j = t$ and $\sigma_k \in S^{\pm 1}$. Write $r = (\gamma_1,\alpha_1,\gamma_2,\alpha_,\dots)$ where the letters of γ_i are in $S^{\pm 1}$ and the letters of α_i are all t. There is a homotopy $\mathcal{H}$ of r_v to r obtained by stacking homotopies H_i as in Figure 3.2. Each H_i combines a finite number of homotopies (one for each edge of γ_i) as in Figure 3.1.

Any compact $K \subset \widetilde{X}$ has bounded image $L(K) \subset \mathbb{R}$ and so only finitely many of the proper homotopies H_i have image that can intersect K. Hence $\mathcal{H}$ is proper. □

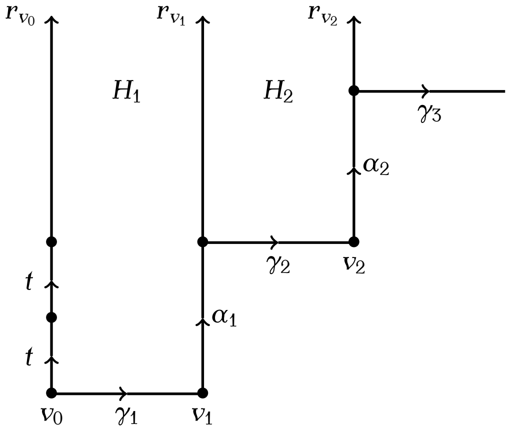

Figure 3.2. Stacking Homotopies

As above, let $\widehat{X} = \Gamma_H(S, t; R_0, t^{-1}st = \phi(s)$ (for $s \in S$)) (with R_0 finite) and assume r is a proper edge path ray based at v in $\widehat{X}$ which is represented as a sequence $r = (\gamma_1, \alpha_1, \gamma_2, \alpha_2, \dots)$ where each γ_k is a (maximal) finite edge path in r all of whose edges are labeled by elements in $S^{\pm 1}$ (or perhaps the constant path, in the case of γ_1), and each α_k is a (maximal) finite edge path in r all of whose edges are either labeled t or t^{-1} (i.e., of the form $(t, t, \dots, t)$ or $(t^{-1}, t^{-1}, \dots, t^{-1})$). By [11, Lem. 3.2], if $A < B$ are integers and K is compact in $\widehat{X}$, then there are at most finitely many edges e in level A of $\widehat{X}$ that cannot be moved to level B by a conjugation homotopy avoiding K (i.e., by means of the 2-cells coming from the corresponding conjugation relations as in Figure 3.1). Since $r = (\gamma_1, \alpha_1, \gamma_2, \alpha_2, \dots)$ is proper, there are only finitely many γ_i of r in level A of $\widehat{X}$ that cannot be moved to level B by a conjugation homotopy avoiding K. Let $K_1 \subset K_2 \subset \cdots$ be a filtration by compact subsets of $\widehat{X}$. Without loss of generality, we may assume that for $i \geqslant 1$, K_i has points in level i.

Suppose γ is a finite edge path of $\widehat{X}$, in the letters $S^{\pm 1}$. Let $L(\gamma)$ be the level containing γ. If γ can be pushed to level $|L(\gamma)| + 1$ by a conjugation homotopy avoiding $K_{|L(\gamma)|+1}$, then let $P(\gamma) (> |L(\gamma)|)$ be the largest integer such that γ can be pushed to level $P(\gamma)$ by a conjugation homotopy avoiding $K_{P(\gamma)}$. Note that $P(\gamma)$ is bounded above by j if the image of γ intersects K_j.

If $P(\gamma_i)$ is defined (and so $P(\gamma_i) > |L(\gamma_i)|$) then let H_i be the conjugation homotopy that moves γ_i to $\overline{\gamma}_i$ in level $P(\gamma_i)$. This means that H_i is the conjugation homotopy of γ_i to $t^{a(i)}\overline{\gamma}_i t^{-a(i)}$, where $a(i) = P(\gamma_i) - L(\gamma_i)$ and $im(H_i) \cap K_{P(\gamma_i)} = \emptyset$.

