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Learning to Solve Counting Problems: Challenges and Opportunities for Non-Math Majors

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ABSTRACT

Our study stems from the belief that meaningful and challenging mathematics can be made accessible to all students. To this end, we designed and implemented a special course for undergraduate non-mathematics and non-mathematics education majors. The course entitled *Counting and Chance* satisfied the requirement of a core course in quantitative reasoning. The content chosen as the focus of this course was basic combinatorics (i.e., counting problems) – a topic that is non-procedural in nature and could be made accessible to students with limited mathematical background. The design of the course was inspired by inclusive pedagogical principles that motivate and support students' conceptual learning. We present the characteristics of the learning environment that was developed within the framework of this course, point to the rich and sophisticated kinds of reasoning that students developed throughout the course, and discuss how students' collaborative engagement in the learning process seemed to have contributed to their views of themselves as math learners and their conceptions of what math is. Our findings point to the feasibility of such a course for non-math students, and its potential merit in helping students: (*i*) develop appreciation of mathematics as a topic that requires thinking, reasoning, and convincing; (*ii*) become more confident in their ability to do math.

Key Words: Counting problems, Collaborative problem solving, Reasoning, Inclusion, Dispositions, Equity

INTRODUCTION

Mathematics is generally viewed as difficult and accessible only to few (Boaler & Greeno, 2000). It is often taught as a high-stake performance subject that aims at determining who can and who cannot do mathematics (Boaler, 2015). Moreover, math classrooms are "unusually narrow and ritualistic, leading students to reject the discipline at a sensitive stage of their identity development" (p.171, Boaler & Greeno, 2000). In spite of calls for enhancing sense making and reasoning (CCSSM, 2010; NCTM, 2000), students' experiences of learning mathematics in secondary school are known to be mostly procedural, answer-based, and rely to a large extent on rote memorization. Consequently, many students develop a rather narrow conception of what math is as well as low expectations regarding their ability and likelihood to do well in math.

Our study stems from the belief that meaningful and challenging math can be made accessible to all students. We strive to offer students opportunities to experience different kinds of math learning than

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those described above. To this end, we designed and implemented a special course for undergraduate students who consider themselves "non-math" people, which satisfied the requirement of a core course in quantitative reasoning. The content chosen as the focus of this course was basic combinatorics (i.e., counting problems) – a topic that is non-procedural in nature and could be made accessible to all students, or at least to a vast majority. The design of the course was inspired by inclusive pedagogical principles that motivate and support students' conceptual learning.

In this paper we present the characteristics of the learning environment that was developed within the framework of this course. We then analyze a lesson that represents the kind of learning process that the students encountered, point to the rich and sophisticated kinds of reasoning that they developed throughout the course, and discuss how this process seemed to have contributed to their views of themselves as math learners and their conceptions of what math is.

Our findings point to the feasibility of such a course for non-math students, and its potential merit in helping students: (*i*) develop appreciation of mathematics as a topic that requires thinking, reasoning, and convincing; (*ii*) become more confident in their ability to do math.

INCLUSION, DISPOSITIONS, AND CLASSROOM NORMS in MATHEMATICS EDUCATION

The need for inclusion and equity in mathematics has been a serious concern of the community of mathematics education. For example, Boaler (2015) attributes mathematics inequality to stereotyped ideas about who can achieve well in mathematics. Stemming from this concern, there have been several calls for refining practices of teaching mathematics to ensure that all students have the opportunity to learn mathematics (e.g., Davis et al, 2018; Cobb & Hodges, 2007). These practices are expected to be responsive and attend to the sense(s) that the individual learner is making. According to Davis et al, teaching should be "both inclusive and deliberate in pedagogy — inclusive, in that one intentionally prioritizes the voices that constitute the collective, and deliberate, in that one purposefully welcomes so-called deviations from a lesson timeline to engage in critical conversations with learners" (ibid, p. 99). Further, it also calls for listening to students' responses in a broad and genuine sense, that is, listening to the ideas they build up, and responding in ways that foster their understanding of and interest in mathematics, not focusing solely on correctness and efficiency (Schulman, 2013).

Inherent to the concern of inclusion is the concern about the dispositions that students develop towards mathematics as they engage in classroom activities (Gresalfi & Cobb, 2006). According to Gresalfi and Cobb (2006), disposition encompasses ideas about, values of, and ways of participating with mathematics that students develop. This view of disposition is closely related to notions of identity and agency (Boaler, 2002). Students' development of interest or a sense of affiliation with mathematics is closely related to the ways in which they engage with mathematics (Lampert, 1990; Nasir, 2002). Moreover, pedagogical practices have the power to turn disaffected students away from mathematics for reasons unrelated to the nature of the content (Boaler, 2002). Thus, the kinds of opportunities students have to participate in classroom practices have bearing on the access they have to doing mathematics meaningfully and on the disposition they develop. For this reason, the nurturing of students' dispositions should be an intentional part of instructional design and pedagogy (Cobb & Hodge, 2007).

Related to issues of inclusion and dispositions are the ways in which classroom norms get constructed and established and how individual students engage in those practices (Yackel & Cobb, 1996; Lampert, 1990). Classroom norms have implications for students' interest, motivation, and sense of confidence. Moreover, the meaning of "doing math" in a particular classroom may account for students' dispositions (Solomon, 2007). Gresalfi and Cobb (2006) reflect on their work and suggest that analyses of classroom learning environment should focus on "the ways in which students interact with classroom resources — such as other students, instructional materials, and associated tools — the teacher, and the ways that these interactions shape and are shaped by the development of classroom norms" (ibid, p. 51).

Classroom norms join with sociomathematical norms (Yackel & Cobb, 1996) to form the construct of the normative identity as a doer of mathematics (Cobb & Hodge, 2007). The normative identity "established in a particular classroom indicates the identity that students would have to develop in order to affiliate with mathematical activity as it is realized in that classroom" (ibid, p. 166). A student's view of what it means to be good at mathematics may shape their disposition towards the subject; however, such personal views can be challenged and altered by the normative identity established in their mathematics classroom. Ruef (2017) reported on the feasibility of such changes, although she did not investigate their causes.

STUDENTS' STRENGTHS AND DIFFICULTIES IN SOLVING COMBINATORIAL PROBLEMS

Combinatorics is an important area of discrete mathematics, which is "an active branch of contemporary mathematics that is widely used in business and industry" (NCTM, 2000, p. 31). Not only is it connected to people's everyday experience and often to their professional practice as well, it also has the potential of creating a rich and motivating problem-solving context for students to explore (Lockwood, 2013). Consequently, there have been calls to include combinatorics as an integral part of the mathematics curriculum, from the early elementary grades through to the senior high school level (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; English, 1993; NCTM, 2000). Interestingly, on the one hand there is evidence that combinatorics can be taught at a wide range of levels, from an informal level to a formal and highly rigorous one. There are studies that show that young children with no prior learning history of any counting methods, can come up with sophisticated ways of solving counting problems (e.g., Maher, Powell, & Uptegrove, 2011a; English, 1991, 1993). On the other hand, combinatorics is considered one of the more difficult mathematical topics to teach and learn (Mashiach-Eizenberg & Zaslavsky, 2004).

Combinatorial problems are non-procedural in nature, most problems do not have readily available solution methods, and create much uncertainty regarding how to approach them and what method to employ. Two different solutions yielding different answers to the same problem may both seem equally convincing, and there are no straightforward verification strategies (Mashiach-Eizenberg & Zaslavsky, 2004). As Wassermen (2019) noted, "…although counting problems can be easy to state, they can be deceptively difficult to solve" (p. 16).

Elementary combinatorial problems deal with counting and accounting for all possible outcomes of an (actual or thought) experiment. A common thread across all combinatorial problems is the need to be cautious not to overcount and not to undercount, that is, we must "cover" all possible outcomes that are relevant to the problem, and make sure we do not count any outcome more than once. This is a concern from the start, even when one tries to count by listing all the outcomes. Usually, unless you have a systematic way of counting, you may easily miss an outcome or double-count it.

Combinatorial problems vary along several dimensions, including: the kinds of objects that are being counted and whether they are distinguishable or not (for example, people, numbers, physical objects); the relevance of order (does order matter? For example, seating people in a row implies that order matters, while selecting a committee with no designated roles, implies that order does not matter); the possibility of repetition or replacement (can the same object appear more than once? For example, if we explore 4-digit numbers, do we have a restriction on how many times a digit can appear in the number?); the underlying model – Selection, Distribution, and Partition (for details see Batanaro et al, 1997); the mathematical expression that is associated with the solution (see Batanaro et al, 1997, and Lockwood, Wasserman, & McGuffey, 2018). Related to the latter, counting problems may examine permutations, arrangements, and/or combinations [see Glossary].

Some of the above factors contribute to difficulties students encounter in solving combinatorial problems. Among these factors are the type of problem, the nature of the elements that comprise the outcomes, and the implicit combinatorial model associated with the problem (Batanero et al, 1997; Fischbein & Gazit, 1988). Lockwood (2013, 2014) provided a model for student combinatorial thinking that has three main elements: the set of outcomes, the counting process, and the mathematical expression associated with the solution of the problem. According to this model, there is a bidirectional relationship between every pair of the elements of the model. Students engage with these relationships when solving counting problems. For example, a set of outcomes can illuminate a particular counting process; a particular counting process can result in a specific mathematical expression, etc. Lockwood (2013) found that students have difficulty in making connections between expressions and sets of outcomes. To address this concern, Lockwood (2014) argued for a set-oriented perspective toward counting, in which a focus on sets of outcomes is an intrinsic part to solving counting problems. Wasserman (2019) attributes students' struggles in solving combinatorial problems to the mathematical notation and symbolism used, which could inhibit students' development of set-oriented perspectives.

