

Integrability of a generalized short pulse equation revisited

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Abstract

We further generalize the generalized short pulse equation studied recently in [Commun. Nonlinear Sci. Numer. Simulat. 39 (2016) 21–28; arXiv:1510.08822], and find in this way two new integrable nonlinear wave equations which are transformable to linear Klein–Gordon equations.

1 Introduction

In this paper, we study the integrability of the nonlinear wave equation

$$u_{xt} = au^2u_{xx} + buu_x^2 \quad (1)$$

containing two arbitrary parameters, a and b , not equal zero simultaneously. Actually, there is only one essential parameter in (1), the ratio a/b or b/a , which is invariant under the scale transformations of u , x and t , while the values of a and b are not invariant. We show that this equation (1) is integrable in two (and, most probably, only two) distinct cases, namely, when $a/b = 1/2$ and $a/b = 1$, which correspond via scale transformations of variables to the equations

$$u_{xt} = \frac{1}{6}(u^3)_{xx} \quad (2)$$

and

$$u_{xt} = \frac{1}{2}u(u^2)_{xx}, \quad (3)$$

respectively.

There is the following reason to study the nonlinear equation (1). Recently, in [1], we studied the integrability of the generalized short pulse equation

$$u_{xt} = u + au^2u_{xx} + buu_x^2 \quad (4)$$

containing two arbitrary parameters, a and b , not equal zero simultaneously. We showed that there are two (and, most probably, only two) integrable cases of (4), namely, those with $a/b = 1/2$ and $a/b = 1$, which can be written as

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx} \quad (5)$$

and

$$u_{xt} = u + \frac{1}{2}u(u^2)_{xx} \quad (6)$$

via scale transformations of variables. The nonlinear equation (5) is the celebrated short pulse equation which appeared first in differential geometry [2, 3], was later rediscovered in nonlinear optics [4, 5], and since then has been studied in almost any aspect of its integrability [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. The nonlinear equation (6), called the single-cycle pulse equation (due to the property of its smooth envelope soliton solution) [1] or the modified short pulse equation [18], is a scalar reduction of the integrable system of coupled short pulse equations of Feng [19]. One may wonder, looking at (4), why not to generalize this equation further, as

$$u_{xt} = au^2u_{xx} + buu_x^2 + cu \quad (7)$$

with arbitrary parameters a , b and c , in order to find new integrable nonlinear wave equations in this way. It is easy to see, however, that there are only two essentially different values of the parameter c in (7), namely, $c = 0$ and (without loss of generality) $c = 1$, because one can always make $c = 1$ by a scale transformation of variables if $c \neq 0$. Since the case of (7) with $c = 1$ is the nonlinear equation (4) studied in [1], we concentrate in the present paper on the remaining case of (7) with $c = 0$, which is the nonlinear equation (1).

In Section 2 of this paper, we transform the nonlinear equation (1) with any finite value of a/b to a corresponding (in general, nonlinear) Klein–Gordon equation whose nonlinearity depends on a/b , and we bring (1) with $b = 0$ into a form suitable for the Painlevé analysis. In Section 3, using the known results on integrability of nonlinear Klein–Gordon equations (for $b \neq 0$) and the Painlevé test for integrability (for $b = 0$), we show that the nonlinear equation (1) is integrable if (and, most probably, only if) $a/b = 1/2$ or $a/b = 1$, that is, when the nonlinear equation (1) is transformable to linear Klein–Gordon equations. This allows us to obtain parametric representations for general solutions of the nonlinear equations (2) and (3) and discuss their properties. Section 4 contains concluding remarks.

2 Transformation

In our experience, a transformation found to relate a new nonlinear equation with a known old one is a powerful tool to derive the fact and character of integrability or non-integrability of the new equation from what is known on integrability or non-integrability of the old equation [1, 20, 21, 22, 23]. By means of transformations relating new equations with known old ones, it is possible to derive analytic properties of solutions [24], expressions for special and general solutions [1, 9, 10, 25, 26], Lax pairs, Hamiltonian structures and recursion operators [27, 28, 29, 30] of the new equations from the corresponding known properties and objects of the old equations.