Given an integer J, [11, Lem. 3.2] implies there are only finitely many γ_i in level J such that $P(\gamma_i)$ is not defined. In particular, H_i is defined for all but finitely many γ_i in level J. If $P(\gamma_i)$ is not defined, then let $\overline{\gamma}_i = \gamma_i$.

Proposition 3.3. — *Suppose $\widehat{X} = \Gamma_H(S, t; R_0, t^{-1}st = \phi(s)$ (for $s \in S$)) (with R_0 finite) and r is a proper edge path ray in $\widehat{X}$. Then r is properly homotopic relative to $r(0)$, to a proper edge path ray r' such that for any integer J, only finitely many $S^{\pm 1}$ maximal subpaths of r' belong to level J.*

Proof. — Assume that r is as above. Let r' be obtained from r by combining the homotopies H_i above to a single homotopy $\mathcal{H}$. We must show that $\mathcal{H}$ is proper and that for any level J, only finitely many of the $\overline{\gamma}_i$ belong to level J.

Suppose $\overline{\gamma}_i$ belongs to level J. By [11, Lem. 3.2], only finitely many γ_i in level J are such that $\gamma_i = \overline{\gamma}_i$. Hence we only need concern ourselves with the γ_i such that $J = P(\gamma_i) > |L(\gamma_i)|$. For each such γ_i, $L(\gamma_i) \in \{-J+1, -J+2, \dots, J-1\}$. Again, by [11, Lem. 3.2], there are only finitely many such γ_i in any given level such that $P(\gamma_i) = J$.

In order to see that $\mathcal{H}$ is proper, it is enough to show that for each integer $n > 0$, only finitely many of the H_i have images that intersect K_n. If $P(\gamma_j) > |L(\gamma_j)|$ (so that H_j is defined) and $|L(\gamma_j)| > n$, then $im(H_j) \cap K_{P(\gamma_j)} = \emptyset$ and so $im(H_j) \cap K_n = \emptyset$. Hence we need only be concerned with γ_j with $L(\gamma_j) \in [-n, n]$. [11, Lem. 3.2] implies that for a given compact set, and level, there are only finitely many edges e in that level such the image of any conjugation homotopy for e intersects that compact set. The same statement is true if e is replaced by γ_j and so $\mathcal{H}$ is proper. □

4. Proof of Theorem 1.1

We now proceed with the proof of the main result. Recall the ascending HNN extension $H = \langle S, t; R, t^{-1}st = \phi(s) \text{ (for } s \in S)\rangle$ is 1-ended and semistable at infinity. For any group G and subset $A \subset G$, let $N_G(A)$ be the normal closure of A in G. Then $\overline{R} = N_{F(\{t\}\cup S)}(R \cup C)$ where

$$C = \{t^{-1}st = \phi(s) \text{ (for } s \in S)\}.$$

Let R_0 be a finite subset of $\overline{R}$ such that $\widehat{X} = \Gamma_H(S,t;R_0,C)$ is semistable at infinity. We show that R_0 may be taken as a finite subset of R, and any ascending HNN-extension $H_1 = \langle S,t;R_1,C\rangle$, with $R_0 \subseteq R_1 \subseteq \overline{R}$ is semistable at infinity. In fact we show that $\widehat{X}_1 = \Gamma_{H_1}(S,t;R_0,C)$ is semistable at infinity.

$$\begin{array}{c}
\widetilde{X}_0 = \Gamma_{H_0}(t,S;R_0,C) \\
\downarrow \\
\widehat{X}_1 = \Gamma_{H_1}(t,S;R_0,C) = \widetilde{X}_0/N_{H_0}(R_1) \\
\downarrow q \\
\widehat{X} = \Gamma_H(t,S;R_0,C) = \widetilde{X}_0/N_{H_0}(R) = \widehat{X}_1/N_{H_1}(R) \\
\downarrow \\
X_0
\end{array}$$

$$H_0 = \pi_1(X_0) = \langle t,S : R_0,C\rangle \;\; H_1 = \langle t,S : R_1,C\rangle, \;\; H = \langle t,S;R,C\rangle$$

Let X_0 be the standard finite complex for the presentation $\langle S,t;R_0,C\rangle \equiv H_0$ and let $\widetilde{X}_0$ be its universal cover. We have the above diagram where each map is a regular covering map.

