FILTERED HIRSCH ALGEBRAS

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Abstract. Motivated by the cohomology theory of loop spaces, we consider a special class of higher order homotopy commutative differential graded algebras and construct the filtered Hirsch model for such an algebra A. When $x \in H(A)$ with $\mathbb{Z}$ coefficients and $x^2 = 0$, the symmetric Massey products $\langle x \rangle^n$ with $n \geq 3$ have a finite order (whenever defined). However, if $\Bbbk$ is a field of characteristic zero, $\langle x \rangle^n$ is defined and vanishes in $H(A \otimes \Bbbk)$ for all n. If p is an odd prime, the Kraines formula $\langle x \rangle^p = -\beta \mathcal{P}_1(x)$ lifts to $H^*(A \otimes \mathbb{Z}_p)$. Applications of the existence of polynomial generators in the loop homology and the Hochschild cohomology with a G-algebra structure are given.

Contents

1. Introduction

In this paper we investigate a special class of homotopy commutative algebras called *Hirsch algebras* [20]. When the structural operations of a Hirsch algebra A agree component-wise with those of a homotopy G-algebra (HGA), the pre-Jacobi axiom can fail [7], [8], [19], [37] and the induced product on the bar construction

2000 *Mathematics Subject Classification.* Primary 55P35, 55S30; Secondary 55S05, 55U20.
Key words and phrases. Hirsch algebra, filtered model, multiplicative resolution, symmetric Massey product, Steenrod operation, Hochschild cohomology.
This research described in this publication was made possible in part by the grant GNF/ST06/3-007 of the Georgian National Science Foundation.

BA is not necessarily associative. Indeed, the theory of loop space cohomology suggests that it is impossible in general, to construct a small model for $H^*(\Omega X)$ in the category of HGAs. The investigation here applies a perturbation theory that extends the well-developed perturbation theories for differential graded modules and differential graded algebras (dgas) [3], [9], [13], [11], [27], [28].

One difficulty encountered when constructing a theory of homological algebra for Hirsch algebras is that the Steenrod cochain product $a \smile_1 b$ fails to be a cocycle even for cocycles a and b. Consequently $a \smile_1 b$ does not necessarily lift to cohomology. We control such difficulties by introducing the notion of a *filtered* Hirsch algebra, which can be thought of as a specialization of a distinguished resolution in the sense of [10] (see also [14]). On the other hand, the filtered Hirsch model $(RH, d+h)$ of a Hirsch algebra A is itself a Hirsch algebra whose structural operations $E_{p,q} : RH^{\otimes p} \otimes RH^{\otimes q} \longrightarrow RH$ are completely determined by the commutative graded algebra (cga) structure of $H = H(A, d_A)$; furthermore, the perturbation $h : RH \to RH$ of the resolution differential d is determined by the Hirsch algebra structure on A (Theorem 1). Thus by ignoring the operations $E_{p,q}$ we obtain a multiplicative resolution $(RH, d) \to (H, 0)$ of the cga H thought of as a non-commutative version of its Tate-Jozefiak resolution ([35], [16]) and the filtered model of the dga A is the perturbation $(RH, d+h) \to (A, d_A)$ in [27] (such a filtered model in the category of cdgas over a field of characteristic zero was constructed by Halperin and Stasheff in [11]).

A Hirsch resolution always admits a binary operation $\cup_2$, which can be viewed as *divided* Steenrod $\smile_2$-operation. This leads to the notion of a *quasi-homotopy commutative* Hirsch algebra (QHHA) introduced here. We note that in general, the construction of a Hirsch map $(RH, d+h) \to A$ compatible with a QHHA structure on A is obstructed by the non-free action of Sq_1 on its cohomology $H(A)$.

Every cdga H can be thought of as a trivial Hirsch algebra in which the operations $E_{p,q} \equiv 0$ for all $p, q \geq 1$. However, we exhibit an example of a cohomology algebra $H = H(A)$ with a non-trivial Hirsch algebra structure determined by Sq_1.

For a Hirsch algebra A over the integers, we establish some formulas relating the structural operations $E_{p,q}$ with syzygies in (RH, d) that arise from a single element $x \in H(A)$ with $x^2 = 0$. Whereas the n-fold symmetric Massey product $\langle x \rangle^n$ with $n \geq 3$ is defined in $H(A)$ ([23], [22]), our formulas imply that $\langle x \rangle^n$ has finite order. Note that when A is an algebra over a field $\Bbbk$ of characteristic zero, $\langle x \rangle^n$ is defined and vanishes for all $n \geq 3$ (Theorem 2). As a consequence we have (compare [4]):

Theorem A. *Let X be a simply connected space, let $\Bbbk$ be a field of characteristic zero and let $\sigma_* : H_*(\Omega X; \Bbbk) \to H_{*+1}(X; \Bbbk)$ be the suspension map. If $y \notin \operatorname{Ker} \sigma_*$ and $y^2 \neq 0$, then $y^n \neq 0$ for all $n \geq 2$.*

Given an odd prime p, consider the Hirsch algebra $A \otimes \mathbb{Z}_p$, let $x \in H^{2m+1}(A \otimes \mathbb{Z}_p)$, and let β be the Bockstein operator. We obtain the formula

$$\langle x \rangle^p = -\beta \mathcal{P}_1(x), \tag{1.1}$$

which has the same form as Kraines's formula in [23], however, the cohomology operation $\mathcal{P}_1 : H^{2m+1}(A \otimes \mathbb{Z}_p) \to H^{2mp+1}(A \otimes \mathbb{Z}_p)$ in (1.1) is canonically determined by the iteration of the $\smile_1$-product on $A \otimes \mathbb{Z}_p$ (Theorem 3). Dually, if A is the singular chains on the triple loop space $\Omega^3 X$, we can identify $\mathcal{P}_1$ with the Dyer-Lashof operation (see [22]). In fact the validity of (1.1) in a general algebraic framework is conjectured by May [25, Section 6]. Furthermore, when $X = BF_4$, the classifying space of the exceptional group F_4, we exhibit explicit perturbations in the filtered model of X and recover formula (1.1) in $H^*(X;\mathbb{Z}_3)$.

Although Theorem 1 provides a theoretical model of a Hirsch algebra A endowed with higher order operations $E_{p,q}$, in practice one can construct a small *multiplicative* model for recognizing $H^*(BA)$ as an algebra in which the product is determined only by the binary operation $E_{1,1}=\smile_1$. Thus, a (minimal) multiplicative resolution of $H^*(A)$ endowed with a $\smile_1$-product provides an economical way to calculate the algebra $H^*(BA)$. We apply this technique to the Hochschild cochain complex $A=C^\bullet(P;P)$ of an associative algebra P over a field $\Bbbk$ of characteristic zero to establish the following

Theorem B. *If the Hochschild cohomology $H^*=H(C^\bullet(P;P))$ is a free algebra, then the Lie algebra structure on $Tor_*^A(\Bbbk,\Bbbk)$ is completely determined by that of the G-algebra H^*. Consequently, the product μ^* on $Tor_*^A(\Bbbk,\Bbbk)$ is commutative if and only if the G-product on $H^*$is trivial.*

Some applications of filtered Hirsch algebras considered in an earlier version of this paper are also considered in [31], [32] (see also [29], [33]).

I wish to thank Jim Stasheff for helpful comments and suggestions. I am also indebted to the referee for a number of helpful comments and for having suggested many improvements of the exposition.

2. The category of Hirsch algebras

This section defines the generalized notion of a Hirsch algebra applied here, the morphisms between them, and the notion of a Hirsch resolution.

Let $\Bbbk$ be a commutative ring with unity 1 and characteristic ν; in the applications, $\Bbbk$ will be the integers $\mathbb{Z}$, a finite field $\mathbb{Z}_p=\mathbb{Z}/p\mathbb{Z}$ with p prime, or a field of characteristic zero. Graded $\Bbbk$-modules A^* are assumed to be graded over $\mathbb{Z}$. A module A^* is connected if $A^0=\Bbbk$, and a non-negatively graded, connected module A^* is 1-*reduced* if $A^1=0$.

For a module A, let $T(A)=\bigoplus_{i=0}^{\infty}A^{\otimes i}$, where $A^0=\Bbbk$, be the tensor module of A. An element $a_1\otimes\cdots\otimes a_n\in A^{\otimes n}$ is denoted by $[a_1|\cdots|a_n]$ when $T(A)$ is viewed as the tensor coalgebra or by $a_1\cdots a_n$ when $T(A)$ is viewed as the tensor algebra. We denote by $s^{-1}A$ the desuspension of A, i.e., $(s^{-1}A)^i=A^{i+1}$.

A dga (A,d_A) is assumed to be supplemented; in particular, it has the form $A=\tilde{A}\oplus\Bbbk$. The (reduced) bar construction BA on A is the tensor coalgebra $T(\bar{A})$, $\bar{A}=s^{-1}\tilde{A}$, with differential $d=d_1+d_2$ given for $[\bar{a}_1|\cdots|\bar{a}_n]\in T^n(\bar{A})$ by

$$d_1[\bar{a}_1|\cdots|\bar{a}_n]=-\sum_{1\le i\le n}(-1)^{\epsilon^a_{i-1}}[\bar{a}_1|\cdots|\overline{d_A(a_i)}|\cdots|\bar{a}_n]$$

and

$$d_2[\bar{a}_1|\cdots|\bar{a}_n]=-\sum_{1\le i<n}(-1)^{\epsilon^a_{i}}[\bar{a}_1|\cdots|\overline{a_ia_{i+1}}|\cdots|\bar{a}_n],$$

where $\epsilon^x_i=|x_1|+\cdots+|x_i|+i$.

Let us generalize (slightly) the definition of a Hirsch algebra [20]. Let A be a dga and consider the dg module $(Hom(BA\otimes BA,A),\nabla)$, where ∇ is the canonical Hom differential. Since the tensor product $BA\otimes BA$ is a dgc with the standard coalgebra structure, the $\smile$-product induces a dga structure on $(Hom(BA\otimes BA,A),\nabla,\smile)$.

Definition 1. *A **Hirsch algebra** is an associative dga A equipped with multilinear maps*

$$E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A,\ \ p,q \geq 0,\ \ p+q>0,$$

satisfying the following conditions:

(i) $\deg E_{p,q} = 1-p-q$;
(ii) $E_{1,0} = Id = E_{0,1}$ *and* $E_{p>1,0} = 0 = E_{0,q>1}$;
(iii) *The homomorphism $E : BA \otimes BA \to A$ defined by*

$$E([\bar{a}_1|\cdots|\bar{a}_p] \otimes [\bar{b}_1|\cdots|\bar{b}_q]) = E_{p,q}(a_1,...,a_p;b_1,...,b_q) \tag{2.1}$$

is a twisting cochain in the dga $(Hom(BA\otimes BA, A), \nabla, \smile)$, i.e., $\nabla E = -E \smile E$.

A morphism $f : A \to B$ *between two Hirsch algebras is a dga map f that commutes with $E_{p,q}$ for all p,q.*

Condition (iii) implies that $\mu_E : BA \otimes BA \to BA$ is a chain map; thus BA is a dg bialgebra whose multiplication μ_E is not necessarily associative (compare [8], [37], [5], [21], [26]); in particular, $\mu_{_{E_{10}+E_{01}}}$ is the shuffle product on BA, and a Hirsch algebra with $E_{p,q} \equiv 0$ for all $p,q \geq 1$ is just a cdga (cf. (2.3)). It is useful to express equation (2.1) component-wise:

$$\begin{aligned} dE_{p,q}(a_1,...,a_p;b_1,...,b_q) = &\sum_{1\leq i\leq p} (-1)^{\epsilon^a_{i-1}} E_{p,q}(a_1,...,da_i,...,a_p;b_1,...,b_q) \\ &+ \sum_{1\leq j\leq q} (-1)^{\epsilon^a_p+\epsilon^b_{j-1}} E_{p,q}(a_1,...,a_p;b_1,...,db_j,...,b_q) \\ &+ \sum_{1\leq i<p} (-1)^{\epsilon^a_i} E_{p-1,q}(a_1,...,a_ia_{i+1},...,a_p;b_1,...,b_q) \\ &+ \sum_{1\leq j<q} (-1)^{\epsilon^a_p+\epsilon^b_j} E_{p,q-1}(a_1,...,a_p;b_1,...,b_jb_{j+1},...,b_q) \\ + \sum_{\substack{0\leq i\leq p\\ 0\leq j\leq q\\ (i,j)\neq(0,0)}} (-1)^{\epsilon_{i,j}} & E_{i,j}(a_1,...,a_i;b_1,...,b_j)\cdot E_{p-i,q-j}(a_{i+1},...,a_p;b_{j+1},...,b_q), \\ &\epsilon_{i,j} = \epsilon^a_i + \epsilon^b_j + (\epsilon^a_i+\epsilon^a_p)\epsilon^b_j + 1. \end{aligned} \tag{2.2}$$

In particular, the operation $E_{1,1}$ satisfies conditions similar to Steenrod's cochain $\smile_1$-product:

$$dE_{1,1}(a;b) - E_{1,1}(da;b) + (-1)^{|a|}E_{1,1}(a;db) = (-1)^{|a|}ab - (-1)^{|a|(|b|+1)}ba; \tag{2.3}$$

consequently, $E_{1,1}$ measures the non-commutativity of the product $\cdot$ on A. We shall use the notation $a \smile_1 b = E_{1,1}(a;b)$ interchangeably. The following special cases will also be important for us, so we write them explicitly:

The Hirsch formulas up to homotopy

$$\begin{aligned} dE_{2,1}(a,b;c) = &E_{2,1}(da,b;c) - (-1)^{|a|}E_{2,1}(a,db;c) + (-1)^{|a|+|b|}E_{2,1}(a,b;dc) - \\ &(-1)^{|a|}(ab)\smile_1 c + (-1)^{|a|+|b|+|b||c|}(a\smile_1 c)b + (-1)^{|a|}a(b\smile_1 c) \end{aligned}$$

and

$$dE_{1,2}(a;b,c) = E_{1,2}(da;b,c) - (-1)^{|a|}E_{1,2}(a;db,c) + (-1)^{|a|+|b|}E_{1,2}(a;b,dc) + (-1)^{|a|+|b|}a \smile_1 (bc) - (-1)^{|a|+|b|}(a \smile_1 b)c - (-1)^{|a|(|b|-1)}b(a \smile_1 c)$$

tell us that the deviations of the binary operation $\smile_1$ from left and right derivation of the $\cdot$ product are measured by the respective boundaries of the operations $E_{1,2}$ and $E_{2,1}$ on three variables.

The following definition describes a class of Hirsch algebras in which the $\smile_1$-product itself is homotopy commutative (cf. (2.5) below).