Lockwood et al. (2018) examined further students' solutions to combination counting problems. They distinguish between *combination* problems that are based on an unordered selection of distinguishable objects and those that are based on (ordered) sequences of two or more indistinguishable objects. They termed the former type Category I and hypothesized that this type of combination problems' encoding of sets of outcomes is fairly straightforward. The latter type of combination problems was called Category II. Problems of Category II required encoding the outcomes (the sets of indistinguishable objects) with a corresponding set of distinct objects. Based on this difference in encoding the sets of outcomes, a survey, consisting of combination problems (problems that could be solved by using binomial coefficients) that had either of the two characteristics, was given to 126 students who did not take a combinatoric-specific class. They found that participants were (statistically) significantly less likely to use a combination approach on Category II problems, and when using a combination approach, they were more likely to do so incorrectly. The authors suggest that students may not view these types of problems in a similar way and based on Lockwood's (2014) set-oriented perspective towards counting perhaps students do not recognize that Category II outcomes can be encoded as sets of distinct objects.

As mentioned earlier, while there is evidence of difficulties that students encounter in solving combinatorial problems, there is also evidence of students' strengths, with or without formal instruction. For example, Fischbein and Gazit (1988) found that students as young as ten years old were able to learn some combinatorial idea with appropriate instruction that facilitated the use of tree diagrams. Interestingly, the teaching they employed using tree diagrams changes the relative level of difficulty of problems students had exhibited before instruction. While at first combination problems were easier than permutation problems, after instruction, permutation problems became much easier compared to combination problems. The explanation the researchers gave for these findings is that, in addition to the tree diagram being a helpful way to think about permutations, the formula for permutations is simpler and more intuitive than the formula for combinations.

Maher et al. (2011a) followed a cohort of students engaging in solving counting problems, in a longitudinal study, from their elementary grades till high-school. Their study was based on none to minimal researcher intervention (e.g., to suggest a solution pathway) as students were solving counting problems collaboratively. At both stages – elementary and high-school – students were found to naturally use listing methods by types of outcomes in order to solve combination problems and, with time, effort, and minimal guidance, the high-school students explored and were able to explain binomial coefficients through the context of concrete problem situations. Maher et al.'s (2011a) findings suggest that counting problems are naturally accessible to various aged students and that the abstract combinatorial concepts (such as the notions of combinations, or isomorphism) take time to develop. One way is through a rich opportunity to explore and revisit counting situations and tasks before introducing abstract combinatorial concepts and once introduced, to encourage explanations of these abstract concepts in terms of concrete task situations.

THE POTENTIAL OF COMBINATORIAL PROBLEMS FOR AN EQUITABLE LEARNING ENVIRONMENT

Boaler (2015) recommends a number of teaching strategies that she considers equitable, which include: offering all students high-level content, encouraging students to think deeply about mathematics, teaching students to work together, helping students become reflective about their learning, and working to change ideas about who can achieve well in math. Clearly, the mathematical content itself is not sufficient to create such an equitable learning environment, however, certain content lends itself particularly well as a basis for it. The studies described above, indicate that the terrain of counting problems could be made accessible to students of a wide range of age levels, mathematics achievement levels, and levels of interest in mathematics. In particular, there are several characteristics of counting problems that could be used as a springboard for the design of an equitable learning environment for students who identify themselves as "non-math" persons.

Counting problems can be **accessible** to many learners with no prior combinatorial background. Unlike some other mathematics domains that require formal prerequisites, the learning environment around counting problems may take non-traditional forms (e.g., exploration or discovery learning versus lecture), thus, offering multiple entry points. Counting problems can be presented in terms of real-life or contextually familiar situations. The literature suggests that combinatorial problems of a variety of complexity can be solved by students as young as in their elementary years and that they facilitate the development of natural enumeration processes (Maher et al., 2011a; English, 1991). For example, some tasks adapted from Maher and her colleagues for undergraduate students in the current study were given to and solved successfully by fourth and fifth graders (Maher, Sran, & Yankelewitz, 2011b). Although the reasoning and methods employed to solve the tasks may vary for different students, familiarity of contexts and the natural ability to count, organize, list, etc. make entry accessible to most learners.

Although accessible, counting problems can also be rather **challenging** for a wide range of students. They qualify as high cognitive demand tasks (according to Stein, Grover, & Henningsen, 1996) as they: require complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach explicitly suggested by the task, task instructions, or a worked-out example); require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions; require students to explore and to understand the nature of mathematical processes, relationships, and concepts; demand self-monitoring or self-regulation of one's own cognitive processes, as it may be hard to verify the solution with no readily available verification methods, particularly to ensure not to over or under count. Verification of solutions to counting problems becomes even more challenging when students are encouraged to try to come up with multiple solutions (as recommended by Bass (2017) and Leikin (2011)). This, in turn, may evoke uncertainty and conflict, the resolution of which can lead to a deeper understanding (Zaslavsky, 2005). If counting problems are carefully chosen for context, size (in count required), and underlying methods (complexity), they may be both accessible for a given population, yet challenging enough to elicit verification methods.

The accessibility of counting problems on the one hand, and their challenging aspects on the other hand, allow for moving from **intuitive/concrete** approaches (e.g., listing) to **more formal**, efficient, and sophisticated ways. Intuitive and informal methods include, for example, enumerative methods that produce concrete unordered (e.g., by randomly selecting) or ordered (e.g., listing by cases or types) sets of outcomes. Formal methods might utilize counting principles and additive or multiplicative operations, that may emerge from concrete experiences and processes of generalization and abstraction. Informal or formal methods do not imply correctness nor subjectivity/value of one over the other, but rather they are used to describe when a solution contains an enumeration of concrete outcomes and when it does not. Related to this feature, counting problems do not necessarily include notation and so they elicit devising plans for effective notation. Although there are formal mathematical expressions (or formulas) that can be applied when solving counting problems (some of them appear in the Glossary), it is still possible and even beneficial to use personal notation to represent outcomes and methods of counting (e.g., lists, tree diagrams) that make sense to the student. Personal notation could help students to reason with the problem and come up with a solution without using the formal mathematical expressions. Making it legitimate for students to operate on an intuitive level while encouraging them to move towards more formal methods is another way of offering an inclusive learning environment.

Counting problems can vary based on their solution strategies and often require non-procedural or a mix of procedural techniques. Thus, memorization of counting concepts (e.g., addition or multiplicative counting principles) is unlikely to be sufficient to solve a problem, particularly with multiple steps. Depending on the opportunities for exploration and learning environment, counting problems can foster a new experience of mathematics that includes **genuine collaboration** and debate. In studying a strand of related counting problems, Krupnik (2020) found that, given an opportunity to explore tasks with minimal researcher intervention, learning of young children occurred collaboratively when they were attentive to the counter examples and arguments posed by others and worked to convince others about their arguments. This suggests that the learning environment is inextricably related to the potentiality of the aforementioned characteristics of counting problems.

THE STUDY

1. Goals

The main goal of the study was to design and implement an inclusive yet mathematically challenging, quantitative reasoning learning environment for non-math undergraduate students, and to examine how the learning environment facilitates the development of students' mathematical reasoning and dispositions toward mathematics. More specifically, we aim at:

- Characterizing the kinds of reasoning students manifest as they solve counting problems;
- Identifying possible elements in the learning environment that may have contributed to the development of these kinds of reasoning;
- Making connections between students' struggles and successes in the course of solving counting problems and their views of themselves as learners of mathematics and of mathematics;
- Establishing the feasibility and merits of this Learning Environment for "non-math" students

Inspired by the work described above, we designed and implemented a learning environment within the framework of the "Counting and Chance" course with the intention not only to offer students the opportunity to learn the content meaningfully but even more importantly, to nurture their dispositions. In our analysis, we focus on how students interact with each other and with the available tools they have as they solve counting problems, in light of the norms that were established throughout the semester. We attempt to make connections to the dispositions they develop in this context.

2. Characteristics of the Learning Environment

As mentioned, the study was carried out within an undergraduate problem-based course on counting and chance (elementary combinatorics and probability) for non-mathematics and non-mathematics education majors, who took it as a core course in quantitative reasoning. The course ran 14 weeks, and met twice a week for 75 minutes. In addition, there was an optional 50-minute recitation session, on a different day.

There was no required textbook for the course, though relevant resources were made available for students on a regular basis. In particular, after every class, the instructor posted an enhanced power point that included the activities for that lesson, including comments, hints, solutions that came up in class. These power points also included snapshots of students' work in class, mostly those that were presented on the board. The problems chosen for the course and their sequencing were inspired by and adapted from some research publications (e.g., Maher et al, 2011a; Lockwood, Swinyard, & Caughman, 2015), other publications, and personal communications with Lockwood who shared some of the problems she used in her undergraduate courses in combinatorics. Typically, these problems require coming up with the total number of possible outcomes of a particular situation. For example, how many domino blocks make a full set? Or, how many different choices of pizza do we have, if we can choose up to 3 toppings from a given list of 3 toppings? Or, in how many ways can we seat 5 people in a row? In more conventional terms, we included problems that deal with permutations, arrangements, and combinations.

The specific characteristics of the learning environment stemmed from the view that what is learned cannot be separated from how it is learned and from the studies described in previous sections. Thus, the learning environment was designed with the following guiding principles in mind: (*i*) build on students' (intuitive) ideas and gradually facilitate the development of more formal methods; (*ii*) emphasize process vs. final answer; (*iii*) establish an atmosphere conductive of sharing and communicating ideas and critiquing them ("risk taking") as well as norms of accountability and justification; (*iv*) foster students' genuine and productive collaborative work; (*v*) encourage multiple solutions; (*vi*) seize opportunities to evoke uncertainty and cognitive conflict and facilitate the resolution of them; (*vii*) keep teacher intervention in the learning process to a minimal, mainly as a facilitator providing some guidance according to the classroom situation.

3. Overview of the Course

The implementation of each one of the lessons followed a similar structure: (a) whole group task introduction, (b) small-group collaboration on the task, and (c) whole group discussion of students' approaches and solutions presented on the board, (d) continued small-group work on the task, if necessary, (e) concluding results of the reasoning to solve the task.