Recall that $R_0 \subset R_1 \subset N_{F(\{t\}\cup S)}(R \cup C) = \overline{R}$. By$N_{H_0}(R_1)$ we mean the normal closure in H_0 of the *image* of R_1 in H_0 under the obvious map from $F(\{t\} \cup S)$. The group H_0 maps onto H_1 with kernel isomorphic to $N_{H_0}(R_1)$ and H_1 maps onto H with kernel $N_{H_1}(R)$.

Proof of Theorem 1.1. — Since H is semistable at infinity, there is a finite set $T \subset F(S \cup \{t\})$ of H-relations such that the corresponding locally finite 2-complex $\Gamma_H(t,S;T)$ is semistable at infinity (see Section 3). As $\Gamma_H(t,S;T')$ is also semistable at infinity for any other finite set of relations $T' \supset T$, we may always assume that T contains the set $C = \{t^{-1}st\phi(s)^{-1}, s \in S\}$ of all conjugation relations in the above presentation. Observe that each relator $\theta \in T$ can be expressed as a product ω_θ (in the letters of $\{t^{\pm 1}\} \cup S^{\pm 1}$) of conjugates of elements in $R \cup C$, and let R_0 be the finite set of relators in R which occur in ω_θ, for every $\theta \in T$. Let $\widehat{R} = C \cup R_0$.

The attaching map of each 2-cell in $\Gamma_H(t,S;T \cup \widehat{R})$ corresponding to any relation in T is null homotopic in $\Gamma_H(t,S;\widehat{R}) \subset \Gamma_H(t,S;T \cup \widehat{R})$, and we can perform all these homotopies simultaneously in an equivariant way (since H acts freely and properly discontinously). This implies there is

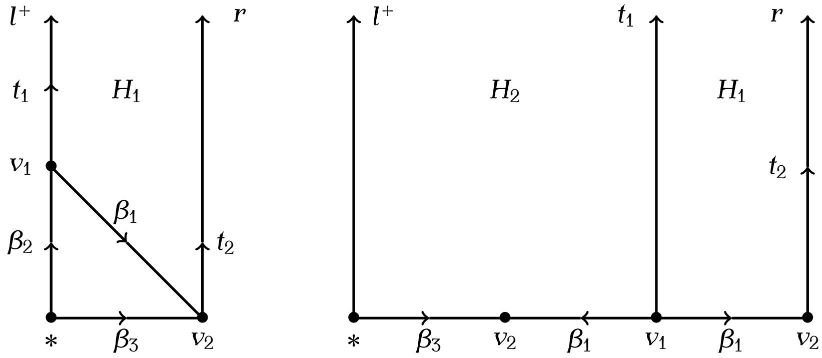

Figure 4.1. Combining the H_i

a proper (cellular) map $p : \Gamma_H(t,S;T\cup\widehat{R}) \to \Gamma_H(t,S;\widehat{R})$ that is the identity on the 1-skeletons. By Proposition 16.2.3 of [2], $\Gamma_H(t,S;T\cup\widehat{R})$ is semistable at infinity if and only if $\Gamma_H(t,S;\widehat{R})$ is semistable at infinity. Since $\Gamma_H(t,S;T)$ is semistable at infinity, $\Gamma_H(t,S;T\cup\widehat{R})$ and hence $\Gamma_H(t,S;\widehat{R}) = \Gamma_H(t,S;R_0,C)$ is semistable at infinity.

We need three lemmas. Lemma 4.3 is the Main Lemma of the paper. Our first lemma is an improvement to Lemma 2 of [8].