Definition 2. *A **quasi-homotopy commutative Hirsch algebra** (QHHA) is a Hirsch algebra A equipped with a binary product $\cup_2 : A \otimes A \to A$ such that*

$$d(a \cup_2 b) = da \cup_2 b + (-1)^{|a|}a \cup_2 db + (-1)^{|a|}a \smile_1 b + (-1)^{(|a|+1)|b|}b \smile_1 a - q(a;b), \tag{2.4}$$

where $q(a;b)$ satisfies:

$(2.4)_1$ *Leibniz rule:* $dq(a;b) = -q(da;b) - (-1)^{|a|}q(a;db)$;
$(2.4)_2$ *Acyclicity:* $[q(a,b)] = 0 \in H(A,d)$ *for* $da = db = 0$.

Note that $(2.4)_1$ follows from the equalities (2.2) and $d^2 = 0$. Obviously, discarding the parameter $q(a;b)$, the above formula just becomes the Steenrod formula for the $\smile_2$ -cochain product:

$$d(a \smile_2 b) = da \smile_2 b + (-1)^{|a|}a \smile_2 db + (-1)^{|a|}a \smile_1 b + (-1)^{(|a|+1)|b|}b \smile_1 a. \tag{2.5}$$

However, $q(-;-)$ may be non-zero when passing to models constructed via cohomology as below. In the following four examples, the first is a naturally occurring example of a *cochain* Hirsch algebra (compare Example 5); in the second example QHHA structures are considered for certain Hirsch algebras; in the third and fourth examples a Hirsch algebra structure is lifted to the cohomology level. In fact, the fourth example was the original motivation for this paper.

Example 1. *The primary examples of Hirsch algebras for topological spaces X are their cubical or simplicial cochain complexes* [20], [19], [21]. *In the simplicial case one can choose $E_{p,q} = 0$ for $q \geq 2$ and obtain an HGA structure on the simplicial cochains $C^*(X;\Bbbk)$* [2] *(see also* [19]*). Furthermore, the product μ_E on $BC^*(X;\Bbbk)$ gives the multiplicative structure of the loop space cohomology $H^*(\Omega X;\Bbbk)$.*

Here the cochain complex $C^*(X;\Bbbk)$ of a space X is 1-reduced, since by definition $C^*(X;\Bbbk) = C^*(\mathrm{Sing}^1 X;\Bbbk)/C^{>0}(\mathrm{Sing}\ x;\Bbbk)$ where $\mathrm{Sing}^1 X \subset \mathrm{Sing} X$ is the Eilenberg 1-subcomplex generated by the singular simplices that send the 1-skeleton of the standard n-simplex Δ^n to the base point x of X. Unlike the cubical cochains, the Hirsch algebra structure of the simplicial cochains is associative*, i.e., the above product μ_E is associative.*

Example 2. *First, note that the Hirsch algebras from the previous example are also QHHA's by setting $\cup_2 = \smile_2$ and $q(-;-) = 0$. Let A be a* special Hirsch algebra*, i.e., A is an associative Hirsch algebra and BA also admits a Hirsch algebra structure. Then A is a QHHA since it admits a $\cup_2$-product satisfying (2.5) (cf.* [18]*). An important example of a special Hirsch algebra is $A = C^*(X;\Bbbk)$ from the previous* *example (cf.* [20], [34]*). Finally, for a QHHA A with ν to be zero or odd and $\smile_2$-product satisfying (2.5), define the* divided *$\smile_2$-operation $\cup_2$ as*

$$a \cup_2 b = \begin{cases} \frac{1}{2}\, a \smile_2 a, & a = b \\ a \smile_2 b, & \textit{otherwise.} \end{cases}$$

Then A with this $\cup_2$-operation is again a QHHA.

Example 3. *Let $(H, d = 0)$ be a free cga $H = S\langle \mathcal{H}^* \rangle$ generated by a graded set $\mathcal{H}^*$. Then any map of sets $\tilde{E}_{p,q} : \mathcal{H}^{\times p} \times \mathcal{H}^{\times q} \to H$ of degree $1 - p - q$ extends to a Hirsch algebra structure $E_{p,q} : H^{\otimes p} \otimes H^{\otimes q} \to H$ on H. Indeed, using formula (2.2) the construction goes by induction on the sum $p + q$. In particular, if only $\tilde{E}_{1,1}$ is non-zero then the image of $E_{p,q}$ for $p+q \geq 3$ is into the submodule of H spanned by the monomials of the form $\tilde{E}_{1,1}(a_1; b_1) \cdots \tilde{E}_{1,1}(a_k; b_k) \cdot x$ for $a_i, b_i \in \mathcal{H}$, $x \in H$, and $k \geq 1$.*

Example 4. *The argument in Example 3 suggests how to lift a Hirsch $\mathbb{Z}_2$-algebra structure from the cochain level to cohomology. Given a Hirsch algebra A, let $H = H^*(A)$. For a cocycle $a \in A^m$, one has $d_A E_{1,1}(a, a) = 0$ and $Sq_1 : H^m \to H^{2m-1}$ is defined by*

$$[a] \to [E_{1,1}(a, a)].$$

The trick here is to convert the Hirsch formulas up to homotopy on A to the Cartan formula $Sq_1(ab) = Sq_1 a \cdot Sq_0 b + Sq_0 a \cdot Sq_1 b$ on H by fixing a set of multiplicative generators $\mathcal{H} \subset H$. Define the map $\tilde{Sq}_{1,1} : \mathcal{H} \times \mathcal{H} \to H$ for $a, b \in \mathcal{H}$ by

$$\tilde{Sq}_{1,1}(a; b) = \begin{cases} Sq_1 a, & a = b, \\ 0, & \textit{otherwise} \end{cases}$$

and extend to the operation $Sq_{1,1} : H \otimes H \to H$ as a (two-sided) derivation with respect to the $\cdot$ product; then in particular, $Sq_{1,1}(u; u) = Sq_1 u$ for all $u \in H$. Define $Sq_{p,q} = E_{p,q} : H^{\otimes p} \otimes H^{\otimes q} \to H$ for $p + q \geq 3$ by means of (2.2). Note that if the multiplicative structure on H is not free, such an extension might not exist. This procedure gives a Hirsch algebra structure $\{Sq_{p,q}\}$ on the cohomology algebra H in the following situations:

- *(i) H has trivial multiplication (e.g. the cohomology of a suspension).*
- *(ii) H is a polynomial algebra.*
- *(iii) H has the following property: If $a \cdot b = 0$, then $Sq_1 a \cdot b = 0 = Sq_1 a \cdot Sq_1 b$ for all $a, b \in H$.*

Obviously we have the following proposition:

Proposition 1. *A morphism $f : A \to A'$ of Hirsch algebras induces a Hopf dga map of the bar constructions*

$$Bf : BA \to BA'.$$

If the modules A, A' are $\Bbbk$-free and f is a homology isomorphism, so is Bf.

This proposition is useful when applying special models for a Hirsch algebra A to calculate the cohomology algebra $H^*(BA) = Tor^A(\Bbbk, \Bbbk)$ (see Subsection 3.4 below), and consequently, the loop space cohomology $H^*(\Omega X; \Bbbk)$ when $A = C^*(X; \Bbbk)$ (see, for example, [31]).

Given a Hirsch algebra A with cohomology $H = H(A)$, let us construct a Hirsch algebra model of A. The commutative algebra H admits a special *multiplicative* resolution (RH, d), which is endowed with the Hirsch algebra structure $\{E_{p,q}\}$. The perturbed differential d_h on RH gives the desired Hirsch algebra model (RH, d_h) of A.

2.1. **Hirsch resolution.** Let H^* be a graded algebra and recall that a multiplicative resolution (R^*H^*, d) of H^* is the bigraded tensor algebra $T(V)$ generated by the bigraded free $\Bbbk$-module

$$V = \bigoplus_{j,m\geq 0} V^{-j,m},$$

where $V^{-j,m} \subset R^{-j}H^m$. The total degree of $R^{-j}H^m$ is the sum $-j+m$, d is of bidegree $(1,0)$ and $\rho : (RH, d) \to H$ is a map of bigraded algebras inducing an isomorphism $\rho^* : H^*(RH, d) \overset{\approx}{\to} H^*$ where H^* is bigraded via $H^{0,*} = H^*$ and $H^{<0,*} = 0$ ([27]; compare [11], [13]). In other words,

$$\left((R^*H^m, d) \overset{\rho}{\to} H^m\right) = (\cdots \overset{d}{\to} R^{-2}H^m \overset{d}{\to} R^{-1}H^m \overset{d}{\to} R^0H^m \overset{\rho}{\to} H^m)$$

is a usual free (additive) resolution of the $\Bbbk$-module H^m for each m, and there is a multiplication on the family $\{R^*H^m\}_{m\in\mathbb{Z}}$, which is compatible with both d and the bidegree. When each H^m is $\Bbbk$-free, ΩBH (the cobar-bar construction of H) is an example of RH with $V = BH$. In general, the multiplicative structure of H^* gives rise to (additively) *non-minimal* submodules (R^*H^m, d) even for H^m to be $\Bbbk$-free or $H^m = 0$. The reason for this is that a (multiplicative) relation in H involving elements of degree $< m$ can produce an element $a \in R^{-1}H^k$ with $k < m$, say $m = kn$, some $n \geq 2$, and since the multiplication on R^*H^* respects the bidegree, the non-zero element a^n, the n^{th} power of a, ultimately belongs to $R^{-n}H^m$, the n^{th} component of a $\Bbbk$-module resolution of H^m (see the proof of Proposition 3 below). Furthermore, even for H to be a free cga over a field $\Bbbk$, the non-commutative nature of RH fails to imply R^*H^m to be a minimal $\Bbbk$-module resolution of H^m, i.e.,

$$R^0H^m = H^m \ \text{ and } \ R^{-i}H^m = 0,\, i > 0;$$

this is quite different from the situation in [11].

For example, consider the polynomial algebra $H = \mathbb{Z}_2[x, y]$ with $x, y \in H^2$ and $x_0, y_0 \in R^0H^2$ satisfying $\rho x_0 = x$ and $\rho y_0 = y$. Then $R^{-1}H^4 \neq 0$ since there is an element $a \in R^{-1}H^4$ such that $da = x_0y_0 + y_0x_0$. In particular, if H is the cohomology of a dga A with a *non-commutative* $\smile_1$-product (and perhaps higher order operations $E_{p,q}$; cf. Examples 1 and 5), then the construction of a *Hirsch algebra* model of A using RH requires to add another element b in $R^{-1}H^4$ with $db = x_0y_0 + y_0x_0$. Then denote $a = x_0 \smile_1 y_0$ and $b = y_0 \smile_1 x_0$ respectively (see Theorem 1). Furthermore, if H^* is 1-reduced and we wish to have a 1-reduced multiplicative resolution RH, we must restrict the resolution length of R^*H^m so that $R^{-i}H^m = 0$ for $i \geq m-1$ (e.g. H^m is $\Bbbk$-free for all m or H^2 is $\Bbbk$ -free and $\Bbbk$ is a principal ideal domain). This motivates the following definition:

Definition 3. *Let H^* be a cga. An absolute Hirsch resolution of H is a multiplicative resolution*

$$\rho : R^*H^* \to H^*, \quad RH = T(V), \quad V = \langle \mathcal{V} \rangle,$$

endowed with the Hirsch algebra structural operations

$$E_{p,q} : RH^{\otimes p} \otimes RH^{\otimes q} \to V \subset RH$$

such that V is decomposed as $V^{*,*}=\mathcal{E}^{*,*}\oplus U^{*,*}$ in which $\mathcal{E}^{0,*}=0$, $U^{0,*}=V^{0,*}$ and $\mathcal{E}^{*,*}=\bigoplus_{p,q\geq 1}\mathcal{E}^{<0,*}_{p,q}$ is distinguished by an isomorphism of modules

$$E_{p,q}:\bigoplus_{\substack{i_{(p)}+j_{(q)}=s\\ k_{(p)}+\ell_{(q)}=t}}\left(\underset{1\leq r\leq p}{\otimes}R^{i_r}H^{k_r}\bigotimes\underset{1\leq n\leq q}{\otimes}R^{j_n}H^{\ell_n}\right)\xrightarrow{\approx}\mathcal{E}^{s-p-q+1\,,\,t}_{p,q}\subset V^{*,*}$$

where $x_{(r)}=x_1+\cdots+x_r$.

Given a Hirsch algebra $(A,\{E_{p,q}\},d)$, a submodule $J\subset A$ is a *Hirsch ideal* of A if it is an ideal with $E_{p,q}(a_1,...,a_p;a_{p+1},...,a_{p+q})\in J$ whenever $a_i\in J$ for some i.

Definition 4. *Let $\rho_a:(R_aH,d)\to H$ be an absolute Hirsch resolution and $J\subset R_aH$ be a Hirsch ideal such that $d:J\to J$ and the quotient map $g:R_aH\to R_aH/J$ is a homology isomorphism. A Hirsch resolution of H is the Hirsch algebra $RH=R_aH/J$ with a map $\rho:RH\to H$ such that $\rho_a=\rho\circ g$.*

Thus an absolute Hirsch resolution is a Hirsch resolution by taking $J=0$.

Proposition 2. *Every cga H^* has an (absolute) Hirsch resolution $\rho:R^*H^*\to H^*$.*

Proof. We build a Hirsch resolution of H^* by induction on the resolution degree. Let $\mathcal{H}^*\subset H^*$ be a set of multiplicative generators. Denote $\mathcal{V}^{0,*}=\mathcal{H}^*$; let $V^{0,*}=\langle\mathcal{V}^{0,*}\rangle$ be the free $\Bbbk$-module span of $\mathcal{V}^{0,*}$ and form the free (tensor) graded algebra $R^0H^*=T(V^{0,*})$. Obviously, there is a dga epimorphism $\rho^0:(R^0H^*,0)\to H^*$. Inductively, given $n\geq 0$, assume we have constructed a $\Bbbk$-module $R^{(-n)}H^*=\oplus_{0\leq r\leq n}R^{-r}H^*$ with a map $\rho^{(n)}:(R^{(-n)}H^*,d)\to H^*$ with $\rho^r(R^{-r}H^*)=0$ for $1\leq r\leq n$, where $d:R^{-r}H^*\to R^{-r+1}H^*$ is a differential of bidegree $(1,0)$ defined for $1\leq r\leq n$ and acyclic in resolution degrees $-r$ for $1\leq r<n$; $R^{-r}H^*$ is a component of bidegree $(-r,*)$ of $T(V^{(-r),*})$ for $V^{(-r),*}=V^{0,*}\oplus\cdots\oplus V^{-r,*}$, so that

$$R^{-r}H^*=V^{-r,*}\oplus\mathcal{D}^{-r,*}=\mathcal{E}^{-r,*}\oplus U^{-r,*}\oplus\mathcal{D}^{-r,*}$$

where $\mathcal{E}^{-r,*}=\bigoplus_{p,q\geq 1}\mathcal{E}^{-r,*}_{p,q}$ and $\mathcal{E}^{-r,*}_{p,q}$ spans the set of (formal) expressions $E_{p,q}(a_1,...,a_p;b_1,...,b_q)$, $a_j\in R^{-i_k}H^*$, $b_\ell\in R^{-j_\ell}H^*$, $r=i_{(p)}+j_{(q)}+p+q-1$, while $\mathcal{D}^{-r,*}$ is the module of decomposables of bidegree $(-r,*)$ in $T(V^{(-r),*})$; d is given by formula (2.2) on $\mathcal{E}^{-r,*}$, while acts as a derivation on $\mathcal{D}^{-r,*}$.