As mentioned, the course was problem-based. Typically, students would work in class on one or two problems per lesson. There were cases where one problem took more than one lesson to explore. For most of the problems, students were deliberately asked to try to solve the problem in multiple ways, unless multiple solutions were suggested spontaneously. Rather than giving students formulas to apply, or rules of thumb, they were offered ways to think about certain situations. For example, the "addition principle" was connected to the idea of dividing the outcome set into cases (i.e., disjoint subsets). It made sense to them that instead of counting the number of all possible outcomes, they could count the numbers in each case separately and then add them up to get the total. Thus, the choice of operation, in this case – addition, seemed very reasonable. Similarly, they were introduced to a "branching" model, which is basically the idea of a tree diagram, where they could picture why multiplying the numbers would make sense. The additive model with cases helped students focus on the outcome set while the idea of branching was closely related to the counting process. In terms of Lockwood (2013, 2014), students had opportunities to develop a set-oriented perspective, as well as conceptualizing of the counting process. While students unknowingly developed and used common combinatorics formulas throughout the first half of the semester, no formula was explicitly presented until the combination notation, $\binom{n}{k}$, was introduced half-way through the course (in Lesson #15). Overall, there was a gradual development of more formal and generalizable tools, as questions of efficiency in reaching a solution began to evolve.

For this paper we focus mainly on Lesson #17 (out of 28 lessons). Students were introduced to the notation and meaning of $\binom{n}{k}$ only in Lesson #15, after working on a number of combination problems, of both Category I and Category II (Lockwood et al, 2018). In Lesson #16, students worked on the Four-Topping Whole Pizza Problem below (adapted from Maher et al, 2011b).

<u>Four-Topping Whole Pizza Problem (4TWP)</u>

> A pizza shop offers a basic cheese pizza with tomato sauce. A customer can then select from the following toppings to add to the basic, cheese pizza: Onions, Sausage, Mushrooms, and Pepperoni.
> How many different choices for pizza does a customer have?
> Note that cheese is NOT an additional topping; all pizzas have cheese as a topping. Also, the order of toppings does not matter either.

They were able to solve the 4TWP problem and reason about it in two different ways (by cases involving binomial coefficients, or as a power of 2). Approaching it by cases, required splitting the outcomes set into 5 cases: Case 1: Pizzas with exactly 0 toppings; Case 2: Pizzas with exactly 1 topping; Case 3: Pizzas with exactly 2 toppings; Case 4: Pizzas with exactly 3 toppings; Case 5: Pizzas with exactly 4 toppings. Approaching it by powers of 2 would suggest that for every topping there are 2 choices: Choosing it or not choosing it. We can describe this counting process as a "branching" model, yielding 2^4.

In previous lessons students solved similar (actually isomorphic) problems, e.g., "The Towers Problem" (adapted from Maher and Martino, 1996), and with some guidance were able to see the connections between these problems.

In Lesson #17 students were introduced to a more advanced pizza problem, that allowed selecting toppings just to half the pizza (also adapted from Maher et al, 2011b). The reason for including this problem at this point, was that they could build on their findings for the whole pizza problem, yet there was an additional complexity that required additional thinking. Nonetheless, students had sufficient tools to build on, but could not apply them automatically.

<u>Three-Topping Pizza with Halves (3THP)</u>

> A local pizza shop has asked us to help them design a form to keep track of certain pizza sales. Their standard "plain" pizza contains cheese. On this cheese pizza, one, two or three toppings could be added to either half of the plain pizza or the whole pie.
> How many choices do customers have if they could choose from three different toppings (Anchovies, Mushrooms and Pepperoni)?

There are several ways to approach this problem. One way would be by listing in a systematic and concrete way each possible outcome. This approach would require a clear representation of the outcomes and a way to monitor that they were not under or over counted. The total number of outcomes is 36.

Another approach could be to treat each half pizza separately as a whole pizza and build on what had been established previously for the whole pizza problem (4TWP). This approach yields 8 different combinations of toppings for a given half (allowing "no topping"). Then the question remains, how the number of possible combinations on each half separately helps in finding the total number of combinations for the whole pie. At first glance, it may appear that multiplying the number of options for each half would yield the correct answer, as the "branching" way of thinking can be applied here (to a certain extent). This would lead to 8×8, which in fact is overcounting. Taking into account that the order of the halves does not matter (we can turn around the pie), would suggest dividing 64 by 2.

However, this would be undercounting. Figuring out how to proceed and what the correct solution might be is challenging though accessible. Clearly, there are other approaches to solve the problem as well.

4. Participants

Participants were undergraduate non-mathematics and non-mathematics education majors. Although for this paper we focus on the first semester in which the course was taught, this course has been offered three semesters in a row, with students of similar backgrounds and aspirations (altogether, 49 students participated in the first three semesters). In the first semester there were 12 undergraduate participants representing four majors: early childhood and special education (3), childhood and special education (2), communications (6), and applied psychology (1), and were a mix of freshmen (8) and juniors (4).

5. Data Sources

The purpose of the data collection was to capture the development of students' mathematical reasoning, disposition, and confidence, and to this end we collected a breadth of data sources, each with its own purpose.

- *Videotaped lessons*

All class sessions (except for the recitation sessions) were videotaped using two cameras to capture both small group and whole group interactions. The video recordings provide us with hindsight access related to students and the course. Video recordings were transcribed as needed.

- *Audio recorded lessons*

Midway through the semester (Week 8, Lesson #13), we began to audio record small group discussions to capture conversations that might not be audible on video. The audio recorders helped us to minimize crosstalk incidents by providing isolated audio of individual group discussions that allowed us to better recreate simultaneous conversations that occur during the small group class segments. The audio recorders also pick up small talk, utterances, and whispered reactions that video cameras miss during whole class presentations. Audio recordings were transcribed as needed.

- *Field notes*

Field notes were taken in all classes and provide a contemporaneous record of student discussions and interactions that took place before, during, and after class, some of which was not captured by video and/or audio.

- *Course material*

Course material included power point presentations of each lesson, classroom and homework assignments, students' written work in class and as part of their homework assignments, and quizzes. Students' written work provide us with an opportunity to examine students' mathematical thought processes as they worked through counting problems.

- *Mathematics Autobiography*

Students' mathematics histories were collected through a mathematics autobiography and were used as a baseline to identify students' dispositions towards mathematics and their confidence as mathematics learners. The mathematics autobiography was the first homework assignment and asked students to respond to the following prompts:

1) If you were asked to describe mathematics to someone who had never studied or has minimal experience with math, how would you respond?
2) How do you feel about mathematics? What experiences led you to this feeling?
3) How confident are you about being able to do well in math? What do you attribute your confidence level to?

- *Weekly feedback questionnaires*

As part of the course requirements students were asked to respond to a weekly questionnaire. The purpose of the weekly questionnaire was twofold: 1. to encourage students' timely reflection on their learning, and 2. to provide the instructor with student feedback that helped to shape future lessons. The weekly questionnaires included questions such as:

- Please reflect and specify what has been working well for you and what may not have been working well for you.
- Reflect on this week's lessons and discuss what, if anything, was interesting, redundant, and/or contributed to your understanding.
- Reflect on today's lesson and discuss to what extent it was challenging and how it contributed to your learning. Please provide a detailed explanation and/or examples.

- To what extent do you feel that your performance on the quiz reflects the level of understanding that you have developed throughout the course? Please provide a detailed explanation and/or examples.

• *End-of-Semester written questionnaire*

During the last class students were asked to respond in writing to a questionnaire that was used as a point of comparison to identify any potential changes to students' dispositions towards mathematics and confidence as mathematics learners. The questionnaire also provided the instructor with student feedback that could be used to help shape the course in future semesters. The end-of-semester written questionnaire included the following questions:

1. Course activities
   1.1 Which course activities (if at all) did you find particularly motivating and/or helpful in developing your thinking about counting problems? Why?
   1.2 What course activities (if at all) did you find the least motivating and helpful in developing your thinking about counting problems? Why?
   1.3 What suggestions can you offer us for improving the course in terms of content, structure, activities, management, etc.?

2. Disposition towards mathematics
   2.1 How confident were you about being able to do well in math before the course began? In what ways, if at all, has your response changed over this semester? Please prob=vide specific examples to identify how this change (or non-change) might have occurred.
   2.2 How do you feel about mathematics? What experiences led you to this feeling? Please consider experiences, both good and bad. In what ways, if at all, have these feelings changed? Please provide specific examples to illustrate this change (or non-change).

3. Classroom environment
   3.1 Think back to when you took traditional math courses in high school and discuss how the learning environment (classroom setup, lesson delivery, etc.) was structured in comparison to the Counting and Chance course. What was similar and what different?
   3.2 In what ways did the Counting and Chance learning environment promote (or impede) your learning?
   3.3 Throughout the semester there were many opportunities to collaborate with each other. How would you characterize the nature of these collaborations? To what extent do you feel they were productive? In what ways was your collaboration in Counting and Chance similar to or different from previous experiences you have had with working in groups during math class?

• *Anonymous Course Evaluation*

The university asks students to complete an anonymous course evaluation in the closing two weeks of each semester, the results of which remain unknown to the instructor until the respective semester is over and all grades have been posted.

6. Analysis

As mentioned above, all lessons were video-taped, audio-taped, and transcribed. For the purpose of this paper, we selected a lesson that captured elements that we observed throughout the course. Therefore, the task and lesson can be considered generic, rather than special cases in our analysis and were chosen to exemplify the nature of the entire semester course for this population over the different semesters.

We then divided it into chronological *segments* that reflected the typical structure of the lessons, according to the setting (whole class – opening; initial pairs/groups – problem-solving; whole class – sharing; return to groups or changes in grouping – problem-solving, discussion, formulation of a complete solution and justification, etc.; whole class – closure). For each segment each author individually annotated them to highlight the development of each group's interaction with group members/instructors pertaining to the solution of the task and their argumentation in support of their reasoning. We discussed our annotations to come to a unified understanding of our observations of the development of the class' problem-solving.