Lemma 4.1. — *Suppose G is a finitely generated group that is 1-ended and semistable at ∞. Let S be a finite generating set for G and R a finite set of S-relations such that $\Gamma_G(S;R)$ is semistable at ∞. Then two rays r and s in $\Gamma_G(S;R)$, with $r(0)=s(0)$ are properly homotopic rel$\{r(0)\}$.*

Proof. — Let $X=\Gamma_G(S,R)$ and let l be a geodesic edge path line in X through the identity $*$, parametrized so that $l(0)=*$. Let $l^+(n)=l(n)$ for $n\geqslant 0$ and $l^-(n)=l(n)$ for all $n\leqslant 0$. Since X is semistable at infinity, there is a compact ball such that any two rays r and s in $X-B_K(*)$ (with $r(0)=s(0)$) are properly homotopic relative to $r(0)$. For $v\in G(\equiv X^0)$, let vl^+ and vl^- be the translates of l^+ and l^- respectively to the vertex v. Given any vertex $v\in X-B_{2K+1}(*)$ only one of the geodesic rays vl^+ or vl^- can intersect $B_K(*)$. Hence either there are infinitely many vertices $v\in X$ such that vl^+ avoids B_K or infinitely many vertices $v\in X$ such that vl^- avoids B_K. Without loss, assume the former holds. It is enough to show that every proper ray r at $*$ is properly homotopic to l^+ relative to $*$.

Choose a path β_1 from a vertex v_1 of l^+ to a vertex v_2 of r such that the image of β_1 and the tail of r at v_2 avoid $B_K(*)$. By the definition of

K, t_1 (the tail of l^+ at v_1) is properly homotopic (relative to v_1) to β_1 followed by t_2 (the tail of r at v_2) by a proper homotopy H_1. Let β_2 be the initial segment of l^+ from $*$ to v_1 and let β_3 be the initial segment of r from $*$ to v_2. Let β be the loop $(\beta_3, \beta_1^{-1}, \beta_2^{-1})$. Choose a vertex (infinitely many exist) $w \in X$ such that the translate wl^+ of l^+ to w and the translate $w\beta$ both avoid $B_K(*)$. By the definition of K, wl^+ is properly homotopic to $(w\beta, wl^+)$ relative to w. Translating by w^{-1}, l^+ is properly homotopic to (β, l^+) relative to $*$. Equivalently, l^+ is properly homotopic to $(\beta_3, w\beta_2^{-1}, t_1)$ relative to $*$ by a proper homotopy H_2. Combining H_1 and H_2 as in Figure 4.1 (and canceling (β_1, β_1^{-1})) shows that l^+ and r are properly homotopic relative to $*$. □

Lemma 4.2. — *Suppose A is an integer and K is compact in $\widehat{X}_1$. Then for all but finitely many points v in level A of $\widehat{X}_1$, there is a proper $S^{\pm 1}$ ray s_v at v such that $q(s_v)$ is proper in $\widehat{X}$ and the image of the (proper) conjugation homotopy of $r_v = (t, t, \dots)$ to s_v does not intersect K.*

Proof. — Say H has base group B and H_1 has base group B_1. Then for each vertex v of $\widehat{X}_1$, the complex $\widehat{X}_1$ contains a copy Γ_v^1 of the Cayley graph of B_1 (with respect to S) which contains v (and $L(\Gamma_v^1) = L(v)$). Similarly there are copies Γ_v of the Cayley graph of B, for vertices $v \in \widehat{X}$. Let E_1 be the (finite) set of S edges $e = (a, b)$ in level A of $\widehat{X}_1$, such that the image of the conjugation homotopy of r_a to (e, r_b) intersects K. If v is a vertex of $\widehat{X}$, then for each vertex w of Γ_v there is a proper $S^{\pm 1}$ ray a_w in Γ_v such that a_w begins at w and only finitely many a_w have image that intersect any given compact subset of Γ_v - this is an elementary fact that generalizes to arbitrary connected, locally finite, infinite graphs (see [7, proof of Lem. 2]). Suppose $v \in \widehat{X}_1$ is a vertex of level A and Γ_v^1 contains no edge of E_1. If $w \in \Gamma_v^1$, then let s_w be the lift of the proper ray $a_{q(w)}$ (in $\Gamma_{q(v)}$) to w. Since Γ_v^1 contains no edge of E_1, the conjugation homotopy of s_w to r_w avoids K.

