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THE LINEAR DISCREPANCY OF A PRODUCT OF TWO POSETS

Minseok Cheong

Abstract. For a poset $P = (X, \leq_P)$, the linear discrepancy of P is the minimum value of maximal differences of all incomparable elements for all possible labelings. In this paper, we find a lower bound and an upper bound of the linear discrepancy of a product of two posets. In order to give a lower bound, we use the known result, $\mathrm{ld}(\mathbf{m} \times \mathbf{n}) = \lceil \frac{mn}{2} \rceil - 2$. Next, we use Dilworth's chain decomposition to obtain an upper bound of the linear discrepancy of a product of a poset and a chain. Finally, we give an example touching this upper bound.

1. Introduction

Let $P = (X, \leq_P)$ be a partially ordered set (shortly, poset) with a finite ground set X and its partial order relation $\leq_P$. For convenience, we use P instead of a ground set X if there is no confusion. The notation $\|_P$ or $\|$ is used for the incomparable relation. For a positive integer n, the *chain* of order n, denoted by $\mathbf{n} = (X, \leq_{\mathbf{n}})$, is a poset such that $|X| = n$ and $x \leq_{\mathbf{n}} y$ or $y \leq_{\mathbf{n}} x$ for all $x, y \in X$. An injective map $f : X \longrightarrow \{1, 2, \ldots, |X|\}$ is called a *natural labeling* (simply say a *labeling*) of P if $f(x) \leq f(y)$ for x and y in P with $x \leq_P y$.

In 2001, the linear discrepancy of a poset was firstly introduced by P. Tanenbaum, A. Trenk, and P. Fishburn [9]. The *linear discrepancy* of a poset P, $\mathrm{ld}(P)$, is defined as

$$\mathrm{ld}(P) = \min \left\{ \max\{|f(x) - f(y)| \ : \ x, \ y \in P \text{ with } x \| y\} \ : \ f \in \mathscr{F} \right\},$$

where $\mathscr{F}$ is the set of all possible labelings.

The problem of determining the linear discrepancy of a poset is known as NP-complete [6], so that the linear discrepancy of a few posets are known such as the standard example S_n, a disjoint sum of chains, semiorder, and the boolean lattice B_n.

Among these examples, B_n has relatively complicate structure, which is a product of n 2-element-chains. A product of general posets has more complicate

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structure than B_n so that determining the linear discrepancy of a product of posets can be said to be more difficult.

In 2005, S. P. Hong, J. Y. Hyun, H. K. Kim, and S.-M. Kim determined the linear discrepancy of a product of two chains, which is the first step for determining the linear discrepancy of a product of posets [7]. In 2008, M. Cheong and S.-M. Kim gave the linear discrepancy of a product of three chains of each size $2k$ for a positive integer k [4]. In [1], M. Cheong, G.-B. Chae, and S.-M. Kim determined the exact linear discrepancy of $\mathbf{3} \times \mathbf{3} \times \mathbf{3}$. Based on this result, they gave the linear discrepancy of a product of three chains of each size $2k+1$ for a positive integer k in [2]. In [3], Cheong, Chae, and Kim suggest an asymptotic linear discrepancy of a product of chains.

In this paper, we intend to find bounds of the linear discrepancy of a product of two posets. Firstly, we investigate some useful properties in Section 2 such as the linear discrepancy of a disjoint sum of two posets. In Section 3, using the result in [7], we give a lower bound of the linear discrepancy of a product of two posets. Next, we give an upper bound of the linear discrepancy of a product of a poset and a chain. Then, we use Dilworth's chain covering theorem in order to find an extension of a product of two posets, and then we give an upper bound of the linear discrepancy of a product of two posets.

Now, we close this section with some definitions of notations and terminologies.

Definition 1. Let P be a poset, and x, $y \in P$.

(1) If $x \leq_P y$, and there is no $z \in P$ such that $x \leq_P z$ and $z \leq_P y$, then we say that y *covers* x, and write this as $x \prec y$.
(2) $D(x) = \{z \in P \setminus \{x\} : z \leq_P x\}$, and $U(x) = \{z \in P \setminus \{x\} : x \leq_P z\}$.
(3) A pair (x, y) is a *critical pair* in P if $x \| y$, $D(x) \subseteq D(y)$, and $U(y) \subseteq U(x)$.

Definition 2. For a poset P, a labeling f of P is called *optimal* if $T_f(P) = \mathrm{ld}(P)$.

Definition 3. Let $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ be posets.

(1) For disjoint X and Y, a *disjoint sum* of P and Q, written as $P + Q$, is the poset $S = (X \cup Y, \leq_S)$ such that $x \leq_S y$ if and only if either $x \leq_P y$ or $x \leq_Q y$.
(2) The *Cartesian product* or *product* of P and Q, written as $P \times Q$, is the poset $R = (Z, \leq_R)$ where (i) $Z = X \times Y$, and (ii) $(x_1, y_1) \leq_R (x_2, y_2)$ if and only if $x_1 \leq_P x_2$, and $y_1 \leq_Q y_2$ for (x_1, y_1) and $(x_2, y_2) \in Z$.