Then *episodes* were identified that conveyed the kinds of thinking, reasoning, struggles and accomplishments students manifested within the segment of a particular class setting. Once the episodes were identified and compared within a segment and across segments, themes about the mathematical learning environment, and the nature of and the reasoning that was elicited by the counting problem emerged. We purposefully chose an excerpt from an episode to focus on and illustrate a theme from each segment (although not only one theme existed in each segment and each theme was not unique to a particular segment or episode).

In order to examine whether there were changes in students' dispositions towards mathematics and their views of themselves as learners of mathematics, we first did a side-by-side comparison of the two questions that appeared on both the mathematical autobiography and end-of-semester questionnaire. For each of these two questions, we identified words and phrases used by students that were indicative of their affective characteristics and mathematics learner confidence pre- and post-course. Following this initial comparison, we turned to the end-of-semester written questionnaires to examine to what each student attributed their change(s) in disposition and/or confidence. Once the attributes were identified, we revisited the segments and episodes presented below to determine if and where such attributes were present during the lesson. Finally, we examined the anonymous course evaluation to determine whether there was any further indication of changes in students' dispositions towards mathematics and their views of themselves as learners of mathematics, again by examining the language used by students.

FINDINGS

The lesson we present had a similar class structure (i.e., segment) as described earlier. We use this lesson to convey how the learning environment unfolded, to show what students were able to do, and how they supported their reasoning.

The findings are separated into two parts. Part 1 is organized by three themes that emerged as students' reasoning developed within each segment. Themes are presented in the same chronological order as the segments. Each theme begins with a general explanation, followed by a context of the segment(s) and a transcripted excerpt (with commentary) of an episode occurring within a segment(s). Utterances from transcripts are denoted by "L" and the corresponding number (e.g., L5).

Part 2 presents the ways in which students characterized their pre- and post-course dispositions towards mathematics and their views of themselves as learners of mathematics. Part 2 is organized in three parts: students' disposition towards mathematics, students' confidence as mathematics learners, and the affordances they attribute to the learning environment in shaping both.

Part 1: Students' Reasoning

The following occurred as the lesson unfolded into four distinct segments:

Segment 1. Initially four pairs solved the problem in the following ways, with flexibility to move around and talk to other groups:

i. **Pair 1:** Yesenia and Carly counted by types of outcomes and came up with a list that seemed to them to cover all outcomes (i.e., 8 whole pizzas with same toppings on both halves, 8 pizzas on one half 0 topping and on the other the 8 topping combinations, 3 pizzas with one different topping on each half, 3 pizzas with two topping on one half and one topping on the other)
   - Solution: 21 outcomes

ii. **Pair 2:** Rosalind and Doug also counted by types, though different than Pair 1, and came up with a list that seemed comprehensive to them.
   - Solution: 21 outcomes (interestingly, they also listed at first exactly 21 outcomes as Pair 1 did, but the specific set of outcomes they listed was different than the set of outcomes Pair 1 listed)

iii. **Pair 3:** Elena and Lisa built on the previous class' ideas treating each half pizza as a whole pizza and were able to justify why each half of a pizza had 8 options. They multiplied 8 options by 8 options to find 64 outcomes. They adjusted their solution to remove duplicates by dividing 64 by 2 to find 32 outcomes after they compared their solution with Pair 2.
   - Solutions: First 64, then 32 outcomes.

iv. **Pair 4:** Veronica and Ethan built on the previous class' ideas, identifying and justifying the 8 options for each half, then multiplying 8 by 8 and dividing by 2.
   - Solution: 32 outcomes.

Segment 2. Different approaches and results selected for presentation to the whole class: first, Carly's list, arriving at 21 outcomes by cases (Pair 1), and then Veronica's multiplication/division calculations, arriving at 32 outcomes (Pair 4). Students, including Rosalind who had different outcomes in her own solution list, added to Carly's list to arrive at 36 outcomes. The class then analyzed Veronica's calculations for over or under counting.

Segment 3. Students returned to pair or small group collaboration about the two solutions presented to resolve discrepancies of which solution to trust, 36 by a case list (to which we refer as Solution A) versus 32 by calculations (to which we refer as Solution B), and why 32 had missing outcomes. Some groups decided on 36 as a final answer and searched for the missing 4 in the calculated solution that gave 32.

Segment 4. Whole class resolution of the undercounted outcomes, putting it all together, and closure.

Note that the entire lesson was devoted to solving the 3THP.

The following themes emerged (in parentheses are the lesson segments that will be presented to illustrate the theme):

1. The accessible yet challenging nature of the counting task (Segment 1);
2. Conflict and uncertainty (Segments 2 and 3);
3. Resolution of the conflict (Segment 4)

Note that these themes were present throughout the lesson.

Theme 1: The accessible yet challenging nature of the counting task

At the start, when the students were presented with the 3THP, they did not have any obvious way to solve it, nor a reliable way to verify their solution. Participants did not have a readily available procedure or prescribed mathematical tool. Thus, the task naturally elicited the groups to reason about which strategies (e.g., would listing by cases be a long process?) and which mathematical tools to use (e.g., choosing between counting operations of addition or of multiplication). On the other hand, the task was accessible and required minimal prior knowledge. As we will see below, they had available tools and ways to try out. Each student could begin working on the problem in a meaningful way, and could make some progress towards a complete solution, regardless of the approach s/he took. The careful choice of "size" (e.g., number of toppings) was small enough to be manageable if explored by listing all possible outcomes (the total of 36), yet – was not too small to make the listing trivial. Moreover, students could utilize counting tools that were established in prior lessons or different task situations, but the procedures to using those tools were not readily available or automated. For example, they could build on their solutions to the whole-pizza problem in the previous lesson. While the problem situation appears to be very similar, these two problems are not isomorphic, even if you treat each half pizza separately as a whole pizza.

Segment 1: Initial working in pairs in collaboration

This class segment illustrates the accessibility of the task. Pairs 1 and 2 and Pairs 3 and 4 entered the problem in different ways. In Episode 1 (*Table 1*), building on a solution to the 4TWP approached by cases (0, 1, 2, 3 topping cases), Elena, from Pair 3, identified the case options for each half. Specifically, she established that for each half, separately, similar to the whole pizza problem with up to 3 toppings, there were altogether 8 combinations/options of toppings. Then, using tools established in the previous lessons, she and Lisa multiplied the 8 options of one half by the same 8 options for the other half of a pizza to obtain 64 outcomes. Rosalind and Doug, from Pair 2, drew on a listing strategy by cases to begin this task. They constructed 21 outcomes.

Episode 1 illustrates the challenging yet accessible nature of the task. Each pair identified a method to arrive at a solution. During this segment, not illustrated by this episode, the instructor suggested to Pair 3 and 2 to discuss their differences. In discussion with Pair 2, Lisa recognized double-counting in their methodology. Lisa and Elena adjusted their solution by dividing 64 by 2 to arrive at 32 outcomes and the groups discussed the possibility that Rosalind and Doug's list of 21 may have missed 11 outcomes. The two different answers and the pedagogical move to ask the pairs to compare, elicited exploration to find the missing outcomes together. In the next episode, a new solution is presented to the whole class and a conflict in different solutions prompts students to reason about the meaning of their methods.

Theme 2: Conflict and uncertainty

Conflict arose naturally and elicited tension about what solutions to trust. In some cases, answers were the same, but each method required justification to verify if the answers were correct (e.g., two different strategies produced 21 number of outcomes, yet not the same 21 outcomes, as in Segment 1, Pairs 1 & 2). An inherent norm in this particular learning environment was that even if the answers were the same, it did not mean the reasoning was correct. In other cases, answers were different, using similar approaches (e.g., by listing one group produces 21 outcomes and another 36 outcomes) or differed approaches (e.g., using mathematical tools versus listing). There was an iterative nature to examining and "correcting" initial "answers" and a culture of encouraging multiple ways to solve a problem with supportive reasoning. Students provided arguments or methods of verification for solutions and strategies (such as to make meaning to strategies in the context of the task situation). Students dealt with ideas of under or over counting their outcome solutions. Furthermore, students were encouraged throughout the semester to examine the advantage of generalizable, formal, or more efficient methods. The search to verify or need to prove and understand formal methods that had potential for generalizability became a norm.

In this lesson, conflict naturally arose about different final answers between to two appearing *convincing* solutions. The segments will illustrate how conflict elicited student examination and reconciliation. At the first stage, students were uncertain about which answer to trust. An episode for Segment 2 (whole class sharing), shows students examining the formal methods used in Solution B. They analyzed if Solution B was an over or under estimate by examining the multiplication and division operations used. The students attempted to understand the meaning of the generalizable solution (the math tools used) in terms of concrete pizzas.

An episode for Segment 3 (return to group problem-solving) shows a shift of focus in one group's attempt to advance their understanding of that approach. As they became more confident about the validity of the concrete list in Solution A, they progressed to a more focused goal that dealt with overcounting or undercounting to overcome uncertainty of the formal methods. This group began to trust Solution A and tried to bridge the two solutions by addressing how Solution B undercounted.

<u>*Segment 2: Carly's and Veronica's presentations of two different solutions and class reactions*</u>

We present Segment 2 where Carly presented her solution with a list of 21 combinations by partial cases. Rosalind, Elena, Yesenia, and others added to the list, thereby providing "counterexamples" to eliminate the answer of 21 pizzas (see *Figure 1*). With new additions to the list, the class collaboratively arrived at 36 outcomes (Solution A).

