Page and Fu *Fixed Point Theory and Applications* (2020) 2020:14
https://doi.org/10.1186/s13663-020-00681-1

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

K-Correspondences, USCOs, and fixed point problems arising in discounted stochastic games

Frank H. Page[1,2] and Jing Fu[3*]

*Correspondence: j.fu@fit.ac.jp
[3]Department of System Management, Fukuoka Institute of Technology, 3-30-1 Wajiro-higashi, Higashi-ku, Fukuoka, 811-0295, Japan
Full list of author information is available at the end of the article

Abstract

We establish a fixed point theorem for the composition of nonconvex, measurable selection valued correspondences with Banach space valued selections. We show that if the underlying probability space of states is *nonatomic* and if the selection correspondences in the composition are K-correspondences (meaning correspondences having graphs that contain their Komlos limits), then the induced measurable selection valued composition correspondence takes *contractible values* and therefore has fixed points. As an application we use our fixed point result to show that all nonatomic uncountable-compact discounted stochastic games have stationary Markov perfect equilibria – thus resolving a long-standing open question in game theory.

Keywords: K-correspondences; USCOs with contractible values; Approximable USCOs; Fixed points of nonconvex measurable selection valued correspondences; Composition correspondences; Komlos limits; Weak*-compact sets of Banach-space-valued Bochner integrable selections; Stationary Markov perfect equilibria; Discounted stochastic games

1 Introduction

We make two contributions. First, we establish a fixed point theorem for the composition of measurable selection valued correspondences with Banach space valued selections.[a] We show that if the underlying probability space of states is *nonatomic* and if the selection correspondences in the composition are K-correspondences (meaning correspondences having graphs that contain their Komlos limits), then the induced nonconvex, measurable selection valued composition correspondence is *approximable* and therefore has fixed points.[b]

Second, we apply our fixed point result to show that all *nonatomic*, uncountable-compact discounted stochastic games ($\mathcal{DSG}s$) satisfying the assumptions of the Nowak–Raghavan $\mathcal{DSG}$ model have stationary Markov perfect equilibria (SMPE) – thus, resolving a long standing open question in game theory (see Nowak and Raghavan [1]).[c]

Regarding our first contribution, the key step allowing us to establish our fixed point result is to show that if the probability measure on the state space is nonatomic and if the selection correspondences in the composition are K-correspondences (or equivalently, are weak* upper semicontinuous correspondences taking nonempty compact values (USCOs), as we show here), then the composition correspondence takes contractible values in its set of selections where contractibility is with respect to the compatibly metrized weak* topologies.[d] It then follows from results in Gorniewicz, Granas, and Kryszewski [2] that the composition correspondence is approximable.

Regarding our second contribution, we first use our fixed point result to show, under mild conditions on primitives, that all nonatomic, parameterized state-contingent games ($\mathcal{PSG}s$) have Blackwell equilibria. An excellent example of a nonatomic $\mathcal{PSG}$ is the one-shot game underlying an uncountable-compact nonatomic discounted stochastic game ($\mathcal{DSG}$) satisfying the assumptions of Nowak and Raghavan [1]. By Blackwell's classical result [3] (extended to games), a $\mathcal{DSG}$ has a *stationary Markov perfect equilibrium* if and only if its underlying one-shot $\mathcal{PSG}$ has a Blackwell equilibrium. By our fixed point result any one-shot $\mathcal{PSG}$ belonging to a nonatomic Nowak–Raghavan $\mathcal{DSG}$ has a Blackwell equilibrium – therefore, by Blackwell [3], all nonatomic Nowak–Raghavan $\mathcal{DSG}$ have stationary Markov perfect equilibria.

We note that in a nonatomic $\mathcal{DSG}$, because the underlying Nash selection correspondence is contractibly valued, the key pathology underlying the recent counterexamples to existence due to Levy [4] and Levy and McLennan [5] is ruled out. In particular, because the Nash selection correspondence is contractibly valued, Nash equilibria homeomorphic to the unit circle cannot arise. Also, we note that in the Levy and McLennan [5] discounted stochastic game model, the state space is *not* nonatomic. However, as noted by Jaskiewicz and Nowak [6], it is easy to modify the Levy–McLennan model so as to allow for a nonatomic state space while preserving the nonapproximability of the Nash payoff selection correspondence – thereby extending the Levy–McLennan counterexamples to the nonatomic case (also see Jaskiewicz and Nowak [7]). Here we will assume that the state space is nonatomic, and under this assumption, together with our result showing that the Nash payoff selection correspondence belonging to any Nowak–Raghavan $\mathcal{DSG}s$ is a K-correspondence, we will establish that the Nash payoff selection correspondence belonging to any Nowak–Raghavan $\mathcal{DSG}s$ is approximable and therefore has fixed points. Finally, we note that if a $\mathcal{DSG}$ has a Nash payoff selection correspondence that is approximable, then it will have fixed points, and therefore, by Blackwell's theorem [3] the $\mathcal{DSG}$ will have stationary Markov perfect equilibria (e.g., see Page [8, 9]). By constructing a $\mathcal{DSG}$ having no stationary Markov perfect equilibria, and therefore, a $\mathcal{DSG}$ having a Nash payoff selection correspondence without fixed points, we can infer that the Nash payoff selection correspondence belonging to a Levy–McLennan $\mathcal{DSG}$ model is not approximable. Thus, while not all nonapproximable $\mathcal{DSG}s$ have SMPE, as shown by Levy [4] and Levy and McLennan [5], all approximable $\mathcal{DSG}s$ do, as shown by Page [8, 9]. Moreover, as we show here, all $\mathcal{DSG}s$ satisfying the Nowak–Raghavan assumptions are approximable, and as a consequence, all such $\mathcal{DSG}s$ escape the Levy–McLennan counterexample, and possess stationary Markov perfect equilibria in behavioral strategies.

2 Preliminaries

2.1 Spaces

Let (Ω, B_Ω, μ) be a probability space of states ω, where Ω is a complete, separable metric space with probability measure μ defined on the Borel σ-field B_Ω, and let Y be a norm-bounded, weak*-closed (i.e., w^*-closed), convex subset of F^*, the separable norm dual of a separable Banach space F. Equip Y with metric ρ_Y^* compatible with the w^*-topology on Y inherited from F^*.

We will often assume that the probability space (Ω, B_Ω, μ) is *nonatomic*. We say that $E \subset \Omega$ is an atom of Ω relative to $\mu(\cdot)$ if the following implication holds: if $\mu(E) > 0$, then $H \subset E$ implies that $\mu(H) = 0$ or $\mu(E-H) = 0$. If Ω contains no atoms relative to $\mu(\cdot)$, Ω is said to be atomless or *nonatomic*. Because Ω is a complete, separable metric space, $\mu(\cdot)$ is atomless (or nonatomic) if and only if $\mu(\{\omega\}) = 0$ for all $\omega \in \Omega$ (see Hildenbrand [10], pp. 44–45).

Next, let $\mathcal{L}_Y^\infty$ denote the set of μ-equivalence classes of F^*-valued, Bochner integrable functions, ν, with $\nu(\omega) \in Y$ a.e. $[\mu]$.[e] Denote by L_Y^∞ the set consisting of *all* F^*-valued, Bochner integrable functions, ν, with $\nu(\omega) \in Y$ a.e. $[\mu]$ (i.e., L_Y^∞ is the prequotient of $\mathcal{L}_Y^\infty$). We will equip $\mathcal{L}_Y^\infty$ with a metric $\rho^*_{\mathcal{L}_Y^\infty}$ compatible with the weak* topology inherited from $\mathcal{L}_{F^*}^\infty$. We note that the set $\mathcal{L}_Y^\infty$ is decomposable, meaning that if the functions $\nu^0(\cdot)$ and $\nu^1(\cdot)$ are in $\mathcal{L}_Y^\infty$, then for any $E \in B_\Omega$, with indicator function $I_E(\cdot)$,

$$\nu^0(\cdot)I_E(\cdot) + \nu^1(\cdot)I_{\Omega\setminus E}(\cdot) \in \mathcal{L}_Y^\infty. \tag{1}$$

Similarly, let X be a norm-bounded, weak*-closed (i.e., w^*-closed), convex subset of E^*, the separable norm dual of a separable Banach space E, and equip X with metric ρ_X^* compatible with the w^*-topology on X inherited from E^*.[f] Also, let $\mathcal{L}_X^\infty$ denote the set of μ-equivalence classes of E^*-valued, Bochner integrable functions $x_{(\cdot)}$, with $x_\omega \in X$ a.e. $[\mu]$. Denote by L_X^∞ the set consisting of *all* E^*-valued, Bochner integrable functions $x_{(\cdot)}$ with $x_\omega \in X$ a.e. $[\mu]$ (i.e., L_X^∞ is the prequotient of $\mathcal{L}_X^\infty$). We will equip $\mathcal{L}_X^\infty$ with a metric $\rho^*_{\mathcal{L}_X^\infty}$ compatible with the weak* topology inherited from $\mathcal{L}_{E^*}^\infty$. We note that the set $\mathcal{L}_X^\infty$ is decomposable.

Finally, equip the spaces Y and X with the Borel σ-fields B_Y^* and B_X^*, generated by the ρ_Y^*- and ρ_X^*-open sets in Y and X, respectively.

2.2 Convergences

We will begin by discussing W^*-convergence in $\mathcal{L}_{F^*}^\infty$ and w^*-K-convergence in its prequotient space, $L_{F^*}^\infty$. Then we will present a result clarifying how the two are related.

2.2.1 W^*-Convergence in $\mathcal{L}_{F^*}^\infty$

By Diestel and Uhl (p. 98) [11], because F^* is separable, $\mathcal{L}_{F^*}^\infty$ is the norm dual of $\mathcal{L}_F^1$. We have the following definitions:

Definition 1 (W^*-Convergence in $\mathcal{L}_{F^*}^\infty$ and W^*-limits) A sequence of F^*-valued functions $\{\nu^n(\cdot)\}_n$ in $\mathcal{L}_{F^*}^\infty$ converges weak*, or W^*-converges to a function $\nu^*(\cdot) \in \mathcal{L}_{F^*}^\infty$ if for every function $l(\cdot) \in \mathcal{L}_F^1$,

$$\langle \nu^n(\cdot), l(\cdot)\rangle := \int_\Omega \langle \nu^n(\omega), l(\omega)\rangle \, d\mu(\omega) \quad \longrightarrow \quad \int_\Omega \langle \nu^*(\omega), l(\omega)\rangle \, d\mu(\omega) := \langle \nu^*(\cdot), l(\cdot)\rangle. \tag{2}$$

In $\mathcal{L}_Y^\infty$ a sequence $\{\nu^n(\cdot)\}_n$ W^*-converges to $\nu^*(\cdot)$ if and only $\rho^*_{\mathcal{L}_Y^\infty}(\nu^n, \nu^*) \longrightarrow 0$.

2.2.2 w^*-K-convergence in $L^\infty_{F^*}$

For any sequence $\{v^n\}_n \subset L^\infty_{F^*}$ let

$$\{\widehat{v}^{n_k}\}_k := \left\{\frac{1}{k}\sum_{r=1}^{k} v^{n_r}\right\}_k \tag{3}$$

be the corresponding sequence of arithmetic mean functions. Because F^* is convex for any subsequence$\{v^{n_k}\}_k \subset L^\infty_{F^*}$, $\widehat{v}^{n_k} \in L^\infty_{F^*}$.

Definition 2 (w^*-K-Sequences, w^*-K-Convergence, and w^*-K-Limits) We say that a sequence $\{v^n(\cdot)\}_n \subset L^\infty_{F^*}$ w^*-K-converges to a w^*-K-limit function $\widehat{v}(\cdot) \in L^\infty_{F^*}$ provided every subsequence $\{v^{n_k}(\cdot)\}_k$ of $\{v^n(\cdot)\}_n$ has an arithmetic mean sequence $\{\widehat{v}^{n_k}(\cdot)\}_k$ that w^*-converges pointwise a.e. $[\mu]$ to $\widehat{v}(\cdot)$, that is,

$$\widehat{v}^{n_k}(\omega) \underset{w^*}{\longrightarrow} \widehat{v}(\omega) \quad \text{a.e. } [\mu]. \tag{4}$$

By Theorem 3.2 in Balder [12], if the sequence, $\{v^n(\cdot)\}_n \subset L^\infty_{F^*}$, is such that

$$\sup_n \int_\Omega \|v^n(\omega)\|_\infty \, d\mu(\omega) < +\infty,$$

then there exists a subsequence $\{v^{n_k}(\cdot)\}_k$ of $\{v^n(\cdot)\}_n$ and a function $\widehat{v}(\cdot) \in L^\infty_{F^*}$ such that any subsequence$\{v^{n_{k_r}}(\cdot)\}_r$ of $\{v^{n_k}(\cdot)\}_k$ w^*-K-converges to w^*-K-limit, $\widehat{v}(\cdot)$.

2.2.3 W^*-Convergence and w^*-K-convergence

Our first result is about the relationship between W^*-convergence and w^*-K-convergence in Σ^∞_Y.

Theorem 1 (W^*-Convergence and w^*-K-Convergence) *Let $\{v^n(\cdot)\}_n$ be any sequence in $\mathcal{L}^\infty_Y$. Then the following statements are true*:

(1) *If $\{v^n(\cdot)\}_n$ w^*-K-converges to $\widehat{v}(\cdot) \in \mathcal{L}^\infty_Y$, then $\{v^n(\cdot)\}_n \subset \mathcal{L}^\infty_Y$ W^*-converges to $\widehat{v}(\cdot) \in \mathcal{L}^\infty_Y$.*

(2) *If $\{v^n(\cdot)\}_n$ W^*-converges to $v^*(\cdot) \in \mathcal{L}^\infty_{F^*}$, then each subsequence $\{v^{n_k}(\cdot)\}_k$ has a further subsequence $\{v^{n_{k_q}}(\cdot)\}_q$, which is w^*-K-convergent to some w^*-K-limit $\widehat{v}(\cdot) \in \mathcal{L}^\infty_Y$ such that $\widehat{v}(\omega) = v^*(\omega)$ a.e. $[\mu]$.*

Proof We will prove part (2) first. Assume that $\{v^n(\cdot)\}_n \subset \mathcal{L}^\infty_Y$ W^*-converges to $v^*(\cdot) \in \mathcal{L}^\infty_{F^*}$, and that the subsequence, $\{v^{n_k}(\cdot)\}_k$, w^*-K-converges to $\widehat{v}(\cdot) \in \Sigma^\infty_Y$. For each $l \in \mathcal{L}^1_F$, we have

$$\frac{1}{k}\sum_{r=1}^{k}\langle v^{n_r}(\omega), l(\omega)\rangle \longrightarrow \langle \widehat{v}(\omega), l(\omega)\rangle \quad \text{a.e. } [\mu]$$

and by the dominated convergence theorem we have for each $l \in \mathcal{L}^1_F$,

$$\begin{aligned}\lim_{k\longrightarrow\infty}\int_\Omega \frac{1}{k}\sum_{r=1}^{k}\langle v^{n_r}(\omega), l(\omega)\rangle \, d\mu(\omega) &= \lim_{k\longrightarrow\infty}\frac{1}{k}\sum_{r=1}^{k}\int_\Omega \langle v^{n_r}(\omega), l(\omega)\rangle \, d\mu(\omega) \\ &= \int_\Omega \langle \widehat{v}(\omega), l(\omega)\rangle \, d\mu(\omega).\end{aligned}$$

Because $\{\nu^{n_k}(\cdot)\}_k$ W^*-converges to $\nu^*(\cdot)$, we have for each $l \in \mathcal{L}^1_F$,

$$\lim_{k\longrightarrow\infty} \int_\Omega \bigl\langle \nu^{n_k}(\omega), l(\omega)\bigr\rangle \, d\mu(\omega) = \int_\Omega \bigl\langle \nu^*(\omega), l(\omega)\bigr\rangle \, d\mu(\omega),$$

implying that for each $l \in \mathcal{L}^1_F$,

$$\lim_{k\longrightarrow\infty} \frac{1}{k}\sum_{r=1}^{k} \int_\Omega \bigl\langle \nu^{n_r}(\omega), l(\omega)\bigr\rangle \, d\mu(\omega) = \int_\Omega \bigl\langle \nu^*(\omega), l(\omega)\bigr\rangle \, d\mu(\omega).$$

Thus, for each $l \in \mathcal{L}^1_F$,

$$\int_\Omega \bigl\langle \widehat{\nu}(\omega), l(\omega)\bigr\rangle \, d\mu(\omega) = \int_\Omega \bigl\langle \nu^*(\omega), l(\omega)\bigr\rangle \, d\mu(\omega),$$

implying that $\widehat{\nu}(\omega) = \nu^*(\omega)$ a.e. $[\mu]$.