Let $\mathcal{E}^{-n-1,*}=\bigoplus_{p,q\geq 1}\mathcal{E}^{-n-1,*}_{p,q}$ where $\mathcal{E}^{-n-1,*}_{p,q}$ spans the set of expressions $E_{p,q}(a_1,...,a_p;b_1,...,b_q)$, $a_k\in R^{-i_k}H^*$, $b_\ell\in R^{-j_\ell}H^*$, $n+1=i_{(p)}+j_{(q)}+p+q-1$, and let $\mathcal{D}^{-n-1,*}$ be the module of decomposables of bidegree $(-n-1,*)$ in $T\left(V^{(-n),*}\oplus\mathcal{E}^{-n-1,*}\right)$; Define d by formula (2.2) on $\mathcal{E}^{-n-1,*}$ and as a derivation on $\mathcal{D}^{-n-1,*}$ so that

$$\mathcal{E}^{-n-1,*}\oplus\mathcal{D}^{-n-1,*}\xrightarrow{d}R^{-n}H^*\xrightarrow{d}R^{-n+1}H^*.$$

Define a free $\Bbbk$-module $U^{-n-1,*}$ and d on it to achieve acyclicity in resolution degree $-n$, i.e, denoting $V^{-n-1,*}=\mathcal{E}^{-n-1,*}\oplus U^{-n-1,*}$, we obtain a partial resolution for each $m\in\mathbb{Z}$

$$V^{-n-1,m}\oplus\mathcal{D}^{-n-1,m}\xrightarrow{d}R^{-n}H^m\xrightarrow{d}R^{-n+1}H^m\xrightarrow{d}\cdots\xrightarrow{d}R^{-1}H^m\xrightarrow{d}R^0H^m\xrightarrow{\rho}H^m.$$

Define $R^{-n-1}H^*=V^{-n-1,*}\oplus\mathcal{D}^{-n-1,*}$ and $\rho^{n+1}:R^{-n-1}H^*\to H^*$ to be trivial. This completes the inductive step.

Finally, set $R^*H^* = \oplus_n R^{(-n)}H^*$ with $V^{*,*} = \langle \mathcal{V}^{*,*}\rangle$, $\mathcal{E}^{*,*} = \oplus_n \mathcal{E}^{-n,*}$, $U^{*,*} = \oplus_n U^{-n,*}$, $\rho|_{R^0H^*} = \rho^0$ and $\rho|_{R^{-n}H^*} = 0$ for $n > 0$ to obtain the desired resolution map $\rho : RH \to H$. □

Note that in a Hirsch resolution $(RH, \{E_{p,q}\}, d)$, we may have relations among $E_{p,q}$'s (e.g. $E_{p,q} = 0$ for some $p, q \geq 1$; cf. Subsection 2.6). For example, the Hirsch structure of RH is *associative* if the product $\mu_{_E}$ on the bar construction $B(RH)$ is associative and is equivalent to the equalities among $E_{p,q}$'s as follows.

Given a Hirsch algebra A and an arbitrary triple

$$(\mathbf{a};\mathbf{b};\mathbf{c}) = (a_1, ..., a_k\,; b_1, ..., b_\ell\,; c_1, ..., c_r), \quad a_i, b_j, c_s \in A,$$

denote

$$\mathcal{R}_{k,\ell,r}((\mathbf{a};\mathbf{b});\mathbf{c}) = \sum_{\substack{k_{(p)}=k;\ell_{(p)}=\ell \\ 1\leq p\leq k+\ell}} (-1)^{\varepsilon} E_{p,r}(E_{k_1,\ell_1}(a_1, ..., a_{k_1}; b_1, ..., b_{\ell_1}),$$
$$..., E_{k_p,\ell_p}(a_{_{k-k_p+1}}, ..., a_k; b_{_{\ell-\ell_p+1}}, ..., b_p)\,; c_1, ..., c_r)$$

and

$$\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c})) = \sum_{\substack{\ell_{(q)}=\ell;r_{(q)}=r \\ 1\leq q\leq \ell+r}} (-1)^{\delta} E_{k,q}(a_1, ..., a_k; E_{\ell_1,r_1}(b_1, ..., b_{\ell_1}; c_1, ..., c_{r_1}),$$
$$..., E_{\ell_q,r_q}(b_{_{\ell-\ell_q+1}}, ..., b_\ell; c_{_{r-r_q+1}}, ..., c_q)),$$

where we use the convention that $E_{0,1}(-;a) = E_{1,0}(a;-) = a$, $E_{0,m}(-;a_1, ..., a_m) = E_{m,0}(a_1, ..., a_m;-) = 0, m \geq 2$, and $x_{(n)} = x_1 + \cdots + x_n$, while the signs ε and δ are induced by permutations of symbols a_i, b_j, c_s (cf. [37]). Then the associativity of A is equivalent to the equalities

$$\mathcal{R}_{k,\ell,r}((\mathbf{a};\mathbf{b});\mathbf{c}) = \mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c})),\ k, \ell, r \geq 1. \tag{2.6}$$

Now consider the expression

$$\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c})) - \mathcal{R}_{k,\ell,r}((\mathbf{a};\mathbf{b});\mathbf{c}) \ \in\ \mathcal{E}^{1-k-\ell-r\,,*} \tag{2.7}$$

in an absolute Hirsch resolution RH. We have that this expression belongs to $\mathcal{E}^{-2,*}$ and is a cocycle for $(\mathbf{a};\mathbf{b};\mathbf{c}) = (a;b;c)$, $a, b, c \in R^0H$ (see (2.8) and Fig. 1 below in which the boundaries of both hexagons are labelled by the 6 components of $d\mathcal{R}_{1,1,1}(a;(b;c)) = d\mathcal{R}_{1,1,1}((a;b);c)$). So there is an element, denoted by $s(\mathcal{R}_{1,1,1}\,(a;(b;c))) \in V^{-3,*}$ such that $ds(\mathcal{R}_{1,1,1}\,(a;(b;c))) = \mathcal{R}_{1,1,1}(a;(b;c)) - \mathcal{R}_{1,1,1}((a;b);c)$. In general, define elements $s(\mathcal{R}_{k,\ell,r}\,(\mathbf{a};(\mathbf{b};\mathbf{c}))) \in V$ such that

$$\begin{aligned}
&ds(\mathcal{R}_{k,\ell,r}\,(\mathbf{a};(\mathbf{b};\mathbf{c}))) + \\
&s(\mathcal{R}_{k,\ell,r}\,(d\mathbf{a};(\mathbf{b};\mathbf{c}))) + (-1)^{\varepsilon_1} s(\mathcal{R}_{k,\ell,r}\,(\mathbf{a};(d\mathbf{b};\mathbf{c}))) + (-1)^{\varepsilon_2} s(\mathcal{R}_{k,\ell,r}\,(\mathbf{a};(\mathbf{b};d\mathbf{c}))) \\
&\qquad = \mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c})) - \mathcal{R}_{k,\ell,r}((\mathbf{a};\mathbf{b});\mathbf{c}) \\
&\qquad\qquad \varepsilon_1 = |\mathbf{a}| + k,\ \varepsilon_2 = |\mathbf{a}| + |\mathbf{b}| + k + \ell.
\end{aligned}$$

Consequently, $RH = R_aH/J_{ass}$ is an associative Hirsch resolution, where $J_{ass} \subset R_aH$ is a Hirsch ideal generated by

$$\{\mathcal{R}_{k,\ell,r}(\mathbf{a};(\mathbf{b};\mathbf{c})) - \mathcal{R}_{k,\ell,r}((\mathbf{a};\mathbf{b});\mathbf{c}),\ s(\mathcal{R}_{k,\ell,r}\,(\mathbf{a};(\mathbf{b};\mathbf{c})))\}.$$

In particular, for $(\mathbf{a};\mathbf{b};\mathbf{c}) = (a;b;c)$ the associativity of a Hirsch resolution implies the following

Proposition 3. *For $a,b,c\in RH$, there is the equality*

$$(a\smile_1 b)\smile_1 c+E_{2,1}(a,b;c)+(-1)^{(|a|+1)(|b|+1)}E_{2,1}(b,a;c) = a\smile_1(b\smile_1 c)+E_{1,2}(a;b,c)+(-1)^{(|b|+1)(|c|+1)}E_{1,2}(a;c,b). \tag{2.8}$$

A Hirsch resolution (RH,d) is *minimal* if

$$d(u)\in\mathcal{E}+\mathcal{D}+\kappa_u\cdot V \quad \text{for} \quad \text{all } u\in U,$$

where $\mathcal{D}^{*,*}\subset R^*H^*$ denotes the submodule of decomposables $RH^+\cdot RH^+$ (RH^+ denotes RH modulo the unital component) and $\kappa_u\in\Bbbk$ is non-invertible. For example, when $\Bbbk=\mathbb{Z}$ we have $\kappa_u\in\mathbb{Z}\setminus\{-1,1\}$; when $\Bbbk$ is a field we have $\kappa_u=0$ for all u. Note that a minimal Hirsch resolution is *not* minimal in the category of dgas since the resolution differential does not send multiplicative generators into $\mathcal{D}$ even when $\Bbbk$ is a field. Furthermore, the notion of minimality of RH does not depend upon whether some operation $E_{p,q}$ is zero (cf. Subsection 2.6). On the other hand, in order to define a $\smile_2$-operation in a simple way on RH we have to consider a non-minimal Hirsch resolution in the next subsection.

Such a flexibility of choice of RH is due to the trivial Hirsch structure of H, and, in practice, the choice is suggested by a Hirsch algebra A that realizes H as the cohomology algebra.

2.2. **QHHA structures on Hirsch algebras.** First, note that one can introduce a $\smile_2$-product on a Hirsch resolution that satisfies (2.5). However, such a QHHA structure on RH in not always satisfactory, and we shall consider a $\cup_2$-operation simultaneously for the reasons explained below. For an even dimensional a, or for any a whenever $\nu=2$, we have that $a\smile_1 a$ is cocycle for $da=0$; hence, there is an element $x\in RH$ with $dx=a\smile_1 a$. But we can not identify x with $a\smile_2 a$ because $d(a\smile_2 a)=0$ according to (2.5). On the other hand, it is helpful to denote $x:=a\cup_2 a$ since certain formulas are conveniently expressed in terms of the binary operation $\cup_2$ (see, for example, Proposition 5 or Remark 7). Furthermore, we can identify $a\cup_2 a$ with $\frac{1}{2}a\smile_2 a$ for $|a|$ even and 2 invertible in $\Bbbk$.

By construction of a Hirsch resolution in Proposition 2, the definition of $\smile_2$ mimics that of $\smile_1$. We start with the consideration of the expression

$$(-1)^a a\smile_1 b+(-1)^{(|a|+1)|b|}b\smile_1 a\ \in\mathcal{E}^{-1,*} \quad \text{for} \quad a,b\in\mathcal{V}^{0,*}.$$

It is a cocycle in (RH,d), and hence, must be killed by a multiplicative generator; denote this generator by $a\smile_2 b\in U^{-2,*}$. Inductively, assume that the right-hand side of (2.5) has been defined as an element of $U^{-n+1,*}$. Then it is bounded by a multiplicative generator $a\smile_2 b\in U^{-n,*}$. Thus, $a\smile_2 b\in U$ for all $a,b\in RH$. In particular, if $dx=0$, then $d(x\smile_2 x)=0$ or $d(\frac{\nu}{2}x\smile_2 x)=0$ for $|x|$ to be odd or for both $|x|$ and ν to be even respectively in which case a multiplicative generator $y\in U$ with $dy=x\smile_2 x$ is denoted by $x\cup_3 x$.

Now define a $\cup_2$-operation by

$$a\cup_2 b=\begin{cases} a\smile_2 b, & a\neq b,\quad a,b \text{ are in a basis of } RH\\ 0, & a=b,\quad |a| \text{ and } \nu \text{ are odd },\end{cases} \tag{2.9}$$

while, otherwise, define $a\cup_2 a\in U$ by

$$d(a\cup_2 a)=\begin{cases} a\smile_1 a+a\smile_2 da+da\cup_3 da, & |a| \text{ is even}\\ \frac{\nu}{2}(a\smile_1 a+a\smile_2 da)+da\cup_3 da, & |a| \text{ is odd},\quad \nu \text{ is even.}\end{cases} \tag{2.10}$$

Hence, $a \cup_2 b \in U$ for any $a, b \in RH$, and let

$$\mathcal{T} = \{a \cup_2 b \in U \mid a, b \in RH\}.$$

Thus, we obtain the decomposition $U = \mathcal{T} \oplus \mathcal{M}$, some $\mathcal{M}$, and, hence, the decomposition

$$V = \mathcal{E} \oplus U = \mathcal{E} \oplus \mathcal{T} \oplus \mathcal{M}.$$

In particular, $\mathcal{T}$ contains elements of the form $a_1 \cup_2 \cdots \cup_2 a_n$, $a_i \in RH$, obtained by the iteration of the $\cup_2$-product for $n \geq 2$. In particular, for $a_i \in V^{0,2r}$ we have the following equality

$$d(a_1 \cup_2 \cdots \cup_2 a_n) = \sum_{(\mathbf{i};\mathbf{j})} sgn(\mathbf{i};\mathbf{j})(a_{i_1} \cup_2 \cdots \cup_2 a_{i_k}) \smile_1 (a_{j_1} \cup_2 \cdots \cup_2 a_{j_\ell}),$$

where the summation is over unshuffles $(\mathbf{i};\mathbf{j}) = (i_1 < \cdots < i_k\,; j_1 < \cdots < j_\ell)$ of $\underline{n}$ with $(a_{i_1}, ..., a_{i_k}) = (a_{i'_1}, ..., a_{i'_k})$ if and only if $\mathbf{i} = \mathbf{i}'$ and $sgn(\mathbf{i};\mathbf{j})$ is induced by the permutation sign $a_i \cup_2 a_j = (-1)^{|a_i||a_j|} a_j \cup_2 a_i$ (see also Fig. 1 for $n = 3$); consequently, for $a_1 = \cdots = a_n = a$ and $a^{\cup_2 n} := a \cup_2 \cdots \cup_2 a$, we get

$$da^{\cup_2 n} = \sum_{k+\ell=n} a^{\cup_2 k} \smile_1 a^{\cup_2 \ell}, \qquad k, \ell \geq 1. \tag{2.11}$$

Note that the above equalities do not depend on the parity of a_i's when $\nu = 2$.