When $a = 0$ in (1), we have $b \neq 0$, and we make $b = 1$ by a scale transformation of variables, without loss of generality,

$$u_{xt} = uu_x^2. \quad (8)$$

If $u_x \neq 0$, we rewrite (8) as

$$\left(\frac{1}{u_x}\right)_t + u = 0, \quad (9)$$

introduce the new dependent variable $v(x, t)$,

$$v = \frac{1}{u_x}, \quad (10)$$

and get the nonlinear Klein–Gordon equation

$$v_{xt} = -\frac{1}{v}. \quad (11)$$

The inverse transformation from (11) to (8),

$$u = -v_t, \quad (12)$$

is also a local transformation, that is, like (10), it requires no integration. Note that the transformations (10) and (12) between the equations (8) and (11) do not cover the case of $u_x = 0$. However, $u = u(t)$ with any function $u(t)$ satisfies the nonlinear equation (1) with any values of a and b , and this set of special solutions tells nothing about the integrability of (8).

When $a \neq 0$ in (1), we introduce the new independent variable y ,

$$x = x(y, t), \quad u(x, t) = p(y, t), \quad (13)$$

and impose the condition

$$x_t = -ap^2 \quad (14)$$

on the function $x(y, t)$ to considerably simplify the result. Then the studied equation (1) takes the form

$$x_y p_{yt} + (2a - b) p p_y^2 = 0. \quad (15)$$

This equation (15) is invariant under the transformation $y \mapsto Y(y)$ with any function Y , which means that solutions of the system (14) and (15) determine solutions $u(x, t)$ of (1) parametrically, with y being the parameter. Next, we make use of the new dependent variable $q(y, t)$, such that

$$x_y = \frac{1}{q} p_y, \quad (16)$$

which means that $q(y, t) = u_x(x, t)$. Since $q \neq 0$ in (16), our transformation does not cover the evident special solutions of (1) with $u_x = 0$. The compatibility condition $(x_t)_y = (x_y)_t$ for (14) and (16) reads

$$p_{yt} = \frac{1}{q} p_y q_t - 2apq p_y. \quad (17)$$

Eliminating x_y from (15) and (16), and using (17), we get

$$q_t = bpq^2. \quad (18)$$

Due to (18), we have to consider the cases of $b \neq 0$ and $b = 0$ separately.

If $b \neq 0$, we make $b = 1$ by a scale transformation of variables, without loss of generality. Using the new dependent variable $r(y, t)$,

$$r = \frac{1}{q}, \quad (19)$$

we get

$$p = -r_t \quad (20)$$

from (18), and

$$r_{ytt} = \frac{2a-1}{r} r_t r_{yt} \quad (21)$$

from (17). Dividing the left- and right-hand sides of (21) by r_{yt} ($r_{yt} \neq 0$ if $x_y \neq 0$, owing to (20) and (16)), and integrating over t , we get

$$r_{yt} = h(y)r^{2a-1} \quad (22)$$

with any nonzero function $h(y)$, which appeared as the (exponent of) “constant” of integration. Finally, we make $h(y) = 1$ in (22) by the transformation $y \mapsto Y(y)$ with a properly chosen $Y(y)$ (thus suppressing the arbitrariness of the parameter y down to $y \mapsto y + \text{constant}$), and obtain the following result. All solutions of the considered case of (1),

$$u_{xt} = au^2 u_{xx} + uu_x^2, \quad (23)$$

except for solutions with $u_x = 0$, are determined parametrically by solutions of the nonlinear Klein–Gordon equation

$$r_{yt} = r^{2a-1} \quad (24)$$

via the relations

$$\begin{aligned} u(x, t) &= -r_t(y, t), \\ x &= x(y, t) : \quad x_y = -r^{2a}, \quad x_t = -ar_t^2, \end{aligned} \quad (25)$$

where y serves as the parameter, and a is an arbitrary nonzero constant.

If $b = 0$, we have $a \neq 0$, and we make $a = 1$ by a scale transformation of variables, without loss of generality. In this case, we get $q_t = 0$ from (18), that is, $q = q(y)$ with any nonzero function $q(y)$, and the equation (17) takes the form

$$p_{yt} + 2q(y)pp_y = 0. \quad (26)$$

Consequently, all solutions of the considered case of (1),

$$u_{xt} = u^2 u_{xx}, \quad (27)$$

except for solutions with $u_x = 0$, are determined parametrically by solutions of the nonlinear equation (26) with any $q(y) \neq 0$ via the relations

$$\begin{aligned} u(x, t) &= p(y, t), \\ x &= x(y, t) : \quad x_y = \frac{1}{q(y)}p_y, \quad x_t = -p^2, \end{aligned} \quad (28)$$

where y serves as the parameter. Note that the arbitrariness of $q(y)$ in (26) cannot be suppressed by the change of parametrization $y \mapsto Y(y)$.