Now we will prove part (1). By part (2), we can conclude that $\mathcal{L}^\infty_Y$ is convex and W^*-compact. Assume that $\{\nu^n(\cdot)\}_n \subset \mathcal{L}^\infty_Y$ w^*-K-converges to some $\widehat{\nu}(\cdot) \in \mathcal{L}^\infty_Y$. In order to show that $\{\nu^n(\cdot)\}_n$ W^*-converges to $\nu^*(\cdot)$ where $\nu^*(\omega) = \widehat{\nu}(\omega)$ a.e. $[\mu]$, it suffices to show that the μ-equivalence class in $\mathcal{L}^\infty_Y$ containing $\widehat{\nu}(\cdot)$ is the only limit point of the sequence of μ-equivalence classes, $\{\nu^n(\cdot)\}_n \subset \mathcal{L}^\infty_Y$. Let $\nu^\infty(\cdot)$ be any W^*-limit point of the sequence $\{\nu^n(\cdot)\}_n$ and let $\{\nu^{n_k}(\cdot)\}_k$ be a subsequence W^*-converging to $\nu^\infty(\cdot)$. By w^*-K-convergence we know that this subsequence also w^*-K-converges to $\widehat{\nu}(\cdot)$, and hence by part (2) we know that $\nu^\infty(\omega) = \widehat{\nu}(\omega)$ a.e. $[\mu]$. □

2.3 Contractible continua

Let (X, ρ_X), (Y, ρ_Y), and (Z, ρ_Z) be metric spaces. Consider a function, $f : Z \longrightarrow X$, from Z *onto* X. If f is *continuous* and *one-to-one*, and if its *inverse* f^{-1} *is also continuous*, then we say that f is a homeomorphism and that the metric spaces Z and X are homeomorphic. If (Z, ρ_Z) is compact, then any continuous, one-to-one mapping f from Z onto X is a homeomorphism. A continuous function $f : Z \longrightarrow X$ is an *embedding* if $f : Z \longrightarrow f(Z)$ is a homeomorphism.

Given metric space (Z, ρ_Z), a set $E \subseteq Z$ is *connected* if E cannot be written as the union of two disjoint open sets (or two disjoint closed sets). A set $E \subseteq Z$ is *locally connected* at $e \in E$ if each neighborhood U_e of e contains a connected neighborhood V_e of e. Also E is locally connected if it is locally connected at each $e \in E$.[g] We note that in any metric space (Z, ρ_Z), the condition of being (i) a locally connected continuum and (ii) the continuous image of an interval are equivalent (this is the Mazurkiewicz–Moore theorem – see Kuratowski [13]).

If the metric space (Z, ρ_Z) is compact and connected, it is called a *continuum*. A nonempty, closed, connected subset of Z is called a *subcontinuum*. We will denote by $C(Z)$ the collection of all subcontinua of Z. If in addition, the continuum (Z, ρ_Z) is locally connected, it is called a *Peano continuum*. Finally, (i) if the Peano continuum (Z, ρ_Z) is unicoherent, meaning that for all subcontinua A and B of Z such that $Z = A \cup B$, the intersection $A \cap B$ is connected, and (ii) if all subcontinua of Z are unicoherent (i.e., if property (i) is hereditary), then (Z, ρ_Z) is a *dendrite*. Thus, (Z, ρ_Z) is a dendrite if it is a compact metric space that is connected, locally connected, and hereditarily unicoherent (see Charatonik and Charatonik [14] for more details).[h]

A retraction $r(\cdot)$ is a continuous function from a space Z into Z such that $r(\cdot)$ is the identity on its range (i.e., $r(r(z)) = r(z)$ for all $z \in Z$). A subset W of Z is said to be a retract of Z provided there is a retraction of Z onto W.

The notion of a homotopy will be important in what follows. A homotopy is a function that essentially provides us with a way to index a set of continuous functions. We have the following formal definition:

Definition 3 (Homotopies) Let (Z, ρ_Z) and (X, ρ_X) be compact metric spaces and let $\mathcal{C}(Z \times [0,1], X)$ denote the collection of all continuous functions, $h : Z \times [0,1] \longrightarrow X$, defined on $Z \times [0,1]$ taking values in X. A function $h \in \mathcal{C}(Z \times [0,1], X)$ is called a homotopy, and each homotopy h specifies a continuously indexed set of continuous functions,

$$\mathcal{C}_h(Z, X) := \{h(\cdot, t) \in \mathcal{C}(Z, X) : t \in [0,1]\}.$$

The indexed collection, $\mathcal{C}_h(Z, X)$, can be thought of as an arc, α_h, in the continuum of continuous functions, $\mathcal{C}(Z, X)$, equipped with the sup metric. The continuous functions f and g in $\mathcal{C}(Z, X)$ are homotopically related, or homotopic, if f and g are the endpoints of an arc α_h whose arc type is identified by some function, $h \in \mathcal{C}(Z \times [0,1], X)$. In particular, if $f, g \in \mathcal{C}(Z, X)$ are homotopic, then there is an arc of type $h \in \mathcal{C}(Z \times [0,1], X)$ running from continuous function $f(\cdot) = h(\cdot, 0)$ to continuous function $g(\cdot) = h(\cdot, 1)$. We denote this h-arc from f to g by writing $g \in [f]_h$ or by writing $f \stackrel{h}{\longrightarrow} g$ (and if the orientation is in the opposite direction, then we write $f \in [g]_h$ or $g \stackrel{h}{\longrightarrow} f$). Constant functions form a special class of homotopy arc end points. Let $g_{\overline{x}} \in \mathcal{C}(Z, X)$ denote the constant function (i.e., $g_{\overline{x}}(z) = \overline{x}$ for all $z \in Z$). If f and $g_{\overline{x}}$ are homotopic (i.e., if $g_{\overline{x}} \in [f]_h$, that is, if $f \stackrel{h}{\longrightarrow} g_{\overline{x}}$ for some $\overline{x} \in X$), then f is said to be *inessential*. Moreover, if for some pair of compact metric spaces (Z, ρ_Z) and (X, ρ_X), all pairs of functions $f, g \in \mathcal{C}(Z, X)$ are homotopic, then, in particular, $f, g_{\overline{x}} \in \mathcal{C}(Z, X)$ are homotopic for some h-arc and some $\overline{x} \in X$ – and this means that for this pair of compact metric spaces, (Z, ρ_Z) and (X, ρ_X), all functions $f \in \mathcal{C}(Z, X)$ are inessential (i.e., for each $f \in \mathcal{C}(Z, X)$, there is $(h(\cdot,\cdot), x) \in (\mathcal{C}(Z \times [0,1], X), X)$, $f \stackrel{h}{\longrightarrow} g_x$).

If $Z \subseteq X$, then Z is *contractible* in X if for some homotopy $h \in \mathcal{C}(Z \times [0,1], X)$, there is an h-arc running from the identity (or inclusion) mapping $f_{id} \in \mathcal{C}(Z, X)$ to a constant mapping $g_{\overline{x}} \in \mathcal{C}(Z, X)$, for some $\overline{x} \in X$. Thus, $f_{id}(\cdot) = h(\cdot, 0)$ where $f_{id}(z) = z$ for all $z \in Z$ is the inclusion mapping (i.e., $f_{id}(z) = z = h(z, 0)$ for all $z \in Z$) and $h(\cdot, 1)$ is the constant mapping (i.e., $h(z, 1) = \overline{x}$ for all $z \in Z$ for some $\overline{x} \in X$).

We say that X is contractible if X is contractible in X. Note that if X is contractible, then for any $Z \subseteq X$, Z is contractible in X. Two useful facts related to the contractibility of continua are the following:

(1) If X is contractible and $Z \subseteq X$ is a retraction of X, then Z is also contractible. Thus if $r : X \longrightarrow Z$, $r \in \mathcal{C}(X, Z)$ where $r(z) = z$ for all $z \in Z$, then Z is also contractible.

(2) If X is contractible, then X is unicoherent (see Corollary A.12.10 in van Mill [15]) – implying that all pairs of functions, $f, g \in \mathcal{C}(X, S^1)$, are homotopic for the unit circle, $S^1 := \{x = (x_1, x_2) : (x_1)^2 + (x_2)^2 = 1\}$. Thus, if X is contractible, then all continuous functions $f : X \longrightarrow S^1$ are inessential, and *we can conclude that X contains no simple closed curves.*

A compact metric space K is called an *absolute retract (AR)* provided that whenever K is embedded in a metric space Y, the embedded copy of K is a retract of Y. A compact metric space is called an *absolute extensor (AE)* provided whenever B is a closed subset of a metric space M and $f : B \longrightarrow K$ is continuous, then f can be extended to a continuous function $F : M \longrightarrow K$ (F being an extension of f means that $F|_B = f$). By Borsuk's theorem, a compact metric space K is an AR if and only if K is an AE (see Borsuk [16], see also, 9.1 in Illanes and Nadler [17]).

A compact metric space K is called an *absolute neighborhood retract (ANR)* provided that whenever K is embedded in a metric space Y the embedded copy K' of K is a retract of some neighborhood of K' in Y. A compact metric space K is called an *absolute neighborhood extensor (ANE)* provided that whenever B is a closed subset of a metric space M and $f : B \longrightarrow K$ is continuous, then there is a neighborhood U of B in M such that f can be extended to a continuous function $F : U \longrightarrow K$. In parallel to Borsuk's theorem, a compact metric space K is an ANR if and only if K is an ANE (see Borsuk [16], see also, 19.5 in Illanes and Nadler [17]).

2.4 Correspondences

2.4.1 USCOs and approximability

For compact metric spaces (X, ρ_X) and (Y, ρ_Y), let $\mathcal{U}_{\rho_Y\text{-}\rho_X}$ denote the collection of all upper semicontinuous correspondences $\Gamma(\cdot)$ defined on Y taking nonempty, ρ_X-closed (and hence ρ_X-compact) values in X. Equip the product space $Y \times X$ with the sum metric, $\rho_{Y\times X} := \rho_Y + \rho_X$. Following the literature, all such mappings are USCOs (e.g., see Hola and Holy [18]). We say that the USCO $\Gamma(\cdot) \in \mathcal{U}_{\rho_Y\text{-}\rho_X}$ is *ρ_Y–ρ_X-approximable if for each n, there is a ρ_Y–ρ_X-continuous function, $g^n(\cdot) : Y \longrightarrow X$, such that for each $(y^n, x^n) \in Grg^n \subset Y \times X$ (i.e., for each $(y^n, x^n) \in Y \times X$, with $x^n = g^n(y^n) \in X$), there exists $(\bar{y}^n, \bar{x}^n) \in Gr\Gamma(\cdot)$ such that*

$$\rho_Y\bigl(y^n, \bar{y}^n\bigr) + \rho_X\bigl(x^n, \bar{x}^n\bigr) < \frac{1}{n}.$$

Bringing together the relevant parts of (5.6) and (5.12) from Gorniewicz, Granas, and Kryszewski [2], we have the following result concerning the approximability of USCOs:

Theorem 2 (The Approximability of USCOs) *Let (X, ρ_X) and (Y, ρ_Y) be compact metric ANR spaces and let $\Gamma(\cdot) \in \mathcal{U}_{\rho_Y\text{-}\rho_X}$. If for each $y \in Y$, $\Gamma(y)$ is contractible, then $\Gamma(\cdot)$ is ρ_Y–ρ_X-approximable.*

For compact metric spaces $(\mathcal{L}_Y^\infty, \rho^*_{\mathcal{L}_Y^\infty})$ and $(\mathcal{L}_X^\infty, \rho^*_{\mathcal{L}_X^\infty})$, let $\mathcal{U}^*_{\mathcal{L}_Y^\infty\text{-}\mathcal{L}_X^\infty}$ denote the collection of all upper semicontinuous correspondences defined on $\mathcal{L}_Y^\infty$ taking nonempty, $\rho^*_{\mathcal{L}_X^\infty}$-closed (and hence $\rho^*_{\mathcal{L}_X^\infty}$-compact) values in $\mathcal{L}_X^\infty$. We will denote this special set of USCOs – USCOs with respect to weak* topologies – by $\mathcal{U}^*_{\mathcal{L}_Y^\infty\text{-}\mathcal{L}_X^\infty}$ or by USCO*. Also, we have the collection of all USCO*s with convex values (CUSCO*).

*2.4.2 K-correspondences and USCO*s*

The formal definition of a K-correspondence is the following:

Definition 4 (K-Limit Property and K-Correspondences) We say that a correspondence $\mathcal{E}(\cdot) : \mathcal{L}_Y^\infty \longrightarrow P(\mathcal{L}_X^\infty)$ has the K-limit property, or is a K-correspondence, if for any sequence

$$\bigl\{\bigl(v^n, x^n_{(\cdot)}\bigr)\bigr\}_n \subset Gr\mathcal{E}(\cdot) := \bigl\{(v, x_{(\cdot)}) \in \mathcal{L}_Y^\infty \times \mathcal{L}_X^\infty : x_{(\cdot)} \in \mathcal{E}(v)\bigr\}$$

with $v^n \underset{\rho^*_Y\text{-}K}{\longrightarrow} \widehat{v} \in \mathcal{L}^\infty_Y$ and $x^n_{(\cdot)} \underset{\rho^*_X\text{-}K}{\longrightarrow} \widehat{x}_{(\cdot)} \in \mathcal{L}^\infty_X$, we have $(\widehat{v},\widehat{x}_{(\cdot)}) \in Gr\mathcal{E}(\cdot)$. We will denote the collection of all K-correspondences defined on $\mathcal{L}^\infty_Y$with nonempty values in $\mathcal{L}^\infty_X$ by $\mathcal{UK}_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_X}$.

As our next result makes clear, because of the near equivalence of w^*-K-convergence and W^*-convergence in $\mathcal{L}^\infty_Y$ and $\mathcal{L}^\infty_X$, K-correspondences and USCO*s are equivalent. We note that each $\mathcal{E}(\cdot) \in \mathcal{U}^*_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_X}$ takes decomposable values in $\mathcal{L}^\infty_X$.

Lemma 1 (Equivalence of $\mathcal{UK}_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_X}$ and $\mathcal{U}^*_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_X}$) *Let $\mathcal{E}(\cdot) : \mathcal{L}^\infty_Y \longrightarrow P(\mathcal{L}^\infty_X)$ be a correspondence. Then $\mathcal{E}(\cdot)$ is a K-correspondence if and only if $\mathcal{E}(\cdot)$ is an USCO*.*

Proof Given the compactness of $\mathcal{L}^\infty_Y$ and $\mathcal{L}^\infty_X$, $\mathcal{E}(\cdot)$ is an USCO* if and only if $Gr\mathcal{E}(\cdot)$ is $\rho^*_{\mathcal{L}^\infty_Y\times\mathcal{L}^\infty_X}$-closed in $\mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X$ ($\rho^*_{\mathcal{L}^\infty_Y\times\mathcal{L}^\infty_X} := \rho^*_{\mathcal{L}^\infty_Y} + \rho^*_{\mathcal{L}^\infty_X}$). By Theorem 1 above, if $\mathcal{E}(\cdot)$ is a K-correspondence, then $Gr\mathcal{E}(\cdot)$ is $\rho^*_{\mathcal{L}^\infty_Y\times\mathcal{L}^\infty_X}$-closed in $\mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X$. Moreover, if $Gr\mathcal{E}(\cdot)$ is $\rho^*_{\mathcal{L}^\infty_Y\times\mathcal{L}^\infty_X}$-closed in $\mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X$ and $\{(v^n, x^n_{(\cdot)})\}_n \subset Gr\mathcal{E}(\cdot)$ is such that $v^n \underset{\rho^*_Y\text{-}K}{\longrightarrow} \widehat{v} \in \mathcal{L}^\infty_Y$ and $x^n_{(\cdot)} \underset{\rho^*_X\text{-}K}{\longrightarrow} \widehat{x}_{(\cdot)} \in \mathcal{L}^\infty_X$, then we have $v^n \underset{\rho^*_{\mathcal{L}^\infty_Y}}{\longrightarrow} \widehat{v}$ and $x^n_{(\cdot)} \underset{\rho^*_{\mathcal{L}^\infty_X}}{\longrightarrow} \widehat{x}_{(\cdot)}$ and because $Gr\mathcal{E}(\cdot)$ is $\rho^*_{\mathcal{L}^\infty_Y\times\mathcal{L}^\infty_X}$-closed in $\mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X$, $(\widehat{v},\widehat{x}_{(\cdot)}) \in Gr\mathcal{E}(\cdot)$, implying that $\mathcal{E}(\cdot)$ is a K-correspondence. □

3 Main results

3.1 Contractibility and approximability

3.1.1 Contractibility

Our key result is the following:

Theorem 3 (The Contractibility Theorem) *Let $\mathcal{E}(\cdot) \in \mathcal{UK}_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_X}$. If $(\Omega, B_\Omega, \mu)$is nonatomic, then $\mathcal{E}(v)$ is contractible in $\mathcal{L}^\infty_X$ for each $v \in \mathcal{L}^\infty_Y$.*

Proof By Lemma 1 above, $\mathcal{E}(\cdot) \in \mathcal{U}^*_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_X}$. Given that μ is nonatomic, we know by Fryszkowski [19] that Lyapunov's theorem [20] on the range of a vector measure guarantees the existence of a family of measurable sets $\{A_t\}_{t\in[0,1]}$ such that

$$\left.\begin{array}{l} t' \leq t \Rightarrow A_{t'} \subseteq A_t, \qquad A_0 = \varnothing \quad \text{and} \quad A_1 = \Omega, \quad \text{and} \\ \mu(A_t) = t\mu(\Omega) = t. \end{array}\right\} \tag{5}$$

Using the properties of the system $\{A_t\}_{t\in[0,1]}$, and the decomposability of $\mathcal{E}(v)$ for each $v \in \mathcal{L}^\infty_Y$, we will show that for each v the function $h_v(\cdot,\cdot)$ given by

$$h_v(x_{(\cdot)}, t) := x^1_{(\cdot)} I_{A_t}(\cdot) + x_{(\cdot)} I_{\Omega\setminus A_t}(\cdot) \tag{6}$$

for all $(x_{(\cdot)}, t) \in \mathcal{E}(v) \times [0, 1]$ is a homotopy (and in particular, a contraction of the selections in $\mathcal{E}(v)$ to $x^1_{(\cdot)}$). Here $v \in \mathcal{L}^\infty_Y$ is fixed, $I_A(\cdot)$ is the indicator function of set A, and $x^1_{(\cdot)}$ is any fixed selection in $\mathcal{E}(v)$.