Definition 4. A poset $P = (X, \leq_P)$ is isomorphic to a poset $Q = (Y, \leq_Q)$ if there is a one to one correspondence f from X to Y such that f and f^{-1} are order-preserving, i.e., $f(x) \leq_Q f(y)$ for all x, $y \in X$ with $x \leq_P y$, and $f^{-1}(w) \leq_P f^{-1}(z)$ for all w, $z \in Y$ with $w \leq_Q z$. In this case the function f is called an *isomorphism* from P to Q. We write this as $P \cong Q$.

2. The linear discrepancy of a disjoint sum of two posets

The following lemma is immediately obtained from Definition 4.

Lemma 5. *For two isomorphic posets P and Q, we have* $\mathrm{ld}(P) = \mathrm{ld}(Q)$.

The distributive law for a product of posets and disjoint sum of posets holds, as follows.

Lemma 6. *Let $P = (X, \leq_P)$, $Q = (Y, \leq_Q)$ and $R = (Z, \leq_R)$ be three posets with disjoint Y and Z. Then $P \times (Q + R) = (P \times Q) + (P \times R)$, and* $\mathrm{ld}(P \times (Q + R)) = \mathrm{ld}((P \times Q) + (P \times R))$.

Proof. Clearly, we have $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$.

Now, for $(x_1, x_2) \leq_{P\times(Q+R)} (y_1, y_2)$, we have $x_1 \leq_P y_1$ and $x_2 \leq_{Q+R} y_2$, i.e., $x_1 \leq_P y_1$ and $x_2 \leq_Q y_2$, or $x_1 \leq_P y_1$ and $x_2 \leq_R y_2$ so that

$$(x_1, x_2) \leq_{(P\times Q)+(P\times R)} (y_1, y_2).$$

Hence, the relation set of $(P \times Q) + (P \times R)$ includes that of $P \times (Q + R)$. The converse inclusion can be easily obtained with the similar way. Thus, two relation sets of $(P \times Q) + (P \times R)$ and $P \times (Q + R)$ are same.

Therefore, $\mathrm{ld}(P \times (Q + R)) = \mathrm{ld}((P \times Q) + (P \times R))$. □

In [5], we can find an important result, called *Dilworth's chain decomposition* or *Dilworth's chain covering theorem*, stated as follows.

Theorem 7 ([5])**.** *For a poset $P = (X, \leq_P)$ with $\mathrm{width}(P) = w \geq 1$, there exists a partition $X = C_1 \cup \cdots \cup C_w$, where C_i is a chain for $i = 1, 2, \ldots, w$.*

For the disjoint sum of chains, the linear discrepancy is given from [9], as the following theorem.

Theorem 8 ([9])**.** *For an integer t with $t \geq 2$, if P is a disjoint sum $\mathbf{r_1} + \mathbf{r_2} + \cdots + \mathbf{r_t}$ of t chains with $r_1 \geq r_2 \geq \cdots \geq r_t$, then* $\mathrm{ld}(P) = \lceil \frac{r_1}{2} \rceil + r_2 + \cdots + r_t - 1$.

The following lemma gives us a lower bound for the linear discrepancy of a disjoint sum of two posets.

Lemma 9. *Let P be a disjoint sum of two posets Q_1 and Q_2, and let $l_i = \max\left\{ \left\lceil \frac{|Q_i|}{2} \right\rceil + |P| - |Q_i| - 1, \mathrm{ld}(Q_i) + |P| - |Q_i| \right\}$ for $i = 1, 2$. Then*

$$\mathrm{ld}(P) \geq \min\{l_1, l_2\}.$$

Proof. For two posets Q_1 and Q_2, let $P = Q_1 + Q_2$, and $|P| = n$ for a positive integer n. Then $n = |Q_1| + |Q_2|$. Suppose f is an optimal labeling of P, and let x_1 and x_n be elements such that $f(x_1) = 1$ and $f(x_n) = n$. We consider two cases: (I) $x_1 \in Q_i$ and $x_n \in Q_j$ for $i \neq j$ in $\{1, 2\}$, and (II) either $x_1, x_n \in Q_1$ or $x_1, x_n \in Q_2$.

Case (I): Without loss of generality, suppose that $x_1 \in Q_1$ and $x_n \in Q_2$. Then, $T_f(P) = n - 1$. Since f is optimal, we obtain that P is an antichain so that Q_1 and Q_2 are all antichains. Then $\left\lceil \frac{|Q_1|}{2} \right\rceil + |P| - |Q_1| - 1 \le \mathrm{ld}(Q_1) + |P| - |Q_1| = n-1$. Hence $l_1 = n-1$. Similarly, $l_2 = n-1$ which leads to the desired result.

Case (II): Suppose that either x_1, $x_n \in Q_1$ or x_1, $x_n \in Q_2$. For $x \in Q_1$, let $n_1(x) = |\{u \in Q_2 : f(u) < f(x)\}|$, and define a map $f_1 : Q_1 \to \{1, 2, \ldots, |Q_1|\}$ as $f_1(x) = f(x) - n_1(x)$ for $x \in Q_1$. Then f_1 is clearly well-defined. Suppose that $y \neq z \in Q_1$. Then $f(y) \neq f(z)$ since f is a labeling of P. Without loss of generality, we may assume that $f(y) < f(z)$. Then $f(z) - f(y) \ge n_1(z) - n_1(y) + 1$ so that

$$\begin{aligned} f_1(z) - f_1(y) &= (f(z) - n_1(z)) - (f(y) - n_1(y)) \\ &\ge f(z) - f(y) - (n_1(z) - n_1(y)) \ge 1, \end{aligned}$$

i.e., $f_1(y) \neq f_1(z)$. Hence, f_1 is one to one.