3 Integrability

Integrability of nonlinear Klein–Gordon equations is very well studied. It was shown in [31] that the equation

$$z_{\xi\eta} = w(z) \quad (29)$$

possesses a nontrivial group of higher symmetries if and only if the function $w(z)$ satisfies either the condition

$$w' = \alpha w \quad (30)$$

or the condition

$$w'' = \alpha w + \beta w', \quad (31)$$

where $z = z(\xi, \eta)$, the prime denotes the derivative with respect to z , the constant α in (30) is arbitrary, while the constants α and β in (31) must satisfy the condition

$$\beta(\alpha - 2\beta^2) = 0. \quad (32)$$

No more integrable cases of (29) have been discovered by various methods as yet.

The right-hand side of the nonlinear Klein–Gordon equation (11) satisfies neither (30) nor (31). Therefore this equation, together with the corresponding case (8) of the studied equation (1), must be non-integrable. The right-hand side of the nonlinear Klein–Gordon equation (24) satisfies (30) or (31) for two values of a only, $a = 1/2$ or $a = 1$, when (24) is actually a linear equation, while the corresponding nonlinear equation (23) takes the form (2) or (3), respectively.

The case of (24) with $a = 1/2$ is the Darboux integrable linear equation

$$r_{yt} = 1 \quad (33)$$

whose solutions parametrically determine all solutions (except for solutions with $u_x = 0$) of the nonlinear equation (2) via the relations

$$\begin{aligned} u(x, t) &= -r_t(y, t), \\ x = x(y, t) : \quad x_y &= -r, \quad x_t = -\frac{1}{2}r_t^2, \end{aligned} \quad (34)$$

where y serves as the parameter. Taking the general solution of (33)

$$r = yt + f(y) + g(t), \quad (35)$$

where $f(y)$ and $g(t)$ are arbitrary functions, we obtain via (34) the following parametric representation for the general solution of the nonlinear equation (2):

$$\begin{aligned} u(x, t) &= -y - g'(t), \\ x &= -\frac{1}{2}y^2t - \int f(y) dy - yg(t) - \frac{1}{2} \int [g'(t)]^2 dt, \end{aligned} \quad (36)$$

where the prime stands for the derivative. It follows from (36) that

$$u_x = \frac{1}{yt + f(y) + g(t)}, \quad (37)$$

which shows that the general solution (36) does not cover the evident special solutions of (2) with $u_x = 0$. Also, due to (37), there are apparently no solutions of (2) without singularities of the type $u_x \rightarrow \infty$, besides the solutions with $u_x = 0$. We do not see how to choose the functions $f(y)$ and $g(t)$ to make the denominator in (37) not equal zero for all values of y and t .

The case of (24) with $a = 1$ is the Fourier integrable linear equation

$$r_{yt} = r \tag{38}$$

whose solutions parametrically determine all solutions (except for solutions with $u_x = 0$) of the nonlinear equation (3) via the relations

$$\begin{aligned} u(x, t) &= -r_t(y, t), \\ x = x(y, t) : \quad x_y &= -r^2, \quad x_t = -r_t^2, \end{aligned} \tag{39}$$

where y serves as the parameter. Since

$$u_x = \frac{1}{r} \tag{40}$$

due to (39), the parametric representation (39) of the general solution of (3) does not cover the evident special solutions of (3) with $u_x = 0$. It is easy to see from (40) that a solution $u(x, t)$ of (3) contains singularities of the type $u_x \rightarrow \infty$ if the corresponding solution $r(y, t)$ of (38) contains zeroes. For example, if we take

$$r = \sin(y - t), \tag{41}$$

we get from (39) the solution

$$u = \cos(y - t), \quad x = -\frac{1}{2}(y + t) + \frac{1}{4} \sin 2(y - t) \tag{42}$$

containing singularities, as shown in Figure 1. On the contrary, taking

$$r = \cosh(y + t), \tag{43}$$

we get the smooth solution

$$u = -\sinh(y + t), \quad x = -\frac{1}{2}(y - t) - \frac{1}{4} \sinh 2(y + t), \tag{44}$$

shown in Figure 2. Note that, in (42) and (44), the constant of integration in x has been fixed so that $x|_{y=t=0} = 0$.