Remark 1. *1. The definition of $\mathcal{T}$ does not depend on the (Hirsch) associativity of RH.*

2. In a minimal Hirsch resolution one can also minimize the module $\mathcal{T}$ as

$$\mathcal{T} = \{a \cup_2 b \in U \mid a, b \in \mathcal{M}\},$$

while $a \cup_2 b$ for $a, b \in RH$ is extended by certain derivation *formulas. These formulas are rather complicated, but they could be written down if necessary.*

3. The module $\mathcal{M}$ reflects the complexity of the multiplicative relations of the commutative algebra H.

For example, if H is a polynomial algebra and RH is a minimal Hirsch resolution, then $\mathcal{M} = \mathcal{M}^{0,*} = V^{0,*}$ and, consequently, RH is completely determined by the $\smile_1$- and $\cup_2$-operations [31] (see also Theorem 4 below).

2.3. **Some canonical syzygies in the Hirsch resolution.** Below we give topological interpretation of some canonical syzygies in the Hirsch resolution RH. In particular these syzygies reflect the non-associativity of the $\smile_1$-product. Remark that higher order canonical syzygies should be also related with the combinatorics of permutahedra. In practice, such relations are helpful to construct small Hirsch resolutions RH (cf. [31], see also Remark 1 above).

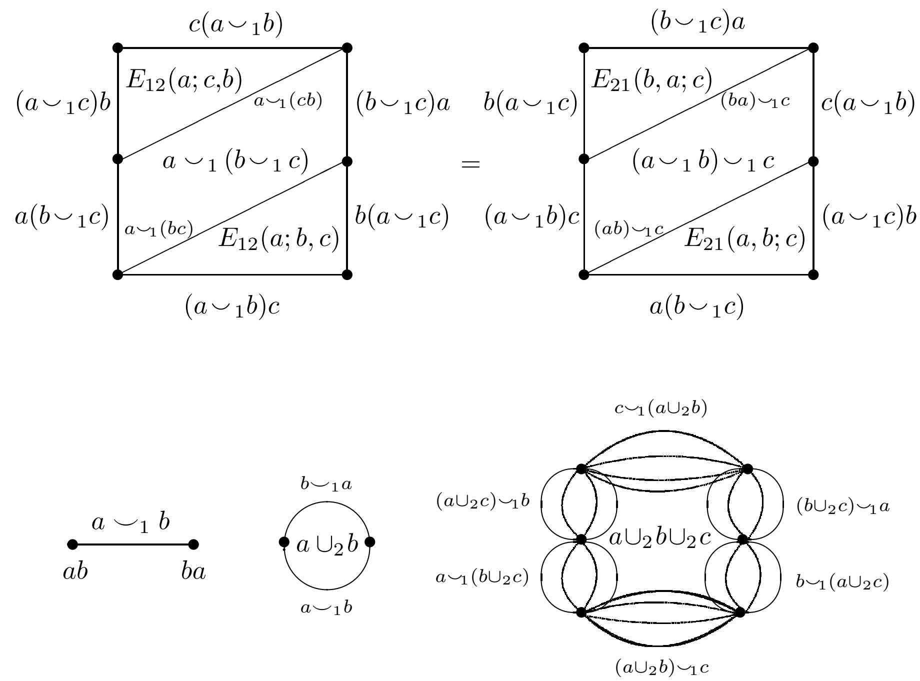

Figure 1. Topological interpretation of some canonical syzygies in the Hirsch resolution RH.

The symbol "=" in the figure above assumes equality (2.8); the picture for $a\cup_2 b\cup_2 c$ is in fact 4-dimensional and must be understood as follows: Whence $a\cup_2 b$ corresponds to the 2-ball, the boundary of $a\cup_2 b\cup_2 c$ consists of the six 3-balls each of which is subdivided into four 3-cells by fixing two equators (these cells just correspond to the four summand components of the differential evaluated on the compositions of the $\smile_1$- and $\cup_2$-products). Then given a 3-ball, two cells from these four cells are glued to the ones of the boundary of the (diagonally) opposite 3-ball, and the other cells are glued to the ones of the boundaries of the neighboring 3-balls according to the relation

$$x\smile_1(y\smile_1 z)+(x\smile_1 y)\smile_1 z=y\smile_1(x\smile_1 z)+(y\smile_1 x)\smile_1 z.$$

2.4. **Filtered Hirsch model.** Recall that a dga (A^*,d) is *multialgebra* if it is bigraded $A^n=\underset{n=i+j}{\oplus}A^{i,j}$, $i\leq 0,\ j\geq 0$, and $d=d^0+d^1+\cdots+d^n+\cdots$ with $d^n:A^{p,q}\to A^{p+n,q-n+1}$ [12]. A dga A is bigraded via $A^{0,*}=A^*$ and $A^{i,*}=0$ for $i\neq 0$; consequently, A is a multialgebra. A multialgebra A is *homological* if $d^0=0$ (hence $d^1d^1=0$) and

$$H^i(\cdots\xrightarrow{d^1}A^{i,*}\xrightarrow{d^1}A^{i+1,*}\xrightarrow{d^1}\cdots\xrightarrow{d^1}A^{0,*})=0,\quad i<0.$$

For a homological multialgebra the sum $d^2+d^3+\cdots+d^n+\cdots$ is called a *perturbation* of d^1. In the sequel we always consider homological multialgebras, d^1 is denoted by d, d^r is denoted by h^r, and the sum $h^2+h^3+\cdots+h^n+\cdots$ is denoted by h. We sometimes denote $d+h$ by d_h.

A *multialgebra morphism* $\zeta : A \to B$ between two multialgebras A and B is a dga map of total degree zero that preserves the resolution (column) filtration, so that ζ has the components $\zeta = \zeta^0 + \cdots + \zeta^i + \cdots , \quad \zeta^i : A^{s,t} \to B^{s+i,t-i}$. A chain homotopy $s : A \to B$ between two multiplicative maps $f, g : A \to B$ is an (f,g)-*derivation* homotopy if $s(ab) = s(a)g(b) + (-1)^{|a|} f(a)s(b)$. A *homotopy* between two morphisms $f, g : A \to B$ of multialgebras is an (f,g) -derivation homotopy $s : A \to B$ of total degree -1 that lowers the column filtration by 1.

A multialgebra is *quasi-free* if it is a tensor algebra over a bigraded $\Bbbk$-module. Given $m \geq 2$, the map $h^m|_{A^{-m,*}} : A^{-m,*} \to A^{0,*}$ is referred to as the *transgressive* component of h and is denoted by h^{tr}. A multialgebra A with a Hirsch algebra structure

$$E_{p,q} : \otimes_{r=1}^{p} A^{i_r,k_r} \bigotimes \otimes_{n=1}^{q} A^{j_k,\ell_n} \longrightarrow A^{s-p-q+1\,,\,t}$$

with $(s,t) = \left(i_{(p)} + j_{(q)}\,,\, k_{(p)} + \ell_{(q)}\right)$, $p, q \geq 1$, is called *Hirsch multialgebra*. A *homotopy* between two morphisms $f, g : A \to A'$ of Hirsch (multi)algebras is a homotopy $s : A \to A'$ of underlying (multi)algebras and

$$\begin{aligned}
&s(E_{p,q}(a_1,...,a_p\,;b_1,...,b_q)) \\
&\quad = \sum_{1\leq \ell\leq q} (-1)^{\epsilon^a_p+\epsilon^b_{\ell-1}} E_{p,q}(fa_1,...,fa_p\,;fb_1,...,fb_{\ell-1},sb_\ell,gb_{\ell+1},...,gb_q) \\
&\quad + \sum_{1\leq k\leq p} (-1)^{\epsilon^a_{k-1}} E_{p,q}(fa_1,...,fa_{k-1},sa_k,ga_{k+1},...,ga_p\,;gb_1,...,gb_q) \\
&\quad - \sum_{\substack{1\leq i\leq p\\ 1<\ell\leq j\leq q}} (-1)^{\epsilon_{i,j,\ell}} E_{i,j}(fa_1,...,fa_i\,;fb_1,...,fb_{\ell-1},sb_\ell,gb_{\ell+1},...,gb_j) \\
&\qquad\qquad\qquad \cdot E_{p-i,q-j}(fa_{i+1},...,fa_{p-1},sa_p\,;gb_{j+1},...,gb_q) \\
&\quad - \sum_{\substack{0\leq i<k\leq p\\ 1\leq j\leq q}} (-1)^{\epsilon_{i,j,k}} E_{i,j}(fa_1,...,fa_i\,;sb_1,gb_2,...,gb_j) \\
&\qquad\qquad \cdot E_{p-i,q-j}(fa_{i+1},...,fa_{k-1},sa_k,ga_{k+1},...,ga_p\,;gb_{j+1},...,gb_q), \\
&\qquad\qquad\qquad \epsilon_{i,j,m} = \epsilon^a_{p-1} + \epsilon^b_{m-1} + (\epsilon^a_p + \epsilon^a_i)\epsilon^b_j,\ p,q\geq 1,
\end{aligned} \tag{2.12}$$

in which the first equality is

$$s(a \smile_1 b) = (-1)^{|a|+1} fa \smile_1 sb + sa \smile_1 gb - (-1)^{(|a|+1)(|b|+1)} sb \cdot sa.$$

Denote the homotopy classes of morphisms between two Hirsch (multi)algebras by $[-,-]$.

Definition 5. *A quasi-free Hirsch homological multialgebra $(A, \{E_{p,q}\}, d+h)$ is a **filtered Hirsch algebra** if it has the following additional properties:*

(i) In $A = T(V)$ a decomposition

$$V^{*,*} = \mathcal{E}^{*,*} \oplus U^{*,*}$$

is fixed where $\mathcal{E}^{*,*} = \bigoplus_{p,q\geq 1} \mathcal{E}^{<0\,,*}_{p,q}$ is distinguished by an isomorphism of modules

$$E_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \xrightarrow{\approx} \mathcal{E}_{p,q} \subset V,\ \ p,q \geq 1;$$

(ii) The restriction of the perturbation h to $\mathcal{E}$ has no transgressive components h^{tr}, i.e., $h^{tr}|_{\mathcal{E}} = 0$.

Given a Hirsch algebra B, a *filtered Hirsch model* for B is a filtered Hirsch algebra A together with a Hirsch algebra map $A \to B$ that induces an isomorphism on cohomology. Our next proposition, which is a Adams-Hilton type of statement, exhibits a basic property of filtered Hirsch algebras:

Proposition 4. *Let $\zeta : B \to C$ be a map of (filtered)Hirsch algebras that induces an isomorphism on cohomology. If A is a filtered Hirsch algebra, there is a bijection of sets of homotopy classes of (filtered) Hirsch algebra maps*

$$\zeta_{\#} : [A, B] \xrightarrow{\approx} [A, C].$$

Proof. Discarding Hirsch algebra structures, the proof goes by induction on the resolution grading and is similar to that of Theorem 2.5 in [12] (see also [28]). The Hirsch algebra structure serves to specify a choice of homotopy s on the multiplicative generators $\mathcal{E} \subset V$. When constructing a chain homotopy $s : A \to C$ between two multiplicative maps $f, g : A \to C$, we can choose an s on $\mathcal{E}^{i,*}$ that satisfies formula (2.12) in each step of the induction. □

The basic examples of a filtered Hirsch algebra are provided by the following theorem, which states our main result on Hirsch algebras:

Theorem 1. *Let H be a cga and let $\rho : (RH, d) \to H$ be an absolute Hirsch resolution. Given a Hirsch algebra A, assume there exists an isomorphism $i_A : H \approx H(A, d)$. Then*

(i) *Existence. There is a pair (h, f) where $h : RH \to RH$ is a perturbation of the resolution differential d on RH and*

$$f : (RH, d + h) \to A$$

is a filtered Hirsch model of A such that $(f|_{R^0H})^* = i_A\rho|_{R^0H} : R^0H \to H(A)$.

(ii) *Uniqueness. If $(\bar{h}, \bar{f})$ and $\bar{f} : (RH, d + \bar{h}) \to A$ satisfy the conditions of (i), there is an isomorphism of filtered Hirsch models*

$$\zeta : (RH, d + h) \xrightarrow{\approx} (RH, d + \bar{h})$$

of the form $\zeta = Id + \zeta^1 + \cdots + \zeta^r + \cdots$ with $\zeta^r : R^{-s}H^t \to R^{-s+r}H^{t-r}$ such that f is homotopic to $\bar{f} \circ \zeta$.

Note that the proof of the theorem uses an induction on resolution grading as it is used by the construction of filtered model due to Halperin-Stasheff [11] (compare also [27], [28]); although in the rational case for the existence and the uniqueness of a pair (h, f) the zero characteristic of $\Bbbk$ is essentially involved, the proof below shows that such a restriction can be simply avoided. Here a technical subtlety is that we have certain canonically chosen multiplicative generators on which (h, f) must act by a canonical rule.

Proof. Existence. Let $RH = T(V)$ with $V = \mathcal{E} \oplus U$. We define a perturbation h and a Hirsch algebra map $f : (RH, d + h) \to (A, d)$ by induction on resolution (column) grading. First consider $R^0H = T(V^{0,*})\,(= T(U^{0,*}))$. Define a chain map $\mathfrak{f}^0 : (V^{0,*}, 0) \to (A, d)$ by $(\mathfrak{f}^0)^* = i_A\rho|_{V^{0,*}} : V^{0,*} \to H(A)$. Extend $\mathfrak{f}^0$ multiplicatively to obtain a dga map $f^0 : R^0H \to A$. There is a map $\mathfrak{f}^1 : V^{-1,*} \to A^{*-1}$ with $f^0d|_{V^{-1,*}} = d\mathfrak{f}^1$; in particular, choose $\mathfrak{f}^1$ on $\mathcal{E}^{-1,*}\,(= \mathcal{E}^{-1,*}_{1,1})$ defined by the formula $\mathfrak{f}^1(a \smile_1 b) = f^0a \smile_1 f^0b$ for $a, b \in R^0H$. Then extend $\mathfrak{f}^0 + \mathfrak{f}^1$ multiplicatively to obtain a dga map $\mathfrak{f}^{(1)}_{\#}: T(V^{(-1),*}) \to (A,d)$; then denote the restriction of $\mathfrak{f}^{(1)}_{\#}$ to $R^{(-1)}H$ by $f^{(1)}: (R^{(-1)}H, d) \to (A,d)$.