For y and z in Q_1 with $y \le z$, we have $f(y) < f(z)$ since f is a labeling of P, and $f(z) - f(y) = n_1(z) - n_1(y) + 1$ so that $f_1(z) - f_1(y) = (f(z) - n_1(z)) - (f(y) - n_1(y)) \ge 1$ immediately. Hence, f_1 is a labeling of Q_1.

For $x \in Q_2$, let $n_2(x) = |\{u \in Q_1 : f(u) < f(x)\}|$, and define a map $f_2 : Q_2 \to \{1, 2, \ldots, |Q_2|\}$ as $f_2(x) = f(x) - n_2(x)$ for $x \in Q_2$. Then, f_2 is also a labeling of Q_2 with a similar way to f_1.

For $i = 1$, 2, let y_i and $z_i \in Q_i$ satisfy that $f_i(z_i) - f_i(y_i) = T_{f_i}(Q_i)$.

Suppose that x_1, $x_n \in Q_1$, and let

$$M_2 = \max\{f(x) : x \in Q_2\} \quad \text{and} \quad m_2 = \min\{f(x) : x \in Q_2\}.$$

Then $1 < m_2 < M_2 < n$, and $M_2 - m_2 \ge |Q_2| - 1$. Since x_1 and x_n are incomparable to all the elements in Q_2, we have

$$\mathrm{ld}(P) \ge \max\{n - m_2, M_2 - 1\} \ge \left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1. \tag{1}$$

Now, there are three cases to consider: (i) $f(y_1) < m_2 < M_2 < f(z_1)$, (ii) $f(z_1) < M_2$, and (iii) $m_2 < f(y_1)$.

Subcase (i): Suppose that $f(y_1) < m_2 < M_2 < f(z_1)$. Since f is optimal, we have

$$\begin{aligned} \mathrm{ld}(P) &\ge f(z_1) - f(y_1) = (f_1(z_1) + |Q_2|) - f_1(y_1) = T_{f_1}(Q_1) + |Q_2| \\ &\ge \mathrm{ld}(Q_1) + |Q_2|. \end{aligned}$$

Subcase (ii): Suppose that $f(z_1) < M_2$. Then

$$\begin{aligned} \mathrm{ld}(P) &\ge M_2 - 1 \\ &= (M_2 - f(z_1)) + (f(z_1) - f(y_1)) + (f(y_1) - 1) \\ &\ge (|Q_2| - n_1(z_1)) + ((f_1(z_1) + n_1(z_1)) - (f_1(y_1) + n_1(y_1))) + n_1(y_1) \\ &\ge (f_1(z_1) - f_1(y_1)) + |Q_2| = T_{f_1}(Q_1) + |Q_2| \\ &\ge \mathrm{ld}(Q_1) + |Q_2|. \end{aligned}$$

Subcase (iii): Suppose that $m_2 < f(y_1)$. Then

$$\begin{aligned}
\mathrm{ld}(P) &\geq n - m_2 \\
&= (n - f(z_1)) + (f(z_1) - f(y_1)) + (f(y_1) - m_2) \\
&\geq (|Q_2| - n_1(z_1)) + ((f_1(z_1) + n_1(z_1)) - (f_1(y_1) + n_1(y_1))) + n_1(y_1) \\
&\geq (f_1(z_1) - f_1(y_1)) + |Q_2| = T_{f_1}(Q_1) + |Q_2| \\
&\geq \mathrm{ld}(Q_1) + |Q_2|.
\end{aligned}$$

Hence, we obtain that

$$\mathrm{ld}(P) \geq ld(Q_1) + |Q_2|. \tag{2}$$

By (1) and (2), we have

$$\mathrm{ld}(P) \geq \max\left\{ \left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1, \mathrm{ld}(Q_1) + |Q_2| \right\}. \tag{3}$$

For the case that x_1, $x_n \in Q_2$, changing the roles of Q_1 and Q_2 induces that

$$\mathrm{ld}(P) \geq \max\left\{ \left\lceil \frac{|Q_2|}{2} \right\rceil + |Q_1| - 1, \mathrm{ld}(Q_2) + |Q_1| \right\} \tag{4}$$

with a similar way to use for obtaining (3).

Note that the case x_1, $x_n \in Q_1$, and the case x_1, $x_n \in Q_2$ do not happen simultaneously. Therefore, we have

$$\mathrm{ld}(P) \geq \min\{l_1, l_2\},$$

where $l_i = \max\left\{ \left\lceil \frac{|Q_i|}{2} \right\rceil + |P| - |Q_i| - 1, \mathrm{ld}(Q_i) + |P| - |Q_i| \right\}$ for $i = 1$, and 2. $\square$

Using Lemma 9, we can obtain the linear discrepancy of a disjoint sum of two posets as follows.

Theorem 10. *Let P be a disjoint sum of two posets Q_1 and Q_2, and let $l_i = \max\left\{ \left\lceil \frac{|Q_i|}{2} \right\rceil + |P| - |Q_i| - 1, \mathrm{ld}(Q_i) + |P| - |Q_i| \right\}$ for $i = 1, 2$. Then*

$$\mathrm{ld}(P) = \min\{l_1, l_2\}.$$

Proof. Let $P = Q_1 + Q_2$, and let f_1 and f_2 be optimal labelings of Q_1 and Q_2, respectively. Now, for upper bounds, we construct a labeling g of P using f_1 and f_2. We consider two cases: (I) $l_1 \leq l_2$ and (II) $l_1 > l_2$.

