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Research Article

Direct Solution of nth-Order IVPs by Homotopy Analysis Method

A. Sami Bataineh,[1] M. S. M. Noorani,[2] and I. Hashim[2]

[1] *Department of Mathematics, Irbid National University, Irbid 2600, Jordan*
[2] *Centre for Modelling and Data Analysis, School of Mathematical Sciences, Universiti Kebangsaan Malaysia (National University of Malaysia), 43600 Bangi Selangor, Malaysia*

Correspondence should be addressed to I. Hashim, ishak_h@ukm.my

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Direct solution of a class of nth-order initial value problems (IVPs) is considered based on the homotopy analysis method (HAM). The HAM solutions contain an auxiliary parameter which provides a convenient way of controlling the convergence region of the series solutions. The HAM gives approximate analytical solutions which are of comparable accuracy to the seven- and eight-order Runge-Kutta method (RK78).

1. Introduction

Higher-order initial value problems (IVPs) arise in mathematical models for problems in physics and engineering. Generally, second- and higher-order IVPs are more difficult to solve than first-order IVPs. It is possible to integrate a special nth-order IVP by reducing it to a first-order system and applying one of the established methods available for such system. However, it seems more natural to provide direct numerical methods for solving the nth-order IVPs.

It is the purpose of the present paper to present an alternative approach for the direct solution of nth-order IVPs based on the homotopy analysis method (HAM). The analytic homotopy analysis method (HAM), initially proposed by Professor Liao in his Ph.D. thesis [1], is a powerful method for solving both linear and nonlinear problems. (The interested reader can refer to the much-cited book [2] for a systematic and clear exposition on this method.) In recent years, this method has been successfully employed to solve many types of nonlinear problems in science and engineering [3–17]. All of these successful applications verified the validity, effectiveness and flexibility of the HAM. More recently, Bataineh et al. [18–25] employed the standard HAM to solve some problems in engineering sciences. HAM yields a very rapid convergence of the solution series and in most cases, usually only a few iterations leading to very accurate solutions. Thus Liao's HAM is a universal one which can solve various kinds of nonlinear equations. Bataineh et al. [18] first presented a modified HAM called (MHAM) to solve systems of second-order BVPs. Another new approach in HAM was presented by Yabushita et al. [26] who applied HAM not only to the governing differential equations, but also to algebraic equation. We call this new variant of HAM as NHAM.

In this work, we consider a class of nth-order IVPs of the form

$$x^{(n)}(t) = f\left(t, x(t), x'(t), \dots, x^{(n-1)}(t)\right), \tag{1.1}$$

subject to the initial conditions

$$x(t_0) = x_0, \qquad x'(t_0) = x'_0, \dots, x^{(n-1)}(t_0) = x_0^{(n-1)}, \tag{1.2}$$

where f represents a continuous, real linear/nonlinear function, and x_0, x'_0, …, $x_0^{(n-1)}$ are prescribed. Some of the more recent direct (purely) numerical methods for solving second-order IVPs were developed by Cash [27], Ramos and Vigo-Aguiar [28, 29]. Recently Yahaya et al. [30] applied the seminumeric multistage modified Adomian decomposition method to solve the nth-order IVPs (1.1)-(1.2). Very recently, Chowdhury and Hashim [31] demonstrated the applicability of the analytic homotopy-perturbation method for solving nth-order IVPs.

The aim of this paper is to apply HAM and NHAM for the first time to obtain approximate solutions of nth-order IVPs directly. We demonstrate the accuracy of the HAM and NHAM through some test examples. Numerical comparison will be made against the seven- and eight-order Runge-Kutta method (RK78).

2. Basic Ideas of HAM

To describe the basic ideas of the HAM, we consider the following differential equation:

$$N[x(t)] = 0, \tag{2.1}$$

where N is a nonlinear operator, t denotes the independent variable, $x(t)$ is an unknown function. By means of generalizing the traditional homotopy method, Liao [2] constructs the so-called *zero-order deformation equation*

$$(1-q)L[\phi(t;q) - x_0(t)] = q\hbar\{N[\phi(t;q)]\}, \tag{2.2}$$

where $q \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary function, L is an auxiliary linear operator, $x_0(t)$ is an initial guess of $x(t)$ and $\phi(t;q)$ is an unknown function. It is important to note that one has great freedom to choose auxiliary objects such as $\hbar$ and L in HAM. Obviously, when $q = 0$ and $q = 1$, both