It only remains to test the integrability of the nonlinear equation (27), because the case of (1) with $b = 0$ could not be transformed into a Klein-Gordon equation. We have found the transformation (28) which relates (27) with the nonlinear equation (26). Let us study the integrability of (26) by means of the Painlevé analysis [32, 33, 34], which is, in our experience, a reliable and easy-to-use tool to test the integrability of nonlinear equations [35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50]. The reliability of the Painlevé test for integrability has been empirically verified in numerous studies of multi-parameter families of nonlinear equations, including the fifth-order

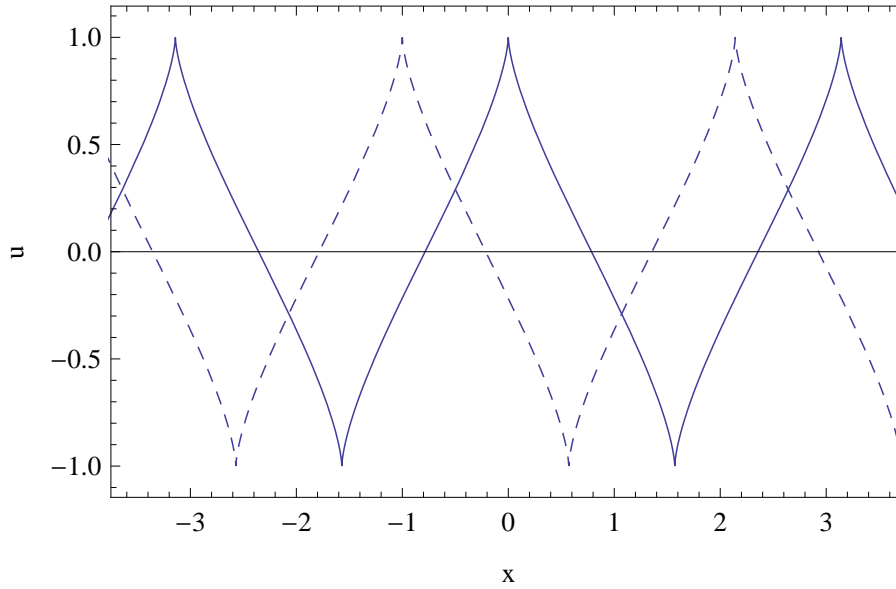


Figure 1: The singular solution (42): $t = 0$ (solid) and $t = 1$ (dashed).

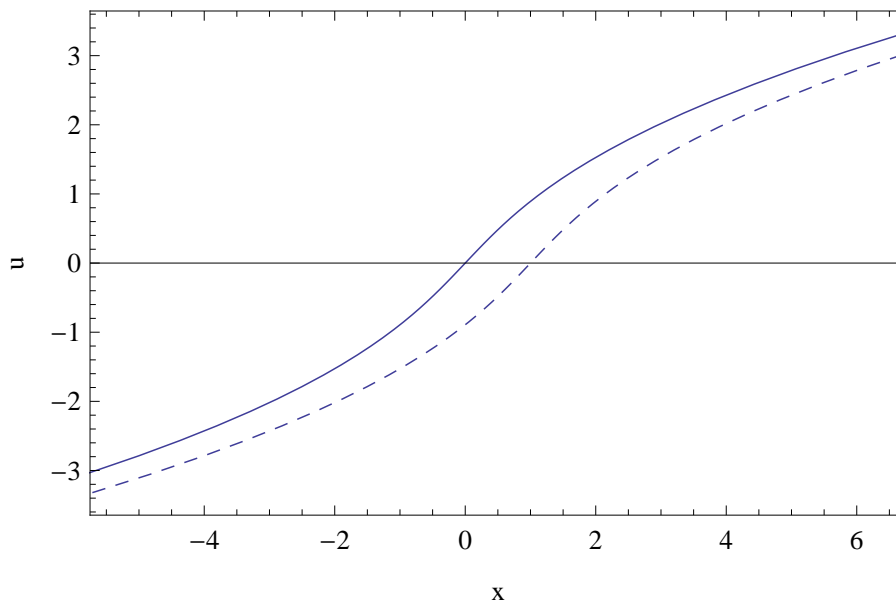


Figure 2: The smooth solution (44): $t = 0$ (solid) and $t = 1$ (dashed).