Inductively, assume that a pair $(h^{(n)}, f^{(n)})$ has been constructed that satisfies the following conditions:

(1) $h^{(n)} = h^2 + \cdots + h^n$ is a derivation on RH,
(2) Equality (2.2) holds on $R^{(-n)}H$ for $d + h^{(n)}$ in which

$$h^r E_{p,q}(a_1,...,a_p\,;b_1,...,b_q) = \sum_{i=1}^{p}(-1)^{\epsilon^a_{i-1}} E_{p,q}(a_1,...,h^r a_i,...,a_q\,;b_1,...,b_q)$$
$$+\sum_{j=1}^{q}(-1)^{\epsilon^a_p+\epsilon^b_{j-1}} E_{p,q}(a_1,...,a_p\,;b_1,...,h^r b_j,...,b_q),\ 2\le r\le n,$$

(3) $dh^n + h^n d + \sum_{i+j=n+1} h^i h^j = 0$,
(4) $f^{(n)}: R^{(-n)}H \to A$ is the restriction of a dga map $\mathfrak{f}^{(n)}_{\#}: T\big(V^{(-n),*}\big) \to A$ to $R^{(-n)}H$ for $f^{(n)} = f^0 + \cdots + f^n$;
(5) $f^{(n)}(d + h^{(n)}) = df^{(n)}$ on $R^{(-n)}H$, and
(6) $f^{(n)}$ is compatible with the maps $E_{p,q}$ on $\mathcal{E}^{(-n),*}$.

Consider
$$f^{(n)}(d+h^{(n)})|_{V^{-n-1,*}}: V^{-n-1,*} \to A^{*-n-1};$$
clearly $df^{(n)}(d+h^{(n)}) = 0$. Define a linear map $h^{n+1}: U^{-n-1,*} \to R^0H^{*-n}$ with $\rho h^{n+1} = i_A^{-1}[f^{(n)}(d+h^{(n)})]$ and extend h^{n+1} on RH as a derivation (denoting by the same symbol) with
$$dh^{n+1} + h^{n+1}d + \sum_{i+j=n+2} h^i h^j = 0$$
and
$$h^{n+1} E_{p,q}(a_1,...,a_p\,;b_1,...,b_q) = \sum_{i=1}^{p}(-1)^{\epsilon^a_{i-1}} E_{p,q}(a_1,...,h^{n+1} a_i,...,a_q\,;b_1,...,b_q)$$
$$+\sum_{j=1}^{q}(-1)^{\epsilon^a_p+\epsilon^b_{j-1}} E_{p,q}(a_1,...,a_p\,;b_1,...,h^{n+1} b_j,...,b_q).$$
Then there is a map $\mathfrak{f}^{n+1}: V^{-n-1,*} \to A^{*-n-1}$ such that it is compatible with $E_{p,q}$ on $\mathcal{E}^{-n-1,*}$ and
$$f^{(n)}(d+h^{(n+1)})|_{V^{-n-1,*}} = d\mathfrak{f}^{n+1}.$$
Extend $\mathfrak{f}^{(n+1)} := \mathfrak{f}^0 + \cdots + \mathfrak{f}^{n+1}$ multiplicatively to obtain a dga map $\mathfrak{f}^{(n+1)}_{\#}$: $T(V^{(-n-1),*}) \to A$; the restriction of $\mathfrak{f}^{(n+1)}_{\#}$ to $R^{(-n-1)}H$ denote by
$$f^{(n+1)}: R^{(-n-1)}H \to A.$$
Thus the construction of the pair $\big(h^{(n+1)}, f^{(n+1)}\big)$ completes the inductive step. Finally, a perturbation $h = h^2 + \cdots + h^n + \cdots$ and a Hirsch algebra map f such that $f = f^0 + \cdots + f^n + \cdots$ are obtained as desired.

Uniqueness. Using Proposition 4 we construct a multialgebra morphism
$$\zeta: (RH, d+h) \to (RH, d+\bar{h}),$$
$\zeta = \zeta^0 + \zeta^1 + \cdots$, with $\bar{f}\circ\zeta \simeq f$; in addition, it is easy to choose ζ with $\zeta^0 = Id$. $\square$

2.5. **Filtered model for a QHHA.** Referring to Subsection 2.2, this section considers the compatibility of the perturbation h and the Hirsch map f with the $\cup_2$-product of RH in Theorem 1. Even if A is a QHHA in the theorem, it is impossible to obtain a *QHHA map* f which commutes with $\cup_2$-products because the compatibility of parameters $q(-;-)$ under f is obstructed. When A is a $\mathbb{Z}_2$-algebra, for example, the obstruction is caused by the non-free action of Sq_1 on H. However, when $q(-;-)=0$ for the $\cup_2$-operation in A (cf. Example 2), one can refine the perturbation h in Theorem 1 as it is stated in Proposition 5 below (in particular, item (i) of this proposition is an essential detail of the proof of the main result in [33]).

Let $\mathtt{T}\subset\mathcal{T}$ be a submodule defined by

$$\mathtt{T}=\langle a\cup_2 b\in\mathcal{T}\mid a\neq b \text{ in a basis of } \mathcal{M}\rangle.$$

For $\nu=2$, let $Sq_1: H^m(A)\to H^{2m-1}(A)$ be the map from Example 4.

Proposition 5. *Let A be a QHHA with $\cup_2$-operation satisfying (2.5) (e.g. A is a special Hirsch algebra from Example 2). Then in the filtered Hirsch model $f:(RH,d_h)\to A$ given by Theorem 1, the perturbation h can be chosen such that*

(i) $h^{tr}|_{\mathtt{T}}=0;$

(ii) Let $\nu=2$. Then for $z_i=h^{tr}(a^{\cup_2 2^i})$ with $a\in R^0H$,

$$\rho z_1=Sq_1(\rho a)\quad\textit{and}\quad h(a^{\cup_2 2^n})=\sum_{1\leq i<n} z_i\cup_2 a^{\cup_2(2^n-2i)}+z_n.$$

Proof. (i) First, remark that any element of $\mathtt{T}$ satisfies (2.5) (cf. (2.9)). Following the construction of a pair (h,f) in the proof of Theorem 1, define f for $a\cup_2 b\in\mathtt{T}^{-2,*}$ with $a,b\in\mathcal{V}^{0,*}$ by the formula

$$f(a\cup_2 b)=fa\cup_2 fb. \tag{2.13}$$

Since (2.5), f is chain with respect to the resolution differential d of RH, so we can set $h^2(a\cup_2 b)=0$. Inductively, assume that for $a\cup_2 b\in\mathtt{T}^{-r,*}$, $2\leq r<n$, the map f is defined by (2.13), while h is defined by

$$h(a\cup_2 b)=ha\cup_2 b+(-1)^{|a|}a\cup_2 hb. \tag{2.14}$$

Then for $a\cup_2 b\in\mathtt{T}^{-n,*}$ define h again by (2.14). Clearly, $fd_h(a\cup_2 b)$ is a cocycle in A and is bounded by $fa\cup_2 fb$. Therefore, we can define f on $a\cup_2 b$ by (2.13). Consequently, we set $h^{tr}(a\cup_2 b)=0$ as required.

(ii) Since f is a Hirsch map, it commutes with $\smile_1$-products and the first equality follows from the definition of Sq_1. The verification of the second equality follows immediately from (2.11). □

Remark 2. *Whereas Sq_1 induces the product on $H(BA)$, the transgressive values z_i in item (ii) of Proposition 5 are closely related with the existence of the symmetric Massey products of the element $\sigma^*(\rho a)\in H(BA)$ for the suspension map $\sigma^*: H^*(A)\to H^{*-1}(BA)$ (compare Theorem 3 and Remark 7 below): When $\sigma^*(\rho z_k)=0$ for $k<i$ (e.g. $z_k\in\mathcal{D}^{0,*}$), the cohomology class $\sigma^*(\rho z_i)$ is automatically identified with the symmetric Massey product $\langle\sigma^*(\rho a)\rangle^{2^i}$.*

Unlike Example 1, the Hirsch algebra A provided by the following example does not have a $\smile_2$-product. This fact allows us to lift a combination $a\smile_1 b\pm b\smile_1 a$ for cocycles $a,b\in A$ to the cohomology level as a non-trivial (binary) product (see also Subsection 3.4).

Example 5. *It is known that the Hochschild cochain complex $C^{\bullet}(P;P)$ of an associative algebra P admits an HGA structure* ([17] , [8]), *which is a particular Hirsch algebra. Furthermore, whereas the Hochschild cohomology $H = H\left(C^{\bullet}(P;P)\right)$ is a cga, H is also endowed with the binary operation $x*y$ defined for $x=[a]$ and $y=[b]$ by $x*y = [a\circ b - (-1)^{(|a|+1)(|b|+1)} b\circ a]$, where $\circ$ $(=\smile_1)$ is Gerstenhaber's operation on the Hochschild cochain complex. The $*$ product on the Hochschild cohomology is referred to as the G-algebra structure. Since H is a cga, we can apply Theorem 1 for $A = C^{\bullet}(P;P)$ and obtain the filtered Hirsch model $f:(RH, d+h)\to C^{\bullet}(P;P)$. Given $a,b\in V^{0,*}$, obviously we have $\rho h^2(a\cup_2 b) = \rho a * \rho b$ (since $f^1(a\smile_1 b) = f^0 a\circ f^0 b$). In other words, the non-triviality of the G-algebra structure on H implies the non-triviality of perturbation h^2 restricted to the submodule $\mathcal{T}\subset V$. Consequently, the operation $a\cup_2 b$ with $q(a,b)$ satisfying item* $(2.4)_2$ *does not exist on the filtered Hirsch model of $C^{\bullet}(P;P)$ in general.*

2.6. **A small Hirsch resolution $R_{\varsigma}H$.** Let A be a Hirsch algebra over $\Bbbk$. Whereas $(RH, d_h) = (T(V), d_h)$ in a filtered Hirsch model $f:(RH,d_h)\to A$, the calculation of $H(BA)$ can be carried out in terms of V as follows. Denote $\bar V = s^{-1}(V^{>0})\oplus\Bbbk$ and define the differential $\bar d_h$ on $\bar V$ by the restriction of $d+h$ to V to obtain the cochain complex $(\bar V, \bar d_h)$. There are isomorphisms

$$H^*(\bar V,\bar d_h)\approx H^*(B(RH), d_{_{B(RH)}}) \overset{Bf^*}{\approx} H^*(BA, d_{_{BA}}) \approx Tor^A(\Bbbk;\Bbbk). \tag{2.15}$$

In particular, for $A = C^*(X;\Bbbk)$ with X simply connected (cf. Example 1),

$$H^*(\bar V,\bar d_h)\approx H^*(BC^*(X;\Bbbk), d_{_{BC}})\approx H^*(\Omega X;\Bbbk).$$

Remark 3. *Note that the first isomorphism of (2.15) is a consequence of a general fact about tensor algebras* [6], *while the second follows from Proposition 1.*

Furthermore, to conveniently involve the multiplicative structure of (2.15), one can reduce V at the cost of $\mathcal{E}\subset V$ in the manner we shall describe. Let $J_{\varsigma}\subset R_aH$ be the Hirsch ideal of an absolute Hirsch resolution R_aH generated by

$$\{E_{p,q}(a_1,...,a_p;a_{p+1},...,a_{p+q}),\, dE_{p,q}(a_1,...,a_p;a_{p+1},...,a_{p+q})\,|\, p+q\geq 3\}$$

with

$$\begin{array}{lll} a_1,...,a_p\in R_aH, & a_{p+1}\in V, & p\geq 1,\, q=1\\ a_1,...,a_{p+q}\in R_aH, & & p\geq 1,\, q>1. \end{array}$$

Then

$$R_{\varsigma}H = R_aH/J_{\varsigma}$$

is a Hirsch resolution of H. Indeed, using (2.2) we see that $d: J_{\varsigma}\to J_{\varsigma}$ and $H(J_{\varsigma}, d)=0$. Thus $g_{\varsigma}:(R_aH,d)\to(R_{\varsigma}H,d)$ is a homology isomorphism. We have an obvious projection $\rho_{\varsigma}:(R_{\varsigma}H,d)\to H$ such that $\rho=\rho_{\varsigma}\circ g_{\varsigma}$. Consequently, ρ_{ς} is also a resolution map. Furthermore, we have $h:J_{\varsigma}\to J_{\varsigma}$ so that $(R_{\varsigma}H, d_h)$ is a Hirsch algebra (in fact an HGA) and g_{ς} extends to a quasi-isomorphism of filtered Hirsch algebras

$$g_{\varsigma}:(R_aH,d_h)\to(R_{\varsigma}H,d_h). \tag{2.16}$$

Thus, the Hirsch (HGA) structure of $R_{\varsigma}H = T(V_{\varsigma})$ is generated by the $\smile_1$-product and (2.2) is equivalent to the following two equalities:

(1) *The (left) Hirsch formula.* For $a,b,c\in R_{\varsigma}H$:

$$c\smile_1 ab = (c\smile_1 a)b + (-1)^{(|c|+1)|a|}a(c\smile_1 b) \tag{2.17}$$

(2) *The (right) generalized Hirsch formula.* For $a, b \in R_\varsigma H$ and $c \in V_\varsigma$ with $d_h(c) = \sum c_1 \cdots c_q$, $c_i \in V_\varsigma$:

$$
ab \smile_1 c = \begin{cases} a(b \smile_1 c) + (-1)^{|b|(|c|+1)}(a \smile_1 c)\, b, & q = 1, \\ \\ a(b \smile_1 c) + (-1)^{|b|(|c|+1)}(a \smile_1 c)\, b \\ + \sum\limits_{1 \le i < j \le q} (-1)^{\varepsilon}\, c_1 \cdots c_{i-1}(a \smile_1 c_i)\, c_{i+1} \\ \cdots c_{j-1}(b \smile_1 c_j)\, c_{j+1} \cdots c_q, & q \ge 2, \end{cases} \tag{2.18}
$$

where $\varepsilon = (|a|+1)\left(\epsilon^c_{i-1} + i + 1\right) - (|b|+1)\left(\epsilon^c_{j-1} + j\right)$.

Remark 4. *First, Formula* (2.18) *can be thought of as a generalization of Adams' formula for the* $\smile_1$*-product in the cobar construction* [1, p. 36] *from* $q = 2$ *to any* $q \ge 2$*. Second, the usage of* $R_\varsigma H$ *shows that the multiplication* μ^*_E *on* $H^*(BA) \approx H^*(\bar{V}_\varsigma, \bar{d}_h)$ *is in fact determined only by the* $\smile_1$*-product on* V_ς*.*

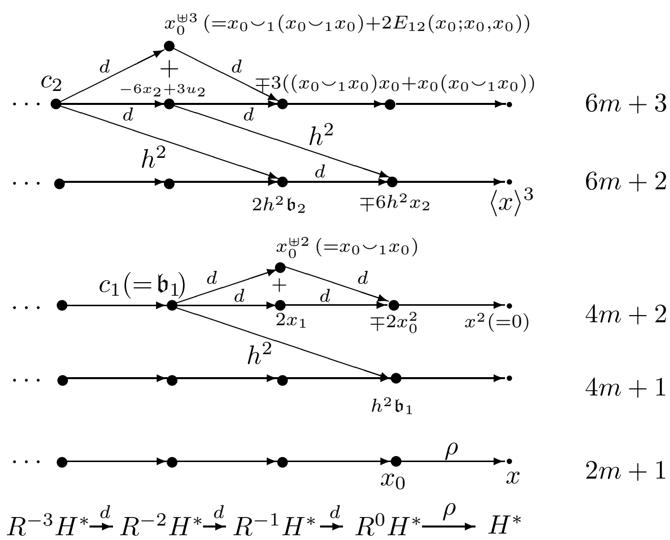

Figure 2. A fragment of the filtered Hirsch $\mathbb{Z}$-algebra obtained as a perturbed resolution $(RH, d + h)$ of a cga H.