$$\phi(t;0) = x_0(t), \qquad \phi(t;1) = x(t), \tag{2.3}$$

hold. Thus as q increases from 0 to 1, the solution $\phi(t;q)$ varies from the initial guess $x_0(t)$ to the solution $x(t)$. Expanding $\phi(t;q)$ in Taylor series with respect to q, one has

$$\phi(t;q) = x_0(t) + \sum_{m=1}^{+\infty} x_m(t)\, q^m, \tag{2.4}$$

where

$$x_m = \frac{1}{m!} \left.\frac{\partial^m \phi(t;q)}{\partial q^m}\right|_{q=0}. \tag{2.5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, then the series (2.4) converges at $q = 1$ and one has

$$\phi(t;1) = x_0(t) + \sum_{m=1}^{+\infty} x_m(t), \tag{2.6}$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao [2]. If $\hbar = -1$, (2.2) becomes

$$(1-q)\, L\left[\phi(t;q) - x_0(t)\right] + q\left\{N\left[\phi(t;q)\right]\right\} = 0, \tag{2.7}$$

which is used mostly in the HPM [32].

According to (2.5), the governing equations can be deduced from the *zero-order deformation equations* (2.2). We define the vectors

$$\vec{x}_i = \{x_0(t), x_1(t), \ldots, x_i(t)\}. \tag{2.8}$$

Differentiating (2.2) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called *mth-order deformation equation*

$$L\left[x_m(t) - \chi_m x_{m-1}(t)\right] = \hbar R_m\left(\vec{x}_{m-1}\right), \tag{2.9}$$

where

$$\begin{gathered} R_m\left(\vec{x}_{m-1}\right) = \frac{1}{(m-1)!} \left.\frac{\partial^{m-1}\left\{N\left[\phi(t;q)\right]\right\}}{\partial q^{m-1}}\right|_{q=0}, \\ \chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \end{gathered} \tag{2.10}$$

It should be emphasized that $x_m(t)$ $(m \geq 1)$ are governed by the linear equation (2.9) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation softwares such as Maple and Mathematica.

A new approach in the HAM was proposed by Yabushita et al. [26]. We will call this method NHAM. Yabushita et al. [26] considered the following projectile problem:

$$\begin{aligned} &\frac{\mathrm{d}u}{\mathrm{d}t} + f(t)\,u = 0, \\ &\frac{\mathrm{d}v}{\mathrm{d}t} + f(t)\,v + 1 = 0, \end{aligned} \tag{2.11}$$

where

$$f(t) = \sqrt{u^2 + v^2}. \tag{2.12}$$

The standard HAM applied to this problem yields a divergent solution on some part of the solution domain. In NHAM, the zeroth-order deformation equations were constructed for not only (2.11), but also for (2.12). This slight modification in the NHAM gives a more accurate solution.

3. Numerical Experiments

To illustrate the effectiveness of the HAM we will consider four examples of nth-order IVPs (1.1)-(1.2).

3.1. Example 1

We first consider the nonlinear second-order IVP

$$x'' + x'^2 = 0, \tag{3.1}$$

subject to the initial conditions

$$x(0) = 1, \qquad x'(0) = 2. \tag{3.2}$$

The exact solution is

$$x(t) = 1 + \ln(1 + 2t). \tag{3.3}$$

To solve (3.1)-(3.2) by means of HAM, we choose the initial approximation

$$x_0(t) = 1 + 2t \tag{3.4}$$

and the linear operator

$$L[\Phi(t;q)] = \frac{\partial^2 \phi(t;q)}{\partial t^2}, \tag{3.5}$$

with the property

$$L\left[c_1 + tc_2\right] = 0, \tag{3.6}$$

where c_i $(i = 1, 2)$ are constants of integration. Furthermore, (3.1) suggests that we define the nonlinear operator as

$$N\left[\phi\left(x; q\right)\right] = \frac{\partial^2 \phi\left(t; q\right)}{\partial t^2} + \frac{\partial \phi^2\left(t; q\right)}{\partial t}. \tag{3.7}$$

