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NONLINEAR PHYSICS AND MECHANICS

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The Motion of a Balanced Circular Cylinder in an Ideal Fluid Under the Action of External Periodic Force and Torque

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The motion of a circular cylinder in a fluid in the presence of circulation and external periodic force and torque is studied. It is shown that for a suitable choice of the frequency of external action for motion in an ideal fluid the translational velocity components of the body undergo oscillations with increasing amplitude due to resonance. During motion in a viscous fluid no resonance arises. Explicit integration of the equations of motion has shown that the unbounded propulsion of the body in a viscous fluid is impossible in the absence of external torque. In the general case, the solution of the equations is represented in the form of a multiple series.

Keywords: rigid body dynamics, ideal fluid, viscous fluid, propulsion in a fluid, resonance

1. Introduction

There has been much discussion recently of problems concerning the propulsion of rigid bodies in a fluid without using external moving elements, for example, by means of moving internal masses or rotors. To describe the motion of such systems, two approaches are used: those based on various finite-dimensional models and those based on the joint numerical solution of the Navier – Stokes equations and equations of body motion. Depending on the formulation of the problem in finite-dimensional models one takes into account the influence of inertial properties of the fluid (the effect of added masses), circulation, gravity force, vortices, and viscous friction. Below we give references to a number of publications devoted to the mathematical modeling of the motion of rigid bodies in a fluid using a specific approach.

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In [25], within the framework of the Kirchhoff equations [20], the problem of the propulsion of a rigid body using the motion of the internal mass was considered for the first time, first integrals were found and necessary conditions for unbounded propulsion of the body were obtained. The ideas of [25] were developed in [26, 39], and a qualitative experimental confirmation of these ideas was presented in [21]. In [11, 42, 43], also within the framework of the Kirchhoff equations, the motion of an unbalanced ellipsoid and a helical body by means of rotors was considered. The results of experimental investigations of the dynamics of an ellipsoid of zero buoyancy with rotors were presented in [18].

The motion of a smooth body under the action of constant circulation and gravity force was examined in [8]. An analysis of the asymptotic behavior of rigid bodies in an ideal fluid in a gravitational field was presented in [4]. The motion of an ellipsoid with internal rotors in an ideal fluid in a gravitational field was considered in [41]. In [40], problems of controlling the motion of a smooth body by means of the internal mass and the rotor with constant circulation were discussed. Also in [40], analysis was made of the problem of compensation for the drift, arising due to circulation, using controls. The focus of [37] was on the problem of controlling the motion of a smooth body using the internal fly-wheel and the Flettner rotor creating variable circulation. The motion of the smooth body by means of oscillations of the rotor and by periodically changing circulation was studied in [10].

It should be noted that in the case of a smooth body the law of change in circulation is either postulated or defined on the basis of some auxiliary mathematical models. On the contrary, for bodies with a sharp edge, within the framework of the model of an ideal fluid, circulation can be calculated on the basis of the Kutta – Chaplygin condition [14, 27] and for a moving body will depend on its translational and angular velocities (see, e.g., [33]). The Kutta – Chaplygin condition can also be used as a basis for the modeling of vortex shedding from sharp edges of the body [34, 38].

When it comes to modeling the motion of rigid bodies in a viscous fluid, the most detailed description of the system dynamics can be obtained by using a joint numerical solution of the Navier-Stokes equations and equations of body motion [15, 36, 44]. Such an approach involves rather laborious calculations, but turns out to be useful in constructing various finite-dimensional models of body motion in a fluid [5, 19, 31]. A qualitative investigation of the motion of smooth bodies in a viscous fluid by means of internal mechanisms was carried out, for example, in [9, 10]. In [32] the problem of self-propulsion of a body with a sharp edge in a viscous fluid by oscillations of the rotor was discussed. It should be noted that the application of different phenomenological models for the same system can lead to qualitatively different results [28].

In the case of strongly anisotropic friction the motion of a rigid body in a fluid can be approximately described by nonholonomic models [12, 17, 24]. For the plane-parallel motion of a rigid body in a fluid such an approach leads to equations of the Chaplygin sleigh, the controlled motion of which was considered, for example, in [1–3]. Approximation by nonholonomic models turns out to be valid only for some time interval, and numerical examples demonstrating this can be found in [6, 9, 16].