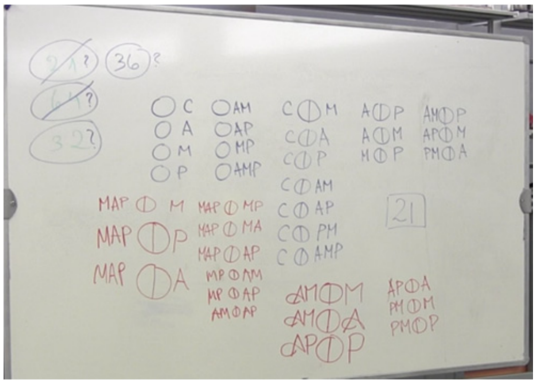


Figure 1. Carly's Solution with added outcomes by classmates

Then, Veronica presented Solution B with 32 outcomes obtained by multiplying 8 options for one half of a pizza by 8 options for the other half of a pizza and dividing by 2! to remove duplicates (see *Figure 2*), since the "order" of each half did not matter. She used mathematical tools in a similar way as Elena and Lisa (see Episode 1, *Table 1*).

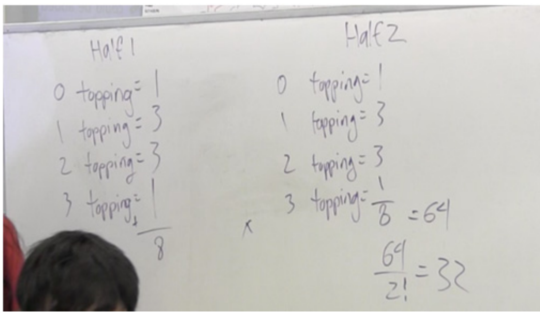


Figure 2. Veronica's Solution in Segment 1

After the presentations of the two solutions, the teacher prompted the class to comment on them. The students offered their arguments to the class. We present a transcript of Episode 2 (*Table 2*) of Elena and Veronica analyzing the meaning of the 8 options on one half multiplied by the same 8 options on the other half divided by 2! in the context of half-pizzas.

Elena initially argued by example (the 0 topping cheese pizza and the 3 same toppings on both halves pizza) that some pizzas were not duplicated by the multiplied count to justify the need to replace removed pizzas. Veronica argued by example (one topping of mushroom on both halves of a pizza) that a single topping pizza was duplicated in the count. Her reasoning for the duplicated pizzas was that the sides of the halves made a difference (this reasoning is invalid).

The purpose of illustrating this segment and episode was to show the focus moved from comparing two solutions to analysis of one method. Interestingly, the uncertainty elicited some students to try to make sense of the solution that was more formal by utilizing counting operations and tools (see *Table 2*). Recall, a goal in this

Table 1. Episode 1 – two pairs' initial problem solving strategies

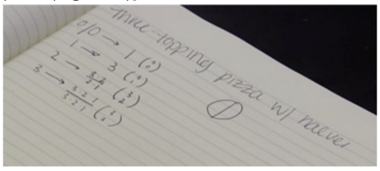
Transcript : Elena and Lisa (Pair 3)	Transcript : Rosalind and Doug (Pair 2)	Commentary
Elena: I really think we have to solve this the same way as we did the last time, but instead of adding [the cases] we multiply the possibilities of one side of the pizza by, I don't know, the other side because for every cheese half you can have like Anchovies, mushrooms, pepperoni [Elena returns to her Lesson #16 notes as she writes case possibilities for one side of a pizza (Figure 3)]  Figure 3. Elena's Lesson #17 notes	Rosalind: I wasn't here the last time, so first I decided I was just gonna draw them all out, but then I realized that for let's say a pizza that like half cheese and then half toppings, for the toppings you have like, I'm thinking, three different toppings options, right? Doug: yeah Rosalind: I don't know, I was just gonna write it out. I'm confused. Doug: I think we should do that first. Write every possibility out Rosalind: Yeah, should we do that? Doug: Yeah.	Elena and Lisa use the cases of pizzas with 0, 1, 2, and 3 toppings to make progress in the new half pizza situation. They identify 8 options for each half. Not shown in this transcript, they then multiply 8 options by 8 options to obtain 64 outcomes at first. Rosalind and Doug decide to make a list. Not illustrated in this transcript, they list by types of pizzas to obtain 21 outcomes.

Table 2. Episode 2 – Students' arguments about the presented solutions

Transcript	Commentary
1. Elena: I was wrong *[referring to her 32 when seeing 36 listed outcomes-page 13 of main]*… I think it's 36… I think when we divide by 2, it takes away like the full pizza. Does that make sense?	Elena's first argument was that the "full pizza" occurred only once and thus, dividing by 2 removed non-duplicated outcomes.
2. Veronica: Like all cheese? 3. Elena: Yeah. Because if you do like the branching way – if you have like one [pizza] with the 0 toppings is just cheese that's only happening once [counted once] when you branch it off. So, if we divide by 2–. And it's the same thing [counted once] for the 3-topping pizza.	Elena's terminology, "branching way," is a reference to the multiplicative counting principle and how the tree diagram representation of combinations "branch out." Combinatorial principles such as the Fundamental Counting Principle (Product Rule) were never explicitly presented in the course, so students had their own descriptors.)
4. Veronica: But if you say you get one pizza of just mushroom from half, you can have mushroom 1 and mushroom 2. That's the same thing as mushroom 2 and mushroom 1. That's also one whole mushroom. 5. Elena: Yeah. You're right. That's true. I don't know.	Veronica counters Elena's claim with an example that "mushroom 1 and mushroom 2…[is] the same thing as mushroom 2 and mushroom 1" which was also the same as "one whole mushroom." Based on evidence of her prior solution presentation, she is reiterating her claim here that the whole mushroom pizza is double counted in the multiplication.
6. Veronica: That's why in my head I was like "oh, you have one whole pizza of just one topping" but that's essentially the same thing as picking one topping here [*points to half 1*] and then picking the same topping again [*points to half 2*] 7. Elena: Yeah like the whole pizzas happen twice which is why we're dividing it. But then I don't know where the other four come from.	Notice Elena agrees with Veronica that "whole pizzas happen twice" and indicates conflict as to the missing four pizzas that were counted in the solution of 36 (Note that Elena would later return to her initial claim that whole pizzas were not duplicated when she engaged in the small group with Lisa, Rosalind, and Doug).

mathematics course was to have students move towards using mathematical tools. Although listing was not very efficient, by listing carefully in a group's effort students were able to find all outcomes without overcounting. The other groups used math tools that were built from the previous 4TWP Problem. When they trusted the listing, they began to focus on the meaning behind their operations within the context of this new situation. Segment 3 shows their attempt to identify the missing outcomes in the undercounted Solution B as compared to the list solution (Solution A).

Segment 3: Small group collaboration about the two solutions presented

The class was prompted by the instructor to address the tension between under and overcounting outcomes. The instructor offered method to resolve the conflict by asking the class to study the two solutions and identify the over or under counted outcomes. She also reminded the class that "order" of each half did not matter in this counting problem. Note that also in the previous 4TWP Problem the order of the toppings did not matter. However, in this problem, an additional aspect of "order" needed to be considered:

> "Maybe there's something there that needs to be thought out. You already accounted for the fact that it doesn't matter which half, right? Because you know when we get a pizza and turn it around, you don't know which half is which, right? So, now I want you to think more and come up with some resolutions...If you think it's 32, then you have to find four that they overcounted. And if you think they're right [*Carly & Yesenia's listing of 36*], you have to find out where you've undercounted."

Elena, Lisa, Rosalind, Ethan, and Doug from Segment 1 communicate their ideas, scrutinizing the multiplication operation to make sense of and account for the types of pizza combinations that is "counted" in the multiplication strategy. They continued their collaboration to explore whether certain pizza combinations were duplicated in the "8×8". It is evident that they began their search for a resolution with the assumption that there were 36 outcomes, and so, they attempted to identify why their solution is four fewer. Lisa returned the conversation back to Elena's argument that not all outcomes were double-counted (*Table 3*, L3 – L7). Elena provided examples of pizzas that were and were not duplicated (this opposed her initial agreement in Episode 2 with Veronica that pizzas with single same-toppings on both halves were double-counted). Notice the examples that Elena provided were of no or one same topping on both halves. Rosalind supported her argument and claimed that the four missing outcomes were the result of dividing those pizzas.

Note that in the 3THP, there were 8 pizza outcomes with same combination of toppings on both halves that were undercounted by the division of 2 from 8×8. The students tended to look at a subset of those undercounted outcomes. They identified a discrepancy of 4 between the Solutions 1 and 2. When they trusted Solution 1 they tried to detect the missing outcomes. They identified the missing outcomes to be the 0-topping cheese pizza and the pizza with the single same-toppings on both halves to address the discrepancy of the difference (of 4) between the two solutions. They did not realize there were 4 other outcomes that also were undercounted (same combination of two or three toppings on both halves).

Theme 3: Resolution of the conflict

In a mathematics classroom a balance and tradeoff exist between students solving a challenging, accessible task and some turning point or closure. On the one hand the instructor was only offering prompts to identify the uncertainties students might deal with in their small groups. On the other hand, when an impasse occurred there was a need for more guidance or interference. The role of the teacher became guided to help students move forward, while still maintaining student notation and solution ideas.