Using the above definition, we construct the *zeroth-order deformation equation* as in (2.2) and the *mth-order deformation equation* for $m \geq 1$ is as in (2.9) with the initial conditions

$$x_m\left(0\right) = 0, \qquad x'_m\left(0\right) = 0, \tag{3.8}$$

where

$$R_m\left(\overrightarrow{x}_{m-1}\right) = x''_{m-1}\left(t\right) + \sum_{j=0}^{m-1} x'_j\left(t\right) x'_{m-1-j}\left(t\right), \tag{3.9}$$

now the solution of the mth-order deformation for $m \geq 1$ is

$$x_m\left(t\right) = \chi_m x_{m-1}\left(t\right) + \hbar L^{-1} R_m\left(\overrightarrow{x}_{m-1}\right). \tag{3.10}$$

We now successively obtain

$$\begin{aligned}
x_1\left(t\right) &= 2\hbar t^2, \\
x_2\left(t\right) &= 2\hbar t^2 + 2\hbar^2 t^2 + \frac{8}{3}\hbar^2 t^3, \\
x_3\left(t\right) &= 2\hbar t^2 + 4\hbar^2 t^2 + 2\hbar^2 t^2 + \frac{16}{3}\hbar^2 t^3 + \frac{16}{3}\hbar^3 t^3 + 4\hbar^3 t^4, \\
&\vdots
\end{aligned} \tag{3.11}$$

Then the series solution expression can be written in the form

$$x\left(t\right) = x_0\left(t\right) + x_1\left(t\right) + x_2\left(t\right) + \cdots, \tag{3.12}$$

and so forth. Hence, the series solution when $\hbar = -1$ is

$$x\left(t\right) \simeq 1 + 2t - 2t^2 + \frac{8}{3}t^3 - 4t^4 + \frac{32}{5}t^5 - \cdots, \tag{3.13}$$

which converges to the closed-form solution (3.3).

3.2. Example 2

Consider the linear fourth-order IVP,

$$x^{(4)} = -5x'' - 4x, \tag{3.14}$$

subject to the initial conditions

$$x(0) = 1, \qquad x'(0) = 0, \qquad x''(0) = 0, \qquad x'''(0) = 1. \tag{3.15}$$

The exact solution is

$$x(t) = \frac{4}{3}\cos t + \frac{1}{3}\sin t - \frac{1}{3}\cos 2t - \frac{1}{6}\sin 2t. \tag{3.16}$$

To solve (3.14)-(3.15) by means of HAM, we choose the initial approximation

$$x_0(t) = 1 + \frac{1}{6}t^3, \tag{3.17}$$

and the linear operator

$$L[\Phi(t;q)] = \frac{\partial^4 \phi(t;q)}{\partial t^4}, \tag{3.18}$$

with the property

$$L\left[c_1 + tc_2 + t^2 c_3 + t^3 c_4\right] = 0, \tag{3.19}$$

where c_i $(i = 1, 2, 3, 4)$ are constants of integration. According to the *zeroth-order deformation equation* (2.2) and the *mth-order deformation equation* for $m \geq 1$ (2.9) with the initial conditions

$$x_m(0) = 0, \qquad x'_m(0) = 0, \qquad x''_m(0) = 0, \qquad x'''_m(0) = 0, \tag{3.20}$$

where

$$R_m(\vec{x}_{m-1}) = x''''_{m-1}(t) + 5x''_{m-1}(t) + 4x_{m-1}(t), \tag{3.21}$$

the solution of the mth-order deformation for $m \geq 1$ is the same as (3.10).

We now successively obtain

$$
\begin{aligned}
x_1(t) &= \frac{1}{6}\hbar t^4 + \frac{1}{24}\hbar t^5 + \frac{1}{1260}\hbar t^7, \\
x_2(t) &= \frac{1}{6}\hbar t^4 + \frac{1}{6}\hbar^2 t^4 + \frac{1}{24}\hbar t^5 + \frac{1}{36}\hbar^2 t^6 + \frac{1}{1260}\hbar t^7 + \frac{29}{5040}\hbar^2 t^7 \\
&\quad + \frac{1}{2520}\hbar^2 t^8 + \frac{1}{9072}\hbar^2 t^9 + \frac{1}{2494800}\hbar t^{11}, \\
&\vdots
\end{aligned}
\tag{3.22}
$$