Along with internal mechanisms, periodic external actions can be used to set a body in motion. For example, in [29] the dynamics of a body with a fixed point and the dynamics of a body in a fluid under the action of periodic force and torque of small amplitude were examined. In [30] the self-propulsion of a sphere in a fluid at low Reynolds numbers by periodic deformations was studied and the force acting on the body were calculated.

This paper is concerned with the problem of the plane-parallel motion of a circular cylinder under the action of external periodic force and torque. Such a periodic action can be implemented, for example, by means of water-jet motors capable of pulling and ejecting water. Section 2 presents a derivation of equations of motion. Section 3 addresses motion in the case of an ideal fluid. It is shown that in the absence of friction the components of the translational velocity of the body undergo oscillations with increasing amplitude due to resonance. Section 4 is concerned with motion in a viscous fluid. In the general case, the solution of the equations of motion can be represented in the form of a multiple series. It is shown that, in the absence of external torque, propulsion in a viscous fluid is impossible.

2. Equations of motion

2.1. The main assumptions and kinematic relations

Let us consider the plane-parallel motion of a circular cylinder in a fluid in the presence of circulation. We make the following assumptions for the system of interest:

- from the fluid, the body is acted upon by forces and torques due to the effect of added masses, circulation and viscous friction. We assume that viscous forces and torque are proportional to the corresponding velocity components of the body as in [23];
- the body is acted upon by the force and torque changing periodically in the coordinate system attached to the body;
- the airfoil is balanced. Thus, the gravity force is balanced by the buoyancy force;
- the airfoil possesses zero buoyancy. Thus, the moment of the gravity force is zero.

For the system described above we consider the following problem:

investigate the possibility of the speed-up of the system and unbounded propulsion.

To describe the motion of the body, we introduce two coordinate systems: a fixed frame Oxy and a moving frame $O_bx_1x_2$ atached to the body (see Fig. 1).

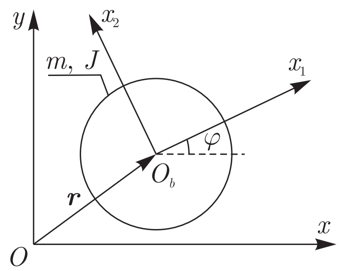


Fig. 1. Oxy — a fixed coordinate system, $O_bx_1x_2$ — a moving coordinate system.

We will specify the position of the body relative to the fixed coordinate system using the radius vector $\boldsymbol{r} = (x,\, y)$ of the origin O_b of the moving coordinate system, and the orientation, using the angle φ between the positive directions of the axes Ox and O_bx_1, measured from the axis Ox. Thus, the configuration space of the system $\mathcal{Q} = \{\boldsymbol{q} = (x,\, y,\, \varphi)\}$ coincides with the motion group of the plane $SE(2)$.

Let $\boldsymbol{v} = (v_1,\, v_2)$ denote the velocity of point O_b in the moving coordinate system and ω the angular velocity of the body. Then the following kinematic relations hold [7]:

$$\dot{x} = v_1\cos\varphi - v_2\sin\varphi, \quad \dot{y} = v_1\sin\varphi + v_2\cos\varphi, \quad \dot{\varphi} = \omega. \tag{2.1}$$

2.2. Equations of motion

The motion of a rigid body in an ideal fluid in the presence of circulation is described by the Chaplygin equations [13]. We supplement the Chaplygin equations with dissipative terms for the modeling of viscous friction and with periodic terms defining the external periodic action:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial v_1}\right) &= \omega\frac{\partial T}{\partial v_2} - \Gamma' v_2 - \frac{\partial R}{\partial v_1} + \delta_1' \sin\Omega t,\\
\frac{d}{dt}\left(\frac{\partial T}{\partial v_2}\right) &= -\omega\frac{\partial T}{\partial v_1} + \Gamma' v_1 - \frac{\partial R}{\partial v_2} + \delta_2' \sin\Omega t,\\
\frac{d}{dt}\left(\frac{\partial T}{\partial \omega}\right) &= v_2\frac{\partial T}{\partial v_1} - v_1\frac{\partial T}{\partial v_2} - \frac{\partial R}{\partial \omega} + \varepsilon' \sin\Omega t.
\end{aligned} \tag{2.2}$$

where T is the kinetic energy of the body–fluid system, Γ' is the circulation of the fluid velocity around the body, $R = \frac{1}{2}\left(\mu_1' v_1^2 + \mu_1' v_2^2 + \mu_2' \omega^2\right)$ is the dissipative Rayleigh function, μ_1', μ_2' are the viscous resistance coefficients, δ_1', δ_2', ε' are the oscillation amplitudes of external force components and torque, and Ω is the circular oscillation frequency of external force and torque.