Case (I): Suppose that $l_1 \leq l_2$. We have two cases: (i) $\left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1 \geq \mathrm{ld}(Q_1) + |Q_2|$ and (ii) $\left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1 < \mathrm{ld}(Q_1) + |Q_2|$.

Subcase (i): If $\left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1 \geq \mathrm{ld}(Q_1) + |Q_2|$, then $l_1 = \left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1$. Define a map $g : P \to \{1, 2, \ldots, |P|\}$ as follows.

$$g(x) = \begin{cases} f_1(x) & \text{if } x \in Q_1 \text{ and } f_1(x) \leq \left\lceil \frac{|Q_1|}{2} \right\rceil, \\ \left\lceil \frac{|Q_1|}{2} \right\rceil + f_2(x) & \text{if } x \in Q_2, \\ f_1(x) + |Q_2| & \text{if } x \in Q_1 \text{ and } f_1(x) \geq \left\lceil \frac{|Q_1|}{2} \right\rceil + 1. \end{cases}$$

Then g is clearly a labeling of P. For x_1, $x_2 \in P$ with $x_1 \| x_2$, we suppose that $g(x_1) < g(x_2)$. Then, if x_1, $x_2 \in Q_1$, we have

$$\begin{aligned} g(x_2) - g(x_1) &\leq \max\{f_1(x_2) - (f_1(x_1), (f_1(x_2) + |Q_2|) - f_1(x_1)\} \\ &\leq \mathrm{ld}(Q_1) + |Q_2|. \end{aligned}$$

If x_1, $x_2 \in Q_2$, we have

$$g(x_2) - g(x_1) = f_2(x_2) - f_2(x_1) \leq |Q_2|.$$

If $x_1 \in Q_1$ and $x_2 \in Q_2$, then

$$g(x_2) - g(x_1) = f_2(x_2) - f_1(x_1) \leq \left(\left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| \right) - 1.$$

If $x_1 \in Q_2$ and $x_2 \in Q_1$, then

$$\begin{aligned} g(x_2) - g(x_1) &= (f_1(x_2) + |Q_2|) - \left(f_2(x_1) + \left\lceil \frac{|Q_1|}{2} \right\rceil \right) \\ &\leq (|Q_1| + |Q_2|) - \left(1 + \left\lceil \frac{|Q_1|}{2} \right\rceil \right) \\ &\leq \left(\left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| \right) - 1. \end{aligned}$$

Since $\left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1 \geq \mathrm{ld}(Q_1) + |Q_2|$, we have

$$T_g(P) \leq \left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1.$$

Hence,

$$\mathrm{ld}(P) \leq T_g(P) \leq l_1 = \left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1.$$

From Lemma 9, we have $\mathrm{ld}(P) \geq l_1 = \left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1$ so that

$$\mathrm{ld}(P) = l_1.$$

Subcase (ii): If $\left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1 < \mathrm{ld}(Q_1) + |Q_2|$, then $l_1 = \mathrm{ld}(Q_1) + |Q_2|$. Define a map $g : P \to \{1, 2, \ldots, |P|\}$ as follows.

$$g(x) = \begin{cases} f_1(x) & \text{if } x \in Q_1 \text{ and } f_1(x) \le f_1(y_1), \\ f_1(y_1) + f_2(x) & \text{if } x \in Q_2, \\ f_1(x) + |Q_2| & \text{if } x \in Q_1 \text{ and } g_1(x) \ge g_1(y_1) + 1. \end{cases}$$

Then this g is also a labeling of P. With similar reason for Subcases (i), we have

$$g(x_2) - g(x_1) \le \max\left\{ \left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1, \mathrm{ld}(P) + |Q_2| \right\}$$

for x_1, $x_2 \in P$ with $x_1 \| x_2$ and $g(x_1) < g(x_2)$. Since $\left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1 < \mathrm{ld}(Q_1) + |Q_2|$, we have

$$T_g(P) = \mathrm{ld}(Q_1) + |Q_2|$$

so that

$$\mathrm{ld}(P) \le l_1 = \mathrm{ld}(Q_1) + |Q_2|.$$

From Lemma 9, we have $\mathrm{ld}(P) \ge l_1 = \mathrm{ld}(Q_1) + |Q_2|$ so that

$$\mathrm{ld}(P) = l_1.$$

From Subcase (i) and (ii), we have

$$\mathrm{ld}(P) = \max\left\{ \left\lceil \frac{|Q_1|}{2} \right\rceil + |Q_2| - 1, \mathrm{ld}(Q_1) + |Q_2| \right\}. \tag{5}$$

Case (II): Suppose that $l_1 > l_2$. Then we have $\mathrm{ld}(P) \ge l_2$. With changing the roles of Q_1 and Q_2, and changing the roles of g_1 and g_2 in Case (I), we can obtain that

$$\mathrm{ld}(P) = \max\left\{ \left\lceil \frac{|Q_2|}{2} \right\rceil + |Q_1| - 1, \mathrm{ld}(Q_2) + |Q_1| \right\}. \tag{6}$$

Therefore, from (5) and (6), we have

$$\mathrm{ld}(P) = \min\{l_1, l_2\}.$$ □

From Theorem 10, we obtain the following result for the disjoint sum of posets.