Then the series solution expression can be written in the form

$$
x(t) = x_0(t) + x_1(t) + x_2(t) + \cdots, \tag{3.23}
$$

and so forth. Hence, the series solution when $\hbar = -1$ is

$$
\begin{aligned}
x_1(t) &= -\frac{1}{6}t^4 - \frac{1}{24}t^5 - \frac{1}{1260}t^7, \\
x_2(t) &= \frac{1}{36}t^6 + \frac{5}{1008}t^7 + \frac{1}{2520}t^8 + \frac{1}{9072}t^9 + \frac{1}{2494800}t^{11}, \\
x_3(t) &= -\frac{5}{2016}t^8 - \frac{25}{72576}t^9 - \frac{1}{22680}t^{10} - \frac{1}{133056}t^{11} - \frac{1}{7484400}t^{12} \\
&\quad - \frac{1}{25945920}t^{13} - \frac{1}{20432412000}t^{15}, \\
x_4(t) &= \frac{5}{36288}t^{10} + \frac{25}{1596672}t^{11} + \frac{1}{399168}t^{12} + \frac{5}{15567552}t^{13} \\
&\quad + \frac{1}{90810720}t^{14} + \frac{1}{544864320}t^{15} + \frac{1}{81729648000}t^{16} \\
&\quad + \frac{1}{277880803200}t^{17} + \frac{1}{475176173472000}t^{19},
\end{aligned}
\tag{3.24}
$$

and so forth. Hence, the series solution is

$$
\begin{aligned}
x(t) &\simeq 1 + \frac{1}{6}t^3 - \frac{1}{6}t^4 - \frac{1}{24}t^5 + \frac{1}{36}t^6 + \frac{1}{240}t^7 - \frac{1}{480}t^8 \\
&\quad - \frac{17}{72576}t^9 + \frac{17}{181440}t^{10} + \cdots,
\end{aligned}
\tag{3.25}
$$

which converges to the closed-form solution (3.16).

3.3. Example 3

Now consider the nonlinear fourth-order IVP,

$$x^{(4)} = -xx'' + x'^{2}, \tag{3.26}$$

subject to the initial conditions

$$x(0) = 0, \qquad x'(0) = 1, \qquad x''(0) = 1, \qquad x'''(0) = 1. \tag{3.27}$$

The exact solution is

$$x(t) = e^{t} - 1. \tag{3.28}$$

According to the HAM, the initial approximation is

$$x_0(t) = t + \frac{1}{2}t^2 + \frac{1}{6}t^3, \tag{3.29}$$

and the linear operator is (3.18) with the property (3.19) where c_i $(i = 1, 2, 3, 4)$ are constants of integration. According to the *zeroth-order deformation equation* (2.2) and the *mth-order deformation equation* (2.9) with the initial conditions (3.20) with

$$R_m\left(\vec{x}_{m-1}\right) = x''''_{m-1} + \sum_{j=0}^{m-1} x_j(t)\, x''_{m-1-j}(t) - \sum_{j=0}^{m-1} x'_j(t)\, x'_{m-1-j}(t), \tag{3.30}$$

the solution of the mth-order deformation for $m \geq 1$ is the same as (3.10).

We now successively obtain

$$\begin{aligned} x_1(t) &= -\frac{1}{24}\hbar t^4 - \frac{1}{120}\hbar t^5 - \frac{1}{270}\hbar t^6 - \frac{1}{2520}\hbar t^7 - \frac{1}{20160}\hbar t^8 \\ &\vdots \end{aligned} \tag{3.31}$$

Then the series solution expression can be written in the form

$$x(t) = x_0(t) + x_1(t) + x_2(t) + \cdots, \tag{3.32}$$

and so forth. Hence, the series solution when $\hbar = -1$ is

$$\begin{aligned}
x_1(t) &= \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6 + \frac{1}{2520}t^7 + \frac{1}{20160}t^8, \\
x_2(t) &= -\frac{1}{5040}t^7 - \frac{1}{40320}t^8 + \frac{1}{362880}t^9 - \frac{1}{453600}t^{10} \\
&\quad - \frac{13}{19958400}t^{11} - \frac{1}{11404800}t^{12} - \frac{1}{148262400}t^{13}, \\
x_3(t) &= \frac{1}{403200}t^{10} + \frac{1}{1478400}t^{11} + \frac{43}{479001600}t^{12} \\
&\quad + \frac{1}{55598400}t^{13} + \frac{11}{1981324800}t^{14} + \frac{53}{32691859200}t^{15} \\
&\quad + \frac{29}{106748928000}t^{16} + \frac{829}{29640619008000}t^{17} + \frac{829}{533531142144000}t^{18}
\end{aligned} \tag{3.33}$$

and so forth.