It suffices to show that $h_v(\cdot,\cdot)$ is $\rho^*_{\mathcal{L}^\infty_X\times[0,1]} - \rho^*_{\mathcal{L}^\infty_X}$-continuous. Let $\{(x^n_{(\cdot)}, t^n)\}_n$ be a sequence such that

$$x^n_{(\cdot)} \underset{\rho^*_{\mathcal{L}^\infty_X}}{\longrightarrow} x^*_{(\cdot)} \quad \text{and} \quad t^n \underset{|\cdot|}{\longrightarrow} t^*.$$

We must show that

$$h_\nu\big(x^n_{(\cdot)}, t^n\big) \underset{\rho^*_{\mathcal{L}^\infty_X}}{\longrightarrow} h_\nu\big(x^*_{(\cdot)}, t^*\big) \in \mathcal{E}(\nu). \tag{7}$$

In particular, we must show that for each $l(\cdot) \in \mathcal{L}^1_E$,

$$\left| \int_\Omega \big\langle \big(h_\nu\big(x^n_\omega, t^n\big) - h_\nu\big(x^*_\omega, t^*\big)\big), l(\omega)\big\rangle \, d\mu(\omega) \right| \longrightarrow 0. \tag{8}$$

For each $l(\cdot) \in \mathcal{L}^1_E$, we have

$$\left.\begin{aligned} &|\textstyle\int_\Omega (\langle h_\nu(x^n_\omega, t^n), l(\omega)\rangle - \langle h_\nu(x^*_\omega, t^*), l(\omega)\rangle)\, d\mu(\omega)| \\ &\quad \leq \underbrace{|\textstyle\int_\Omega \langle (x^1_\omega I_{A_{t^n}}(\omega) - x^1_\omega I_{A_{t^*}}(\omega)), l(\omega)\rangle \, d\mu(\omega)|}_{(a)} \\ &\qquad + \underbrace{|\textstyle\int_\Omega \langle (x^n_\omega I_{\Omega\setminus A_{t^n}}(\omega) - x^n_\omega I_{\Omega\setminus A_{t^*}}(\omega)), l(\omega)\rangle \, d\mu(\omega)|}_{(b)} \\ &\qquad + \underbrace{|\textstyle\int_\Omega \langle (x^n_\omega I_{\Omega\setminus A_{t^*}}(\omega) - x^*_\omega I_{\Omega\setminus A_{t^*}}(\omega)), l(\omega)\rangle \, d\mu(\omega)|}_{(c)}. \end{aligned}\right\} \tag{9}$$

For expressions (9(a)) and (9(b)) above we, have that

$$\left.\begin{aligned} &\underbrace{|\textstyle\int_\Omega \langle (x^1_\omega I_{A_{t^n}}(\omega) - x^1_\omega I_{A_{t^*}}(\omega)), l(\omega)\rangle \, d\mu(\omega)|}_{(a)} \\ &\quad \leq |\textstyle\int_{A_{t^n} \Delta A_{t^*}} \langle x^1_\omega, l(\omega)\rangle \, d\mu(\omega)| \\ &\quad \leq M_X \|l(\cdot)\|_1 \mu(A_{t^n} \Delta A_{t^*}), \end{aligned}\right\} \tag{10}$$

and

$$\left.\begin{aligned} &\underbrace{|\textstyle\int_\Omega \langle (x^n_\omega I_{\Omega\setminus A_{t^n}}(\omega) - x^n_\omega I_{\Omega\setminus A_{t^*}}(\omega)), l(\omega)\rangle \, d\mu(\omega)|}_{(b)} \\ &\quad \leq |\textstyle\int_{A_{t^n} \Delta A_{t^*}} \langle x^n_\omega, l(\omega)\rangle \, d\mu(\omega)| \\ &\quad \leq M_X \|l(\cdot)\|_1 \mu(A_{t^n} \Delta A_{t^*}), \end{aligned}\right\} \tag{11}$$

For expression (9(c)), because $l(\cdot)I_{\Omega\setminus A_{t^*}}(\cdot)$ is also contained in $\mathcal{L}^1_E$ and $x^n_{(\cdot)} \underset{\rho^*_{\mathcal{L}^\infty_X}}{\longrightarrow} x^*_{(\cdot)}$, we have that

$$\left.\begin{aligned} &\underbrace{|\textstyle\int_\Omega \langle (x^n_\omega I_{\Omega\setminus A_{t^*}}(\omega) - x^*_\omega I_{\Omega\setminus A_{t^*}}(\omega)), l(\omega)\rangle \, d\mu(\omega)|}_{(c)} \\ &\quad = |\textstyle\int_\Omega \langle (x^n_\omega - x^*_\omega), l(\omega) I_{\Omega\setminus A_{t^*}}(\omega)\rangle \, d\mu(\omega)| \longrightarrow 0. \end{aligned}\right\} \tag{12}$$

Because $\mu(A_{t^n} \Delta A_{t^*}) \longrightarrow 0$, we have that $M_X \|l(\cdot)\|_1 [\frac{1}{r} \sum_{q=1}^r \mu(A_{t^{n_{k_q}}} \Delta A_{t^*})] \longrightarrow 0$, implying that in expression (9) $(a) \longrightarrow 0$ and $(b) \longrightarrow 0$, and we have already from expression (12) that $(c) \longrightarrow 0$. Thus, we have that for each $l(\cdot) \in \mathcal{L}^1_E$,

$$\left| \int_\Omega \big\langle \big(h_\nu\big(x^n_\omega, t^n\big) - h_\nu\big(x^*_\omega, t^*\big)\big), l(\omega)\big\rangle \, d\mu(\omega) \right| \longrightarrow 0, \tag{13}$$

implying that $h_\nu(x^n_{(\cdot)}, t^n) \underset{\rho^*_{\mathcal{L}^\infty_X}}{\longrightarrow} h_\nu(x^*_{(\cdot)}, t^*)$. Because $\mathcal{E}(\cdot)$ is an USCO*, $\{h_\nu(x^n_{(\cdot)}, t^n)\}_n \subset \mathcal{E}(\nu)\rho^*_{\mathcal{L}^\infty_X}$-converging to $h_\nu(x^*_{(\cdot)}, t^*)$ implies that $h_\nu(x^*_{(\cdot)}, t^*) \in \mathcal{E}(\nu)$.

Thus, given the properties of the Lyapunov system (5) for each ν, the function, $h_\nu(\cdot,\cdot)$, given in (6) is $\rho^*_{\mathcal{L}^\infty_X\times[0,1]}-\rho^*_{\mathcal{L}^\infty_X}$-continuous, implying that for each $\nu \in \mathcal{L}^\infty_Y$, $h_\nu(\cdot,\cdot)$ is a homotopy for the set of measurable selections, $\mathcal{E}(\nu)$, and therefore, for each ν, $\mathcal{E}(\nu)$ is contractible. □

Our proof that $\mathcal{E}(\nu)$ is contractible is inspired by the contractibility result given by Mariconda [21] showing that if the underlying probability space is nonatomic, then any decomposable subset of E-valued, Bochner integrable functions in $\mathcal{L}^1_E$ is contractible (where E is a Banach space). In Mariconda's result, the space of functions is equipped with the norm in $\mathcal{L}^1_E$, while here our space of functions (with each function taking values in $X \subset E^*$) is equipped with the W^* topology – a topology metrized by $\rho^*_{\mathcal{L}^\infty_X}$.

3.1.2 Approximability

The importance of (Ω, B_Ω, μ) being nonatomic and $\mathcal{E}(\cdot)$ being a K-correspondence is that together they guarantee that $\mathcal{E}(\cdot)$ is an USCO* with contractible values (Theorem 3 above), and this in turn guarantees the $\rho^*_{\mathcal{L}^\infty_Y}-\rho^*_{\mathcal{L}^\infty_X}$-approximability of $\mathcal{E}(\cdot)$, as our next result shows.

Theorem 4 (The Approximability Theorem) *Let $\mathcal{E}(\cdot) \in \mathcal{UK}_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_X}$. If (Ω, B_Ω, μ) is nonatomic, then $\mathcal{E}(\cdot)$ is $\rho^*_{\mathcal{L}^\infty_Y}-\rho^*_{\mathcal{L}^\infty_X}$-approximable.*

Proof By Lemma 1 and Theorem 3 above, $\mathcal{E}(\cdot)$ is an USCO* taking contractible values. By Theorem 2 above (see 5.6 and 5.12 in Gorniewicz, Granas, and Kryszewski [2]), because $\mathcal{E}(\cdot)$ is an USCO* defined on the ANR space of value functions $\mathcal{L}^\infty_Y$ taking nonempty, compact, and *contractible* values in the ANR space, $\mathcal{L}^\infty_X$, $\mathcal{E}(\cdot)$ is $\rho^*_{\mathcal{L}^\infty_Y}-\rho^*_{\mathcal{L}^\infty_X}$-approximable. □

3.2 Fixed points for *K*-correspondences and their compositions

3.2.1 K-Correspondences

Consider the correspondence $\mathcal{U}(\cdot)$ from $\mathcal{L}^\infty_Y$ with nonempty set values in $\mathcal{L}^\infty_Y$.

Theorem 5 (Fixed Point Theorem for K-Correspondences) *Let $\mathcal{U}(\cdot) \in \mathcal{UK}_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_Y}$. If (Ω, B_Ω, μ) is nonatomic, then $\mathcal{U}(\cdot)$ has fixed points (i.e., there exists $\nu^* \in \mathcal{L}^\infty_Y$ such that $\nu^* \in \mathcal{U}(\nu^*)$).*

Proof By Theorem 4 above, because (Ω, B_Ω, μ) is nonatomic and $\mathcal{U}(\cdot)$ is a K-correspondence, $\mathcal{U}(\cdot)$ is $\rho^*_{\mathcal{L}^\infty_Y}-\rho^*_{\mathcal{L}^\infty_Y}$-approximable. Therefore, we have for each n, a $\rho^*_{\mathcal{L}^\infty_Y}-\rho^*_{\mathcal{L}^\infty_Y}$-continuous function, $g^n(\cdot): \mathcal{L}^\infty_Y \longrightarrow \mathcal{L}^\infty_Y$, such that for each $(\nu^n, u^n) \in Grg^n \subset \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_Y$ (i.e., for each $(\nu^n, u^n) \in \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_Y$, with $u^n = g^n(\nu^n) \in \mathcal{L}^\infty_Y$) there exists $(\overline{\nu}^n, \overline{u}^n) \in Gr\mathcal{S}^\infty(\Gamma_{(\cdot)})$ such that

$$\rho^*_{\mathcal{L}^\infty_Y}\left(\nu^n, \overline{\nu}^n\right) + \rho^*_{\mathcal{L}^\infty_Y}\left(u^n, \overline{u}^n\right) < \frac{1}{n^2}. \tag{14}$$

Because each of the functions, g^n, is $\rho^*_{\mathcal{L}^\infty_Y}-\rho^*_{\mathcal{L}^\infty_Y}$-continuous and defined on the $\rho^*_{\mathcal{L}^\infty_Y}$-compact and convex subset, $\mathcal{L}^\infty_Y$, in $\mathcal{L}^\infty_{F^*}$, taking values in $\mathcal{L}^\infty_Y$, it follows from the Brouwer–Schauder–Tychonoff fixed point theorem (see Aliprantis and Border [22], 17.56), that each g^n has a fixed point, $v^n \in \mathcal{L}_Y^\infty$ (i.e., for each n there exists some $v^n \in \mathcal{L}_Y^\infty$ such that $v^n = g^n(v^n)$). Let $\{v^n\}_n$ be a fixed point sequence corresponding to the sequence of $\rho^*_{\mathcal{L}_Y^\infty}$–$\rho^*_{\mathcal{L}_Y^\infty}$-continuous approximating functions, $\{g^n(\cdot)\}_n$. Expression (14) can now be rewritten as follows: for each v^n in the fixed point sequence, there is a corresponding pair, $(\overline{v}^n, \overline{u}^n) \in Gr\mathcal{U}(\cdot)$, such that

$$\rho^*_{\mathcal{L}_Y^\infty}\left(v^n, \overline{v}^n\right) + \rho^*_{\mathcal{L}_Y^\infty}\left(g^n\left(v^n\right), \overline{u}^n\right) < \frac{1}{n^2},$$

and therefore such that

$$\underbrace{\rho^*_{\mathcal{L}_Y^\infty}\left(v^n, \overline{v}^n\right)}_{A} + \underbrace{\rho^*_{\mathcal{L}_Y^\infty}\left(v^n, \overline{u}^n\right)}_{B} < \frac{1}{n^2}. \tag{15}$$

By the $\rho^*_{\mathcal{L}_Y^\infty}$-compactness of $\mathcal{L}_Y^\infty$, we can assume WLOG that the fixed point sequence, $\{v^n\}_n \subset \mathcal{L}_Y^\infty$, $\rho^*_{\mathcal{L}_Y^\infty}$-converges to a limit $v^* \in \mathcal{L}_Y^\infty$. Thus, by part A of (15), as $n \longrightarrow \infty$ we have

$$v^n \underset{\rho^*_{\mathcal{L}_Y^\infty}}{\longrightarrow} v^* \quad \text{and} \quad \overline{v}^n \underset{\rho^*_{\mathcal{L}_Y^\infty}}{\longrightarrow} v^*,$$

and therefore by part B of (15), as $n \longrightarrow \infty$ we have $\overline{u}^n \underset{\rho^*_{\mathcal{L}_Y^\infty}}{\longrightarrow} v^*$. Because $\mathcal{U}(\cdot)$ has a $\rho^*_{\mathcal{L}_Y^\infty \times \mathcal{L}_Y^\infty}$-closed graph in $\mathcal{L}_Y^\infty \times \mathcal{L}_Y^\infty$, $\{(\overline{v}^n, \overline{u}^n)\}_n \subset Gr\mathcal{U}(\cdot)$, and $\overline{v}^n \underset{\rho^*_{\mathcal{L}_Y^\infty}}{\longrightarrow} v^*$ and $\overline{u}^n \underset{\rho^*_{\mathcal{L}_Y^\infty}}{\longrightarrow} v^*$ imply that $(v^*, v^*) \in Gr\mathcal{U}(\cdot)$. Therefore, $v^* \in \mathcal{U}(v^*)$. □

3.2.2 Composition correspondences

Finally, let us consider the correspondences,

$$\left.\begin{array}{l}\mathcal{E}(\cdot) : \mathcal{L}_Y^\infty \longrightarrow P(\mathcal{L}_X^\infty), \\ \mathcal{T}(\cdot,\cdot) : \mathcal{L}_Y^\infty \times \mathcal{L}_X^\infty \longrightarrow P(\mathcal{L}_Y^\infty),\end{array}\right\} \tag{16}$$

and the composition correspondence,

$$v \longrightarrow \mathcal{U}(v) := \left\{\mathcal{T}(v, x_{(\cdot)}) : x_{(\cdot)} \in \mathcal{E}(v)\right\}. \tag{17}$$

Here, the notation $P(\mathcal{L}_X^\infty)$ denotes the collections of all nonempty subsets of $\mathcal{L}_X^\infty$ (and similarly for $P(\mathcal{L}_Y^\infty)$). We will use the notation $P_{*f}(\mathcal{L}_X^\infty)$ to denote the hyperspace of all nonempty, W^*-closed (and hence, W^*-compact) subsets of $\mathcal{L}_X^\infty$ (and similarly for $P_{*f}(\mathcal{L}_Y^\infty)$).