Corollary 11. *For posets $Q_1, \ldots, Q_{k-1}$ and Q_k with $|Q_i| \ge |Q_{i+1}|$ for $i = 1, 2, \ldots, k-1$, let $P = Q_1 + \cdots + Q_k$. Then, we have*

$$\mathrm{ld}(P) \le \max\left\{ |P| - \left\lfloor \frac{|Q_1|}{2} \right\rfloor - 1, \mathrm{ld}(Q_1) + |P| - |Q_1| \right\}.$$

Proof. Let $P_1 = Q_2 + \cdots + Q_k$. Then $P = Q_1 + P_1$. From Theorem 10, we have

$$\mathrm{ld}(P) \le \max\left\{ \left\lceil \frac{|Q_1|}{2} \right\rceil + |P_1| - 1, \mathrm{ld}(Q_1) + |P_1| \right\}.$$ □

Lemma 12 ([8]). *For a poset P, and its labeling f, let x and x' be elements in P. Then if (x, x') is a tight pair, i.e., $f(x') - f(x) = T_f(P)$, then (x, x') is a critical pair in P.*

From Lemma 12, we only have to investigate the differences of labels of critical pairs for determining the linear discrepancy.

3. Bounds of the linear discrepancy of a product of two posets

In this section, we find a lower bound and an upper bound of a product of two posets P and Q.

At first, a lower bound can be easily obtained from the known result in [7] that, for two positive integers m and n with m and $n \geq 2$, the linear discrepancy of a product of two chains $\mathbf{m} \times \mathbf{n}$ is $\lceil \frac{mn}{2} \rceil - 2$, as follows.

Theorem 13. *Let P_1 and P_2 be two posets with $|P_1| = m$ and $|P_2| = n$. Then* $\mathrm{ld}(P_1 \times P_2) \geq \lceil \frac{mn}{2} \rceil - 2$.

Proof. Let L_1 and L_2 be linear extensions of P_1 and P_2, respectively. Then, it is clear that $L_1 \times L_2$ is an extension of $P_1 \times P_2$. Therefore, $\mathrm{ld}(P_1 \times P_2) \geq \mathrm{ld}(L_1 \times L_2) = \lceil \frac{mn}{2} \rceil - 2$. □

Now, we find an upper bound of the linear discrepancy of a product of two posets. Firstly, we consider the linear discrepancy of a product of a poset and a two-element chain, as follows.

Lemma 14. *Let C be a two-element chain whose ground set is $\{c_1, c_2\}$ and $c_1 \leq_C c_2$, and P a poset with $|P| = n$ for a positive integer n. Then*

$$\mathrm{ld}(P \times C) \leq \mathrm{ld}(P) + n - 1.$$

Proof. Let f be an optimal labeling of P, and let (x_0, x_0') be a critical pair in P satisfying $f(x_0') - f(x_0) = \mathrm{ld}(P)$. Then $D(x_0) \subseteq D(x_0')$ and $U(x_0') \subseteq U(x_0)$. Since (x_0, x_0') is a critical pair in P, we have $(x_0, c_1) \| (x_0', c_2)$, $D((x_0, c_1)) \subseteq D((x_0, c_2)) \subseteq D((x_0', c_2))$, and $U((x_0', c_2)) \subseteq U((x_0, c_2)) \subseteq U((x_0, c_1))$. Hence, $((x_0, c_1), (x_0', c_2))$ is a critical pair in $P \times C$.

Define $g : P \times C \to \{1, 2, \ldots, 2n\}$ as follows:

$$g(z) = \begin{cases} f(x), & \text{if } z = (x, c_1) \text{ and } f(x) \leq n-1, \\ f(x) + (n-1), & \text{if } z = (x, c_2) \text{ and } f(x) \leq n-1, \\ 2n-1, & \text{if } z = (x, c_1) \text{ and } f(x) = n, \\ 2n, & \text{if } z = (x, c_2) \text{ and } f(x) = n, \end{cases}$$

where $x \in P$. Then, g is clearly a labeling of $P \times C$. For $z = (x, c)$ and $z' = (x', c') \in P \times C$ with $z \| z'$, there are four cases to be considered, i.e., (I) $x \| x'$, (II) $x \leq_P x'$ and $c' \leq_C c$, and (III) $x' \leq_P x$ and $c \leq_C c'$.

Case (I): Suppose that $x \| x'$. Then $(x, c_1) \| (x', c_2)$ so that

$$|g(z') - g(z)| \leq |(f(x') + (n-1)) - f(x)|$$

$$\begin{aligned}&\le |f(x') - f(x)| + (n-1)\\&\le \mathrm{ld}(P) + (n-1).\end{aligned}$$

Case (II): Suppose that $x \le_P x'$ and $c' \le_C c$, i.e., $z = (x, c_2)$ and $z' = (x', c_1)$. Since $x \le_P x'$, we have $f(x) < f(x')$. Hence,

$$\begin{aligned}|g(z') - g(z)| &\le |f(x') - (f(x) + (n-1))|\\&\le |f(x') - f(x)| + (n-1)\\&\le \mathrm{ld}(P) + (n-1).\end{aligned}$$

Case (III): Suppose that $x' \le_P x$ and $c \le_C c'$, i.e., $z = (x, c_1)$ and $z' = (x', c_2)$. Since $x' \le_P x$, we have $f(x') < f(x)$. Hence,

$$\begin{aligned}|g(z') - g(z)| &\le (|f(x') + (n-1)) - f(x)|\\&\le |f(x') - f(x)| + (n-1)\\&\le \mathrm{ld}(P) + (n-1).\end{aligned}$$

Hence, for all $z \| z'$ in $P \times C_2$, we have $|g(z') - g(z)| \le \mathrm{ld}(P) + (n-1)$, i.e., $T_f(P) \le \mathrm{ld}(P) + (n-1)$. Therefore, $\mathrm{ld}(P \times C_2) \le \mathrm{ld}(P) + (n-1)$. □

We can easily find an example which shows that the upper bound in Lemma 14 is sharp.