Segment 4: Whole class discussion and conclusions

After the instructor checked in with each group regarding their findings, she shifted the class back to a whole class format. Episode 4 (*Table 4*) illustrates the conclusion of the class session where resolution took place with real-time class constraints. Building on students' representations and ideas (Ethan's representation question about Carly's list in *Table 4*,

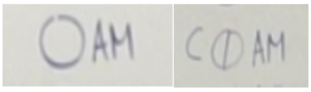
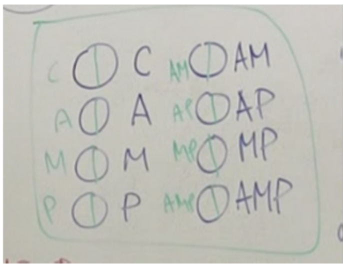
Table 3. Episode 3 – a group debate

Transcript	Commentary
1. Lisa: If we undercounted, we would've undercounted four. What did you [Elena] say, like the whole pizzas? 2. Elena: Yeah.	Lisa references Elena's initial claim from Episode 1 that pizzas with the single same-topping on both halves were undercounted.
3. Rosalind: And that makes sense because there are four whole pizzas. There's a full cheese, there's a full– 4. Lisa: I know it makes sense, but we didn't undercount those ['whole pizzas']. 5. Rosalind: When you divided, don't you eliminate those because you're eliminating all the double counting and mushroom, mushroom is the same.	Rosalind agrees with Elena's initial claim that "whole pizzas" were not double-counted. She supports her reasoning by referencing four pizza combinations with only cheese or a single topping of mushroom, anchovy, or pepperoni on both halves. (Note there are four other pizza combinations of the same toppings on both halves.)
6. Lisa: No because by multiplying 8 by 8, we multiplied mushroom-one/mushroom-two, mushroom-two/mushroom-one.	Lisa explains that the single same-topping on both halves are double-counted in 8×8. She uses an example of a pizza with mushroom on both halves would be duplicated.
7. Elena: But I'm saying cheese, just if you have cheese, the only time the zero topping is occurring is once. You're not gonna have cheese, cheese [pizza with cheese on both halves], cheese, cheese [duplicate pizza with cheese on both halves].	Elena provides an example of a pizza combination (the no topping on both halves) that occurs only once.
8. Lisa: Oh, four of those [Lisa's face indicated a light bulb went off]… But we need to add four to this [to the number 32 to get 36]. 9. Rosalind: You're undercounting according to that [motions to Carly's list].	Lisa notes that four pizzas (it may be the case that these are the zero or one topping on the whole pizza) should be added to the number solution of 32 obtained from the multiplication. Rosalind makes reference to the 36 listed pizzas. They each agree that something is missing, implicitly agreeing that 36 is the solution.
10. Elena: I know. [Long pause] I don't know. I'm ready to give up...I think dividing by 2... 11. Lisa: ...li ke cuts something out. 12. Elena: Because like Mushroom [taps left hand middle finger]-Pepperoni [taps left hand index finger] and then Mushroom [taps right hand middle finger]-Pepperoni [taps right hand index finger] that occurs twice. 13. Lisa: Yeah. 14. Elena: But like mushroom [gesturing with a switch of her index and middle fingers] and mushroom doesn't occur twice? [her statement turns into a question] 15. Lisa: I thought by multiplying it by each other you're like gonna have like this occur twice, so like P-one, P-two, and then P-two, P-one and that's why we divide by half, cause it's like the same thing. I don't know. I get why we would add 4, but I also don't understand. […]	Elena provides an example of a pizza combination (one different topping on each half) that occurs twice and counter argues Lisa's claim (in L6) that the mushroom, mushroom pizza occurs only once. Notice her argument turns into a question. Elena is "ready to give up," before she resorts to using gestures that mimic the symmetry of the outcomes (L12; L14). Lisa responds to this with continued confusion: "I get why we would add four, but I also don't understand."
16. Rosalind: I'm gonna go with this one [pointing to Carly's solution on the board]. Now we're trying to reason why it's four more. Then we're just counting that it's the full pizzas that we eliminated that happens to be exactly four. […] 17. Abigail: Yup, cause that one [Carly's solution on the board] you can see them all, so I feel like that one is going to be right [Carly's list solution], but that's the one [points to Veronica's multiplication solution] that makes sense to me.	Rosalind and Abigail point out that the listed solution of 36 is their reference for reasoning about the difference of four pizzas in the two solutions. Although Abigail claimed the other method made more sense to her, interestingly, Carly's listing solution was relied upon by the students in this episode as the correct solution "cause you can see them all." *Notice that some students refer to four "whole" or "full" pizzas as a single same-topping on the whole pizza – a plain cheese pizza, a pizza with pepperoni, a pizza with mushroom, and a pizza with anchovies. Therefore, when Rosalind claims the "full pizzas…happens to be exactly four" she may be referring to the four aforementioned pizzas. Elena and Lisa also refer to pizzas with the focus of one topping on one half of the pizza and one topping on the other half of the pizza. Note that the latter and former references are the same pizza combinations if the same topping is on both halves.

L1 & L2), the instructor slightly modified the representation to depict what students were referring to as "whole/full pizzas" as being constructed from two identical halves with the same topping on either side. Elena claimed this new representation resonated with her. Elena and Lisa then contend that there were four pizzas that were not double counted. The instructor redirected the class to attend to the modified representations, resulting in Rosalind and Veronica recognizing that there were eight such pizzas with the same toppings on either side.

Due to real-time class constraints and decision- making

Table 4. Episode 4 – whole class discussion and closure

Transcript	Commentary
1. Ethan: Actually, I have a question for you guys [*turns to Carly and Yesenia*]. So, do you see like the second column of blue where its AM [*see Figure 4*]. What's the difference with that and with the fourth row? 2. Carly: AM is on the entire pizza [*second column*], while AM is on half [*fourth row*]. 3. Instructor: Could I please maybe introduce slightly additional notation? I don't know if it's going to be helpful or the opposite, but I'll try. Maybe it's going to help some of you. These eight - right - are really like taking this. I'm writing them a little differently [*see Figure 5 in green*]. Does this make sense? 4. Elena: Yeah, that's how I've been thinking about it. 5. Instructor: So, how many of these cases do we have that are really not double? 6. Lisa: Four 7. Instructor: How many? Look there [*modified representations*]. 8. Elena: Four. 9. Rosalind: Eight. 10. Instructor: We have 8. 11. Unknown: We're all wrong [*laughs*]. 12. Veronica: Oh! 13. Instructor: You see it? 14. Veronica: It's when both sides have the same combo. *[Instructor modifies the calculation by subtracting 8 from 64, then dividing by two, and reintroducing an addition of 8. With the two solutions now with the same answer of 36 outcomes, students audibly react]* 15. Instructor: Magic or not? 16. Elena: No way! 17. Ariana: Kind of. 18. Elena: That's kind of crazy. 19. Instructor: Is it? Or beautiful? 20. Elena: Beautiful. I would say beautiful. 21. [*laughs are heard*] 22. Lisa: I would literally never come to that. I don't think I'd ever come to that by myself.	Following Ethan's question and Carly's response, the instructor introduces a modified notation for the pizza representations.  Figure 4. On the left, Ethan's reference to 2nd column and, on the right, his reference to 4th row (L1).  Figure 5. Representations of pizzas, split into 2 halves, with the same topping on either side According to Elena, this notation provided a visual to her thinking (L4). The instructor then revisits under and overestimating in L5. Through a series of questions, the instructor and students analyze the types of pizzas "when both sides have the same combo (L14) that were not duplicated. New calculations are made using similar approaches of Veronica's and Elena's groups to obtain 36 outcomes. Note the excitement when they fully "got it". An Aha moment, accompanied by surprise and joy.

on the part of the instructor that students could easily follow, the instructor modified the calculations of Veronica's and Elena's groups to arrive at a new total of 36 pizzas. Through the instructor's prompting, students assisted with the new calculation.

Part 2: Students' Dispositions

Data collected via the mathematics autobiographies and final course reflections reveal shifts in students' dispositions towards mathematics and their confidence as a mathematics learner, as well as student insight into the affordances of collaboration in the counting course.

Students' dispositions towards mathematics

In their pre-course mathematics autobiographies, students drew on their K – 12 mathematics experiences when reflecting on their feelings about mathematics. In their post-course final reflections, students were asked to revisit their feelings about mathematics and identify in what ways, if any, their feelings had changed since writing their mathematics autobiography. The students wrote their final reflections in class, without looking back at their initial autobiographies. Note that there were 10 students who responded to both the autobiography and the final questionnaire.

Table 5 presents a sample of pairs of responses that allow us to compare how students described their feelings about mathematics in response to the

Table 5. Pre and post course reflections on dispositions towards mathematics

	Pre-course Mathematics Autobiography	End-of-Semester Written Questionnaire
Charlotte	…I *never grew to love* math as I continued to struggle with it every year…	… prior to this course *maybe the case wasn't that I was bad at math…*
Abigail	…I *have not liked* math...	…I now *appreciate* math…
Yesenia	…I usually *don't find math fun…*	...there are certain concepts & problem types that I do *like or mind less…*
Jennifer	… I never really *was comfortable* with math…	…I think I can *enjoy* math at a certain pace…
Elena	I have always had a *love-hate* relationship with math…	This class does a good job…*taking away that intimidation*…it's okay to make mistakes.
Rosalind	[Although] I don't have the "math brain" I still *enjoy* learning math…	I think my perspective on math has tremendously shifted for [from] a negative one to an *exciting* one…[1]
Doug	I personally *am not a fan* of mathematics…	...I began to *appreciate* the aspects that made it [math] difficult…
Ethan	I *don't have any strong opinions* towards math...	I *feel good* about math….
Veronica	I am quite *comfortable* with math…	...I feel even *more validated* in terms of my skills/knowledge…
Carly	I enjoy mathematics…math is *fun* for me.	I *enjoy* mathematics…

following prompts, before the course began and after it ended. These responses relate to the prompt 2 of the mathematics autobiography and prompt 2.2 of the end-of-semester written questionnaire.

Pre-course mathematics autobiography. In their mathematics autobiographies, a range of feelings about mathematics emerged through the use of phrases such as "am not a fan," "love-hate relationship," and "fun for me." When describing the origin of their feelings about mathematics, students reflected on a variety of experiences: the nature of learning mathematics (e.g., "…filled with a considerable number of rules, formulas, and patterns that you must know..."), teacher pedagogical decisions (e.g., "…my dislike for math comes from the way in which I was taught math…"), evolving experiences (e.g., "I used to love math…[AP Statistics] became overwhelming.", relevance to their lives (e.g., "…I don't find relevant to my day to day life…"), and past performance (e.g. "…I continued to struggle with it every year.") were each noted by multiple students as reasons for their pre-course feelings about mathematics.