The kinetic energy of a homogeneous, balanced circular foil is defined by the following expression:

$$T_b = \frac{1}{2}m\left(v_1^2 + v_2^2\right) + \frac{1}{2}I\omega^2, \tag{2.3}$$

where m is the mass of the foil and I is the central moment of inertia of the body.

The kinetic energy of the fluid surrounding the body is defined by the following expression:

$$T_f = \frac{1}{2}\lambda\left(v_1^2 + v_2^2\right), \tag{2.4}$$

where λ is the added mass of the foil. We note that the added moment of inertia is zero due to the choice of the moving coordinate system and the circular shape of the body [22].

The total kinetic energy of the body–fluid system can be represented as

$$T = T_b + T_f = \frac{1}{2}A\left(v_1^2 + v_2^2\right) + \frac{1}{2}I\omega^2, \tag{2.5}$$

where $A = m + \lambda$.

Let us define the linear momentum of the system $\boldsymbol{p} = (p_1, p_2)$ and the angular momentum M:

$$p_1 = \frac{\partial T}{\partial v_1} = Av_1, \quad p_2 = \frac{\partial T}{\partial v_2} = Av_2, \quad M = \frac{\partial T}{\partial \omega} = I\omega. \tag{2.6}$$

In view of (2.6), *the equations of motion* (2.2) *of the system under consideration take the form*

$$\begin{gathered}
\dot v_1 = (\omega - \Gamma)\, v_2 - \mu_1 v_1 + \delta_1 \sin\Omega t, \quad \dot v_2 = -\,(\omega - \Gamma)\, v_1 - \mu_1 v_2 + \delta_2 \sin\Omega t,\\
\dot\omega = -\mu_2\omega + \varepsilon\sin\Omega t,
\end{gathered} \tag{2.7}$$

$$\Gamma = \frac{\Gamma'}{A}, \quad \mu_1 = \frac{\mu_1'}{A}, \quad \mu_2 = \frac{\mu_2'}{I}, \quad \delta_1 = \frac{\delta_1'}{A}, \quad \delta_2 = \frac{\delta_2'}{A}, \quad \varepsilon = \frac{\varepsilon'}{I}.$$

In the absence of dissipation ($\mu_1 = 0$, $\mu_2 = 0$) and external perturbations ($\delta_1 = \delta_2 = 0$, $\varepsilon = 0$), Eqs. (2.1) and (2.7) admit three first integrals of motion [7, 8]:

$$p_x = v_1 \cos\varphi - v_2 \sin\varphi + \Gamma y = \mathrm{const}, \quad p_y = v_1 \sin\varphi + v_2 \cos\varphi - \Gamma x = \mathrm{const},$$
$$L = \frac{I}{A}\omega + x p_y - y p_x + \frac{\Gamma}{2}(x^2 + y^2) = \mathrm{const}. \tag{2.8}$$

Using the transformation

$$v = v_x + i v_y = (v_1 + i v_2)\exp\bigl(i\varphi(t)\bigr), \quad \varphi(t) = \varphi_0 + \int\limits_0^t \omega(s)ds, \tag{2.9}$$

where $v_x = \dot{x}$, $v_y = \dot{y}$, $\delta = \delta_1 + i\delta_2$, φ_0 is the initial value of angle φ, the first two equations of (2.7) can be written with respect to the fixed coordinate system in complex form:

$$\dot{v} = -\left(\mu_1 - i\Gamma\right) v + \delta \sin\Omega t \exp\bigl(i\varphi(t)\bigr). \tag{2.10}$$

Equation (2.10) is a linear inhomogeneous equation, and its solution greatly depends on the presence of friction and the form of the function $\varphi(t)$. In what follows, we investigate various cases.