Let

$$\mathcal{T}(\cdot,\cdot) : \mathcal{L}_Y^\infty \times \mathcal{L}_X^\infty \longrightarrow P\left(\mathcal{L}_Y^\infty\right)$$

be a $\rho^*_{\mathcal{L}_Y^\infty \times \mathcal{L}_X^\infty}$–$\rho^*_{\mathcal{L}_Y^\infty}$-upper semicontinuous correspondence. Thus, if $\{(v^n, x^n_{(\cdot)}, u^n)\}_n$ is a sequence in $Gr\mathcal{T}(\cdot,\cdot)$ such that $v^n \underset{\rho^*_{\mathcal{L}_Y^\infty}}{\longrightarrow} v^*$, $x^n_{(\cdot)} \underset{\rho^*_{\mathcal{L}_X^\infty}}{\longrightarrow} x^*_{(\cdot)}$, and $u^n \underset{\rho^*_{\mathcal{L}_Y^\infty}}{\longrightarrow} u^*$, then $(v^*, x^*_{(\cdot)}, u^*) \in$

$Gr\mathcal{T}(\cdot,\cdot)$. Let $\mathcal{U}^*_{\mathcal{L}^\infty_Y\times\mathcal{L}^\infty_X\text{-}\mathcal{L}^\infty_Y}$ denote the collection of all such upper semicontinuous correspondences. We note that, by Lemma 1 above, $\mathcal{T}(\cdot,\cdot)$ is a K-correspondence. Next, let $\mathcal{E}(\cdot)\in\mathcal{UK}_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_X}$ and consider the induced composition correspondence,

$$v\longrightarrow\mathcal{U}(v):=\mathcal{T}\big(v,\mathcal{E}(v)\big). \tag{18}$$

We can now state our main fixed point result.

Theorem 6 (Fixed Point Theorem for Compositions of K-Correspondences) *Let* $\mathcal{E}(\cdot)\in\mathcal{UK}_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_X}$ *and* $\mathcal{T}(\cdot,\cdot)\in\mathcal{UK}_{\mathcal{L}^\infty_Y\times\mathcal{L}^\infty_X\text{-}\mathcal{L}^\infty_Y}$. *Consider the composition correspondence*

$$\mathcal{T}\big(\cdot,\mathcal{E}(\cdot)\big):\mathcal{L}^\infty_Y\longrightarrow P\big(\mathcal{L}^\infty_Y\big).$$

The following statements are true:

(1) *The correspondence* $v\longrightarrow\mathcal{U}(v):=\mathcal{T}(v,\mathcal{E}(v))$ *is a* K-*correspondence.*

(2) *If* (Ω,B_Ω,μ) *is nonatomic, then* $\mathcal{U}(\cdot)$ *has fixed points* (*i.e., there exists* $v^*\in\mathcal{L}^\infty_Y$ *such that* $v^*\in\mathcal{T}(v^*,\mathcal{E}(v^*))$).

Proof (1) Because $\mathcal{E}(\cdot)\in\mathcal{UK}_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_X}$ and $\mathcal{T}(\cdot,\cdot)\in\mathcal{UK}_{\mathcal{L}^\infty_Y\times\mathcal{L}^\infty_X\text{-}\mathcal{L}^\infty_Y}$, by Lemma 1, $\mathcal{E}(\cdot)\in\mathcal{U}^*_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_X}$, and $\mathcal{T}(\cdot,\cdot)\in\mathcal{U}^*_{\mathcal{L}^\infty_Y\times\mathcal{L}^\infty_X\text{-}\mathcal{L}^\infty_Y}$. Thus, the composition $\mathcal{U}(\cdot):=\mathcal{T}(\cdot,\mathcal{E}(\cdot))$ is contained in $\mathcal{U}^*_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_Y}$, and, by Lemma 1,

$$\mathcal{U}(\cdot):=\mathcal{T}\big(\cdot,\mathcal{E}(\cdot)\big)\in\mathcal{UK}_{\mathcal{L}^\infty_Y\text{-}\mathcal{L}^\infty_Y}.$$

(2) Because (Ω,B_Ω,μ) is nonatomic and $\mathcal{U}(\cdot)$ is a K-correspondence, we have by Theorem 5 above that $\mathcal{U}(\cdot)$ has fixed points. Therefore, there exists $v^*\in\mathcal{L}^\infty_Y$ such that

$$v^*\in\mathcal{U}\big(v^*\big):=\mathcal{T}\big(v^*,\mathcal{E}\big(v^*\big)\big). \qquad \square$$

4 Applications

4.1 Parameterized, state-contingent games ($\mathcal{PSG}$s)

4.1.1 Primitives, assumptions, and Blackwell equilibria of $\mathcal{PSG}$s

We must first expand the number of spaces to take into account that now we have m players indexed by $d=1,2,\dots,m$. For each player d, let Y_d be a closed bounded interval $[-M,M]$, $M>0$, the same for all players and equip Y_d with the absolute value metric – a metric we will continue to denote by $\rho^*_{Y_d}$.

Let $\mathcal{L}^\infty_R$ denote the Banach space of μ-equivalence classes of real-valued, essentially bounded measurable functions equipped with the weak* topology (i.e., the w^*-topology), and let $\mathcal{L}^\infty_{Y_d}$ be the nonempty, convex, w^*-compact, and metrizable subset of $\mathcal{L}^\infty_R$ consisting of functions v_d with $v_d(\omega)\in Y_d$ a.e. $[\mu]$. Denote by $L^\infty_{Y_d}$ the set consisting of *all* such functions v_d with $v_d(\omega)\in Y_d$ a.e. $[\mu]$ (i.e., $L^\infty_{Y_d}$ is the prequotient of $\mathcal{L}^\infty_{Y_d}$). We will equip $\mathcal{L}^\infty_{Y_d}$ with a metric, $\rho^*_{\mathcal{L}^\infty_{Y_d}}$, compatible with the w^*-topology in $\mathcal{L}^\infty_{Y_d}$ inherited from $\mathcal{L}^\infty_R$.

Next let X_d be a norm-bounded, weak*-closed (i.e., w^*-closed), convex subset of E^*_d, the separable norm dual of a separable Banach space E_d, and equip X_d with metric $\rho^*_{X_d}$ compatible with the w^*-topology on X_d inherited from E^*_d. Also, let $\mathcal{L}^\infty_{E^*_d}$ denote the Banach space of μ-equivalence classes of E^*_d-valued, Bochner integrable functions equipped with the weak* topology, and let $\mathcal{L}^{\infty}_{X_d}$ be the nonempty, convex, weak*-compact, and metrizable subset of μ-equivalence classes of E^*_d-valued, Bochner integrable functions $x_{d(\cdot)}$ with $x_{d\omega} \in X_d$ a.e. $[\mu]$. Denote by $L^{\infty}_{X_d}$ the set consisting of *all* E^*_d-valued, Bochner integrable functions $x_{d(\cdot)}$ with $x_{d\omega} \in X_d$ a.e. $[\mu]$ (i.e., $L^{\infty}_{X_d}$ is the prequotient of $\mathcal{L}^{\infty}_{X_d}$). We will equip $\mathcal{L}^{\infty}_{X_d}$ with a metric, $\rho^*_{\mathcal{L}^{\infty}_{X_d}}$, compatible with the weak* topology in $\mathcal{L}^{\infty}_{X_d}$ inherited from $\mathcal{L}^{\infty}_{E^*_d}$.

Equip the product spaces $Y := Y_1 \times \cdots \times Y_m$ and $X := X_1 \times \cdots \times X_m$ with the sum metrics, denoted by $\rho^*_Y := \sum_d \rho^*_{Y_d}$ and $\rho^*_X := \sum_d \rho^*_{X_d}$, respectively. Also equip the product spaces Y and X with the Borel product σ-fields, $B^*_Y = B^*_{Y_1} \times \cdots \times B^*_{Y_m}$ and $B^*_X = B^*_{X_1} \times \cdots \times B^*_{X_m}$, generated by the ρ^*_Y- and ρ^*_X-open sets in Y and X, respectively.

Finally, let $\mathcal{L}^{\infty}_{F^*} := \mathcal{L}^{\infty}_{F^*_1} \times \cdots \times \mathcal{L}^{\infty}_{F^*_m}$ and $\mathcal{L}^{\infty}_{Y} := \mathcal{L}^{\infty}_{Y_1} \times \cdots \times \mathcal{L}^{\infty}_{Y_m}$ with $Y_d \subset F^*_d$ for all d and let $L^{\infty}_{Y_d}$ and $L^{\infty}_{F^*_d}$ denote their prequotients. Equip $\mathcal{L}^{\infty}_{Y}$ with the sum metric $\rho^*_{\mathcal{L}^{\infty}_Y} := \sum_d \rho^*_{\mathcal{L}^{\infty}_{Y_d}}$, and note that on the prequotient L^{∞}_Y, $\rho^*_{\mathcal{L}^{\infty}_Y}$ is a semimetric. Similarly, let $\mathcal{L}^{\infty}_{E^*} := \mathcal{L}^{\infty}_{E^*_1} \times \cdots \times \mathcal{L}^{\infty}_{E^*_m}$ and $\mathcal{L}^{\infty}_{X} := \mathcal{L}^{\infty}_{X_1} \times \cdots \times \mathcal{L}^{\infty}_{X_m}$ with $X_d \subset E^*_d$ for all d and let $L^{\infty}_{X_d}$ and $L^{\infty}_{E^*_d}$ denote their prequotients. Equip $\mathcal{L}^{\infty}_{X}$ with the sum metric, $\rho^*_{\mathcal{L}^{\infty}_X} := \sum_d \rho^*_{\mathcal{L}^{\infty}_{X_d}}$, and again note that on the prequotient L^{∞}_X, $\rho^*_{\mathcal{L}^{\infty}_X}$ is a semimetric.

Label the assumptions above [A-1].

A parameterized, state-contingent game (denoted by $\mathcal{PSG}$) consists of a ν-parameterized collection of strategic form games, $\mathcal{G}_{\mathcal{L}^{\infty}_Y} := \{\mathcal{G}_\nu : \nu \in \mathcal{L}^{\infty}_Y\}$, with each ν-game $\mathcal{G}_\nu$ being specified by primitives

$$\mathcal{G}_\nu := \big(\mathcal{L}^{\infty}_{X_d}, \mathcal{S}^{\infty}\big(\Phi_d(\cdot, \nu_d)\big), U_d(\nu_d, \cdot, x_{-d(\cdot)})\big)_d. \tag{19}$$

Letting (Ω, B_Ω, μ) be the underlying probability space of states ω, where Ω is a complete, separable metric space, B_Ω is the Borel σ-field, and μ is a *nonatomic* probability measure, in the ν-game $\mathcal{G}_\nu$, each player $d \in D$, seeks to choose a feasible state-contingent action (i.e., feasible strategy)

$$x_{d(\cdot)} \in \mathcal{S}^{\infty}\big(\Phi_d(\cdot, \nu_d)\big) \subset \mathcal{L}^{\infty}_{X_d}, \tag{20}$$

so as to maximize player d''s expected payoff

$$U_d(\nu_d, x_{d(\cdot)}, x_{-d(\cdot)}) := \int_\Omega u_d(\omega, \nu_d, x_{d\omega}, x_{-d\omega})\, d\mu(\omega), \tag{21}$$

given player d's value function $\nu_d \in \mathcal{L}^{\infty}_{Y_d}$, and the state-contingent actions $x_{-d(\cdot)} \in \mathcal{L}^{\infty}_{X_{-d}}$ of the other players.[i] Also $\mathcal{S}^{\infty}(\Phi_d(\cdot, \nu_d))$ denotes the collection of μ-equivalence classes of measurable selections of the state-contingent constraint correspondence, $\Phi_d(\cdot, \nu_d)$. A typical element of a μ-equivalence class of $\mathcal{S}^{\infty}(\Phi_d(\cdot, \nu_d))$ is a function $x_{d(\cdot)} \in \mathcal{L}^{\infty}_{X_d}$ such that $x_{d\omega} \in \Phi_d(\omega, \nu_d)$ a.e. $[\mu]$. Here, $Y := Y_1 \times \cdots \times Y_m \subset R^m$ is the set of all possible player payoff profiles, and for each player $d = 1, 2, \ldots, m$, we have for each possible state-contingent action profile $(x_{d(\cdot)}, x_{-d(\cdot)}) \in \mathcal{L}^{\infty}_{X_d} \times \mathcal{L}^{\infty}_{X_{-d}}$ that

$$U_d(\nu_d, x_{d(\cdot)}, x_{-d(\cdot)}) := \int_\Omega u_d(\omega, \nu_d, x_{d\omega}, x_{-d\omega})\, d\mu(\omega) \in Y_d = [-M, M], \quad M > 0.$$

We will assume the following concerning a parameterized game $\mathcal{G}_{\mathcal{L}^{\infty}_Y} := \{\mathcal{G}_\nu : \nu \in \mathcal{L}^{\infty}_Y\}$. Label these assumptions [$\mathcal{PSG}$-1].

(1) $(\omega, v_d) \longrightarrow \Phi_d(\omega, v_d)$ *is player d's Caratheodory action constraint correspondence, defined on* $\Omega \times \mathcal{L}^\infty_{Y_d}$ *taking nonempty, convex, and* $\rho^*_{X_d}$*-compact values in* X_d, $\rho^*_{\mathcal{L}^\infty_{Y_d}} - \rho^*_{X_d}$*-continuous in* v_d *for each* $\omega \in \Omega$ *and measurable in* ω *for each* $v_d \in \mathcal{L}^\infty_{Y_d}$.

(2) $v_d \longrightarrow \mathcal{S}^\infty(\Phi_d(\cdot, v_d))$ *is player d's* $\rho^*_{\mathcal{L}^\infty_{Y_d}} - \rho^*_{\mathcal{L}^\infty_{X_d}}$*-continuous strategy constraint correspondence, defined on* $\mathcal{L}^\infty_{Y_d}$ *taking nonempty, convex, and* $\rho^*_{\mathcal{L}^\infty_{X_d}}$*-compact values in* $\mathcal{L}^\infty_{X_d}$, *with values consisting of* μ*-equivalence classes of strategies* $x_{d(\cdot)} \in \mathcal{L}^\infty_{X_d}$ *such that* $x_{d\omega} \in \Phi_d(\omega, v_d)$ *a.e.* $[\mu]$.

(3) $x_d \longrightarrow u_d(\omega, v_d, x_d, x_{-d})$ *is player d's Caratheodory payoff function, defined on* $\Omega \times \mathcal{L}^\infty_{Y_d} \times X$ *taking values in* Y_d *(measurable in* ω *and weak* continuous in* (v_d, x) *on* $\mathcal{L}^\infty_{Y_d} \times X$*), with* $u_d(\omega, v_d, \cdot, x_{-d})$ *quasiconcave in* x_d *given state* ω*, value function* $v_d \in \mathcal{L}^\infty_{Y_d}$*, and other players' actions* $x_{-d} \in X_{-d}$*, where for each* $(v_d, x_{(\cdot)}) \in \mathcal{L}^\infty_{Y_d} \times \mathcal{L}^\infty_X$ *player d's induced state-contingent payoff function,* $u_d(\cdot, v_d, x_{(\cdot)})$*, is contained in* $\mathcal{L}^\infty_{Y_d}$*, and is* $\rho^*_{\mathcal{L}^\infty_{Y_d} \times \mathcal{L}^\infty_X} - \rho^*_{\mathcal{L}^\infty_{Y_d}}$*-continuous (i.e.,* $v^n_d \underset{\rho^*_{\mathcal{L}^\infty_{Y_d}}}{\longrightarrow} v^*_d$ *and* $x^n_{(\cdot)} \underset{\rho^*_{\mathcal{L}^\infty_X}}{\longrightarrow} x^*_{(\cdot)}$*, then* $u(\cdot, v^n, x^n_{(\cdot)}) \underset{\rho^*_{\mathcal{L}^\infty_Y}}{\longrightarrow} u(\cdot, v^*, x^*_{(\cdot)})$*).*

An excellent example of a $\mathcal{PSG}$ satisfying assumptions [$\mathcal{PSG}$-1] as well as assumptions [A-1] is the one-shot game underlying any nonatomic discounted stochastic game satisfying the Nowak–Raghavan assumptions (see Nowak and Raghavan [1]). We will provide just such an example below.