Hence, the series solution is

$$\begin{aligned}
x(t) &\simeq t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6 + \frac{1}{5040}t^7 \\
&\quad + \frac{1}{40320}t^8 + \frac{1}{362880}t^9 + \frac{1}{3628800}t^{10} + \cdots,
\end{aligned} \tag{3.34}$$

which converges to the closed-form solution (3.28).

3.4. Example 4

Finally we consider the nonlinear Genesio equation [33]

$$x'''(t) + ax''(t) + bx'(t) - f(x(t)) = 0, \tag{3.35}$$

where

$$f(x(t)) = -cx(t) + x^2(t), \tag{3.36}$$

subject to the initial conditions

$$x(0) = 0.2, \qquad x'(0) = -0.3, \qquad x''(0) = 0.1, \tag{3.37}$$

where $a,\ b,\ c$ are positive constants satisfying $ab < c$.

First we solve (3.35) by means of HAM. According to the HAM, the initial approximation is

$$x_0(t) = 0.2 - 0.3t + 0.05t^2, \tag{3.38}$$

and the linear operator is

$$L\left[\Phi(t;q)\right] = \frac{\partial^3\phi(t;q)}{\partial t^3}, \tag{3.39}$$

with the property

$$L\left[c_1 + tc_2 + t^2 c_3\right] = 0, \tag{3.40}$$

where c_i $(i = 1, 2, 3)$ are constants of integration. According to the *zeroth-order deformation equation* (2.2) and the *mth-order deformation equation* (2.9) with the initial conditions

$$x_m(0) = 0, \qquad x'_m(0) = 0, \qquad x''_m(0) = 0, \tag{3.41}$$

where

$$\begin{aligned} R_m\left(\vec{x}_{m-1}\right) = {} & x'''_{m-1}(t) + a x''_{m-1}(t) + b x'_{m-1}(t) + c x_{m-1}(t) \\ & - \sum_{j=0}^{m-1} x_j(t)\, x_{m-1-j}(t), \end{aligned} \tag{3.42}$$

the solution of the mth-order deformation for $m \geq 1$ is the same as (3.10).

We now successively obtain when $a = 1.2$, $b = 2.92$ and $c = 6$,

$$\begin{aligned} x_1(t) = {} & 0.0673333\hbar t^3 - 0.0578333\hbar t^4 + 0.0031666\hbar t^5 + 0.00025\hbar t^6 \\ & - 0.0000119048\hbar t^7, \\ x_2(t) = {} & 0.0673333\hbar t^3 + 0.0673333\hbar^2 t^3 - 0.0578333\hbar t^4 - 0.0376333\hbar^2 t^4 \\ & + 0.00316667\hbar t^5 - 0.000882667\hbar^2 t^5 + 0.00025\hbar t^6 - 0.001603\hbar^2 t^6 \\ & - 0.0000119048\hbar t^7 - 0.00109873\hbar^2 t^7 - 0.0000592857\hbar^2 t^8 \\ & + 0.0000175397\hbar^2 t^9 - 0.0000000324\hbar^2 t^{10} - 0.0000000032\hbar^2 t^{11} \\ & + 0.0000000009\hbar^2 t^{12}, \end{aligned} \tag{3.43}$$

and so forth.

Now we use the new technique, namely NHAM, of Yabushita et al. [26] to solve (3.35). In this technique, we construct the *zeroth-order deformation equations* for not only (3.35) but also for (3.36) as follows:

$$\begin{aligned} (1-q)\, L\left[\phi(t;q) - x_0(t)\right] &= q\hbar\left[\frac{\partial^3\phi(t;q)}{\partial t^3} + a\frac{\partial^2\phi(t;q)}{\partial t^2} + b\frac{\partial\phi(t;q)}{\partial t} - f(t;q)\right], \\ (1-q)\, L\left[f(t;q) - f_0(x(t))\right] &= q\hbar\left[f(t;q) + c\,\phi(t;q) - \phi^2(t;q)\right], \end{aligned} \tag{3.44}$$

and the *mth-order deformation equation*

$$L\left[x_m(t) - \chi_m x_{m-1}(t)\right] = \hbar R_m\left(\overrightarrow{x}_{m-1}\right),$$
$$L\left[f_m(x(t)) - \chi_m f_{m-1}(x(t))\right] = \hbar R_m\left(\overrightarrow{f}_{m-1}\right), \tag{3.45}$$

with the initial conditions

$$x_m(0) = 0, \qquad x'_m(0) = 0, \qquad x''_m(0) = 0, \tag{3.46}$$

where

$$R_m\left(\overrightarrow{x}_{m-1}\right) = x'''_{m-1}(t) + a x''_{m-1}(t) + b x'_{m-1}(t) - f_{m-1}(x(t)),$$
$$R_m\left(\overrightarrow{f}_{m-1}\right) = c x_{m-1}(t) - \sum_{j=0}^{m-1} x_j(t)\, x_{m-1-j}(t). \tag{3.47}$$