KdV-type equation [51], the coupled KdV equations [52, 53, 54, 55, 56], the symmetrically coupled higher-order nonlinear Schrödinger equations [57, 58, 59], the generalized Ito system [60], the sixth-order bidirectional wave equation [61], and the seventh-order KdV-type equation [62].

A hypersurface $\phi(y, t) = 0$ is non-characteristic for the studied equation (26) if $\phi_y \phi_t \neq 0$, and we choose $\phi_t = 1$ without loss of generality, that is, $\phi = t + \psi(y)$ with $\psi_y \neq 0$. Using the expansion

$$p = p_0(y)\phi^\gamma + \cdots + p_n(y)\phi^{\gamma+n} + \cdots, \quad (45)$$

we find the dominant singular behavior of solutions of (26) near $\phi = 0$,

$$\gamma = -1, \quad p_0 = \frac{1}{q(y)}, \quad (46)$$

together with the corresponding positions of resonances,

$$n = -1, 2, \quad (47)$$

where $n = -1$ refers to the arbitrariness of $\psi(y)$. Substituting the expansion

$$p = p_0(y)\phi^{-1} + p_1(y) + p_2(y)\phi + \cdots \quad (48)$$

to (26), and collecting terms with equal degrees of ϕ , we get the following. The terms with ϕ^{-3} , of course, give the expression (46) for p_0 . The terms with ϕ^{-2} give the expression

$$p_1 = \frac{-q_y}{2q^2\psi_y}. \quad (49)$$

The terms with ϕ^{-1} , however, do not determine $p_2(y)$ (here we have the resonance) but lead to the nontrivial compatibility condition

$$qq_y\psi_{yy} - qq_{yy}\psi_y + 3q_y^2\psi_y = 0. \quad (50)$$

In order to satisfy this condition (50) for all functions $\psi(y)$ ($\psi_y \neq 0$), we must set $q_y = 0$. Otherwise, for $q_y \neq 0$, the compatibility condition (50) is not satisfied identically, and we must introduce logarithmic terms to the expansion (48), starting from the term proportional to $\phi \log \phi$, which is a clear indication of non-integrability. Consequently, the nonlinear equation (26) is integrable for $q = \text{constant}$ only, not for any nonzero function $q(y)$. Therefore the corresponding equation (27) is not integrable. Moreover, since $q(y, t) = u_x(x, t)$, we believe that the only solutions of the nonlinear equation (27) obtainable in a closed form are the evident solutions with $u_x = \text{constant}$.

4 Conclusion

In this paper, we have generalized further the generalized short pulse equation studied recently in [1], and found in this way two new integrable nonlinear wave equations, namely, (2) and (3), which are transformable to linear Klein–Gordon equations. These new equations (2) and (3), due to the absence of the linear term “ u ” in them, can be considered as “massless” counterparts of the short pulse

equation (5) and the single-cycle pulse equation (6), respectively. Let us note, however, that the types of integrability of (2) and (3) are essentially different from the type of integrability of (5) and (6). While the equations (5) and (6) are two “avatars” (in the sense of transformations) of the sine-Gordon equation, the new nonlinear equation (2) is an “avatar” of a Darboux integrable linear Klein–Gordon equation, and the new nonlinear equation (3) is an “avatar” of a Fourier integrable linear Klein–Gordon equation. Taking this into account, we expect that the integrability properties of the new equation (2) are similar to those of the Liouville equation (continual sets of generalized symmetries and conservation laws, and several mutually non-equivalent Lax pairs [63]), whereas the properties of (3) may be similar to those of linear wave equations (a discrete hierarchy of symmetries, a finite set of conservation laws, and no phase shifts in wave interactions). We believe that these new equations (2) and (3) can be useful, as integrable scalar reductions, for classifications of integrable vector short pulse equations.

Let us also note that our equations (2) and (3) did not appear in the most recent integrability classification of generalized short pulse equations of Hone, Novikov and Wang [64] because equations without the linear term “ u ” were not studied there.

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