A Blackwell equilibrium of a parameterized game $\mathcal{G}_{\mathcal{L}^\infty_Y}$ is defined as follows:

Definition 5 (Blackwell Equilibrium – Selection Form) Let $\mathcal{G}_{\mathcal{L}^\infty_Y}$ be a $\mathcal{PSG}$ satisfying assumptions [A-1] and [$\mathcal{PSG}$-1]. A value function-strategy profile $(v^*, x^*_{(\cdot)}) \in \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X$ is a Blackwell equilibrium of $\mathcal{G}_{\mathcal{L}^\infty_Y}$ provided for each player d,

$$\left.\begin{array}{l} v^*_d = u_d(\cdot, v^*_d, x^*_{d(\cdot)}, x^*_{-d(\cdot)}) \\ \text{and} \\ U_d(v^*_d, x^*_{d(\cdot)}, x^*_{-d(\cdot)}) = \max_{x_{d(\cdot)} \in \mathcal{S}^\infty(\Phi_d(\cdot, v^*_d))} U_d(v^*_d, x_{d(\cdot)}, x^*_{-d(\cdot)}). \end{array}\right\} \tag{22}$$

By Theorem 2.2 in Hiai and Umegaki [23], we have that

$$\left.\begin{array}{l} \max_{x_{d(\cdot)} \in \mathcal{S}^\infty(\Phi_d(\cdot, v^*_d))} U_d(v^*_d, x_{d(\cdot)}, x^*_{-d(\cdot)}) \\ \quad := \max_{x_{d(\cdot)} \in \mathcal{S}^\infty(\Phi_d(\cdot, v^*_d))} \int_\Omega u_d(\omega, v^*_d, x_{d\omega}, x^*_{-d\omega})\, d\mu(\omega) \\ \quad = \int_\Omega \max_{x_d \in \Phi_d(\omega, v^*_d)} u_d(\omega, v^*_d, x_d, x^*_{-d\omega})\, d\mu(\omega). \end{array}\right\} \tag{23}$$

Thus, in state-contingent form, an equivalent way of writing down the definition of a Blackwell equilibrium of a parameterized game $\mathcal{G}_{\mathcal{L}^\infty_Y}$ is the following:

Definition 6 (Blackwell Equilibrium – State-Contingent Form) Let $\mathcal{G}_{\mathcal{L}^\infty_Y}$ be a $\mathcal{PSG}$ satisfying assumptions [A-1] and [$\mathcal{PSG}$-1]. A value function-strategy profile $(v^*, x^*_{(\cdot)}) \in \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X$ is a Blackwell equilibrium provided for each player d and a.e. $[\mu]$ in ω,

$$\left.\begin{array}{l} v^*_d(\omega) = u_d(\omega, v^*_d, x^*_{d\omega}, x^*_{-d\omega}) \\ \text{and} \\ u_d(\omega, v^*_d, x^*_{d\omega}, x^*_{-d\omega}) = \max_{x_d \in \Phi_d(\omega, v^*_d)} u_d(\omega, v^*_d, x_d, x^*_{-d\omega}). \end{array}\right\} \tag{24}$$

The first condition in expression (24) above requires that each player's value function $v_d^*(\cdot)$ satisfies state by state a.e. $[\mu]$ the Bellman equation

$$v_d^*(\omega) = u_d\big(\omega, v_d^*, x_\omega^*\big), \tag{25}$$

given players' Nash equilibrium strategy profile

$$x_{(\cdot)}^* := \big(x_{1(\cdot)}^*, \ldots, x_{m(\cdot)}^*\big) \in \mathcal{S}^\infty\big(\Phi_1(\cdot, v_1)\big) \times \cdots \times \mathcal{S}^\infty\big(\Phi_m(\cdot, v_m)\big) \subset \mathcal{L}_X^\infty.$$

The second condition in expression (24) above requires that each player's strategy $x_{d(\cdot)}^* \in \mathcal{S}^\infty(\Phi_d(\cdot, v_d)) \subset \mathcal{L}_{X_d}^\infty$ satisfies state by state a.e. $[\mu]$ the Nash condition

$$u_d\big(\omega, v_d^*, x_{d\omega}^*, x_{-d\omega}^*\big) = \max_{x_d \in \Phi_d(\omega, v_d^*)} u_d\big(\omega, v_d^*, x_d, x_{-d\omega}^*\big), \tag{26}$$

given player $d's$ value function $v_d \in \mathcal{L}_{Y_d}^\infty$, and the strategies of the other players $x_{-d(\cdot)}^* \in \mathcal{S}^\infty(\Phi_{-d}(\cdot, v_{-d})) \subset \mathcal{L}_{X_{-d}}^\infty$.

Under assumptions $[A\text{-}1]$ and $[\mathcal{PSG}\text{-}1]$, we know that each (ω, v)-game,

$$\mathcal{G}_{(\omega,v)} := \big(X_d, \Phi_d(\omega, v_d), u_d(\omega, v_d, \cdot, x_{-d})\big)_d$$

has a nonempty, ρ_X^*-compact set of Nash equilibria, denoted by $\mathcal{N}(\omega, v)$. Moreover, by the Berge maximum theorem and the measurable maximum theorem (see 17.31 and 18.19 in Aliprantis and Border [22]), we know that the collection of (ω, v)-games $\mathcal{G}_{\Omega \times \mathcal{L}_Y^\infty} := \{\mathcal{G}_{(\omega,v)} : (\omega, v) \in \Omega \times \mathcal{L}_Y^\infty\}$ has a *Nash correspondence* $(\omega, v) \longrightarrow \mathcal{N}(\omega, v)$ and an associated *Nash payoff correspondence* $(\omega, v) \longrightarrow \mathcal{P}(\omega, v)$ that are upper Caratheodory (i.e., jointly measurable in (ω, v) and upper semicontinuous in v with nonempty compact values). Here, for each $(\omega, v) \in \Omega \times \mathcal{L}_Y^\infty$, we have that

$$\begin{aligned} \mathcal{P}(\omega, v) &:= u\big(\omega, v, \mathcal{N}(\omega, v)\big) \\ &:= \big\{u \in Y : u = \big(u_1(\omega, v_1, x), \ldots, u_m(\omega, v_m, x)\big) : x \in \mathcal{N}(\omega, v)\big\}. \end{aligned} \tag{27}$$

Let $\mathcal{G}_{\Omega \times \mathcal{L}_Y^\infty}$ denote the collection of (ω, v)-games underlying the parameterized game $\mathcal{G}_{\mathcal{L}_Y^\infty}$. We have that $(v^*, x_{(\cdot)}^*) \in \mathcal{L}_Y^\infty \times \mathcal{L}_X^\infty$ is a Blackwell equilibrium of parameterized game $\mathcal{G}_{\mathcal{L}_Y^\infty}$ if and only if

$$\big(v^*(\omega), x_\omega^*\big) \in \mathcal{P}\big(\omega, v^*\big) \times \mathcal{N}\big(\omega, v^*\big) \text{ a.e. } [\mu]. \tag{28}$$

Moreover, letting

$$\mathcal{S}^\infty(\mathcal{P}_v) := \big\{u \in \mathcal{L}_Y^\infty : u(\omega) \in \mathcal{P}(\omega, v) \text{ a.e. } [\mu]\big\} \tag{29}$$

and

$$\mathcal{S}^\infty(\mathcal{N}_v) := \big\{x_{(\cdot)} \in \mathcal{L}_X^\infty : x_\omega \in \mathcal{N}(\omega, v) \text{ a.e. } [\mu]\big\}, \tag{30}$$

be the Nash selection correspondences, we have that $(v^*, x^*_{(\cdot)}) \in \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X$ is a Blackwell equilibrium of parameterized game $\mathcal{G}_{\mathcal{L}^\infty_Y}$ if and only if

$$\left(v^*, x^*_{(\cdot)}\right) \in \mathcal{S}^\infty(\mathcal{P}_{v^*}) \times \mathcal{S}^\infty(\mathcal{N}_{v^*}), \tag{31}$$

where $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is the Nash payoff selection correspondence and $\mathcal{S}^\infty(\mathcal{N}_{(\cdot)})$ is the Nash selection correspondence.

Finally, we note that for any $u \in \mathcal{S}^\infty(\mathcal{P}_v)$, we can deduce the existence of Nash strategy profile $x_{(\cdot)} \in \mathcal{S}^\infty(\mathcal{N}_v)$ such that $u(\omega) = u(\omega, v, x_\omega)$ a.e. $[\mu]$ using implicit measurable selection (see Theorem 7.1 in Himmelberg [24]). Thus, a necessary and sufficient condition for parameterized game $\mathcal{G}_{\mathcal{L}^\infty_Y}$ to have a Blackwell equilibrium is that there exist a $v^* \in \mathcal{L}^\infty_Y$ such that $v^* \in \mathcal{S}^\infty(\mathcal{P}_{v^*})$.

4.1.2 Best strategy response correspondences

Given a Blackwell equilibrium $(v^*, x^*_{(\cdot)})$, belonging to a parameterized $\mathcal{G}_{\mathcal{L}^\infty_Y}$, we have by the Nash condition above (26) and by Theorem 2.2 in Hiai and Umegaki [23] that for each player d and a.e. $[\mu]$ in ω,

$$\left.\begin{array}{l} x^*_{d\omega} \in \arg\max_{x_d \in \Phi_d(\omega, v^*_d))} u_d(\omega, v^*_d, x_d, x^*_{-d\omega}), \\ \text{or equivalently,} \\ x^*_{d(\cdot)} \in \arg\max_{x_{d(\cdot)} \in \mathcal{S}^\infty(\Phi_d(\cdot, v^*_d))} \int_\Omega u_d(\omega, v^*_d, x_{d\omega}, x^*_{-d\omega})\, d\mu(\omega) \\ \qquad = \arg\max_{x_{d(\cdot)} \in \mathcal{S}^\infty(\Phi_d(\cdot, v^*_d))} U_d(v^*_d, x_{d(\cdot)}, x^*_{-d(\cdot)}). \end{array}\right\} \tag{32}$$

Under assumptions $[A\text{-}1]$ and $[\mathcal{PSG}\text{-}1]$, we have for each player d that the strategy constraint correspondence

$$v_d \longrightarrow S^\infty\left(\Phi_d(\cdot, v_d)\right)$$

is continuous and because $v^n_d \underset{\rho^*_{\mathcal{L}^\infty_{Y_d}}}{\longrightarrow} v_d$ and $x^n_{(\cdot)} \underset{\rho^*_{\mathcal{L}^\infty_X}}{\longrightarrow} x_{(\cdot)}$ imply (via assumption $[\mathcal{PSG}\text{-}1](3)$ above) that $U_d(v^n_d, x^n_{(\cdot)}) \underset{R}{\longrightarrow} U_d(v_d, x_{(\cdot)})$, i.e., because the real-valued function $U_d(\cdot,\cdot)$ is $\rho^*_{\mathcal{L}^\infty_{Y_d} \times \mathcal{L}^\infty_X}$-continuous on $\mathcal{L}^\infty_{Y_d} \times \mathcal{L}^\infty_X$, we have by Berge's maximum theorem (see 17.31 in Aliprantis and Border [22]) that player d''s best strategy response correspondence (i.e., the argmax correspondence)

$$(v_d, x_{-d(\cdot)}) \longrightarrow \mathcal{B}_d(v_d, x_{-d(\cdot)}) := \arg \max_{x'_{d(\cdot)} \in \mathcal{S}^\infty(\Phi_d(\cdot, v_d))} U_d\left(v_d, x'_{d(\cdot)}, x_{-d(\cdot)}\right)$$

is upper semicontinuous and takes nonempty and $\rho^*_{\mathcal{L}^\infty_{X_d}}$-compact values. Moreover, because $U_d(v_d, \cdot, x_{-d(\cdot)})$ is quasiconcave on $\mathcal{L}^\infty_{X_d}$ and because $\mathcal{S}^\infty(\Phi_d(\cdot, v_d))$ is convex-valued, we have that player d''s best strategy response correspondence $\mathcal{B}_d(\cdot,\cdot)$ is a CUSCO* (i.e., is a convex valued USCO*; see Hola and Holy [18]). Thus, the best strategy response correspondence $\mathcal{B}(\cdot,\cdot) : \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X \longrightarrow P_{*fc}(\mathcal{L}^\infty_X)$[j] belonging to the parameterized $\mathcal{G}_{\mathcal{L}^\infty_Y}$, given by

$$\left.\begin{array}{l}\mathcal{B}(v,x_{(\cdot)}) := \mathcal{B}(v_1,x_{-1(\cdot)}) \times \cdots \times \mathcal{B}(v_m,x_{-m(\cdot)})\\ \text{where for each player } d = 1,\ldots,m,\\ \mathcal{B}_d(v_d,x_{-d(\cdot)})\\ \quad := \{x_{d(\cdot)} \in \mathcal{L}^\infty_{X_d} : x_{d(\cdot)} \in \arg\max_{x'_{d(\cdot)} \in \mathcal{S}^\infty(\Phi_d(\cdot,v_d))} \mathcal{U}_d(v_d,x'_{d(\cdot)},x_{-d(\cdot)})\},\end{array}\right\} \tag{33}$$

is also a CUSCO* but one defined on $\mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X$ with nonempty, convex, and $\rho^*_{\mathcal{L}^\infty_X}$-compact values in $\mathcal{S}^\infty(\Phi(\cdot,v_d)) \subset \mathcal{L}^\infty_X$, implying that players' best strategy response correspondence $\mathcal{B}(v,\cdot)$ is a CUSCO* for each possible profile of player value functions v. Moreover, the players' best strategy response CUSCO* has associated with it a fixed point correspondence

$$v \longrightarrow \mathcal{E}_B(v) = \{x_{(\cdot)} \in \mathcal{L}^\infty_X : x_{(\cdot)} \in \mathcal{B}(v,x_{(\cdot)})\}. \tag{34}$$

(see 17.28 in Aliprantis and Border [22]). By the Kakutani–Fan–Glicksberg fixed point theorem (see 17.55 in Aliprantis and Border [22]), we know that each v-BSR CUSCO* $\mathcal{B}(v,\cdot) : \mathcal{L}^\infty_X \longrightarrow P_{*fc}(\mathcal{L}^\infty_X)$ has a nonempty, $\rho^*_{\mathcal{L}^\infty_X}$-compact set of fixed points. Thus, $\mathcal{E}_B(\cdot)$ takes nonempty values, and because $\mathcal{B}(\cdot,\cdot)$ is a CUSCO*, and therefore because $\mathcal{B}(\cdot,\cdot)$ has a weak* closed graph in $\mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X \times \mathcal{L}^\infty_X$, it is easy to show that $\mathcal{E}_B(\cdot)$ has a weak* closed graph in $\mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X$, implying that $\mathcal{E}_B(\cdot)$ is an USCO* (see Hola and Holy [18]) and takes decomposable values in $\mathcal{L}^\infty_X$. By Lemma 1 above $\mathcal{E}_B(\cdot)$ is a K-correspondence. It then follows from Theorems 1–6 above that the correspondence $v \longrightarrow \mathcal{U}(v) = \mathcal{T}(v,\mathcal{E}_{\mathcal{B}}(v))$ has fixed points in $\mathcal{L}^\infty_Y$.

Moreover, because $\mathcal{E}_{\mathcal{B}}(v) = \mathcal{S}^\infty(\mathcal{N}_v)$ and $\mathcal{T}(v,\mathcal{E}_{\mathcal{B}}(v)) = \mathcal{S}^\infty(\mathcal{P}_v)$ for all $v \in \mathcal{L}^\infty_Y$, where $\mathcal{S}^\infty(\mathcal{N}_{(\cdot)})$ is the game's Nash selection correspondence and where $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is the game's Nash payoff selection correspondence, we know that the game has *Blackwell equilibria*, provided there exists $(v^*,x^*_{(\cdot)}) \in \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_X$ with

$$(v^*,x^*_{(\cdot)}) \in \mathcal{S}^\infty(\mathcal{P}_{v^*}) \times \mathcal{S}^\infty(\mathcal{N}_{v^*}). \tag{35}$$

Thus, for any Blackwell equilibrium $(v^*,x^*_{(\cdot)})$, we have, for each player d and for μ-almost every state ω, that

$$\left.\begin{array}{l}v^*_d(\omega) = u_d(\omega,v^*_d,x^*_{d\omega},x^*_{-d\omega}),\\ \text{and}\\ u_d(\omega,v^*_d,x^*_{d\omega},x^*_{-d\omega}) = \max_{x_d \in \Phi_d(\omega,v^*_d)} u_d(\omega,v^*_d,x_d,x^*_{-d\omega}).\end{array}\right\} \tag{36}$$

As we will see in the next subsection, the key facts about a one-shot $\mathcal{PSG}$ belonging to a discounted stochastic game ($\mathcal{DSG}$) is that it has an equilibrium strategy correspondence $\mathcal{E}_B(\cdot)$ and an equilibrium payoff function correspondence $\mathcal{T}(\cdot,\mathcal{E}_{\mathcal{B}}(\cdot))$, that are K-correspondences with decomposable values. These facts are particularly important because they guarantee that equilibrium payoff correspondence $\mathcal{T}(\cdot,\mathcal{E}_{\mathcal{B}}(\cdot))$ is an *USCO**, and if μ is nonatomic then $\mathcal{T}(\cdot,\mathcal{E}_{\mathcal{B}}(\cdot))$ is an *USCO** with contractible values – guaranteeing that $\mathcal{T}(\cdot,\mathcal{E}_{\mathcal{B}}(\cdot))$ is approximable – and finally, that $\mathcal{T}(\cdot,\mathcal{E}_{\mathcal{B}}(\cdot))$ has fixed points in $\mathcal{L}^\infty_Y$. Moreover, given that for all $v \in \mathcal{L}^\infty_Y$,

$$\mathcal{S}^\infty(\mathcal{N}_v) = \mathcal{E}_{\mathcal{B}}(v) \quad \text{and} \quad \mathcal{S}^\infty(\mathcal{P}_v) = \mathcal{T}(v,\mathcal{E}_{\mathcal{B}}(v)), \tag{37}$$

for the Nash selection correspondences $\mathcal{S}^{\infty}(\mathcal{N}_{(\cdot)})$ and $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, belonging to the parameterized game $\mathcal{G}_{\mathcal{L}_Y^{\infty}}$, we have that the Nash payoff selection correspondence $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is approximable and has fixed points.