Again, we successively obtain when $a = 1.2$, $b = 2.92$ and $c = 6$,

$$\begin{aligned}
f_0(x(t)) &= -1.16 + 1.68t - 0.19t^2 - 0.03t^3 + 0.0025t^4, \\
f_1(x(t)) &= 0, \\
f_2(x(t)) &= 0.377067\hbar^2 t^3 - 0.283467\hbar^2 t^4 - 0.0237\hbar^2 t^5 + 0.00908333\hbar^2 t^6 \\
&\quad - 0.000233333\hbar^2 t^7 - 0.000032142\hbar^2 t^8 + 0.00000119048\hbar^2 t^9, \\
x_1(t) &= 0.0673333\hbar t^3 - 0.0578333\hbar t^4 + 0.00316667\hbar t^5 + 0.00025\hbar t^6 \\
&\quad - 0.0000119048\hbar t^7, \\
x_2(t) &= 0.0673333\hbar t^3 - 0.0673333\hbar^2 t^3 - 0.0578333\hbar t^4 - 0.0376333\hbar^2 t^4 \\
&\quad + 0.00316667\hbar t^5 - 0.000882667\hbar^2 t^5 + 0.00025\hbar t^6 - 0.00474578\hbar^2 t^6 \\
&\quad - 0.0000119048\hbar t^7 + 0.000251111\hbar^2 t^7 + 0.00001125\hbar^2 t^8 \\
&\quad - 0.000000482\hbar^2 t^9,
\end{aligned} \tag{3.48}$$

and so forth. Then the series solution expression can be written in the form

$$x(t) = x_0(t) + x_1(t) + x_2(t) + \cdots, \tag{3.49}$$

$$f(t) = f_0(t) + f_1(t) + f_2(t) + \cdots. \tag{3.50}$$

The series solutions (3.12), (3.23), (3.32), (3.49) and (3.50) contain the auxiliary parameter $\hbar$. The validity of the method is based on such an assumption that the series (2.4) converges at $q = 1$. It is the auxiliary parameter $\hbar$ which ensures that this assumption can be satisfied. In

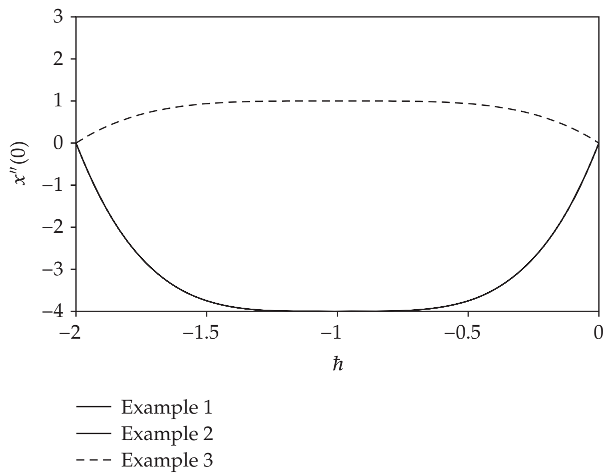

Figure 1: The $\hbar$-curve of $x''(0)$ given by (3.1), (3.14) and (3.26): fifth -order approximation of $x''(0)$.

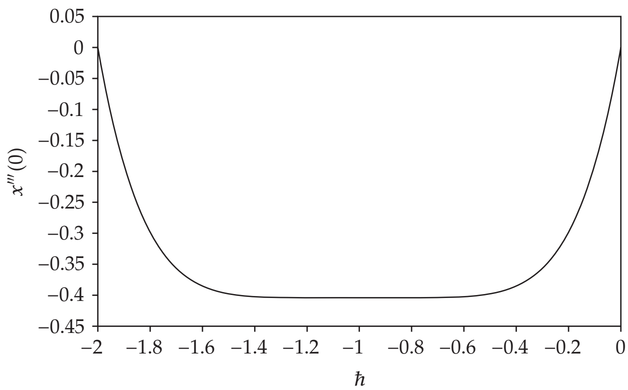

Figure 2: The $\hbar$-curve of $x''(0)$ obtained from the eleventh -order HAM approximation solution of (3.35).