Post-course final reflection. In their final reflections, a shift in feelings about mathematics emerged through the use of phrases such as "different mindset," "less intimidated," and "more validated." When describing the ways in which their pre-course feelings had changed, students returned to the same themes as addressed in their mathematics autobiographies: the nature of learning mathematics (e.g., "…understand the concept behind the math…"), teacher pedagogical decisions (e.g., "…focused on your reasoning…"), evolving experiences (e.g., "…I began to appreciate the aspects that made it difficult…"), relevance to their lives (e.g., "…I'll know how to approach in my everyday life…"), and past performance (e.g., "…maybe the case wasn't that I was bad at math…").

Student confidence as a mathematics learner.

In their pre-course mathematics autobiographies, students reflected on their confidence as a mathematics learner, as well as to what they attributed their confidence. In their post-course final reflections, students were asked to revisit their pre-course confidence and identify in what ways, if any, their confidence had changed since writing their mathematics autobiography.

Table 6 presents a comparison of how each student described their confidence as a mathematics learner in response to prompt 3 of the mathematics autobiography and prompt 2.1 of the end-of-semester written questionnaire.

Pre-course mathematics autobiography. In their mathematics autobiographies, all students described their confidence in their ability to do well in math using qualifiers, such as "pretty," "somewhat," and "not particularly." Students attributed their pre-course

1 Rosalind wrote this in her week 5 reflection. We bring it here for brevity, as she reiterated this in length in the end-of-semester questionnaire

Table 6. Pre and Post course reflections on the ability to do well in mathematics

	Pre-course Mathematics Autobiography	End-of-Semester Written Questionnaire
Charlotte	…my confidence in my math abilities are not high…	… there was always another [method] that I understood.
Abigail	I do not feel very confident …	This course has helped me to view math in a new way…
Yesenia	It depends. My level of confidence can be attributed to the type of math that I am tasked with doing and the manner in which I am expected to solve the problems.	Not very confident at all…I sometimes confuse topics rather easily…
Jennifer	…I'm kind of confident…	…Now I know I can do at least some math. All other kinds of math not so much.
Elena	…I am not particularly confident…	…now I feel much more comfortable…
Rosalind	My confidence is not in the subject but rather has to be in myself…	…reasoning skills that I never thought I could have entering this course…
Doug	…I am not very confident…	…I became more comfortable…
Ethan	I am pretty confident…	I felt pretty comfortable with math coming in; however, this course definitely changed how I look at math. By solving different problems & thinking without formulas or equations was new for me.
Veronica	…I would say I am somewhat confident…	…This class has made me feel more confident…
Carly	To a certain degree, I am confident…	…the contents of this class forced me to challenge myself…

confidence to a variety of reasons: teacher pedagogical decisions (e.g., "…if my past teachers had taken more time to explain…"), particular high school math courses (e.g., "…fairly 'confident'…until I took Pre-Calc…"), past performance (e.g., "…because of my past success in math…"), time since last math class (e.g., "…I haven't taken it [math] in a couple years…"), and individual attitudes (e.g., "…I have gained confidence in my work ethic and depend on this…) were each given as reasons by multiple students to explain their pre-course level of confidence.

Post-course final reflection. In their final reflections, a shift in students' confidence as mathematics learners emerged for those on the lower end and middle of the confidence spectrum, as seen through the use of phrases such as "more comfortable," "more confident," and "I can do at least some math." Students who initially seemed to be on the higher end of the confidence spectrum remained there. Regardless of their initial confidence, students largely attributed their post-course confidence to the nature of the learning environment (e.g., "…class collaboration helped immensely," "…it is very rewarding to share and get feedback…while learning where I messed up…," "…it is not just about the answer …," "...thinking without formulas or equations was new for me.").

Affordances of the learning environment

In the post-course reflection questionnaire, students who shared that they sensed a change in their disposition towards mathematics or about their confidence in being able to do well, were encouraged to reflect on encounters they may have had that contributed to this sense of change. In *Table 7* we present a sample of students' responses to these prompts. Their responses point to elements in the learning environment that they attributed to the shifts they described. For each of these responses, we tried to identify occurrences in Lesson #17 that could account for these responses. In particular, we looked at the parts of the lesson that the reader is already familiar with, from Part 1 of the Findings (see *Tables 1, 2, 3, & 4*).

The first three are attributes of changes in students' feelings about math and the last three are attributes of changes in students' confidence as mathematics learners.

Course Evaluation

Course evaluations are done directly through the university. The students fill them online anonymously, voluntarily (there is no sanction for not filling it), and the instructor receives the responses only after the semester is over and all the grades have been posted.

Two open ended questions on the course evaluation results seem relevant to our study and may support some of the claims we have made so far. We bring below a few responses that are particularly relevant.

Table 7. Attributions made in final reflections to the learning environment

	Attributions in final reflections:	Reference to Lesson #17
Elena	… I think the lack of emphasis on final answers helps this [intimidation] because you're focused on your reasoning rather than a number; it's okay to work through misunderstandings.	When Veronica's solution is challenged, the debate is focused on why Elena and Veronica had differing views. (Theme 2, Segment 2, Table 2, L1 – L4) After the conflict arose, students were prompted to examine which solution was flawed, why it was flawed, and what could be modified to address the flaws. (Theme 2, Segment 3, Table 3) The instructor addressed student misunderstandings during the resolution. (Theme 3, Segment 4, Table 4, L3 – L14)
	…now I feel much more comfortable not only in my abilities as a student, but also with the idea of making mistakes. I think class collaboration helped immensely.	Elena changed her answer three times (64, 32, 36) during this task. Each change stemmed from Elena's refining and justifying her (revised) thought process, rather than from "fixing mistakes". (Themes 1 – 2, Segments 1 – 3, Tables 1 – 3) Each segment of the task required student collaboration (Themes 1 – 3, Segments 1 – 4, Tables 1 – 4).
Veronica	… I realized I was able to support/show my logic and math is not simply a right or wrong thing.	Veronica presented a strong justification for a flawed solution, which the instructor never dismissed as "wrong." Instead, students investigated its merits and flaws as they worked to determine if Veronica's solution captured all the outcomes. (Theme 2, Segment 3, L1 – L7) Only at the resolution stage, towards closure, the instructor built on Veronica's ideas, and suggested a modification to her solution by addressing the flaws while validating Veronica's thought process. (Theme 3, Segment 4, Table 4, L5 – L14)
	… This class has made me feel more confident in sharing my logic and ideas because though my answer may be wrong, it is very rewarding to share and get feedback on the parts I logically was able to solve, while learning where I messed up.	Despite Carly's listing of 36 outcomes, Veronica still shared her solution of 32 (Theme 2, Segment 2, Table 2) and defended her reasoning when challenged by Elena (Theme 2, Segment 3, Table 3). During the resolution, the instructor's modification of Carly's representations helped Veronica understand where the flaw in her reasoning was. (Theme 3, Segment 4, Table 4, L13 – L14).
Charlotte	…now my feelings towards mathematics have changed because I understand the concept behind the math…	The instructor suggested a slight modification of Carly's representations to help her and other students uncover on their own which outcomes were missing from Veronica's solution. This led to an understanding of why Veronica's solution was undercounted by four and how to modify it so that it counted all outcomes. (Theme 3, Segment 4, Table 4, L3)
	… approaching problems in many different ways helped me realize that even if I could not grasp one method of finding the solution there was always another that I understood.	There are multiple points of entry and solution paths. (Theme 1, Segment 1, Table 1)

A response to the following question: *What did the instructor do well?*

- … Furthermore, she [the instructor] did a great job of encouraging us and letting us learn and discover on our own before jumping in a giving us the answers. She cared a lot about the reasons why we perform certain math functions and I like how she encouraged us to explain our work - it made me think deeper about the math. She also was very helpful in relieving my test anxiety. She assured me throughout the course that she cares more about my thinking and my explanations than the perfect answers.

Responses to the following question: *What additional comments would you like to share about your experience in this course?*

- This course was one of the most impactful I have taken [at the university] so far. I hope it is offered in the future and would recommend it to anyone who has ever felt that math is a difficult or inaccessible subject, because this course changed my experience of math.

- As a person that struggles in math this class was an enjoyable way to understand the concepts within math.
- I have never been a big fan of math. In fact, I have always struggled in my math classes and have suffered from test anxiety in this subject. This is the first math class that I have ever taken that has successfully relieved my anxiety. We have been trained to focus on the answer in math and whether it is correct or not. In this class, we focused more on the process of getting to the answer and understanding why we need to perform specific functions. As a student looking to go into education, this was very useful to me. I now feel more equip to teach young students the subject of math and that I can ensure that they will understand why they have to perform certain functions and that the answer is not the only thing that matters. This class made me view math in a way that I have not done before and I am really happy that I took it.
- The class was thought provoking and very engaging.

As an anecdote, several months after the semester was over, Rosalind sent an email to the instructor, in which she wrote: "It was such a pleasure being in your class last semester as I felt it was the first math course that I fully understood and enjoyed. Your teaching styles really contributed to my new found confidence in math."

DISCUSSION

Our findings provide a glimpse into how combinatorics may evolve in an authentic situation of a real classroom – a formal undergraduate course for students who are not majoring in mathematics related subjects. These findings establish the feasibility and point to the merits of such a course. They add to findings from other studies that were carried out in informal settings (e.g., the work of Maher et al., 2011a), or in university courses for mathematics or mathematics education majors (e.g., Wasserman & Galarza, 2019).