4.2 Discounted stochastic games ($\mathcal{DSG}s$)

For the one-shot $\mathcal{PSG}$

$$\mathcal{G}_{\mathcal{L}_Y^{\infty}}^{\mathrm{NR}} := \{\mathcal{G}_{\nu}^{\mathrm{NR}} : \nu \in \mathcal{L}_Y^{\infty}\}, \tag{38}$$

underlying any discounted stochastic game $\mathcal{DSG}^{\mathrm{NR}}$, satisfying the Nowak–Raghavan assumptions, each ν-game $\mathcal{G}_{\nu}^{\mathrm{NR}}$ is specified by the primitives

$$\mathcal{G}_{\nu}^{\mathrm{NR}} := \big(\mathcal{L}_{\Delta(A_d)}^{\infty}, \mathcal{S}^{\infty}\big(\Delta\big(\Phi_d(\cdot)\big)\big), U_d\big(\nu_d, \cdot, \sigma_{-d}(\cdot)\big)\big)_d. \tag{39}$$

The following list of Nowak–Raghavan assumptions specializes and extends assumptions $[A\text{-}1]$ *and* $[\mathcal{PSG}\text{-}1]$ above:

(1) $(\Omega, B_{\Omega}, \mu)$ *is a probability space of states* ω*, where* Ω *is a complete, separable metric space with probability measure* μ *defined on the Borel* σ*-field* B_{Ω}.

(2) A_d *is a compact metric space of actions available to player* **d***. We will equip* A_d *with metric* ρ_{A_d} *and we will equip* $A := A_1 \times \cdots \times A_m$ *with the sum metric,* $\rho_A = \sum_d \rho_{A_d}$*. Also* $\Delta(A_d)$ *is the compact metric space of behavioral actions available to player d given by the set of all probability measures with support contained in* A_d*. We will equip* $\Delta(A_d)$ *with a metric* $\rho_{w_{\mathrm{ca}}^{*d}}$ *compatible with its* w_{ca}^{*d}*-topology (classical narrow topology), and we will equip* $\Delta(A) := \Delta(A_1) \times \cdots \times \Delta(A_m)$ *with the sum metric,* $\rho_{w_{\mathrm{ca}}^{*}} = \sum_d \rho_{w_{\mathrm{ca}}^{*d}}$.

(3) $\Phi_d(\cdot)$ *is player d's measurable, state-contingent action constraint correspondence (not dependent on* $\nu_d \in \mathcal{L}_Y^{\infty}$*).*

(4) $\Delta(\Phi_d(\cdot))$ *is player d's measurable, state-contingent behavioral action constraint correspondence with*

$$\Delta\big(\Phi_d(\omega)\big) := \big\{\sigma_d \in \Delta(A_d) : \sigma_d\big(\Phi_d(\omega)\big) = 1 \text{ a.e. } [\mu]\big\}. \tag{40}$$

(5) $\mathcal{L}_{\Delta(A_d)}^{\infty}$ *is the collection of* μ*-equivalences classes of measurable functions* $\sigma_d(\cdot|\cdot) : \Omega \longrightarrow \Delta(A_d)$ *(i.e., for* ω *a.e.* $[\mu]$*,* $\sigma_d(\cdot|\omega) \in \Delta(A_d)$*) and is the norm dual of* $\mathcal{L}_{\mathcal{C}(A_d)}^{1}$*, where* $\mathcal{C}(A_d)$ *is the Banach space of continuous functions on compact metric space* A_d *with the sup-norm (see Theorem V-2 in Castaing and Valadier [25]).*

(6) $\mathcal{S}^{\infty}(\Delta(\Phi_d(\cdot)))$ *is player d''s behavioral strategy set given by the collection of* μ*-equivalences classes of measurable functions* $\sigma_d(\cdot|\cdot) : \Omega \longrightarrow \Delta(A_d)$ *such that for* ω *a.e.* $[\mu]$*,* $\sigma_d(\omega) \in \Delta(A_d)$ *and* $\sigma_d(\Phi_d(\omega)|\omega) = 1$ *(i.e.,* $\sigma_d(\cdot|\omega) \in \Delta(\Phi_d(\omega))$ *a.e.* $[\mu]$*). We will equip* $\mathcal{S}^{\infty}(\Delta(\Phi_d(\cdot)))$ *with a metric* $\rho_{W_{\mathrm{ca}}^{*d}}$ *compatible with its* W_{ca}^{*d}*-topology inherited from* $\mathcal{L}_{\Delta(A_d)}^{\infty}$.

(7) $\mathcal{S}^{\infty}(\Delta(\Phi(\cdot))) := \mathcal{S}^{\infty}(\Delta(\Phi_1(\cdot))) \times \cdots \times \mathcal{S}^{\infty}(\Delta(\Phi_m(\cdot)))$ *is the set of players' strategy profiles. We will equip* $\mathcal{S}^{\infty}(\Delta(\Phi(\cdot)))$ *with the sum metric,* $\rho_{W_{\mathrm{ca}}^{*}} := \sum_d \rho_{W_{\mathrm{ca}}^{*d}}$*, and we will denote a typical element of* $\mathcal{S}^{\infty}(\Delta(\Phi(\cdot)))$ *by*

$$\sigma(da|\cdot) := \big(\sigma_1(da_1|\cdot), \ldots, \sigma_m(da_m|\cdot)\big).$$

Also, given $\sigma(da|\cdot) := (\sigma_1(da_1|\cdot), \ldots, \sigma_m(da_m|\cdot)) \in \mathcal{S}^\infty(\Delta(\Phi(\cdot)))$, *for each state* ω *we will denote by*

$$\pi\sigma(da|\omega) := \pi\big(\sigma_1(da_1|\omega), \ldots, \sigma_m(da_m|\omega)\big) := \bigotimes_{d=1}^{m} \sigma_d(da_d|\omega) \in \Delta\big(\Phi(\omega)\big)$$

the product probability measure determined by the m*-tuple of probability measures,* $(\sigma_1(da_1|\omega), \ldots, \sigma_m(da_m|\omega))$ *(i.e., behavioral actions) chosen by the players in state* ω.

(8) $\mathcal{L}^\infty_{Y_d} := \mathcal{L}^\infty_{[-M,M]} \subset \mathcal{L}^\infty_R$ *and* $\mathcal{L}^\infty_Y := \mathcal{L}^\infty_{[-M,M]^m} \subset \mathcal{L}^\infty_{R^m}$ *is the set of players' state-contingent payoff profiles.*

(9) $r_d(\cdot,\cdot) : \Omega \times A \longrightarrow Y_d$ *is player d''s Caratheodory stage payoff function (i.e.,* $r_d(\omega,\cdot)$ *is* ρ_A*-continuous on* A *and* $r_d(\cdot,a)$ *is* (B_Ω, B_{Y_d})*-measurable on* Ω *for each* a*).*

(10) $q(\cdot|\cdot,\cdot) : \Omega \times A \longrightarrow \Delta(\Omega)$ *is the law of motion defined on* $\Omega \times A$ *taking values in the space of probability measures on* Ω*, having the following properties: (i)* $q(\cdot|\omega,a) \ll \mu$ *for all* $(\omega,a) \in \Omega \times A$ *(i.e., each probability measure* $q(\cdot|\omega,a)$ *is absolutely continuous with respect to the probability measure* μ*), (ii) for each* $E \in B_\Omega$*,* $q(E|\cdot,\cdot)$ *is measurable on* $\Omega \times A$*, and (iii) the collection of probability density functions*

$$H_\mu := \big\{h(\cdot|\omega,a) : (\omega,a) \in \Omega \times A\big\}$$

of $q(\cdot|\omega,a)$ *with respect to* μ *is such that for each state* ω*, the function* $a \longrightarrow h(\omega'|\omega,a)$ *is continuous in* a *a.e.* $[\mu]$ *in* ω'.

(11) $\sigma_d(\cdot) \longrightarrow U_d(v_d, \sigma_d(\cdot), \sigma_{-d}(\cdot))$ *is player d's expected stage payoff function given* $(v_d, \sigma_{-d}(\cdot)) \in \mathcal{L}^\infty_{Y_d} \times \mathcal{L}^\infty_{\Pi_{d' \neq d}\Delta(A_{d'})}$*, where*

$$\left.\begin{aligned}
&U_d(v_d, \sigma(\cdot)) \\
&\quad := \int_\Omega \overbrace{(1-\beta_d)\int_A r_d(\omega,a)\pi\sigma(da|\omega) + \beta_d \int_A \Big[\int_\Omega v_d(\omega')h(\omega'|\omega,a)\,d\mu(\omega')\Big]\pi\sigma(da|\omega)}^{u_d(\omega,v_d,\sigma(\omega))}\,d\mu(\omega) \\
&\quad := \int_\Omega u_d(\omega, v_d, \sigma(\omega))\,d\mu(\omega) \\
&\quad := \int_\Omega \int_A u_d(\omega, v_d, a)\pi\sigma(da|\omega)\,d\mu(\omega),
\end{aligned}\right\} \tag{41}$$

where for each ω*,* $(v_d, a) \longrightarrow u_d(\omega, v_d, a)$ *is jointly continuous in* (v_d, a) *on* $\mathcal{L}^\infty_{Y_d} \times A$ *(see Appendix 2).*

Label the assumptions above $[\mathcal{DSG}^{\text{NR}}]$.

Sometimes we will write $\sigma_1(\omega) \otimes \cdots \otimes \sigma_m(\omega)$ rather than $\sigma_1(da_1|\omega) \otimes \cdots \otimes \sigma_m(da_m|\omega)$ to denote the product probability measure determined by $\sigma(da|\cdot) := (\sigma_1(da_1|\cdot), \ldots, \sigma_m(da_m|\cdot)) \in \mathcal{S}^\infty(\Delta(\Phi(\cdot)))$. Also, we will write

$$\pi\big(\sigma_d(\cdot), \sigma_{-d}(\cdot)\big) = \pi\big(\sigma_{-d}(\cdot)\big) \otimes \sigma_d(\cdot) = \pi\big(\sigma_{-d}(\cdot)\big) \otimes \sigma_d(da_d|\cdot),$$

where $\pi(\sigma_{-d}(\cdot)) := \bigotimes_{d' \in D\setminus\{d\}} \sigma_{d'}(\cdot)$. Note that $U_d(v_d, \sigma_d(\cdot), \sigma_{-d}(\cdot))$ is linear in $\sigma_d(\cdot) \in \mathcal{L}^\infty_{\Delta(A_d)}$.

In order to show that the parameterized, one-shot game $\mathcal{G}^{\text{NR}}_{\mathcal{L}^\infty_Y}$, underlying any Nowak–Raghavan discounted stochastic game $\mathcal{DSG}^{\text{NR}}$, has Blackwell equilibria, it suffices to verify that each player d has a state-contingent, expected stage payoff function

$$\left.\begin{aligned} &u_d(\cdot,v_d,\sigma(\cdot)) \\ &\quad := \int_A \underbrace{\Big[(1-\beta_d)r_d(\cdot,a)+\beta_d\int_\Omega v_d(\omega')h(\omega'|\cdot,a)\,d\mu(\omega')\Big]}_{u_d(\cdot,v_d,a)}\pi\sigma(da|\cdot) \\ &\quad := \int_A u_d(\cdot,v_d,a)\pi\sigma(da|\cdot) \end{aligned}\right\} \tag{42}$$

satisfying assumption $[\mathcal{PSG}\text{-}1](3)$ above.

Theorem 7 (Continuity of Players' State-Contingent Payoff Functions) *Consider a* $\mathcal{DSG}^{\mathrm{NR}}$ *having a one-shot game* $\mathcal{G}_v^{\mathrm{NR}}$ *satisfying assumptions* $[\mathcal{DSG}^{\mathrm{NR}}]$ *with players' state-contingent payoff function profile*

$$\big(v,\sigma(\cdot)\big)\longrightarrow u\big(\cdot,v,\sigma(\cdot)\big):=\big(u_1\big(\cdot,v_1,\sigma(\cdot)\big),\dots,u_m\big(\cdot,v_m,\sigma(\cdot)\big)\in\mathcal{L}_Y^\infty.$$

If $\{(v^n,\sigma^n(\cdot))\}_n$ *is a sequence of value function-strategy profile pairs in* $\mathcal{L}_Y^\infty\times\mathcal{S}^\infty(\Delta(\Phi(\cdot)))$ *such that* $v^n\underset{\rho_{w^*}}{\longrightarrow}v^*$ *and* $\sigma^n(\cdot)\underset{\rho_{W^*_{\mathrm{ca}}}}{\Longrightarrow}\sigma^*(\cdot)$, *then the sequence of state-contingent, expected stage payoff functions in* (42),

$$\big\{u\big(\cdot,v^n,\sigma^n(\cdot)\big)\big\}_n\subset\mathcal{L}_Y^\infty,$$

ρ_{w^*}*-converges to* $u(\cdot,v^*,\sigma^*(\cdot))\in\mathcal{L}_Y^\infty$.

Proof Let $\{(v^n,\sigma^n(\cdot))\}_n\subset\mathcal{L}_Y^\infty\times\mathcal{S}^\infty(\Delta(\Phi(\cdot)))$ be a sequence such that

$$\left.\begin{aligned} &v^n\underset{\rho_{w^*}}{\longrightarrow}v^*\in\mathcal{L}_Y^\infty \\ &\text{and} \\ &\sigma^n(\cdot)\underset{\rho_{W^*_{\mathrm{ca}}}}{\Longrightarrow}\sigma^*(\cdot)\in\mathcal{S}^\infty(\Delta(\Phi(\cdot))). \end{aligned}\right\} \tag{43}$$

By Balder [26], we have for every subsequence $\{(v^{n_k},\sigma^{n_k}(\cdot))\}_k$ of $\{(v^n,\sigma^n(\cdot))\}_n$, a further subsequence $\{(v^{n_{k_r}},\sigma^{n_{k_r}}(\cdot))\}_r$ such that

$$v^{n_{k_r}}\underset{K}{\longrightarrow}v^*\quad\text{and}\quad\sigma^{n_{k_r}}(\cdot)\underset{\rho_{w^*_{\mathrm{ca}}}\text{-}K}{\longrightarrow}\sigma^*(\cdot). \tag{44}$$

First, consider the sequence $\{R_d(\cdot,\sigma^n(\cdot))\}_n\subset\mathcal{L}_{Y_d}^\infty$ where

$$R_d\big(\omega,\sigma^n(\omega)\big):=\int_A r_d(\omega,a)\pi\sigma^n(da|\omega). \tag{45}$$

Because $\sigma^n(\cdot)\underset{\rho_{W^*_{\mathrm{ca}}}}{\Longrightarrow}\sigma^*(\cdot)$, by the linearity of the integral in (45), we have for any subsequence $\{\sigma^{n_k}(\cdot)\}_k$ of $\{\sigma^n(\cdot)\}_n$ that

$$R_d\left(\omega,\frac{1}{k}\sum_{r=1}^k\sigma^{n_r}(\omega)\right)=\frac{1}{k}\sum_{r=1}^k R_d\big(\omega,\sigma^{n_r}(\omega)\big).$$

Thus, because $\sigma^n(\cdot) \underset{\rho_{W^*_{ca}}}{\Longrightarrow} \sigma^*(\cdot)$, we have by the linearity of the integral in (45), that for every subsequence $\{R_d(\cdot,\sigma^{n_k}(\cdot))\}_k$ of $\{R_d(\cdot,\sigma^n(\cdot))\}_n$, there is a further subsequence $\{R_d(\cdot,\sigma^{n_{k_r}}(\cdot))\}_r$ such that $\{R_d(\cdot,\sigma^{n_{k_r}}(\cdot))\}_r$ K converges to $R^*_d(\cdot) = R_d(\cdot,\sigma^*(\cdot)) \in \mathcal{L}^\infty_{Y_d}$. Thus, by (1) $\Longleftrightarrow$ (2) of Balder [26], we have that for each player d,

$$R_d\big(\cdot,\sigma^n(\cdot)\big) \underset{\rho_{w^*_d}}{\longrightarrow} R_d\big(\cdot,\sigma^*(\cdot)\big).$$