general, by means of the so-called $\hbar$-curve, it is straightforward to choose a proper value of $\hbar$ which ensures that the solution series is convergent. Figure 1 show the $\hbar$-curves obtained from the fifth -order HAM approximation solutions of (3.1), (3.14) and (3.26). From this figure, the valid regions of $\hbar$ correspond to the line segments nearly parallel to the horizontal axis. Substituting the special choice $\hbar = -1$ into the series solutions (3.12), (3.23) and (3.32) yields the exact solution (3.3), (3.16) and (3.28). Also Figures 2 and 3 show the $\hbar$-curves obtained from the eleventh -order HAM and NHAM approximation solutions of (3.35) and (3.36). In Figure 4 we obtain numerical solution of the Genesio equation using the eleventh -order HAM and NHAM approximation. It is demonstrated that the HAM and NHAM solutions agree very well with the solutions obtained by the seven- and eight-order Runge-Kutta method (RK78). Moreover we conclude that the proposed algorithm given by NHAM is more stable than the classical HAM.

Remarks 1. Equation (3.35) represented by Genesio [33] as a system includes a simple square part and three simple ordinary differential equations that depend on three positive real parameters. Bataineh et al. [19] discussed the behavior of this system in the interval $t \in [0, 2]$

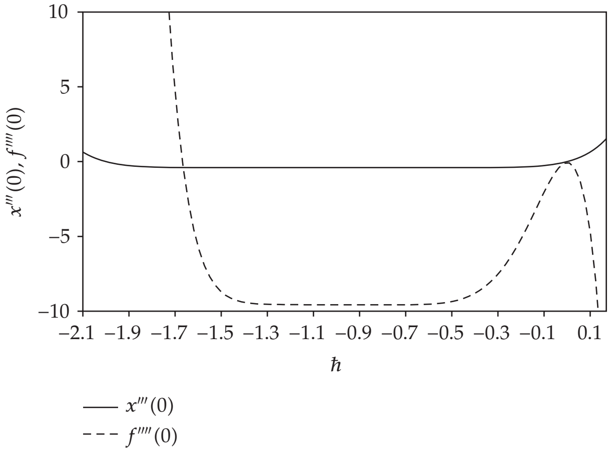

Figure 3: The $\hbar$-curve of $x'''(0)$ and $f''''(0)$ given by (3.49) and (3.50): eleventh -order NHAM approximation of (3.49) and (3.50).

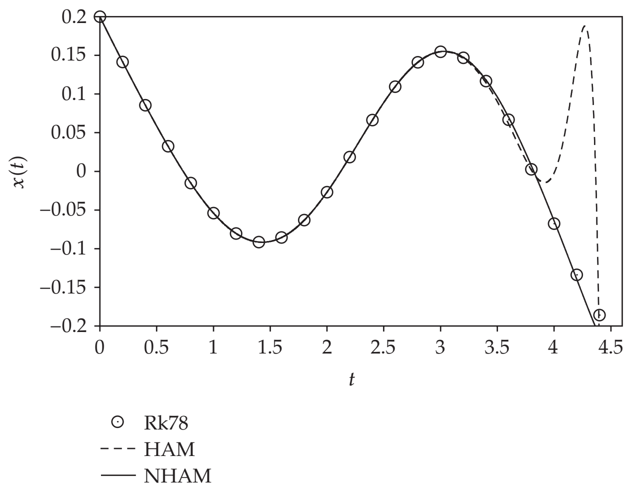

Figure 4: The eleventh -order HAM and NHAM solutions (3.49) with $\hbar = -0.7998$ for HAM solution and $\hbar = -0.7$ for NHAM solution versus RK78 solution for the (3.35) when $a = 1.2$, $b = 2.92$ and $c = 6$.

by using HAM, so according to Figure 4 we conclude that the behavior of numerical solution (3.35) is more stable than the numerical solution obtained by [19] using the classical HAM.

4. Conclusions

In this paper, the homotopy analysis method HAM was applied to solve a class of linear and nonlinear nth-order IVPs and the Genesio equation. HAM provides us with a convenient way of controlling the convergence of approximation series, which is a fundamental qualitative difference in analysis between HAM and other methods. The illustrative examples suggest that HAM is a powerful method for nonlinear problems in science and engineering.

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