In a time when equity in education is a major concern, our study addresses this concern in a unique way. In the spirit of Boaler's (2015) recommendations, we take equitable teaching to mean offering *all* students an opportunity to engage in *challenging* and *meaningful* mathematics and to think deeply about mathematics. This is in contrast to a common assumption that students who do not do well in standard school mathematics will not be able to do well in more sophisticated math. The ways in which students are usually assessed for decisions regarding their tracking builds on this (invalid) assumption (Boaler, 2015). Our findings suggest that students who may have done poorly in mathematics in K–12 grades, can develop sophisticated reasoning and problem-solving strategies in a challenging topic, if made accessible to them. Their experiences in this course changed some students' ideas of who can and what it means to do well in mathematics. This is in line with Ruef's (2017) findings that over a 7-month course that incorporated open tasks and developed a "safe ground for risk-taking that public sensemaking demands", six grade students' perceptions of what it meant to be a 'good math student' shifted from features such as speed, following directions, doing homework assignments, to features such as being brave in presenting their work, feeling comfortable making mistakes, as well as being active agents. Our findings indicate also that the Counting and Chance course led to some shifts in students' dispositions, and that they attributed these shifts not only to the supportive atmosphere and opportunities to interact with their peers, but even more so to the mathematical reasoning and understandings that they developed.

We identified several types of reasoning that students employed, in the context of solving the half-pizza problem (3THP). These occurred in other problems throughout the course as well. One approach focused on listing *outcomes by cases* (Lockwood, 2013), which indicates a set-oriented perspective according to Lockwood (2014), and resonates with Maher et al.'s (2011a) findings. Two pairs (Pair 1 and Pair 2) used a *systematic* method of counting, by looking at subsets of the outcome set (i.e., types of pizzas). Both pairs obtained an incomplete list that included 21 pizzas (out of the total of 36 possible outcomes), though each pair looked at different subsets, and by coincidence got to the same number. In the course of reaching the full list of outcomes, students began reasoning iteratively with *counter-examples* to the (wrong) assertion that the list of 21 outcomes was complete. Because the two lists of 21 were different, each pair was able to find some outcomes (or types of outcomes) that the other pair didn't include. Together as a class, students identified both missing categories of types of pizza and missing outcomes within the categories, until they exhausted all 36 possibilities. This reasoning led to the full outcome set that became the "trusted" solution. Interestingly, different from the children in Maher et al.'s study (2011b), the undergraduate students did not justify their reasoning for the completeness of the co-created list (although the teacher suggested that they deal with the possibility of missing or duplicate outcomes in the list). The other two pairs (Pair 3 and Pair 4) employed what may be considered *analogical reasoning* (Gentner & Maravilla, 2018), by building on solutions to problems that they had solved earlier (e.g., the 4TWP problem), and mapping the previous solution to a whole pizza problem to each half of the pizza in the new problem, obtaining 8 different types of half-pizza. This approach occurred in other cases as well, once students identified problems that seem different (in context) but are the same (in structure), as Maher and Martino (1996) term *isomorphic problems* (for example, there is an isomorphism between the Towers Problems and Pizza Problems mentioned earlier). Once they established the number of combinations for each half-pizza, they applied the Multiplicative Counting Principle, and realized that they had to account for double counting, obtaining 32 as their solution. However, there still remained a difference of 4 between the different approaches, which led them to scrutinize over the differing solutions and move between the "trusted" set of outcomes and the mathematical expression associated with their solution, in search of a convincing resolution (Lockwood, 2013 & 2014).

By looking more holistically at our data, we were able to illuminate the nature of the collaborative problem-solving processes that took place in this course and the progression in students' thinking. Rather than identifying certain patterns of student thinking (whether productive or less productive) or focusing on their strategies, we tried to capture the complexity of this environment, and in particular – paths through which knowledge was collectively constructed. We identified reasoning and justifications, that were continuously challenged, due to the lack of readily verification methods and the role of the instructor as a facilitator. The fact that formulas were introduced rather late in the course, and only after students gained some concrete experiences that helped them make sense of these formulas beforehand, created a situation in which they had to reason within each problem instead of trying to (mindlessly) apply a formula. Moreover, typically, problems were not categorized by the instructor into types, so in order to build on their accumulating experiences, students began looking, on their own, for similarities between problems that they had already solved. Thus, they often asked themselves how problems that appeared to be different (e.g., in the objects under consideration or the situation described) may be related structurally. This led naturally to notions of relatedness and isomorphism that are critical for problem-solving (Silver, 1979; Maher et al., 2011a; Pólya, 1945). The emphasis on the thinking process and reasoning underlying their solutions rather than on the final (correct) answers seems to have contributed to students' willingness to "take risks" and share their ideas openly, even if they are likely to err, which in return led to rich debating and an inner need to justify and convince (Zaslavsky et al., 2012). This need emerged naturally from the context, not in response to a formal requirement (similar to the classrooms described in Yackel & Cobb, 1996 and Lampert, 1999).

The challenging nature of counting problems together with the pedagogy that was employed elicited a lot of encounters of uncertainty, as manifested in the analysis of the episodes of Lesson #17. Fischbein (1987) asserts that the need for certitude is a strong driving force for learning, which explains the students' persistence in trying to reach resolution. Apparently, the learning environment was rich in giving rise to various types of uncertainty. Actually, all three types of uncertainty that Zaslavsky (2005) discusses were evoked spontaneously: uncertainty that arises by competing claims (e.g., which of two different or contradicting solutions is correct, if any); uncertainty that arises when facing an unknown path or questionable conclusion; and uncertainty that arises by non-readily verifiable outcomes. Consequently, students often expressed frustration and confusion. This can be expected, as Stein et al. (1996) maintain that high cognitive demanding tasks (in our case – counting tasks) require considerable cognitive effort and may involve some level of anxiety due to the unpredictable nature of the solution process required. Nonetheless, students seemed resilient to confusion; it did not stop them from perseverance, and to some –overcoming the confusion felt very rewarding.

As conveyed by the excerpts from Lesson #17, collaboration was a key component in the learning environment of this study. There have been calls for teaching students to work together (e.g., Boaler, 2015), and studies that point to the merits of working in pairs or small groups (e.g., Forman, 1989; Leikin & Zaslavsky, 1997; Mashiach-Eizenberg & Zaslavsky, 2003; Yackel, Cobb, & Wood, 1991). In the end-of-semester questionnaire, students indicated that the collaborative part of the course contributed the most to their learning as well as to their enjoyment of the course. This is reflected, for example, in Ethan's words: "I enjoyed and got out a lot out of the collaborative work, because we were encouraged to look at the same problem in different ways, which is different than working together to get an answer", and in Carly's words too: "The collaboration allowed for the formation of a diverse set of ideas and was always productive." It appears that the benefits that Davidson and Kroll (1991) attribute to collaborative work, came to play in the Counting and Chance course, in particular increased conceptual understanding, and improved attitudes or motivation. It should be noted that the genuine collaboration emerged mainly from the nature of the tasks (e.g., evoking uncertainty, lending themselves to different solution paths). There was no need in structuring the work in pairs or small groups (in contrast, for example, to the work of Leikin & Zaslavsky, 1997). In addition, this collaborative context called for considering multiple solutions for a given problem spontaneously, some of which highlighted different counting principles and helped connect them. The value of multiple-solution connecting tasks for facilitating theory building and development of conceptual understanding in mathematics has been widely recognized (Bass, 2017; Kondratieva, 2011; Leikin, 2011). Unlike most work on multiple-solution tasks, in the current study there was no need to carefully select and design a task with multiple solutions, as this is inherent in the nature of counting problems. Moreover, in combinatorics, counting the total number of elements of the same outcome set in two different (valid) ways, could constitute a proof of a mathematical identity (e.g., the sum of the binomial coefficients and the corresponding power of 2).

The focal course of our study follows English's (2011) and other calls for school mathematics to prepare students for their future careers in a society that is constantly changing and to address the increasing complexity of learning and learners in ways that advance their mathematical understanding within and beyond the classroom. She argues for fostering complex learning through activities that encourage knowledge generation and active processing. English considers four features that are especially important in advancing students' mathematical learning: (*i*) constructing important ideas and processes; (*ii*) describing, explaining, comparing, assessing, and justifying; (*iii*) creating multiple representations in format and choice; and (*iv*) using creations to make predictions. Our findings indicate that all four key features were present in the Counting and Chance course.

Finally, a significant implication of our study is that it offers a way to interweave a challenging mathematical content and an equitable learning environment that can be implemented in various contexts. Specifically, it can be an integral part of the high-school curriculum, as recommended by NCTM (2000). As illustrated, there are no prerequisites needed for making good progress in the course. The students that participated in the study did not take any additional mathematics courses since graduating from high-school. They could have studied this course much earlier. A course like this could counter balance the common mathematics experiences many students have in classes in which the focus of teaching mathematics is on carrying out procedures and getting the correct answers. Interestingly, although this was a content course, one of the students (Abigail) who studied in a childhood education program, reflected on her take away from the course, and wrote: "As a future educator, it is important to understand the why – not just the how". There were other students who shared with us spontaneously that this course inspired them to teach differently (not just math) in their future career.

GLOSSARY

- The *Factorial* of a natural number *n* is the product of all positive integers up to *n*. It is denoted by: *n*!
- A *Permutation* of *n* distinct objects is an arrangement, or ordering, of the *n* objects. The number of permutations of n objects is *n*!.
- A *Combination* [of *k* out of *n* distinct objects] is an unordered selection, or subset, of *k* out of the *n* objects. $\binom{n}{k}$ denotes the number of combinations of *k* objects out of a set of *n* objects.
- $\binom{n}{k}$ is also called a binomial coefficient, because of the following identity: $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n-1}+\binom{n}{n}=2^n$ (this forms a typical row in the Pascal's Triangle)
- *Category I* of combination problems (according to Lockwood et al, 2018), lend themselves to considering $\binom{n}{k}$ as: $\binom{n}{k}=\frac{n\cdot(n-1)\cdots(n-k+1)}{k!}$

- *Category II* of combination problems (according to Lockwood et al, 2018), lend themselves to considering $\binom{n}{k}$ as: $\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$

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