Note that for any Hirsch resolution of H considered here, and consequently for any filtered Hirsch model, the first two columns in Figure 2 are the same.

3. Some examples and applications

In the discussion that follows we sometimes abuse notation and denote $R_\varsigma H$ by RH. As we mentioned in the introduction, certain applications of the above material are given in [31], [32]. The applications that appear here are new.

3.1. **Symmetric Massey products.** Recall the definition of the n-fold symmetric Massey product $\langle x\rangle^n$ (cf. [23], [25]). Let $x \in H(A)$ be an element for a dga A, and $x_0 \in A$ be a cocycle with $x = [x_0]$. Given $n \geq 3$, consider a sequence $(x_0, x_1, ..., x_{n-2})$ in A such that

$$dx_k = \sum_{i+j=k-1} (-1)^{|x_i|+1} x_i x_j, \quad 1 \leq k \leq n-2; \tag{3.1}$$

in particular, $dx_1 = -(-1)^{|x_0|} x_0^2$, i.e., $x^2 = 0$. Then $\sum\limits_{i+j=n-2} (-1)^{|x_i|+1} x_i x_j$ is a cocycle, and a subset of $H(A)$ formed by the classes of all such cocycles is denoted by $\langle x\rangle^n$. (In other words, the existence of a sequence $(x_0, x_1, ..., x_k, ...)$ satisfying (3.1) for all k implies that $c := \sum\limits_{k\geq 0} x_k$ is a *twisting* element in A whenever this sum (possibly infinite) has a sense; an element $c \in A$ is twisting if $dc = \pm\, c \cdot c$; cf. [3].)

When $A = C^*(X; \mathbb{Z}_p)$ for p to be an odd prime, and $x \in H^{2m+1}(X; \mathbb{Z}_p)$ is odd dimensional, the following formula is established in [23] (for the dual case see [22]):

$$\langle x\rangle^p = -\beta \mathcal{P}_1(x) \tag{3.2}$$

where $\mathcal{P}_1 : H^{2m+1}(X; \mathbb{Z}_p) \to H^{2mp+1}(X; \mathbb{Z}_p)$ is the Steenrod cohomology operation. Thus, the formulas in [23] and [22] involve the connection of the symmetric Massey products with the Steenrod and Dyer-Lashof (co)homology operations in their respective topological settings (cf. [25]). Below Theorem 3 emphasizes the algebraic content of these formulas and generalizes them using a filtered Hirsch model over the integers.

3.2. **Massey syzygies in the Hirsch resolution.** Let (RH, d) be a Hirsch resolution of H. Given a sequence of relations of the form $da_i = \lambda b_i$ and

$$\begin{gathered} du_i = (-1)^{|a_i|+1} a_i a_{i+1} + \lambda v_i, \quad dv_i = (-1)^{|a_i|} b_i a_{i+1} + a_i b_{i+1}, \\ a_i, u_i, v_i \in RH,\ \lambda \in \mathbb{Z} \setminus \{-1, 1\}, \quad 1 \leq i < n, \end{gathered} \tag{3.3}$$

in (RH, d), there are elements $u_{a_{i_1},...,a_{i_k}} \in RH$, $3 \leq k \leq n$, defined in terms of syzygies that mimic the definition of k-fold Massey products arising from k-tuples $(a_{i_1}, ..., a_{i_k})$ [23]. Precisely, $u_{a_1,...,a_n}$ is defined by

$$\begin{aligned} du_{a_1,...,a_n} &= \sum_{0\leq i<n} (-1)^{\epsilon_i^a} u_{a_1,...,a_i} u_{a_{i+1},...,a_n} + \lambda v_{a_1,...,a_n}, \\ dv_{a_1,...,a_n} &= \sum_{0\leq i<n} ((-1)^{\epsilon_i^a+1} v_{a_1,...,a_i} u_{a_{i+1},...,a_n} + u_{a_1,...,a_i} v_{a_{i+1},...,a_n}), \end{aligned} \tag{3.4}$$

with the convention that $u_{a_i} = a_i$, $u_{a_i,a_{i+1}} = u_i$ and $v_{a_i} = b_i$, $v_{a_i,a_{i+1}} = v_i$. When $b_i = 0$, equation (3.4) reduces to

$$du_{a_1,...,a_n} = \sum_{0\leq i<n} (-1)^{\epsilon_i^a} u_{a_1,...,a_i} u_{a_{i+1},...,a_n}.$$

We are interested in the special case of (3.3) obtained by setting $a_1 = \cdots = a_n$. More precisely, we consider the following situation (see also Example 6 below).

Let A be a torsion free Hirsch algebra over $\mathbb{Z}$ and fix a filtered model $f : (RH, d_h) \to A$. For a module C over $\mathbb{Z}$, let $C_{\Bbbk} := C \otimes_{\mathbb{Z}} \Bbbk$ and let $t_{\Bbbk} : C \to C_{\Bbbk}$ be the standard map; then $A_{\Bbbk} = A \otimes_{\mathbb{Z}} \Bbbk$ and $RH_{\Bbbk} = RH \otimes_{\mathbb{Z}} \Bbbk$. Also let $H_{\Bbbk} := H(A_{\Bbbk})$. There is the Hirsch model of $(A_{\Bbbk}, d_{A_{\Bbbk}})$ given by

$$f_{\Bbbk} = f \otimes 1 : (RH_{\Bbbk}, d_h \otimes 1) \to (A_{\Bbbk}, d_{A_{\Bbbk}}).$$

Given an element $x \in H_{\Bbbk}$, let x_0 be a *representative* of x in RH so that $[t_{\Bbbk} f(x_0)] = x$. In particular, $x_0 \in R^0H^*$ for $\beta(x) = 0, k \geq 1$, and $x_0 \in R^{-1}H^*$ with $dx_0 = \lambda x_0'$, $x_0' \in R^0H^*$, for $\beta(x) \neq 0$, where β denotes the Bockstein cohomology homomorphism associated with the sequence

$$0 \to \mathbb{Z}_\lambda \to \mathbb{Z}_{\lambda^2} \to \mathbb{Z}_\lambda \to 0.$$

If $x \in H = H^*(A)$, then obviously $x_0 \in R^0H^*$. In any case, assuming $x^2 = 0$ we have the corresponding relation in (RH, d) :

$$dx_1 = (-1)^{|x_0|+1} x_0^2 + \lambda x_1'$$

with the convention that $x_1' = 0$ whenever $x_0 \in R^0H^*$. This equality is a special case of (3.3), so (3.4) gives the following sequence of relations in (RH, d) :

$$dx_n = \sum_{\substack{i+j=n-1\\ i,j\geq 0}} (-1)^{|x_i|+1} x_i x_j + \lambda x_n', \quad n \geq 1, \tag{3.5}$$

where $x_n' = 0$ for $x_0 \in R^0H$.

We have the following description of Massey symmetric products in terms of the sequence $\mathbf{x} = \{x_n\}_{n\geq 0}$ in (RH, d_h). Denote $y_i = t_{\Bbbk} x_i$ in $(RH_{\Bbbk}, d_h)$. If $hy_i = 0$ for $0 \leq i < n$, then (3.5) implies $d_h d(y_n) = dd(y_n) = 0$, and consequently, $[dy_n] = -[hy_n]$. Therefore

$$f_{\Bbbk}^*[dy_n] = -f_{\Bbbk}^*[hy_n] \in \langle x \rangle^{n+1}. \tag{3.6}$$

Furthermore, the elements x_n appear in a family of relations in (RH, d). For example, these relations can be deduced from the following observation. For $x \in H$ with $x^2 = 0$, let $\iota : BH \to B(RH, d)$ be a chain map such that $\iota([\bar{x}|...|\bar{x}]) = (-1)^n [\,\overline{x_n}\,]$ for $[\bar{x}|...|\bar{x}] \in B^{n+1}H$, $n \geq 0$. Assuming BH is endowed with the shuffle product sh_H, the map ι will be multiplicative up to a chain homotopy $\mathfrak{b}$. Since $B(RH)$ is cofree, we can choose $\mathfrak{b}$ to be $(\,\mu_E \circ (\iota \otimes \iota)\,, \iota \circ sh_H)$-coderivation. This observation easily extends to the mod λ case when $x_0 \in R^{-1}H$ with $dx_0 = \lambda x_0'$. Now let

$$\bar{\mathfrak{b}}_{k,\ell} := \mathfrak{b}(\,\overbrace{[\bar{x}|...|\bar{x}]}^{k} \otimes \overbrace{[\bar{x}|...|\bar{x}]}^{\ell}\,)\,|_{\overline{RH}} \quad \text{and} \quad i_{[n]} := i_1 + \cdots + i_n + n;$$

then the equality $\mu_E(\iota \otimes \iota) - \iota \circ sh_H = d_{B(RH)}\mathfrak{b} + \mathfrak{b} d_{BH \otimes BH}$ implies in (RH, d) :

For $|x_0|$ odd:

$$\begin{aligned} d\mathfrak{b}_{k,\ell} = {} & (-1)^{k+\ell} \binom{k+\ell}{k} x_{k+\ell-1} \\ & + \sum_{i_{[p]}=k,\, j_{[q]}=\ell} (-1)^{k+\ell+p+q} E_{p,q}(x_{i_1}, ..., x_{i_p}; x_{j_1}, ..., x_{j_q}) \\ & - \sum_{\substack{0\leq r<k, 0\leq m<\ell\\ i_{[s]}=r,\, j_{[t]}=m}} (-1)^{r+m} \Big((-1)^{s+t} E_{s,t}(x_{i_1},..., x_{i_s}; x_{j_1},..., x_{j_t}) \mathfrak{b}_{k-r,\ell-m} \\ & \qquad + \binom{r+m}{r} \mathfrak{b}_{k-r,\ell-m}\, x_{r+m-1} \Big) + \lambda \mathfrak{b}_{k,\ell}' \end{aligned} \tag{3.7}$$

in which $\mathfrak{b}'_{k,\ell}=0$ for $x_0\in R^0H$, and the first equalities are:

$$\begin{aligned}
d\mathfrak{b}_{1,1}&=2x_1+x_0\smile_1 x_0+\lambda\mathfrak{b}'_{1,1},\\
d\mathfrak{b}_{2,1}&=-3x_2+E_{2,1}(x_0,x_0;x_0)-x_1\smile_1 x_0-x_0\mathfrak{b}_{1,1}+\mathfrak{b}_{1,1}x_0+\lambda\mathfrak{b}'_{2,1},\\
d\mathfrak{b}_{1,2}&=-3x_2+E_{1,2}(x_0;x_0,x_0)-x_0\smile_1 x_1-x_0\mathfrak{b}_{1,1}+\mathfrak{b}_{1,1}x_0+\lambda\mathfrak{b}'_{1,2}.
\end{aligned}$$

For $|x_0|$ even:

$$\begin{aligned}
(3.8)\quad d\mathfrak{b}_{k,\ell}&=(-1)^{k+\ell}\alpha_{k,\ell}\,x_{k+\ell-1}\\
&\quad+\sum_{i_{[p]}=k,\,j_{[q]}=\ell}(-1)^{k+\ell+p+q}E_{p,q}(x_{i_1},...,x_{i_p};x_{j_1},...,x_{j_q})\\
&\quad-\sum_{\substack{0\le r<k,0\le m<\ell\\ i_{[s]}=r,\,j_{[t]}=m}}\Big((-1)^{(k+r+1)m+s+r+t}E_{s,t}(x_{i_1},...,x_{i_s};x_{j_1},...,x_{j_t})\mathfrak{b}_{k-r,\ell-m}\\
&\qquad+(-1)^{k+\ell+r(\ell+m)}\alpha_{r,m}\mathfrak{b}_{k-r,\ell-m}\,x_{r+m-1}\Big)+\lambda\mathfrak{b}'_{k,\ell},
\end{aligned}$$

$$\alpha_{i,j}=\begin{cases}\binom{(i+j)/2}{i/2}, & i,j\text{ are even},\\ \binom{(i+j-1)/2}{i/2}, & i\text{ is even}, j\text{ is odd},\\ 0, & i,j\text{ are odd},\end{cases}$$

in which $\mathfrak{b}'_{k,\ell}=0$ for $x_0\in R^0H$, and the first equalities are:

$$\begin{aligned}
d\mathfrak{b}_{1,1}&=x_0\smile_1 x_0+\lambda\mathfrak{b}'_{1,1}\quad(\text{i.e., }\mathfrak{b}_{1,1}=x_0\cup_2 x_0\ \text{ when }\ x_0\in R^0H^*),\\
d\mathfrak{b}_{2,1}&=-x_2+E_{2,1}(x_0,x_0;x_0)-x_1\smile_1 x_0-x_0\mathfrak{b}_{1,1}-\mathfrak{b}_{1,1}x_0+\lambda\mathfrak{b}'_{2,1},\\
d\mathfrak{b}_{1,2}&=-x_2+E_{1,2}(x_0;x_0,x_0)-x_0\smile_1 x_1+x_0\mathfrak{b}_{1,1}+\mathfrak{b}_{1,1}x_0+\lambda\mathfrak{b}'_{1,2}.
\end{aligned}$$

Of course, for the sake of minimality, one can choose only certain $\mathfrak{b}_{k,\ell}$ above to be nontrivial. For example, let $|x|$ be even, let $\mathfrak{b}_{2j+1}:=\mathfrak{b}_{1,2j+1}$, and set x_{2n} in (3.5) as

$$(3.9)\qquad x_{2n}=-x_0\smile_1 x_{2n-1}+\sum_{i+j=n-1}(x_{2i}\mathfrak{b}_{2j+1}-\mathfrak{b}_{2j+1}x_{2i}).$$

Thus one can also set $\mathfrak{b}_{1,2n}=0$ and eliminate $\mathfrak{b}_{1,2n}$ from (3.8); in particular, $\mathfrak{b}_{2,1}$ can be identified with $x_0\smile_2 x_1$ for $n=1$.

Note that for an HGA A (e.g. $A=C^*(X;\mathbb{Z})$) we have that $E_{p,q}=0$ for all $q\ge 2$, that the second Hirsch formula up to homotopy from Section 2 becomes strict, and consequently, the formulas above are much simpler (see also Subsection 2.6).