Next, consider

$$V(\omega,\nu_d,a) := \int_\Omega \nu_d\big(\omega'\big)h\big(\omega'|\omega,a\big)\,d\mu\big(\omega'\big). \tag{46}$$

For $\nu^*_d \in \mathcal{L}^\infty_{Y_d}$, by arguments similar to those immediately above, we have for each player d that

$$\begin{aligned} V\big(\cdot,\nu^*_d,\sigma^n(\cdot)\big) &:= \int_A V\big(\cdot,\nu^*_d,a\big)\pi\sigma^n(da|\cdot) \\ \underset{\rho_{w^*_d}}{\longrightarrow} &\int_A V\big(\cdot,\nu^*_d,a\big)\pi\sigma^*(da|\cdot) := V\big(\cdot,\nu^*_d,\sigma^*(\cdot)\big). \end{aligned} \tag{47}$$

Finally, because $(\mathcal{L}^\infty_{Y_d}\times A,\rho_{w^*_d\times A})$ is a compact metric space and because for each $\omega\in\Omega$, $V(\omega,\cdot,\cdot)$ is continuous on $\mathcal{L}^\infty_{Y_d}\times A$, we have for each $\omega\in\Omega$, that $V(\omega,\cdot,\cdot)$ is uniformly continuous on $\mathcal{L}^\infty_{Y_d}\times A$ for each d (see proof of Theorem A.1 in Appendix 2). This implies that for any $\varepsilon>0$, there exists a $\delta_\varepsilon>0$ such that for any d and any pair, (ν_d,a) and (ν'_d,a') in $\mathcal{L}^\infty_{Y_d}\times A$ with

$$\rho_{w^*_d\times A}\big((\nu_d,a),\big(\nu'_d,a'\big)\big) := \rho_{w^*_d}\big(\nu_d,\nu'_d\big) + \rho_A\big(a,a'\big) < \delta_\varepsilon,$$

we have $|V(\omega,\nu_d,a) - V(\omega,\nu'_d,a')| < \varepsilon$. Because for each player d, $\nu^n_d \underset{\rho_{w^*_d}}{\longrightarrow} \nu^*_d$ (i.e., $\rho_{w^*_d}(\nu^n_d,\nu^*_d) \longrightarrow 0$), we can choose N_ε sufficiently large so that for $n>N_\varepsilon$,

$$\big|V\big(\omega,\nu^n_d,a\big) - V\big(\omega,\nu^*_d,a\big)\big| < \varepsilon \quad \text{for each } d \text{ and for all } a\in A.$$

We have therefore, for each d and for each $n>N_\varepsilon$,

$$\begin{aligned} &\left|\int_A V\big(\omega,\nu^n_d,a\big)\pi\sigma^n(da|\omega) - \int_A V\big(\omega,\nu^*_d,a\big)\pi\sigma^n(da|\omega)\right| \\ &\quad\le \int_A \big|V\big(\omega,\nu^n_d,a\big) - V\big(\omega,\nu^*_d,a\big)\big|\pi\sigma^n(da|\omega) \\ &\quad\le \varepsilon\int_A \pi\sigma^n(da|\omega) \\ &\quad= \varepsilon. \end{aligned}$$

Thus, we have for each d and for each $\omega\in\Omega$,

$$\int_A V\big(\omega,\nu^n_d,a\big)\pi\sigma^n(da|\omega) \longrightarrow \int_A V\big(\omega,\nu^*_d,a\big)\pi\sigma^n(da|\omega),$$

implying that for each d,

$$\int_A V(\cdot, v_d^n, a)\pi\sigma^n(da|\cdot) \underset{\rho_{w_d^*}}{\longrightarrow} \int_A V(\cdot, v_d^*, a)\pi\sigma^n(da|\cdot). \tag{48}$$

Together expressions (47) and (48) imply that for each d,

$$V(\cdot, v_d^n, \sigma^n(\cdot)) := \int_A V(\cdot, v_d^n, a)\pi\sigma^n(da|\cdot) := \int_A \left[\int_\Omega v_d^n(\omega')h(\omega'|\cdot, a)\,d\mu(\omega')\right]\pi\sigma^n(da|\cdot)$$

$$\underset{w_d^*}{\longrightarrow}$$

$$\int_A \left[\int_\Omega v_d^*(\omega')h(\omega'|\cdot, a)\,d\mu(\omega')\right]\pi\sigma^*(da|\cdot) := \int_A V(\cdot, v_d^*, a)\pi\sigma^*(da|\cdot) := V(\cdot, v_d^*, \sigma^*(\cdot)).$$

We have, therefore, for each d that $v_d^n \underset{\rho_{w_d^*}}{\longrightarrow} v_d^*$ and $\sigma^n(\cdot) \underset{\rho_{W_{ca}^*}}{\Longrightarrow} \sigma^*(\cdot)$ imply that

$$\left.\begin{array}{l} u_d(\cdot, v_d^n, \sigma^n(\cdot)) := (1-\beta_d)R_d(\cdot, \sigma^n(\cdot)) + \beta_d V(\cdot, v_d^n, \sigma^n(\cdot)) \\ \underset{w_d^*}{\longrightarrow} \\ (1-\beta_d)R_d(\cdot, \sigma^*(\cdot)) + \beta_d V(\cdot, v_d^*, \sigma^*(\cdot)) := u_d(\cdot, v_d^*, \sigma^*(\cdot)). \end{array}\right\} \tag{49}$$

□

Because Y is bounded and because $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$ and $\sigma^n(\cdot) \underset{\rho_{W_{ca}^*}}{\Longrightarrow} \sigma^*(\cdot)$ imply that

$$u(\cdot, v^n, \sigma^n(\cdot)) \underset{\rho_{w^*}}{\longrightarrow} u(\cdot, v^*, \sigma^*(\cdot)),$$

we have that

$$U(v^n, \sigma^n(\cdot)) = \int_\Omega u(\omega, v^n, \sigma^n(\omega))\,d\mu(\omega) \underset{\rho_Y^*}{\longrightarrow} \int_\Omega u(\omega, v^*, \sigma^*(\omega))\,d\mu(\omega) = U(v^*, \sigma^*(\cdot)).$$

Thus, by Theorems 1–6 above, the correspondence

$$v \longrightarrow \mathcal{T}(v, \mathcal{E}_{\mathcal{B}}(v)) := \{u(\cdot, v, \sigma(\cdot)) \in \mathcal{L}_Y^\infty : \sigma(\cdot) \in \mathcal{E}_{\mathcal{B}}(v)\}$$

has fixed points in $\mathcal{L}_Y^\infty$, and because $\mathcal{T}(v, \mathcal{E}_{\mathcal{B}}(v)) = \mathcal{S}^\infty(\mathcal{P}_v)$ for all $v \in \mathcal{L}_Y^\infty$, where $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ is the one-shot game's (i.e., $\mathcal{G}_{\mathcal{L}_Y^\infty}^{\mathrm{NR}}$) Nash payoff selection correspondence, we know that the game has a *Blackwell equilibrium* – a pair $(v^*, \sigma^*(\cdot)) \in \mathcal{L}_Y^\infty \times \mathcal{L}_{\Delta(A)}^\infty$ such that

$$(v^*, \sigma^*(\cdot)) \in \mathcal{S}^\infty(\mathcal{P}_{v^*}) \times \mathcal{S}^\infty(\mathcal{N}_{v^*}). \tag{50}$$

Thus, we have for each player d and for μ-almost every state ω that

$$\left.\begin{array}{l} v_d^*(\omega) = u_d(\omega, v_d^*, (\sigma_d^*(\omega), \sigma_{-d}^*(\omega))) \\ \text{and} \\ u_d(\omega, v_d^*, (\sigma_d^*(\omega), \sigma_{-d}^*(\omega))) = \max_{\sigma_d \in \Delta(\Phi_d(\omega))} u_d(\omega, v_d^*, (\sigma_d, \sigma_{-d}^*(\omega))). \end{array}\right\} \tag{51}$$

The following theorem, our main result on the existence of stationary Markov perfect equilibria for uncountable-compact $\mathcal{DSG}s$ satisfying the Nowak–Raghavan assumptions, summarizes the discussion above.

Theorem 8 (All Nonatomic Nowak–Raghavan $\mathcal{DSG}s$ Have Stationary Markov Perfect Equilibria) *Consider a $\mathcal{DSG}^{\mathrm{NR}}$ satisfying assumptions $[\mathcal{DSG}^{\mathrm{NR}}]$ with (Ω, B_Ω, μ) nonatomic, having a one-shot game $\mathcal{G}_v^{\mathrm{NR}}$ given by*

$$\mathcal{G}_{\mathcal{L}_Y^\infty}^{\mathrm{NR}} := \left\{ \left(\mathcal{L}_{\Delta(A_d)}^\infty, \mathcal{S}^\infty\left(\Delta\left(\Phi_d(\cdot)\right)\right), U_d\left(v_d, \cdot, \sigma_{-d}(\cdot)\right)\right)_d \right\}_{v \in \mathcal{L}_Y^\infty},$$

where

$$\begin{aligned} &U_d\left(v_d, \sigma_d(\cdot), \sigma_{-d}(\cdot)\right) \\ &\quad := \int_\Omega \overbrace{(1-\beta_d) \int_A r_d(\omega, a) \pi\sigma(da|\omega) + \beta_d \int_A \left[\int_\Omega v_d(\omega') h(\omega'|\omega, a)\, d\mu(\omega') \right] \pi\sigma(da|\omega)}^{u_d(\omega, v_d, \sigma(\omega))}\, d\mu(\omega). \end{aligned}$$

Then $\mathcal{G}_{\mathcal{L}_Y^\infty}^{\mathrm{NR}}$ has Blackwell equilibria and $\mathcal{DSG}^{\mathrm{NR}}$ has stationary Markov perfect equilibria.

5 Main conclusions

Focusing on composition correspondences of the following type:

$$v \longrightarrow \mathcal{U}(v) := \left\{ \mathcal{T}(v, x_{(\cdot)}) : x_{(\cdot)} \in \mathcal{E}(v) \right\} := \mathcal{T}\left(v, \mathcal{E}(v)\right), \tag{52}$$

where $\mathcal{E}(\cdot) : \mathcal{L}_Y^\infty \longrightarrow P(\mathcal{L}_X^\infty)$ and $\mathcal{T}(\cdot,\cdot) : \mathcal{L}_Y^\infty \times \mathcal{L}_X^\infty \longrightarrow P(\mathcal{L}_Y^\infty)$ are K-correspondences, we show that if the probability space of states (Ω, B_Ω, μ), underlying the function spaces $\mathcal{L}_Y^\infty$ and $\mathcal{L}_X^\infty$, is equipped with a nonatomic probability measure μ, then for each $v \in \mathcal{L}_Y^\infty$, $\mathcal{T}(v, \mathcal{E}(v))$ is contractible with respect to the compatibly metrized weak* topologies, and, as a consequence, the composition correspondence $\mathcal{T}(\cdot, \mathcal{E}(\cdot))$ is approximable and has fixed points.

Next, we consider a discounted stochastic game $\mathcal{DSG}$, satisfying the Nowak–Raghavan assumptions, with underlying parameterized, state-contingent, one-shot game $\mathcal{G}_{\mathcal{L}_Y^\infty}^{\mathrm{NR}}$, with Nash selection correspondence $v \longrightarrow \mathcal{S}^\infty(\mathcal{N}_v)$, and Nash payoff selection correspondence $v \longrightarrow \mathcal{S}^\infty(\mathcal{P}_v)$. We show that if the correspondence $\mathcal{E}(\cdot)$ in expression (52) is given by

$$v \longrightarrow \mathcal{E}_{\mathcal{B}}(v) := \left\{ x_{(\cdot)} \in \mathcal{L}_X^\infty : x_{(\cdot)} \in \mathcal{B}(v, x_{(\cdot)}) \right\},$$

where $\mathcal{B}(\cdot,\cdot)$: $\mathcal{L}_Y^\infty \times \mathcal{L}_X^\infty \longrightarrow P_{*f}(\mathcal{L}_X^\infty)$ is the best strategy response correspondence (a CUSCO*) belonging to the $\mathcal{DSG}'s$ parameterized, state-contingent, one-shot game $\mathcal{G}_{\mathcal{L}_Y^\infty}^{\mathrm{NR}}$, then $\mathcal{E}_{\mathcal{B}}(\cdot)$ and $\mathcal{T}(\cdot, \mathcal{E}_{\mathcal{B}}(\cdot))$ are K-correspondences with

$$\mathcal{S}^\infty(\mathcal{N}_v) = \mathcal{E}_{\mathcal{B}}(v) \quad \text{and} \quad \mathcal{S}^\infty(\mathcal{P}_v) = \mathcal{T}\left(v, \mathcal{E}_{\mathcal{B}}(v)\right).$$

Therefore, if the $\mathcal{DSG}$ has a probability space of states (Ω, B_Ω, μ), equipped with a nonatomic probability measure μ, then, by our Theorem 6, the Nash payoff selection correspondence $v \longrightarrow \mathcal{S}^\infty(\mathcal{P}_v) = \mathcal{T}(v, \mathcal{E}_{\mathcal{B}}(v))$ has fixed points. By Blackwell's theorem [3], it then follows that the nonatomic, Nowak–Raghavan $\mathcal{DSG}$ to which $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$ belongs has stationary Markov perfect equilibria. All of this is true provided the $\mathcal{DSG}$ satisfies the

Nowak–Raghavan assumptions. It then suffices to show that each player's state-contingent payoff function

$$u_d(\cdot, v_d, \sigma(\cdot)) := (1-\beta_d)\int_A r_d(\cdot, a)\pi\sigma(da|\cdot) + \beta_d \int_A \left[\int_\Omega v_d(\omega')h(\omega'|\cdot, a)\, d\mu(\omega')\right]\pi\sigma(da|\cdot)$$

is sequentially weak* continuous. Thus, we show in Theorem 7 that if $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$ and $\sigma^n(\cdot) \underset{\rho_{W^*_{ca}}}{\Longrightarrow} \sigma^*(\cdot)$, then $u(\cdot, v^n, \sigma^n(\cdot)) \underset{\rho_{w^*}}{\longrightarrow} u(\cdot, v^*, \sigma^*(\cdot))$.

Appendix 1: Measurability and integrability of Banach space valued functions

Recall the set-up: (Ω, B_Ω, μ) is a probability space of states (not necessarily nonatomic), where Ω is Polish with probability measure μ defined on the Borel σ-field B_Ω, and Y is a norm-bounded, weak*-closed (i.e., w^*-closed), convex subset of F^*, the norm dual of a separable Banach space F, with F^* having the Radon–Nikodym property.

(1) Because F is separable, F^* is (norm-) separable if and only if F^* has the Radon–Nikodym property (Bourgin [27], Theorem 5.2.12). Moreover, F^* has the Radon–Nikodym property if and only if for each μ-continuous vector measure $G : B_\Omega \longrightarrow F^*$ of bounded variation, there exists an integrable function $g \in L^1_{F^*}$ such that for all $A \in B_\Omega$,

$$G(A) = \int_A g(\omega)\, d\mu(\omega)$$

(Diestel and Uhl [11]).

(2) A function $v : \Omega \longrightarrow F^*$ is (a) *strongly measurable* if there exists a sequence $\{\varphi^n\}_n$ of F^*-valued, (B_Ω-measurable) simple functions such that

$$\left\| v(\omega) - \varphi^n(\omega)\right\|^* \longrightarrow 0 \quad \text{a.e. } [\mu],$$

(b) *scalarly or weakly measurable* if $\omega \longrightarrow \langle l, v(\omega)\rangle$ is (B_Ω, B_R)-measurable for all $l \in E$, where B_R is the Borel σ-field in R (the real numbers), and (c) (B_Ω, B_{w^*})*-measurable* if for all Borel sets $B \in B_{w^*}$,

$$v^{-1}(B) := \{\omega \in \Omega : v(\omega) \in B\} \in B_\Omega,$$

where B_{w^*} is the Borel σ-field generated by the w^*-topology in F^*. By Lemma 11.37 in Aliprantis and Border [22], if $v(\cdot)$ is strongly measurable, then $v(\cdot)$ is (B_Ω, B_{w^*})-measurable. By the Pettis measurability theorem (Diestel and Uhl [11], p. 42) if $v(\Omega\backslash N)$ is norm separable for $N \in B_\Omega$ with $\mu(N) = 0$ (i.e., off a set of μ-measure zero), then the range of $v(\cdot)$ is norm separable and if $v(\cdot)$ is (B_Ω, B_{w^*})-measurable, then $v(\cdot)$ is strongly measurable. In addition, by Proposition A.1 in Cornet and Martin-da-Rocha [28], $v(\cdot)$ is (B_Ω, B_{w^*})-measurable if and only if $v(\cdot)$ is scalar measurable. Thus, letting L^∞_Y (the prequotient of $\mathcal{L}^\infty_Y$) be the set of all (B_Ω, B_{w^*})-measurable functions defined on Ω taking values a.e. $[\mu]$ in the w^*-closed and $\|\cdot\|^*$-bounded subset Y of the norm dual F^*, we have for each $v(\cdot) \in L^\infty_Y$ that $v(\cdot)$ is strongly measurable because $v(\Omega\backslash N) \subseteq Y$ for $N \in B_\Omega$ with $\mu(N) = 0$ and, by

Theorem 7.7 in Kahn [29], Y is $\|\cdot\|^*$-separable. Thus, each function $v(\cdot)$ in the prequotient space L_Y^∞ of (B_Ω, B_{w^*})-measurable functions defined on Ω and taking values a.e. $[\mu]$ in the w^*-compact subset Y of the norm dual F^* is not only (B_Ω, B_{w^*})-measurable, but also scalarly measurable, as well as strongly measurable.