3. Motion in an ideal fluid

3.1. Motion in the absence of external torque

When the body moves in an ideal fluid ($\mu_1 = 0$, $\mu_2 = 0$) in the absence of external force and torque ($\delta = 0$, $\varepsilon = 0$), the translational velocity components and the angular velocity are bounded functions of time [32]. In this case, unbounded propulsion of the body turns out to be possible only under the condition $\Gamma = 0$. In the case $\Gamma \neq 0$, the motion of the body occurs in some bounded region of the plane Oxy [40].

For these reasons we consider the motion of a body in the absence of dissipation ($\mu_1 = 0$, $\mu_2 = 0$) and external torque ($\varepsilon = 0$). In this case, under a suitable choice of the frequency of external action a resonance arises, and the velocity of the body undergoes oscillations with an amplitude linearly increasing with time. Moreover, unbounded propulsion of the body turns out to be possible even if $\Gamma \neq 0$.

The third equation of (2.7) takes the following form:

$$\dot{\omega} = 0. \tag{3.1}$$

It can be seen from (3.1) that the angular velocity of the body remains constant: $\omega = \omega_0 = \mathrm{const}$, and Eq. (2.10) can be written as

$$\dot{v} = i\Gamma v - \frac{i\delta \exp(i\varphi_0)}{2}\Bigl(\exp\bigl(i(\Omega + \omega_0)t\bigr) - \exp\bigl(-i(\Omega - \omega_0)t\bigr)\Bigr). \tag{3.2}$$

We note that the dynamics of the system for $\omega_0 = \Gamma = 0$ is not considered, since this case corresponds to the problem of the motion of a material point under the action of a given force.

Depending on the relation of the quantities $\omega_0 - \Gamma$ and Ω, two qualitatively different cases are possible: a nonresonant case with $\omega_0 - \Gamma \neq \pm\Omega$ and a resonant case with $\omega_0 - \Gamma = \pm\Omega$.

1. Nonresonant case with $\omega_0 - \Gamma \neq \pm\Omega$. The solution of Eq. (3.2) can be written as

$$v(t) = v_0 \exp(i\Gamma t) - \frac{\delta \exp(i\varphi_0)}{2}\left(\frac{\exp(i(\Omega+\omega_0)t) - \exp(i\Gamma t)}{\Omega+\omega_0-\Gamma} + \right.$$
$$\left. + \frac{\exp(-i(\Omega-\omega_0)t) - \exp(i\Gamma t)}{\Omega-\omega_0+\Gamma}\right), \quad (3.3)$$

where v_0 is the initial value of the translational velocity of the body. It can be seen from (3.3) that

in the case at hand the projections of the translational velocity of the point O_b of the body onto the axes of the fixed coordinate system are bounded functions of time.

According to (3.3), if $v_0 \neq 0$, the phase trajectory on the plane (v_x, v_y) is periodic if the following conditions are satisfied:

$$\frac{\Gamma}{\Omega} \in \mathbb{Q}, \quad \frac{\omega_0}{\Omega} \in \mathbb{Q}. \quad (3.4)$$

Otherwise the trajectory is quasi-periodic. Examples of phase trajectories for $v_0 = 0$, $\delta = 0.2 - 0.1i$ and different values of ω_0, Γ and Ω are shown in Fig. 2.

The trajectory of the point O_b of the body can be obtained by integrating the expression (3.3), and when $\Omega \neq \pm\omega_0$, the trajectory is described by the expression

$$z = x + iy = z_0 - \frac{iv_0}{\Gamma}\big(\exp(i\Gamma t) - 1\big) +$$
$$+ \frac{i\delta \exp(i\varphi_0)}{2(\Omega+\omega_0-\Gamma)}\left(\frac{\exp(i(\Omega+\omega_0)t) - 1}{\Omega+\omega_0} + \frac{\exp(i\Gamma t) - 1}{\Gamma}\right) -$$
$$- \frac{i\delta \exp(i\varphi_0)}{2(\Omega-\omega_0+\Gamma)}\left(\frac{\exp(-i(\Omega-\omega_0)t) - 1}{\Omega-\omega_0} + \frac{\exp(i\Gamma t) - 1}{\Gamma}\right) \quad (3.5)$$