Theorem 2. *Let A be a Hirsch algebra over $\mathbb{Z}$ and let $\Bbbk$ be a field of characteristic $p\ge 0$.*

- *(i) Let $x\in H(A)$ with $x^2=0$. If $\langle x\rangle^n$ is defined for $n\ge 3$, it has a finite order.*
- *(ii) Let $x\in H_{\Bbbk}$ with $x^2=0$ and $p>0$. Then $\langle x\rangle^n$ is defined for $3\le n\le p$ and vanishes whenever $3\le n<p$.*
- *(iii) Let $x\in H_{\Bbbk}$ with $x^2=0$ and $p=0$. Then $\langle x\rangle^n$ is defined and vanishes for all n.*

Proof. (i) Observe that the inductive construction of the terms $h^r,\ r\ge 2$, of h in (RH,d_h) implies $hx_i=0$ for $0\le i\le n-2$ whenever $\langle x\rangle^n$ is defined. Apply formulas

(3.7)–(3.8) to verify that $m\langle x\rangle^n=0$ with $m=n$ for $|x|$ odd (take $(k,\ell)=(1,n-1)$ in (3.7)), while $m=r-1$ or $m=r$ for $n=2r$ or $n=2r+1$ (take $(k,\ell)=(2,n-2)$ in (3.8)) for $|x|$ even.

(ii)–(iii) The proof follows an argument similar to that in (i). $\square$

Remark 5. *First, regarding Theorem 2, item (i), note that formula* (3.9) *implies that $\langle x\rangle^n=0$ whenever $|x|$ and n are even. Second, if $|x|$ is odd, formulas* (3.7) *–*(3.8) *imply that whenever defined, $\langle x\rangle^n$ consists of a single cohomology class independent of the parity of n (see* [23], [22]*).*

3.3. **The Kraines formula.** Let $p:=\lambda$ be an odd prime. Let $a\in A^{2m+1}$ be an element with $da=0$ or $da=pa'$ for some a'. Given $n\geq 2$, take (the right most) n^{th}-power of $\bar a\in\bar A$ under the $\mu_{_E}$ product on BA and consider its component in $\bar A$. Denote this component by $s^{-1}(a^{\uplus n})$ for $a^{\uplus n}\in A^{2mn+1}$. The element $a^{\uplus n}$ has the form

$$a^{\uplus n}=a^{\smile_1 n}+Q_n(a),$$

where $Q_n(a)$ is expressed in terms of $E_{1,k}$ for $1<k<n$ (for the relations of small degrees involving this power, see also Fig. 2). For example, $Q_2(a)=0$ since $a^{\uplus 2}=a^{\smile_1 2}$ and $Q_3(a)=2E_{1,2}(a;a,a)$. In particular, if A is an HGA, then obviously $a^{\uplus n}=a^{\smile_1 n}$. Thus $da^{\uplus n}$ is divided by an integer $p\geq 2$ if and only if p is a prime and $n=p^i$, some $i\geq 1$. Consequently, the homomorphism

$$\mathcal{P}_1: H^{2m+1}_{\mathbb{Z}_p}\to H^{2mp+1}_{\mathbb{Z}_p},\quad [t_{\mathbb{Z}_p}(a)]\to[t_{\mathbb{Z}_p}(a^{\uplus p})],\quad a\in A,\ d(t_{\mathbb{Z}_p}(a))=0, \tag{3.10}$$

is well defined.

Theorem 3. *Let A be a Hirsch algebra as in Proposition 5. Let A be torsion free and p be an odd prime. Then formula (3.2) holds in $H_{\mathbb{Z}_p}$ for $\mathcal{P}_1$ given by (3.10).*

Proof. Given $n\geq 1$, let $\mathfrak{b}_n:=\mathfrak{b}_{1,n}$ and set $(k,\ell)=(1,n)$ in (3.7) to obtain

$$\begin{aligned} d\mathfrak{b}_n=(-1)^{n+1}((n+1)x_n-&\sum_{\substack{j_{[q]}=n\\1\leq q\leq n}}(-1)^qE_{1,q}(x_0;x_{j_1},...,x_{j_q}))\\ &+\sum_{i+j=n-1}(-1)^i\left(\mathfrak{b}_jx_i-x_i\mathfrak{b}_j\right)+p\mathfrak{b}'_n.\end{aligned} \tag{3.11}$$

By means of the element x_0 and the sequence $\{\mathfrak{b}_n\}_{n\geq 1}$, form the sequence $\{c_n\}_{n\geq 1}$ in RH as follows:

$$c_1=\mathfrak{b}_1\ \text{ and }\ c_n=n!\,\mathfrak{b}_n+x_0\smile_1 c_{n-1},\ n\geq 2.$$

For $n=p-1$, relation (3.11) implies a relation of the form

$$dc_{p-1}=-p!\,x_{p-1}+x_0^{\uplus p}+pu_{p-1}, \tag{3.12}$$

where $u_{p-1}\in RH^+\cdot RH^+$ for $\beta(x)=0$, while $u_{p-1}=w_{p-1}+(p-1)!\,\mathfrak{b}'_{p-1}$ with $w_{p-1}\in RH^+\cdot RH^+$ for $\beta(x)\neq 0$. Hence, from $d^2(c_{p-1})=0$ we get

$$d(x_0^{\uplus p})=p!\,dx_{p-1}-p\,du_{p-1}=p((p-1)!\,dx_{p-1}-du_{p-1}).$$

Obviously, $h(x_0^{\uplus p})=0$ because $h(x_0)=0$ (recall that a perturbation h annihilates $R^{(-1)}H$ and is a derivation on $\mathcal{E}$). Consequently,

$$d_h(x_0^{\uplus p})=p((p-1)!\,dx_{p-1}-du_{p-1}).$$

Taking into account $(p-1)! = -1 \bmod p$, and passing to $H_{\mathbb{Z}_p}$ we obtain

$$\beta\mathcal{P}_1(x) = f^*_{\mathbb{Z}_p}[-dy_{p-1} - dv_{p-1}] = -f^*_{\mathbb{Z}_p}[dy_{p-1}] - f^*_{\mathbb{Z}_p}[dv_{p-1}] \text{ for } v_{p-1} := t_{\mathbb{Z}_p}(u_{p-1}).$$

Since $f^*_{\mathbb{Z}_p}[dy_{p-1}] = \langle x\rangle^p$ by (3.6), it remains to show that $f^*_{\mathbb{Z}_p}[dv_{p-1}] = 0$. Indeed, if $\beta(x) = 0$, then $x_0 \in R^0H$, $u_{p-1} \in RH^+ \cdot RH^+$, and $hv_{p-1} = 0$ by the similar argument as in the proof of Theorem 2 (ii). Consequently, $0 = f^*_{\mathbb{Z}_p}[-hv_{p-1}] = f^*_{\mathbb{Z}_p}[dv_{p-1}]$. If $\beta(x) \neq 0$, then $x_0 \in R^{-1}H$, and let $dx_0 = px_0'$. We have that u_{p-1} contains $\mathfrak{b}'_{p-1}$ as a summand, and $hv_{p-1} = -h\mathfrak{b}'_{p-1}$. Denoting $z_0 = g_\varsigma(x_0)$ and $z_0' = g_\varsigma(x_0')$ in (R_ς, d_h) where g_ς is given by (2.16), we have that $g_\varsigma(x_0^{\uplus p}) = z_0^{\smile_1 p}$ and $g_\varsigma(h\mathfrak{b}'_{p-1})$ is mod p cohomologous to

$$\sum_{0\leq i<p} z_0^{\smile_1 i} \smile_1 z_0' \smile_1 z_0^{\smile_1 p-i-1}, \text{ a summand component of } d(z_0^{\smile_1 p}).$$

But this component bounds $\sum\limits_{0\leq i\leq p-2} z_0^{\smile_1 i} \smile_1 (z_0 \cup_2 z_0') \smile_1 z_0^{\smile_1 p-i-2}$ mod p that finishes the proof. □

Remark 6. *When $p = 2$ the relation $d(x_0 \smile_1 x_0) = -2x_0^2 + 2(x_0' \smile_1 x_0 + x_0 \smile_1 x_0')$ implies the Adem relation $Sq_0(a) = Sq^1Sq_1(a)$ in $H_{\mathbb{Z}_2}$ thought of as the "Kraines formula" $\langle a\rangle^2 = a^2 = \beta Sq_1(a)$.*

Example 6. *Fix a Hirsch filtered model $f : (RH, d_h) \to A$ with $RH = T(V)$. Suppose that we are given a single relation*

$$da = \lambda b, \quad a \in V^{-1,2k+1}, b \in V^{0,2k+1}, \lambda \geq 2, k \geq 1, \tag{3.13}$$

and deduce the following relations in (RH, d) : First, define $c \in V$ by

$$dc = \begin{cases} ab + \frac{\lambda}{2} b \smile_1 b, & \lambda \quad \text{is even} \\ 2ab + \lambda b \smile_1 b, & \lambda \quad \text{is odd.} \end{cases} \tag{3.14}$$

When λ is odd, denote (cf. (3.3))

$$u_{2a,b} := -c, \quad u_{b,2a} := c - 2a \smile_1 b \quad \text{and} \quad u_{2b,b} := 2ab + (\lambda-1)b \smile_1 b$$

and obtain

(3.15)

$$\begin{aligned} du_{a,a} &= -a^2 + \lambda v_{a,a}, \quad v_{a,a} = c - a \smile_1 b, \\ du_{a,2b,b} &= -au_{2b,b} - u_{a,2b}b + \lambda v_{a,2b,b} \\ &= -2a^2b - (\lambda-1)a(b \smile_1 b) + cb + \lambda u_{b,2b,b}, \\ du_{b,2a,b} &= bu_{2a,b} - u_{b,2a}b + \lambda v_{b,2a,b} = bc - (c - 2a \smile_1 b)b + \lambda u_{b,2b,b}, \\ du_{a,2a,b} &= -au_{2a,b} + u_{2a,a}b + \lambda v_{a,2a,b}, \end{aligned}$$

where $v_{a,2b,b} = v_{b,2a,b} = u_{b,2b,b} = 2u_{b,b,b}$. Keeping in mind the fact that $d_h^2 = 0$, there is the following action of the perturbation h on the relations above:

$$\begin{aligned} dh^2u_{a,a} &= -\lambda h^2 c, \\ dh^2u_{a,2b,b} &= -h^2c \cdot b - \lambda h^2 u_{b,2b,b}, \\ dh^2u_{b,2a,b} &= b \cdot h^2c + h^2c \cdot b - \lambda h^2 u_{b,2b,b}, \\ dh^2u_{a,2a,b} &= -a \cdot h^2c - 2h^2u_{a,a} \cdot b - \lambda h^2 v_{a,2a,b}, \\ dh^3u_{a,2a,b} &= -h^3u_{2a,a} \cdot b - \lambda h^3 v_{a,2a,b} - h^2h^2u_{a,2a,b}. \end{aligned}$$

Below we shall exploit the third equality in list of relations above. First, we have

$$d\left(h^2u_{b,2a,b} + b \smile_1 h^2c\right) = -\lambda h^2 u_{b,2b,b}.$$

Suppose that $\Bbbk$ is a ring such that ν divides λ and

$$[t_{\Bbbk}(a)][t_{\Bbbk}(b)] = 0. \tag{3.16}$$

By (3.14) *one has* $[t_{\Bbbk}(ab)] = -[t_{\Bbbk}h^2c]$, *so that* $h^2c = 0 \mod \nu$ *above. Denoting* $[t_{\Bbbk}f(a)] := y$ *and* $[t_{\Bbbk}f(b)] := x$, *we have* $xy = 0$ *by* (3.16). *Thus the triple Massey product* $\langle x, y, x\rangle$ *is defined in* $H_{\Bbbk}$ *and contains* $[t_{\Bbbk}f(bu_{a,b} - u_{b,a}b)]$ $(= -[t_{\Bbbk}f(hu_{b,a,b})])$. *Obviously,* $\langle x\rangle^3$ *is also defined and*

$$\beta_\lambda\langle x, y, x\rangle = -\langle x\rangle^3$$

(here β_λ *denotes the Bockstein map associated with* $0 \to \mathbb{Z}_\nu \to \mathbb{Z}_{\nu\lambda} \to \mathbb{Z}_\lambda \to 0$*). Now let* $p = \lambda = 3$ *and consider* (3.12) *for* x. *Then*

$$c_2 = 2\mathfrak{b}_2 + x_0 \smile_1 \mathfrak{b}_1, \quad x_0^{\uplus 3} = x_0^{\smile_1 3} + 2E_{1,2}(x_0; x_0, x_0), \quad u_2 = \mathfrak{b}_1x_0 - x_0\mathfrak{b}_1$$

and

$$dc_2 = -6x_2 + x_0^{\smile_1 3} + 2E_{1,2}(x_0; x_0, x_0) + 3(\mathfrak{b}_1x_0 - x_0\mathfrak{b}_1).$$

Since $[x_0]^2 = 0$, *one has* $h^2\mathfrak{b}_1 = 0$ *and hence*

$$hc_2 = 2(h^2 + h^3)\mathfrak{b}_2$$

(for the relations above, see also Fig. 2*). In particular,* $dh^2c_2 = 6h^2x_2$. *Let* $a := y_0$, $b := x_0$, $u_{b,b} := x_1$ *and* $u_{b,b,b} := x_2$ *and set* $h^2c_2 = -2h^2u_{x_0,y_0,x_0}$. *Furthermore, if we also have* $h^3c_2 = h^3u_{x_0,y_0,x_0} \mod 3$, *then* $[t_{\Bbbk}f(x_0^{\uplus 3})] = -[t_{\Bbbk}f(hc_2)] = -[t_{\Bbbk}f(hu_{x_0,y_0,x_0})]$ *and, consequently,*

$$\mathcal{P}_1(x) \in \langle x, y, x\rangle. \tag{3.17}$$

For example, let $A = C^*(BF_4; \mathbb{Z}_3)$, *the cochain complex of the classifying space* BF_4 *of the exceptional group* F_4. *Then equality* (3.16) *together with* (3.17) *holds in* $H(BF_4; \mathbb{Z}_3)$. *More precisely, let* $x_i \in H^i(BF_4; \mathbb{Z}_3)$ *be multiplicative generators in notation of* [36] *and recall the following relations among them:* $x_8x_9 = 0 = x_4x_{21}$, $\delta x_8 = x_9$, $\delta x_{25} = x_{26}$; *also* $\mathcal{P}^3(x_9) = x_{21}$ *and* $\mathcal{P}^1(x_{21}) = x_{25}$; *thus* $\mathcal{P}^1\mathcal{P}^3(x_9) = \mathcal{P}_1(x_9) = x_{25}$ *by an application of the Adem relation. Thus the knowledge of both* $H^*(BF_4; \mathbb{Z}_3)$ *and* $H^*(F_4; \mathbb{Z}_3)$ *in low degrees enables us to use the filtered Hirsch model of* BF_4 *to deduce the following: Let* a *and* b *be defined in* (3.13) *by* $[t_{\mathbb{Z}_3}f(a)] = x_8$ *and* $[t_{\mathbb{Z}_3}f(b)] = x_9$. *Then* $[t_{\mathbb{Z}_3}f(hc_2)] = [t_{\mathbb{Z}_3}f(hu_{b,a,b})] = -x_{25}$ *and* $[t_{\mathbb{Z}_3}f(h^2u_{b,b,b})] = x_{26}$ *so that*

$$\langle x_9\rangle^3 = -\beta\mathcal{P}_1(x_9) \quad \textit{with} \quad \mathcal{P}_1(x_9) = \langle x_9, x_8, x_9\rangle.$$

Finally, we remark that the both sides of this formula become trivial under the loop suspension map $\sigma^* : H^*(BF_4; \mathbb{Z}_3) \to H^{*-1}(F_4;\ \mathbb{Z}_3)$ *by a general well-known fact about Massey products* [23], [24] *(compare* $\mathcal{P}_1(i_3)$ *for* $i_3 \in H^3(K(\mathbb{Z}_3; 3); \mathbb{Z}_3)$*).*

3.4. **Hochschild cohomology with the G-algebra structure.** In this section we assume that $\Bbbk$ is a field of characteristic zero. Refer to Example 5 and recall that the HGA structure $E = \{E_{p,q}\}_{p\geq 0; q=0,1}$ on the Hochschild cochain complex $A = C^\bullet(P; P)$ induces an associative product μ_E on the bar construction BA and hence the product μ_E^* on $H^*(BA) = Tor_*^A(\Bbbk, \Bbbk)$. Since $Tor_*^A(\Bbbk, \Bbbk)$ is an associative algebra, it can be converted into a Lie algebra in the standard way.