(3) Given probability space (Ω, B_Ω, μ), a strongly measurable F^*-valued function f is Bochner integrable if there is a sequence of simple functions $\{\varphi^n\}_n$ such that

$$\int_\Omega \|f(\omega) - \varphi^n(\omega)\|^* \, d\mu(\omega) = 0.$$

If f is Bochner integrable, then for each $A \in B_\Omega$, $\int_A f \, d\mu$ is defined by

$$\int_A f(\omega) := \lim_n \int_A \varphi^n(\omega) \, d\mu(\omega),$$

where the last limit is in the norm topology on F^* (i.e., the $\|\cdot\|^*$-topology). By Theorem 2 of Diestel and Uhl [11] (p. 45), a strongly measurable, F^*-valued function f is Bochner integrable if and only if its real-valued norm function $\|f(\cdot)\|^*$ is integrable, i.e., $\int_\Omega \|f(\omega)\|^* \, d\mu(\omega) < \infty$.

Let $\mathcal{L}_{F^*}^\infty$ denote the Banach space of μ-equivalence classes of Bochner integrable functions $v : \Omega \longrightarrow F^*$ such that

$$\|v\|_\infty := \operatorname{ess\,sup}\{\|v(\omega)\|^* : \omega \in \Omega\} < \infty,$$

and let $\mathcal{L}_F^1$ denote the Banach space of μ-equivalence classes of Bochner integrable functions (with prequotient L_F^1) $l : \Omega \longrightarrow F$ such that

$$\|l(\omega)\|_1 := \int_\Omega \|l(\omega)\| \, d\mu(\omega) < \infty.$$

Because F^* has the Radon–Nikodym property, it follows from Theorem 1 of Diestel and Uhl [11] (p. 98) that $\mathcal{L}_{F^*}^\infty$ is the norm dual of $\mathcal{L}_F^1$.

Appendix 2: Continuity of payoff functions

Theorem A.1 (Continuity of Payoff Functions) *Suppose assumptions* $[\mathcal{DSG}^{\text{NR}}]$ *hold. Consider a* $\mathcal{DSG}$ *with underlying state-contingent game* $\mathcal{G}_{\Omega\times\mathcal{L}_Y^\infty}$, *given by*

$$\mathcal{G}_{\Omega\times\mathcal{L}_Y^\infty} := \{(A_d, \Phi_d(\omega), u_d(\omega, v_d, \cdot))_d : (\omega, v) \in \Omega \times \mathcal{L}_Y^\infty\},$$

where

$$u_d(\omega, v_d, a) := (1 - \beta_d) r_d(\omega, a) + \beta_d \int_\Omega v_d(\omega') h(\omega' | \omega, a) \, d\mu(\omega'),$$

and

$$(v, a) \longrightarrow u(\omega, v, a) := (u_1(\omega, v_1, a), \dots, u_m(\omega, v_m, a)).$$

If $\{(v^n, a^n)\}_n$ *is a sequence in* $\mathcal{L}^\infty_Y \times A$ *such that* $v^n \underset{w^*}{\longrightarrow} v^*$ *and* $a^n \underset{\rho_A}{\longrightarrow} a^*$*, then in each state* $\omega \in \Omega$,

$$u(\omega, v^n, a^n) \underset{R^m}{\longrightarrow} u(\omega, v^*, a^*).$$

Proof Let $\{(v^n, a^n)\}_n$ be a sequence such that $v^n \underset{w^*}{\longrightarrow} v^*$ and $a^n \underset{\rho_A}{\longrightarrow} a^*$. Let ω be given and fixed, and observe that for each player d:

$$\left.\begin{aligned} &|u_d(\omega, v^n_d, a^n) - u_d(\omega, v^*_d, a^*)|_R \\ &\quad\le \underbrace{|u_d(\omega, v^n_d, a^n) - u_d(\omega, v^n_d, a^*)|_R}_{A^n} + \underbrace{|u_d(\omega, v^n_d, a^*) - u_d(\omega, v^*_d, a^*)|_R}_{B^n}. \end{aligned}\right\} \tag{53}$$

We will carry out our proof for one player d, keeping in mind that the argument holds for all players simultaneously. Consider B^n first. We have

$$B^n = \beta_d \left| \int_\Omega v^n_d(\omega')q(\omega'|\omega, a^*)\, d\mu(\omega') - \int_\Omega v^*_d(\omega')q(\omega'|\omega, a^*)\, d\mu(\omega') \right|_R. \tag{54}$$

Let $h(\cdot|\omega, a^*)$ be a density of $q(\cdot|\omega, a^*)$ with respect to μ. Given that $h(\cdot|\omega, a^*) \in \mathcal{L}^1_R$ and $v^n_d \underset{w^*_d}{\longrightarrow} v^*_d$, where $\{v^n_d\}_n \subset \mathcal{L}^\infty_{Y_d} \subset \mathcal{L}^\infty_R$ we have

$$\left.\begin{aligned} &\textstyle\int_\Omega v^n_d(\omega')q(\omega'|\omega, a^*)\, d\mu(\omega') = \int_\Omega v^n_d(\omega')h(\omega'|\omega, a^*)\, d\mu(\omega') \\ &\quad\longrightarrow \quad \textstyle\int_\Omega v^*_d(\omega')h(\omega'|\omega, a^*)\, d\mu(\omega') = \int_\Omega v^*_d(\omega')q(\omega'|\omega, a^*)\, d\mu(\omega'). \end{aligned}\right\} \tag{55}$$

Thus, $B^n \overset{n}{\longrightarrow} 0$.

Next, consider A^n. We have

$$\left.\begin{aligned} A^n &\le (1 - \beta_d)\underbrace{|r_d(\omega, a^n) - r_d(\omega, a^*)|_R}_{A^n_1} \\ &\quad + \beta_d \underbrace{\left|\textstyle\int_\Omega v^n_d(\omega')q(\omega'|\omega, a^n)\, d\mu(\omega') - \int_\Omega v^n_d(\omega')q(\omega'|\omega, a^*)\, d\mu(\omega')\right|_R}_{A^n_2}. \end{aligned}\right\} \tag{56}$$

Continuity of $r_d(\omega, \cdot)$ and $a^n \underset{\rho_A}{\longrightarrow} a^*$ imply that $A^n_1 \overset{n}{\longrightarrow} 0$. To see that $A^n_2 \overset{n}{\longrightarrow} 0$, observe that

$$\left.\begin{aligned} &\left| \textstyle\int_\Omega v^n_d(\omega')q(\omega'|\omega, a^n)\, d\mu(\omega') - \int_\Omega v^n_d(\omega')q(\omega'|\omega, a^*)\, d\mu(\omega') \right| \\ &\quad \le M\|q(\cdot|\omega, a^n) - q(\cdot|\omega, a^*)\|_\infty \overset{n}{\longrightarrow} 0. \end{aligned}\right\} \tag{57}$$

□

Acknowledgements

Page is most grateful to Nancy C. Page for her unwavering support during the writing of this paper. Both authors are also grateful to J. P. Zigrand, Jon Danielsson, and Ann Law of the Systemic Risk Centre (SRC) at LSE for all their support and hospitality during many memorable visits to the SRC. Both authors thank J. P. Zigrand, Jon Danielsson, Nicholas Yannelis, Klaus Ritzberger, Erik Balder, Heinrich von Weizsäcker, Ondrej Kalenda, A. S. Nowak, Longin Rybiński, Akira Koyama, and Mark Lynch for many helpful discussions during the writing and revising of the paper.

Funding

Page acknowledges financial support from the Systemic Risk Centre (under ESRC grant number ES/K002309/1). Fu thanks JSPS KAKENHI for financial support under grant number 19K13662.

Abbreviations

$\mathcal{DSG}s$, discounted stochastic games; SMPE, stationary Markov perfect equilibria; USCOs, upper semicontinuous correspondences taking nonempty compact values; USCO*, USCO where upper semicontinuity is with respect to weak* topologies; $\mathcal{PSG}s$, parameterized state-contingent games; AR, absolute retract; AE, absolute extensor; ANR, absolute neighborhood retract; ANE, absolute neighborhood extensor; CUSCO*, the collection of all USCO*s with convex values.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally. Both authors read and approved the final manuscript.

Author details

[1]Systemic Risk Centre, London School of Economics and Political Science, Houghton St, Holborn, London, WC2A 2AE, UK. [2]Department of Economics, Indiana University, 100 S Woodlawn Ave, Bloomington, 47405, USA. [3]Department of System Management, Fukuoka Institute of Technology, 3-30-1 Wajiro-higashi, Higashi-ku, Fukuoka, 811-0295, Japan.

Endnotes

[a] See Appendix 1 for a definition and a discussion of Banach space valued Bochner integrable functions.

[b] A correspondence is approximable if it has a graph about which, for any $\varepsilon > 0$, an open ε-ball can be placed containing the graph of a continuous function (see De Blasi and Myjak [30], and Cellina [31]).

[c] We will often refer to an m-player $\mathcal{DSG}$ having an uncountable state space and compact metric action spaces as an uncountable-compact $\mathcal{DSG}$.

[d] An USCO is an upper semicontinuous correspondence taking nonempty compact values (e.g., see Hola and Holy [18]). A weak* USCO, is an USCO where upper semicontinuity is with respect to weak* topologies (USCO*).

[e] See Appendix 1 for a discussion of measurability and Bochner integrability.

[f] We will assume that the subset X of the dual Banach space E^* is norm bounded by M_X (i.e., each $x \in X$, $\|x\| \leq M_X$).

[g] Local connectedness differs from connectedness. To see this, note, for example, that the set E in R given by $E = [0,1) \cup (1,2]$ is locally connected but not connected (because E is equal to the union of two disjoint, half-open intervals). While the set G in R^2 given by

$$G := \left\{(x,0), \left(x, \frac{1}{n}\right) : 0 \leq x \leq 1 \text{ and } n = \pm 1, \pm 2, \ldots\right\} \cup \left\{(0,y), (1,y) : y \in R\right\}$$

is connected but not locally connected (because only the points $(0,0)$ and $(1,0)$ in G possess a collection of connected neighborhoods). These examples are taken from Willard [32], Chap. 8.

[h] Unicoherent continua are particularly important to game theory because unicoherent continua are not homeomorphic to the unit circle and therefore contain no simple closed curves. In game theory, the presence of simple closed curves can create nonexistence-of-equilibria problems (see Levy [4]).

[i] In a complete separable metric space, μ is nonatomic if and only if $\mu(\{\omega\}) = 0$.

[j] Here, $P_{*fc}(\mathcal{L}_X^\infty)$ denotes the collection of all nonempty, convex, $\rho^*_{\mathcal{L}_X^\infty}$-closed (and hence $\rho^*_{\mathcal{L}_X^\infty}$-compact) subsets of $\mathcal{L}_X^\infty$.

Publisher's Note

Received: 16 January 2020 Accepted: 9 September 2020 Published online: 22 September 2020

References

1. Nowak, A.S., Raghavan, T.E.S.: Existence of stationary correlated equilibria with symmetric information for discounted stochastic games. Math. Oper. Res. **17**(3), 519–526 (1992)
2. Gorniewicz, L., Granas, A., Kryszewski, W.: On the homotopy method in the fixed point index theory of multi-valued mappings of compact absolute neighborhood retracts. J. Math. Anal. Appl. **161**(2), 457–473 (1991)
3. Blackwell, D.: Discounted dynamic programming. Ann. Math. Stat. **36**(1), 226–235 (1965)
4. Levy, Y.: A discounted stochastic game with no stationary Nash equilibrium: two examples. Econometrica **81**(5), 1973–2007 (2013)
5. Levy, Y., McLennan, A.: Corrigendum to "Discounted stochastic games with no stationary Nash equilibrium: two examples". Econometrica **83**(3), 1237–1252 (2015)
6. Jaskiewicz, A., Nowak, A.S.: Nonzero-sum stochastic games. In: Handbook of Dynamic Game Theory, pp. 281–344. Springer, Cham (2018)
7. Jaskiewicz, A., Nowak, A.S.: Stationary almost Markov perfect equilibria in discounted stochastic games. Math. Oper. Res. **41**(2), 430–441 (2016)
8. Page, F.: A fixed point theorem for measurable selection valued correspondences arising in game theory. Discussion Paper 43, Systemic Risk Centre, London School of Economics and Political Science (2015)
9. Page, F.: Stationary markov equilibria for approximable discounted stochastic games. Discussion Paper 60, Systemic Risk Centre, London School of Economics and Political Science (2016)

10. Hildenbrand, W.: Core and Equilibria of a Large Market. Princeton University Press, Princeton (1974)
11. Diestel, J., Uhl, J.J. Jr.: Vector Measures. Math. Surveys, vol. 15. Am. Math. Soc., Providence (1977)
12. Balder, E.J.: New sequential compactness results for spaces of scalarly integrable functions. J. Math. Anal. Appl. **151**(1), 1–16 (1990)
13. Kuratowski, K.: Introduction to Set Theory and Topology, 2nd edn. Pergamon, Warszawa (1972)
14. Charatonik, J., Charatonik, M.: Dendrites. Aportaciones Mat. Comun. **22**, 227–253 (1998)
15. van Mill, J.: The Infinite-Dimensional Topology of Function Spaces. North-Holland Mathematical Library, vol. 64. North-Holland, Amsterdam (2002)
16. Borsuk, K.: Sur les retractes. Fundam. Math. **17**(1), 152–170 (1931)
17. Illanes, A., Nadler, S.: Hyperspaces: Fundamentals and Recent Advances. Dekker, New York (1999)
18. Hola, L., Holy, D.: Minimal usco and minimal cusco maps. Khayyam J. Math. **1**(2), 125–150 (2015)
19. Fryszkowski, A.: Continuous selections from a class of non-convex multivalued maps. Stud. Math. **76**(2), 163–174 (1983)
20. Liapounoff, A.A.: Sur les fonctions-vecteurs completement additives. Izv. Akad. Nauk SSSR, Ser. Mat. **4**(6), 465–478 (1940)
21. Mariconda, C.: Contractibility and fixed point property: the case of decomposable sets. Nonlinear Anal., Theory Methods Appl. **18**(7), 689–695 (1992)
22. Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis: A Hitchhiker's Guide, 3rd edn. Springer, Berlin (2006)
23. Hiai, F., Umegaki, H.: Integrals, conditional expectations, and martingales of multivalued functions. J. Multivar. Anal. **7**(1), 149–182 (1977)
24. Himmelberg, C.J.: Measurable relations. Fundam. Math. **87**(1), 53–72 (1975)
25. Castaing, C., Valadier, M.: Convex Analysis and Measurable Multifunctions. Lect. Notes Math., vol. 580. Springer, Berlin (1977)
26. Balder, E.J.: Lectures on young measure theory and its application in economics. Rend. Ist. Mat. Univ. Trieste **31**, 1–40 (1998)
27. Bourgin, R.: Geometric Aspects of Convex Sets with the Radon–Nikodym Property. Lect. Notes Math., vol. 993. Springer, Berlin (1983)
28. Cornet, B., Martins-da-Rocha, V.F.: Fatou's lemma for unbounded gelfand integrable mappings. Working Paper Series in Theoretical and Applied Economics 200503, Department of Economics, University of Kansas (2005)
29. Kahn, M.A.: On extensions of the Cournot-Nash theorem. In: Advances in Equilibrium Theory. Lect. Notes Econ. Math., vol. 244, pp. 79–106. Springer, Berlin (1985)
30. De Blasi, F.S., Myjak, J.: On continuous approximations for multifunctions. Pac. J. Math. **123**(1), 9–31 (1986)
31. Cellina, A.: Approximation of set valued functions and fixed point theorems. Ann. Mat. Pura Appl. **82**, 17–24 (1969)
32. Willard, S.: General Topology. Addison-Wesley, Reading (1970)