and, when $\Omega = \omega_0$, by the expression

$$z = x + iy = z_0 - \frac{iv_0}{\Gamma}\big(\exp(i\Gamma t) - 1\big) +$$
$$+ \frac{i\delta \exp(i\varphi_0)}{2(\Omega+\omega_0-\Gamma)}\left(\frac{\exp(i(\Omega+\omega_0)t) - 1}{\Omega+\omega_0} + \frac{\exp(i\Gamma t) - 1}{\Gamma}\right) -$$
$$- \frac{\delta \exp(i\varphi_0)}{2(\Omega-\omega_0+\Gamma)}\left(t + \frac{i\exp(i\Gamma t) - i}{\Gamma}\right), \quad (3.6)$$

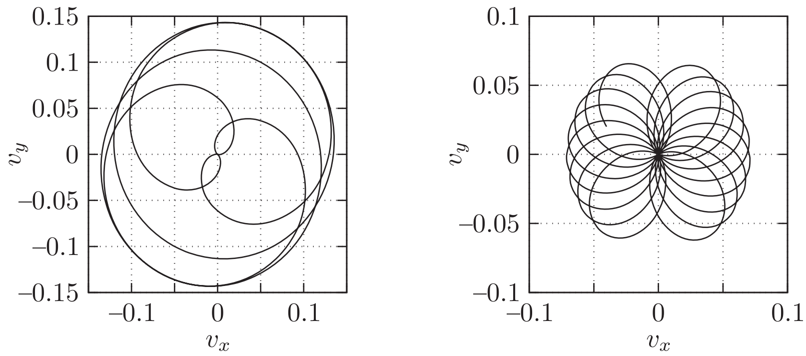
where z_0 is the initial position of point O_b. For $\Omega = -\omega_0$ an expression similar to (3.6) can be obtained.

It can be seen from (3.5) that, when $\Omega \neq \Gamma$, the trajectory of point O_b always lies in a bounded region of the plane Oxy. When conditions (3.4) are satisfied, the trajectory is a periodic curve (see Fig. 3a), otherwise the trajectory is quasi-periodic (see Fig. 3b).

It can be seen from (3.6) that in the case $\Omega = \omega_0$ the trajectory of point O_b is unbounded and the body moves in the direction of the vector $(-\delta_1 \cos\varphi_0 + \delta_2 \sin\varphi_0, -\delta_1 \sin\varphi_0 - \delta_2 \cos\varphi_0)$ with average velocity $\dfrac{|\delta|}{2(\Omega-\omega_0+\Gamma)}$ (see Fig. 3c).

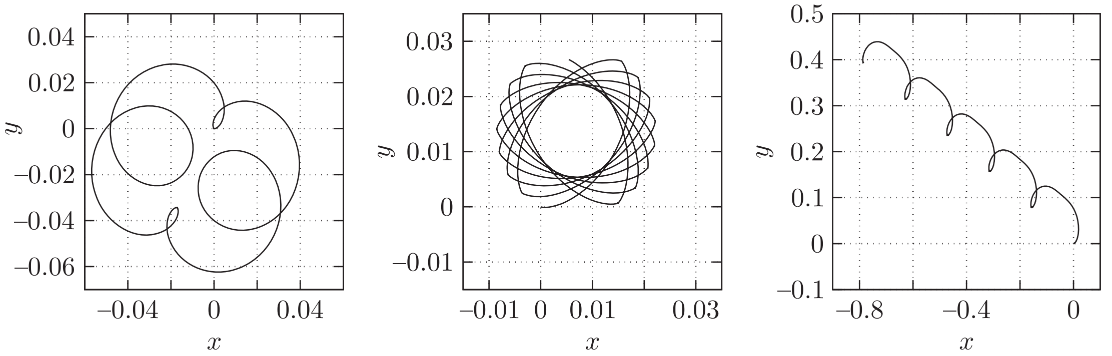
Thus,

when the conditions $\Omega = \pm\omega_0$ and $\omega_0 - \Gamma \neq \pm\Omega$ are satisfied, the mean motion of the system of interest is rectilinear.



(a) $\omega_0 = 4$, $\Gamma = 5$, $\Omega = 3$ (b) $\omega_0 = \Gamma = 3$, $\Omega = 2\pi$

Fig. 2. Phase trajectories of the system for $v_0 = 0$, $\delta = 0.2 - 0.1i$ and different values of ω_0, Γ and Ω.