Theorem 4. *If the Hochschild cohomology* $H^* = H(C^\bullet(P; P))$ *is a free algebra, then the Lie algebra structure on* $Tor_*^A(\Bbbk, \Bbbk)$ *is completely determined by that of the* G*-algebra* H^*. *Consequently, the product* μ_E^* *on* $Tor_*^A(\Bbbk, \Bbbk)$ *is commutative if and only if the* G*-product on* H^* *is trivial.*

Proof. For a free algebra H, the module $\mathcal{M}\subset V$ has simple form in the (minimal) Hirsch resolution (RH,d), i.e., $\mathcal{M}^{<0,*}=0$. Indeed, given an odd dimensional multiplicative generator $x\in H$ and a representative $x_0\in R^0H$ of x, the elements x_n in the sequence (3.5) can be defined as $x_n=\frac{(-1)^n}{(n+1)!}\,x_0^{\smile_1 n+1}$ and hence $x_n\in\mathcal{E}$ for $n\geq 1$. In particular, there is a map of dg algebras $(RH,d)\to A$ and hence an isomorphism of dg coalgebras $H^*(BA)\approx H^*(BH)$ for a dga A with $H=H^*(A)$ (a free $\Bbbk$-algebra H is intrinsically $\Bbbk$-formal). Regarding the filtered Hirsch model (RH,d_h), the perturbation h may be non-zero only on $\mathcal{T}$. More precisely, according to Example 5 the cohomology class $[h(a\cup_2 b)]\in H^*(RH,d_h)$ is defined by $\rho a*\rho b\in H$ for $a,b\in V^{0,*}$. Since $H^*(BH)\approx H^*(BA)\approx H^*(\bar{V},\bar{d}_h)$ (cf. (2.15)), the multiplication μ_E^* on $H^*(BH)$ is induced by the $\smile_1$-product on V (cf. Remark 3). Therefore, the Lie bracket on $H^*(BH)$ is determined by the bracket

$$[a,b]=a\smile_1 b-(-1)^{(|a|+1)(|b|+1)}b\smile_1 a$$

on V. The observation that $s^{-1}[a,b]$ is cohomologous to $s^{-1}h(a\cup_2 b)$ in $\bar{V}$ for all $a,b\in V^{0,*}$ completes the proof. $\square$

Remark 7. *Note that the transgressive component h^{tr} evaluated on the elements $a_1\cup_2\cdots\cup_2 a_n\in\mathcal{T}$ for $a_i\in V^{0,*}$, $n\geq 3$, determines higher order operations on $Tor^A(\Bbbk;\Bbbk)$ that extend the Lie algebra structure to an L_∞-algebra structure.*

For example, a polynomial algebra $P=\Bbbk[x_1,...,x_n]$ provides the case of H^* in the theorem. Indeed, in general, to calculate the Hochschild cohomology of an algebra P construct a small complex $(C_V^\bullet(P),\bar{d})$, which is quasi-isomorphic to $C^\bullet(P;P)$ as follows (compare [15]): Fix an ordinary multiplicative resolution $\rho: RP\to P$ with $RP=T(V)$, view P as an RP-bimodule via ρ, and let $B(\rho)^\bullet: C^\bullet(P;P)\to C^\bullet(RP;P)$ be a quasi-isomorphism induced by $B(\rho):B(RP)\to BP$. Set $(C_V^\bullet(P),\bar{d})=(Hom(\bar{V},P),\bar{d})$ in which $\bar{d}$ is defined for $f\in C_V^\bullet(P)$ by $\bar{d}f=g$,

$$g(\bar{x})=\sum_{1\leq i\leq k}(-1)^{\nu_i}\rho(v_1)\cdots f(\bar{v}_i)\cdots\rho(v_k),\quad dx=\sum v_1\cdots v_k,\ v_i\in V,\ k\geq 1,$$

$\nu_i=(|f|+1)(|v_1|+\cdots+|v_{i-1}|)$, and define a chain map $\chi:C_V^\bullet(P)\to C^\bullet(RP;P)$ by $\chi f=f'$,

$$f'(\bar{x})=\begin{cases} f(\bar{x}), & x\in V,\\ \sum\limits_{1\leq i\leq n}(-1)^{\nu_i}\rho(v_1)\cdots f(\bar{v}_i)\cdots\rho(v_n), & x=\sum v_1\cdots v_n,\ v_i\in V, n\geq 2.\end{cases}$$

Isomorphism (2.15) implies that χ is a homology isomorphism. On the other hand, the $\smile$-product on $C^\bullet(P;P)$ induces a $\smile$-product on $C_V^\bullet(P)$; more precisely, we have that $\bar{V}$ is a coalgebra with the coproduct $\bar{\Delta}:\bar{V}\to\bar{V}\otimes\bar{V}$ induced by the standard coproduct of BP and, consequently, $Hom(\bar{V},P)$ is endowed with the standard $\smile$-product. When P is polynomial, the minimal V^* can be thought of as generated by the iterations of a (commutative) $\smile_1$-product ([30]); consequently, $(\bar{V}^*,\bar{\Delta})$ is an exterior coalgebra. Dually, $\bar{V}_*$ is an exterior algebra on generators $\bar{x}_1,...,\bar{x}_n$. Furthermore, $\bar{d}=0$ and hence $H(C_V^\bullet(P),\bar{d})=C_V^\bullet(P)$. Thus the Hochschild cohomology H^* is isomorphic to the algebra $C_V^\bullet(P)\approx\bar{V}_{*-1}\otimes P^*$, which is the tensor product of an exterior algebra and a polynomial algebra, as required.

3.5. **Symmetric Massey products in $C^*(X;\Bbbk)$ and powers in the loop homology $H_*(\Omega X;\Bbbk)$.** Let A_* be a dg coalgebra over a field $\Bbbk$ and let $A^* = Hom(A_*,\Bbbk)$ be a dg algebra so that $H(A^*) = Hom(H(A_*),\Bbbk)$. Let

$$\iota : H(BA^*) \to Hom(H(\Omega A_*),\Bbbk)),$$

be the canonical map, where ΩA_* denotes the cobar construction of the coalgebra A_*. Given the suspension map $\sigma^* : H^*(A^*) \to H^{*-1}(BA^*)$, let $x \in H_*(A^*)$ and $y \in H_{*-1}(\Omega A_*)$, where y is a basis element with $\iota(\sigma^* x)(y) = 1 \in \Bbbk$, and $\iota(\sigma^* x)(y') = 0$ for any basis element $y' \neq y$.

Suppose that $\langle x\rangle^n$ is defined for x. Let $\{a_i\}_{0\leq i<n}$ be a defining system of $\langle x\rangle^n$ with $a_0 \in A^*$ a representative cocycle of x. Then $\bar{a}_0 \in BA^*$ is a cocycle with $[\bar{a}_0] = \sigma^* x$ and $\{a_i\}_{0\leq i<n}$ lifts to a cocycle $a \in BA^*$ so that the cohomology class $[a] \in H^*(BA^*)$ is represented by the y^n (the n^{th}-power of y) in $H_*(\Omega A_*)$ via the map ι. Then Theorem 2 immediately implies the following:

Theorem 5. *Let X be a simply connected space, let $\Bbbk$ be a field of characteristic zero, and let $\sigma_* : H_*(\Omega X;\Bbbk) \to H_{*+1}(X;\Bbbk)$ be the suspension map. If $y \in H_*(\Omega X;\Bbbk)$ such that $y \notin \operatorname{Ker}\sigma_*$ and $y^2 \neq 0$, then $y^n \neq 0$ in $H_*(\Omega X;\Bbbk)$ for all $n \geq 2$.*

Finally, recalling the connection between symmetric Massey products and twisting elements in A^*, which arise from the sequences $\{a_i\}_{i\geq 0}$ above, we remark that the observation above relates the existence of twisting elements in A^* with the existence of polynomial generators in $H_*(\Omega A_*)$.

References

[1] J.F. Adams, On the non-existence of elements of Hopf invariant one, Ann. Math., 72 (1960), 20–104.
[2] H. J. Baues, The cobar construction as a Hopf algebra, Invent. Math., 132 (1998), 467–489.
[3] N. Berikashvili, On the differentials of spectral sequences (Russian), Proc. Tbilisi Mat. Inst., 51 (1976), 1–105.
[4] W. Browder, Torsion in H-spaces, Ann. Math., 74 (1961), 24–51.
[5] A. Clark, Homotopy commutativity and the Moore spectral sequence, Pacific J. Math., 15 (1965), 65–74.
[6] Y. Felix, S. Halperin and J.-C. Thomas, Adams' cobar equivalence, Trans. AMS, 329 (1992), 531–549.
[7] M. Gerstenhaber and A. A. Voronov, Higher operations on the Hochschild complex, Functional Analysis and its Applications, 29 (1995), 1–5.
[8] E. Getzler and J.D.S. Jones, Operads, homotopy algebra, and iterated integrals for double loop spaces, preprint (1995).
[9] V.K.A.M. Gugenheim, On the chain complex of a fibration, Ill. J. Math., 16 (1972), 398–414.
[10] V.K.A.M. Gugenheim and J.P. May, On the theory and applications of differential torsion products, Memoirs of AMS, 142 (1974), 1–93.
[11] S. Halperin and J. Stasheff, Obstructions to homotopy equivalences, Adv. in Math., 32 (1979), 233–279.
[12] J. Huebschmann, Minimal free multi-models for chain algebras, Georgian Math. J., 11 (2004), 733–752.
[13] J. Huebschmann and T. Kadeishvili, Small models for chain algebras, Math. Z., 207 (1991), 245–280.
[14] D. Husemoller, J.C. Moore and J. Stasheff, Differential homological algebra and homogeneous spaces, J. Pure and Applied Algebra, 5 (1974), 113–185.

[15] J.D.S. Jones and J. McCleary, Hochschild homology, cyclic homology, and the cobar construction, Adams memorial symposium on algebraic topology, 1 (Manchester, 1990), London Math. Soc., Lecture Note Ser., 175 (1992), 53–65.
[16] J. T. Jozefiak, Tate resolutions for commutative graded algebras over a local ring, Fund. Math., 74 (1972), 209–231.
[17] T. Kadeishvili, The $A(\infty)$-algebra structure and cohomology of Hochschild and Harrison (Russian), Proc. Tbilisi Math. Inst., 91 (1988), 19–27.
[18] ————, Cochain operations defining Steenrod $\smile_i$-products in the bar construction, Georgian Math. J., 10 (2003), 115-125.
[19] T. Kadeishvili and S. Saneblidze, A cubical model of a fibration, J. Pure and Applied Algebra, 196 (2005), 203–228.
[20] ————, The twisted Cartesian model for the double path fibration, Georgian Math. J., 22, no. 4 (2015), 489–508.
[21] L. Khelaia, On the homology of the Whitney sum of fibre spaces (Russian), Proc. Tbilisi Math. Inst., 83 (1986), 102–115.
[22] S. Kochman, Symmetric Massey products and a Hirsch formula in homology, Trans. AMS, 163 (1972), 245–260.
[23] D. Kraines, Massey higher products, Trans. AMS, 124 (1966), 431–449.
[24] ———— , The kernel of the loop suspension map, Illinois J. Math., 21 (1977), 91–108.
[25] J. P. May, A general algebraic approach to Steenrod operations, Lect. Notes in Math., 168 (1970), 153-231.
[26] H. J. Munkholm, The Eilenberg-Moore spectral sequence and strongly homotopy multiplicative maps, J. Pure and Applied Algebra, 5 (1974), 1–50.
[27] S. Saneblidze, Perturbation and obstruction theories in fibre spaces, Proc. A. Razmadze Math. Inst., 111 (1994), 1–106.
[28] ————, On derived categories and derived functors, Extracta Mathematicae, 22 (2007), 315–324.
[29] ————, The bitwited Cartesian model for the free loop fibration, Topology and Its Applications, 156 (2009), 897–910.
[30] ————, On the homotopy classification of maps, J. Homotopy and Rel. Struc., 4 (2009), 347–357.
[31] ————, The loop cohomology of a space with the polynomial cohomology algebra, preprint, math.AT/0810.4531.
[32] ————, On the Betti numbers of a loop space, J. Homotopy and Rel. Struc., 5 (2010), 1–13.
[33] ————, On the homology theory of the closed geodesic problem, preprint, math. AT/1110.5233.
[34] S. Saneblidze and R. Umble, Diagonals on the Permutahedra, Multiplihedra and Associahedra, J. Homology, Homotopy and Appl., 6 (2004), 363–411.
[35] J. Tate, Homology of noetherian rings and local rings, Illinois J. Math., 1 (1957), 14–27.
[36] H. Toda, Cohomology mod 3 of the classifying space BF_4 of the exceptional group F_4, J. Math. Kyoto Univ., 13-1 (1973), 97–115.
[37] A.A. Voronov, Homotopy Gerstenhaber algebras, in: Conférence Moshé Flato 1999: Quantization, Deformations, and Symmetries (Dijon 1999), Vol. II, Math.Phys. Stud. 22, Kluwer, Dordrecht (2000), 307–331.

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