Example 15. In [9], $\mathrm{ld}(B_4) = 10$ and $\mathrm{ld}(B_3) = 3$ are given. Note that $B_4 = B_3 \times B_1$, i.e., $B_4 = B_3 \times \mathbf{2}$. From Lemma 14, we have

$$\mathrm{ld}(B_4) = \mathrm{ld}(B_3 \times \mathbf{2}) \le \mathrm{ld}(B_3) + 8 - 1 = 10.$$

Hence, B_4 is a poset whose linear discrepancy touches the upper bound.

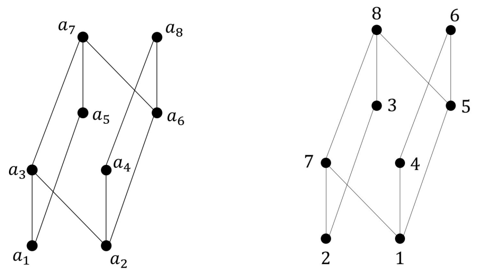
Example 16. For $N \times \mathbf{2}$, we obtain that

$$\mathrm{ld}(N \times \mathbf{2}) \le \mathrm{ld}(N) + 4 - 1 = 4$$

from Lemma 14. Actually, $\mathrm{ld}(N \times \mathbf{2})$ can be determined as follows.

Let f be an optimal labeling of $N \times \mathbf{2}$. In Figure 1(a), either a_1 or a_2 should have the label 1. Suppose that the label 1 is assigned to a_1. Then the label 8 should be assigned to a_7. The label 2 should be assigned to either a_2 or a_5. If the label of a_5 is 2, then $T_f(N \times \mathbf{2}) \ge 5$ since 7 should be assigned to either a_3 or a_8, both of which are incomparable to a_5. If the label of a_2 is 2, then the label 7 should be assigned to a_3. Then the label 3 is assigned to either a_5 or a_6 all of which are incomparable to a_3 so that $T_f(N \times \mathbf{2}) \ge 4$.

Now, we suppose that the label 1 is assigned to a_2. If a_8 has the label 8, then we have $T_f(N \times \mathbf{2}) \ge 4$ with similar way to the case that a_1 has the label 1. Hence, the label 8 is assigned to a_7. The label 2 can be assigned to either a_1 or a_6. If a_1 has 2, then a_3 has 7. In this case, the label 3 can be assigned to either a_5 or a_6. Both elements are incomparable to a_3 which has the label 7 so that $T_f(N \times \mathbf{2}) \ge 4$. If a_6 has the label 2, then a_8 has the label 7. In this case, the label 3 is assigned to a_1 which is incomparable to a_8 so that $T_f(N \times \mathbf{2}) \ge 4$.



(a) (b)

FIGURE 1. (a) $\mathcal{N} \times \mathbf{2}$, and (b) an optimal labeling of $\mathcal{N} \times \mathbf{2}$.

From all cases we considered, we have $\operatorname{ld}(\mathcal{N} \times \mathbf{2}) \geq 4$. Figure 1(b) shows an optimal labeling of $\mathcal{N} \times \mathbf{2}$. This implies that $\operatorname{ld}(\mathcal{N} \times \mathbf{2}) = 4$, and $\mathcal{N} \times \mathbf{2}$ is a poset whose linear discrepancy touches the upper bound.

From Lemma 14, we immediately obtain the following corollary.

Corollary 17. *For a positive integer n, let P be a poset with $|P| = n$, and let k be a positive integer. Then* $\operatorname{ld}(P \times \mathbf{k}) \leq \operatorname{ld}(P) + (k-1)(n-1)$.

From Theorem 7, every poset has a partition consisting of chains, and the original poset of this partition is an extension of the disjoint sum of these chains. Using these properties, we can find an upper bound of the linear discrepancy of two posets, as follows.

Theorem 18. *For two posets P and Q, let C_1 and D_1 be maximum chains of P and Q, respectively, and let*

$$l_1 = \max\left\{|P||Q| - \left\lfloor \frac{|P||D_1|}{2} \right\rfloor - 1, \operatorname{ld}(P) + |P||Q| - |P| - |D_1| + 1)\right\}$$

and

$$l_2 = \max\left\{|P||Q| - \left\lfloor \frac{|C_1||Q|}{2} \right\rfloor - 1, \operatorname{ld}(Q) + |P||Q| - |Q| - |C_1| + 1\right\}.$$

Then, we have

$$\operatorname{ld}(P \times Q) \leq \min\{l_1, l_2\}.$$

Proof. From Theorem 7, there are two chain partitions $\mathscr{C} = \{C_1, \ldots, C_s\}$ and $\mathscr{D} = \{D_1, \ldots, D_t\}$ of P and Q such that C_1 and D_1 are maximum chains of P and Q, respectively. For convenience, we suppose that $|C_i| \geq |C_{i+1}|$ for $i = 1, 2, \ldots, s-1$, and $|D_i| \geq |D_{i+1}|$ for $i = 1, 2, \ldots, t-1$.