Hence it suffices to consider one of the finitely many Γ_v^1 in level A of $\widehat{X}_1$ that does contain an edge of E_1. Let V be the (finite) set of vertices $z \in \Gamma_{q(v)}$ such that $im(a_z)$ intersects $q(E_1)$. Choose an integer N such that for each vertex $w \in V$ there is an $S^{\pm 1}$ edge path p_w of length $\leqslant N$ from w to a vertex of $\Gamma_{q(v)}$ not in V. There are only finitely many vertices of Γ_v^1 within N of a vertex of E_1. Consider a vertex $w \in \Gamma_v^1$ that is not within N of a vertex of E_1. If $q(w) \in V$, consider the path $p_{q(w)}$ in $\Gamma_{q(z)}$ (of length $\leqslant N$) with end point $w' \notin V$ and otherwise, let $p_{q(w)}$ be the trivial path at $q(w)$. Let $\widehat{s}_w$ be the proper ray, $(p_{q(w)}, a_{w'})$. Let s_w be the lift of $\widehat{s}_w$ to w. No edge of s_w belongs to E_1 and so the conjugation homotopy of s_w to r_w avoids K. □

Lemma 4.3. — *If r is a proper ray in $\widehat{X}_1$ then there is a proper ray r' such that r is properly homotopic (relative to $r(0)$) to r' and $q(r')$ is proper in $\widehat{X}$.*

Proof. — If there is a vertex v of r such that all letters of r beyond v are in $S^{\pm 1}$ or all letters are labeled t then r is properly homotopic to r_v. If all letters beyond v are labeled t^{-1} then r is properly homotopic to this tail of r. Hence we assume that r satisfies the conclusion of Proposition 3.3 and has the form $(\gamma_1, \alpha_1, \gamma_2, \alpha_2, \ldots)$ where γ_i is a $S^{\pm 1}$ path and α_i is a t or t^{-1} path. It is convenient to modify notation. If $\alpha_i = (t^{-1}, t^{-1}, \ldots, t^{-1})$ then replace α_i in our representation of r by α_i^{-1}. In this way we have each α_i has the form $(t, t, \ldots, t)$. Let x_i be the initial and y_i the terminal vertex of α_i.

Let A be the set of all i such α_i contains no vertex of level 0. By the conditions of Proposition 3.3, there are only finitely many x_i (y_i) in any given level. Hence for any $N > 0$, there are only finitely many $i \in A$ such that the image of α_i (equivalently, the image of $q(\alpha_i)$) contains a point v with $L(v) \in [-N, N]$. In particular:

(1) *Given any compact set $C \subset \widehat{X}$ there are only finitely many $i \in A$ such that the image of $q(\alpha_i)$ intersects C.*

Recall that $K_1 \subset K_2 \subset \cdots$ is a filtration of $\widehat{X}_1$ by compact sets. Suppose $i \notin A$ is a positive integer and v_i is the vertex of α_i with smallest level in $[-1, 0]$ (so $L(v_i) = 0$ if $v_i = x_i$ and $L(x_i) = 0$ and $L(v_i) = -1$ otherwise). If the image of the conjugation homotopy of s_{v_i} (of Lemma 4.2) to r_{v_i} intersects K_1, then $i \in S_0$. The set S_0 is finite since r is proper and since there are only finitely many vertices w in levels -1 and 0 such that the image of the conjugation homotopy of s_w to r_w meets K_1.

Suppose $i \notin S_0 \cup A$ is a positive integer and v_i is the vertex of α_i with smallest level in $[-2, 0]$ (so either $v_i = x_i$ or $L(v_i) = -2$). If the image of the conjugation homotopy of s_{v_i} to r_{v_i} intersects K_2, then $i \in S_1$. The set S_1 is finite.