Since D_i is a subposet of Q for $i = 1, 2, \ldots, t$, it is clear that Q is an extension of $D_1 + \cdots + D_t$. Hence, $P \times Q$ is an extension of $P \times (D_1 + \cdots + D_t)$. From Lemma 6, we have

$$P \times (D_1 + \cdots + D_t) = (P \times D_1) + \cdots + (P \times D_t).$$

Hence, $\mathrm{ld}(P \times (D_1 + \cdots + D_t)) = \mathrm{ld}((P \times D_1) + \cdots + (P \times D_t))$. Since $P \times Q$ is an extension of $P \times (D_1 + \cdots + D_t)$, the poset $P \times Q$ is also an extension of $(P \times D_1) + \cdots + (P \times D_t)$ so that

$$\mathrm{ld}(P \times Q) \leq \mathrm{ld}((P \times D_1) + \cdots + (P \times D_t)).$$

Note that $|P \times D_i| \geq |P \times D_{i+1}|$ for $i = 1, 2, \ldots, t-1$. From Corollary 11, we have

$$\begin{aligned} \mathrm{ld}(P \times Q) \leq \max\Bigg\{ &|P \times Q| - \left\lfloor \frac{|P \times D_1|}{2} \right\rfloor - 1, \\ &\mathrm{ld}(P \times D_1) + |P \times Q| - |P \times D_1| \Bigg\}. \end{aligned} \tag{7}$$

From Corollary 17, we have $\mathrm{ld}(P \times D_1) \leq \mathrm{ld}(P) + (|P| - 1)(|D_1| - 1)$. Hence, from (7), we have

$$\begin{aligned} \mathrm{ld}(P \times Q) &\leq \max\Bigg\{ |P \times Q| - \left\lfloor \frac{|P \times D_1|}{2} \right\rfloor - 1, \\ &\qquad \mathrm{ld}(P) + (|P| - 1)(|D_1| - 1) + |P||Q| - |P||D_1| \Bigg\} \\ &= \max\Bigg\{ |P||Q| - \left\lfloor \frac{|P||D_1|}{2} \right\rfloor - 1, \\ &\qquad \mathrm{ld}(P) + |P||Q| - |P| - |D_1| + 1) \Bigg\}. \end{aligned} \tag{8}$$

Similarly, we also have

$$\begin{aligned} \mathrm{ld}(P \times Q) \leq \max\Bigg\{ &|P||Q| - \left\lfloor \frac{|C_1||Q|}{2} \right\rfloor - 1, \\ &\mathrm{ld}(Q) + |P||Q| - |Q| - |C_1| + 1 \Bigg\}. \end{aligned} \tag{9}$$

Note that (8) and (9) should hold simultaneously. Therefore, we obtain the result that

$$\mathrm{ld}(P \times Q) \leq \min\{l_1, l_2\},$$

where

$$l_1 = \max\left\{ |P||Q| - \left\lfloor \frac{|P||D_1|}{2} \right\rfloor - 1, \mathrm{ld}(P) + |P||Q| - |P| - |D_1| + 1) \right\},$$

and

$$l_2 = \max\left\{ |P||Q| - \left\lfloor \frac{|C_1||Q|}{2} \right\rfloor - 1, \mathrm{ld}(Q) + |P||Q| - |P| - |C_1| + 1 \right\}. \qquad \square$$

We give an example for Theorem 18, as follows.

Example 19. In [9], Tanenbaum, et al. gave the linear discrepancy of B_4, which is 10. we may consider B_4 as a product of two B_2's. Then two values l_1 and l_2 in Theorem 18 are same, i.e.,

$$\begin{aligned} l_1 = l_2 = \max\Bigg\{ & |B_2||B_2| - \left\lfloor \frac{|B_2||\mathbf{3}|}{2} \right\rfloor - 1, \\ & \mathrm{ld}(B_2) + |B_2||B_2| - |B_2| - |\mathbf{3}| + 1) \Bigg\} \\ = & \max\{9, 11\} = 11. \end{aligned}$$

Clearly, $\mathrm{ld}(P) \leq 11$.

4. An example touching the given upper bound

In this section, we give an example touching the upper bound given in Theorem 18. Applying Theorem 18 to the poset $\mathcal{N}\times\mathcal{N}$, we know the given bound is quite tight, as follows.

Example 20. For the product of two $\mathcal{N}$'s, two values l_1 and l_2 in Theorem 18 are also same, i.e.,

$$\begin{aligned} l_1 = l_2 = \max\Bigg\{ & |\mathcal{N}||\mathcal{N}| - \left\lfloor \frac{|\mathcal{N}||\mathbf{2}|}{2} \right\rfloor - 1, \\ & \mathrm{ld}(\mathcal{N}) + |\mathcal{N}||\mathcal{N}| - |\mathcal{N}| - |\mathbf{2}| + 1) \Bigg\} \\ = & \max\{11, 12\} = 12. \end{aligned}$$

From Theorem 18, we have $\mathrm{ld}(\mathcal{N} \times \mathcal{N}) \leq 12$.

Now, we find a lower bound of $\mathrm{ld}(\mathcal{N} \times \mathcal{N})$ as follows:

Let $P = \mathcal{N}\times\mathcal{N}$, and let f be an optimal labeling of P. In Figure 2, the label 1 can be assigned to one of these a_1, a_2, a_3 or a_4.

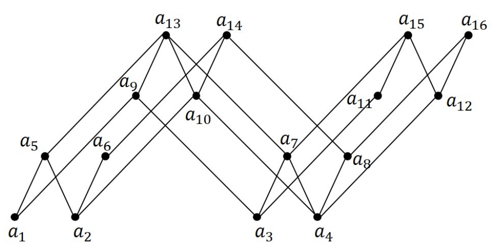


FIGURE 2. $\mathcal{N} \times \mathcal{N}$.

Firstly, we suppose that a_1 has the label 1. Then 16 should be assigned to a_{13}. The label 2 can be assigned to one of these a_2, a_3, or a_4. If a_2 has the label 2, then a_5 has the label 15. However, any element whose label can be 3 is incomparable to a_5 so that $T_f(P) \geq 12$. If a_3 has the label 2, then the label 15 should be assigned to a_9. In this case, any element whose label can be 3 is incomparable to a_9 so that $T_f(P) \geq 12$. If a_4 has the label 2, then any element whose label can be 15 is incomparable to a_4 so that $T_f(P) \geq 13$.

Secondly, suppose that a_2 has the label 1. Then 16 may be assigned to either a_{13} or a_{14}. If a_{13} has 16, then a_1 or a_3 has the label 2. If a_1 has 2, then a_5 has 15 so that the element whose label is 3 is incomparable to a_5. Hence $T_f(P) \geq 12$. If a_3 has the label 2, then the label 15 should be assigned to one of the elements comparable to a_2. However, these elements are incomparable to a_3 so that $T_f(P) \geq 13$. These are all cases for $f(a_{13}) = 16$. If $f(a_{14} = 16$, then a_4 has the label 2 so that a_{10} has the label 15. In this case, any element whose label is 3 is incomparable to a_{10} so that $T_f(P) \geq 12$.

Thirdly, suppose that a_3 has the label 1. Then 16 can be assigned to a_{13} or a_{15}. If $f(a_{13}) = 16$, then the label 2 may assigned to either a_1 or a_4. In any case, the label 15 should be assigned to a_9. Then, any element whose label is 3 is incomparable to a_9 so that $T_f(P) \geq 13$. If $f(a_{15}) = 16$, then the label 2 should be assigned to a_4. The label 15 is assigned to a_7 so that any element whose label is 3 is incomparable to a_7. Hence, $T_f(P) \geq 12$.

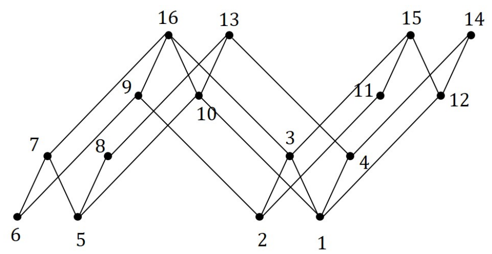


FIGURE 3. An optimal labeling of $\mathbb{N} \times \mathbb{N}$.

Finally, suppose that a_4 has the label 1. Then 16 can be assigned to one of these a_{13}, a_{14}, or a_{16}. If $f(a_{13}) = 16$, then the label 2 can be assigned to one of these a_1, a_2, or a_3. If a_1 has the label 2, then the label 15 can be assigned to one of these a_{14}, a_9, a_{15}, or a_{16}. Note that $a_9 \| a_4$, and the elements a_{14}, a_{15}, and a_{16} are incomparable to a_1. Hence, $T_f(P) \geq 13$. If a_2 has the label 2, then any element whose label is 15 is incomparable to a_2 so that $T_f(P) \geq 13$. If a_3 has the label 2, then the label 15 is assigned to a_{15} so that a_7 has the label 3. Then any element whose label is 14 is incomparable to a_3 so that $T_f(P) \geq 12$. These cases are all for the case $f(a_{13}) = 16$. If $f(a_{14}) = 16$, then 2 and 15 are assigned to a_8 and a_{16}, respectively, so that 14 can be assigned to a_{13} or a_{15} which are incomparable to a_8. Then $T_f(P) \geq 12$. If $f(a_{16}) = 16$, then a_8 and a_{14} have 2 and 15, respectively, so that a_{12} has 3. Since $a_{12} \| a_{14}$, we have $T_f(P) \geq 12$.

Hence, we conclude that $\mathrm{ld}(\mathcal{N} \times \mathcal{N}) \geq 12$. Note that $\mathrm{ld}(\mathcal{N} \times \mathcal{N}) \leq 12$. Therefore, $\mathrm{ld}(\mathcal{N} \times \mathcal{N}) = 12$. In fact, Figure 3 shows an optimal labeling of $\mathcal{N} \times \mathcal{N}$.The poset $\mathcal{N} \times \mathcal{N}$ is an example touching the upper bound.

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Minseok Cheong
Information Security Convergence
College of Informatics
Korea University
Seoul 02841, Korea
E-mail address: poset@korea.ac.kr