Inductively, assume S_{n-1} is defined. Suppose $i \notin A$ is a positive integer, $i \notin S_m$ for $m < n$, and v_i is the vertex of α_i with smallest level in $[-n-1, 0]$ (so either $v_i = x_i$ or $L(v_i) = -n-1$). If the image of the conjugation homotopy of s_{v_i} to r_{v_i} intersects K_{n+1}, then $i \in S_n$. The set S_n is finite.

Next we show:

(2) *The sets S_n for $n \geqslant 0$, partition $\mathbb{Z}^+ - A$ (into a collection of disjoint finite sets).*

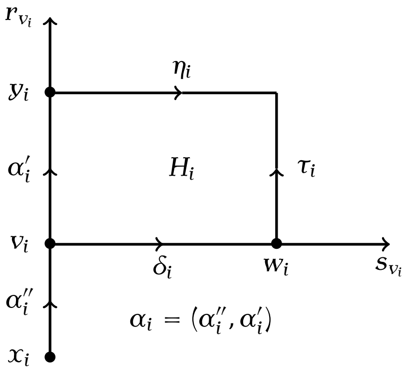

Figure 4.2. Component Homotopies

By definition, $S_n \cap A = \emptyset = S_n \cap S_m$ for all $n > m \geqslant 0$. Say $i \in \mathbb{Z}^+ - A$ and $i \notin \bigcup_{n=0}^{\infty} S_n$. Recall that x_i is the initial vertex of α_i. Since $i \notin A$, $L(x_i) \leqslant 0$. For $-n \in [L(x_i), 0]$, let v_n be the vertex of α_n in level $-n$ and for $-n < L(x_i)$, let $v_n = x_i$. Since $i \notin S_n$, the image of the conjugation homotopy of s_{v_n} to r_{v_n} does not intersect K_{n+1} for all n. But, this is impossible since for large n, K_n contains x_i and $v_n = x_i$ for all $-n \leqslant L(x_i)$. So (2) follows.

At this point we will alter each of the α_i for $i \notin A \cup S_0$ and produce the desired proper ray r'. Suppose $i \in S_n$ then $i \notin S_{n-1}$. So if v_i is the vertex of α_i with smallest level in $[-n, 0]$, then the image of the conjugation homotopy of s_{v_i} to r_{v_i} does not intersects K_n.

Recall that y_i is the terminal point of α_i and $q(s_{v_i})$ is a proper $S^{\pm 1}$ ray in $\widehat{X}$. Let δ_i be an initial segment of s_{v_i} with terminal point w_i, where w_i is such that the t-path at w_i with end point in $L(y_i)$ (call it τ_i) is such that $q(\tau_i)$ avoids $q(K(n))$. Let H_i be the conjugation homotopy of δ_i to the $S^{\pm 1}$ path η_i in level $L(y_i)$ (see Figure 4.2). Since the image of H_i is a subset of the image of the conjugation homotopy of s_{v_i} to r_{v_i} (in $\widehat{X}_1 - K_n$). We have:

(3) *If $i \in S_n$ then the image of H_i does not intersect K_n.*

Now we defined r'. For each $i \in \bigcup_{n=1}^{\infty} S_n$ replace the segment α'_i of α_i from v_i to y_i in r by $(\delta_i, \tau_i, \eta_i^{-1})$. Let H be the homotopy of r to r' obtained by combining all of the H_i. Statement (3) and the fact that there are only finitely many integers in each S_n immediately imply H is proper. We have:

(4) *The ray r is properly homotopic to r' in $\widehat{X}_1$.*

It remains to show that $q(r')$ is proper in $\widehat{X}$. First we show:

(5) *The number of maximal $S^{\pm 1}$ subpaths of r' in any level is finite.*

For $i \in \bigcup_{n=1}^{\infty} S_n$, the path η_i is in level $L(y_i)$, so the $S^{\pm 1}$ subpath (γ_i or γ_{i+1}) of r in this level is simply extended by η_i in r'. Hence we have not increased the number of maximal $S^{\pm 1}$ subpaths of r in level $L(y_i)$ when we add η_i to r'. If δ_j is defined and in level $-n$, then either $j \in S_n$ or $L(x_j) = -n$. There are only finitely many $j \in S_n$ and only finitely many x_j in level $-n$. Hence, (4) is verified. Statement (5) immediately implies:

(6) *The number of $S^{\pm 1}$ edges of r' in any given level is finite.*

We can now show that $q(r')$ is proper. Suppose otherwise. Then r' contains infinitely many edges all of which are mapped by q to the same edge e of $\widehat{X}$. If e is an $S^{\pm 1}$ edge, then since q is level preserving, each of the edges of r' that maps to e is in the same level as e. But by (6) r' has only finitely many $S^{\pm 1}$ edges in any given level.

This means that e has label t. By (1), there are only finitely many $i \in A$ such that the image of $q(\alpha_i)$ intersects any given compact set. This means that r' has infinitely many t edges that map to e, none of which belong to α_i for any $i \in A$. For $i \in \bigcup_{n=1}^{\infty} S_n$, and $v_i \neq x_i$, write α_i as (α_i'', α_i'), otherwise write $\alpha_i = \alpha_i'$. Either there are infinitely i such that τ_i contains an edge that q maps to e (see Figure 4.2) or there are infinitely many i such that α_i'' contains an edge that is mapped by q to e (and this edge precedes v_i in α_i). If $i \in S_n$ then by definition, $q(\tau_i)$ avoids $q(K_n)$ and so the former is not possible. Instead, there must be infinitely many α_i'' containing an edge that is mapped by q to e. If $\alpha_i' \neq \alpha_i$, then δ_i is a maximal $S^{\pm 1}$ subpath of r' in a level $\leqslant 0$. Recall, there are only finitely many maximal $S^{\pm 1}$ subpaths of r' in any level. Since α_i'' lies in levels at and below the level of δ_i (and since γ_i lies in a level < 0), only finitely many α_i'' contain an edge that can be mapped by q to e. We have shown that $q(r')$ is proper. □

We conclude the proof of Theorem 1.1. Suppose r is a proper ray based at $*$ in $\widehat{X}_1$, we will show that it is properly homotopic to the proper edge path ray $r_* \equiv (t, t, \dots)$ based at $*$, and this concludes the proof of Theorem 1.1. By Lemma 4.3, we may assume that the projection $q(r)$ is a proper edge path ray based at $q(*)$ in $\widehat{X}$. Observe that $q(r_*) \equiv (t, t, \dots)$ is also a proper edge path ray based at $q(*) \in \widehat{X}$. Since $\widehat{X}$ is semistable at infinity, Lemma 4.1 implies that $q(r_*)$ is properly homotopic to $q(r)$ relative to $q(*)$. Let $H : [0, \infty) \times [0, 1] \to \widehat{X}$ be a proper homotopy such that $H(t, 0) = q(r_*(t))$ and $H(t, 1) = q(r(t))$ for all $t \in [0, \infty)$ and $H(0, t) = q(*)$ for all $t \in [0, 1]$. Certainly $H|_{[0,\infty)\times\{0\}} (= q(r_*))$ lifts to r_*. The homotopy lifting theorem implies that H lifts to $\overline{H}$, a homotopy of r_* to r, relative to $*$.

Lifting proper homotopies always results in a proper homotopy so $\overline{H}$ is proper. □

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Manuscrit reçu le 27 septembre 2021,
révisé le 14 juillet 2022,
accepté le 23 septembre 2022.

Francisco F. LASHERAS
Departamento de Geometría y Topología
Universidad de Sevilla, Fac. Matemáticas
C/. Tarfia s/n, 41012 Sevilla (Spain)
lasheras@us.es

Michael MIHALIK
Department of Mathematics
Vanderbilt University
Nashville TN, 37240 (USA)
michael.l.mihalik@vanderbilt.edu