(a) $\omega_0 = 4$, $\Gamma = 5$, $\Omega = 3$ (b) $\omega_0 = \Gamma = 3$, $\Omega = 2\pi$ (c) $\omega_0 = 3$, $\Gamma = 2$, $\Omega = 3$

Fig. 3. Trajectories of the point O_b of the body for $v_0 = 0$, $\delta = 0.2 - 0.1i$ and different values of ω_0, Γ and Ω.

2. Resonant case with $\omega_0 - \Gamma = \Omega$.[1] The solution of Eq. (3.2) can be written as

$$v(t) = \left(v_0 + \frac{i\delta \exp(i\varphi_0)}{2} t\right) \exp\left(i\Gamma t\right) - \frac{\delta \exp(i\varphi_0)}{4\Omega} \left(\exp\left(i\left(2\Omega + \Gamma\right)t\right) - \exp\left(i\Gamma t\right)\right). \tag{3.7}$$

It can be seen from (3.7) that

under the action of external periodic force with $\omega_0 - \Gamma = \Omega$ the velocity of the body undergoes oscillations whose amplitude increases linearly with time.

The phase trajectory of the system on the plane $(v_x,\, v_y)$ is a nonuniform spiral (see Fig. 4a). The trajectory of point O_b can be obtained by immediate integration of the expression (3.7):

$$\begin{gathered} z = x + iy = z_0 - \frac{i(v_0\Gamma + \delta \exp(i\varphi_0))}{4\Gamma\Omega}\Big(\exp\big(i\Gamma t\big) - 1\Big) + \\ + \frac{i\delta \exp(i\varphi_0)}{4\Omega(2\Omega + \Gamma)}\Big(\exp\big(i(2\Omega + \Gamma)t\big) - 1\Big) - \frac{i\delta \exp(i\varphi_0)}{2\Gamma^2}\Big(\big(i\Gamma t - 1\big)\exp\big(i\Gamma t\big) + 1\Big). \end{gathered} \tag{3.8}$$

It can be seen from (3.8) that the point O_b of the body moves in a spiral trajectory (see Fig. 4b).

Thus,

in the absence of torque, mean rectilinear motion in an ideal fluid is impossible for the system when the conditions $\omega_0 - \Gamma = \pm\Omega$ are satisfied.

[1]The case $\omega_0 - \Gamma = -\Omega$ is similar to that considered here and will not be presented.

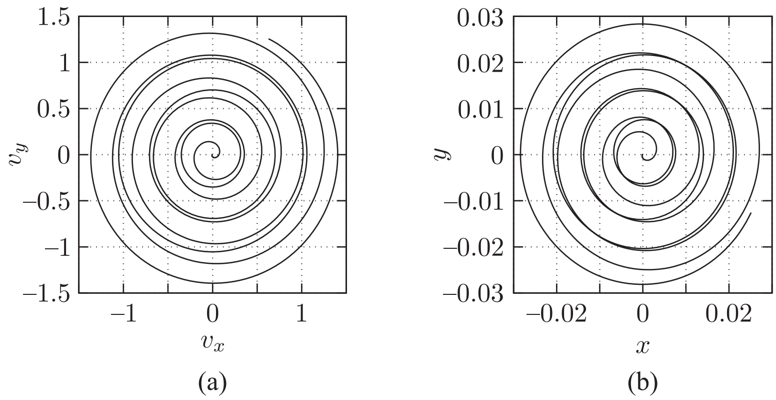


Fig. 4. Phase trajectory of the system (a) and the trajectory of the point O_b of the body (b) for $v_0 = 0$, $\delta = 0.2 - 0.1i$, $\omega_0 = 4$, $\Gamma = 5$, $\Omega = -1$.

3.2. Motion in the presence of external torque

Consider the motion of a body in the absence of dissipation ($\mu_1 = 0$, $\mu_2 = 0$) and in the presence of external periodic torque ($\varepsilon \neq 0$). In this case, the solution of Eq. (2.10) can be obtained in the form of a Fourier series. In addition, resonant frequencies can be found at which the velocity of the body undergoes oscillations with increasing amplitude.

According to the third equation of (2.7), the angular velocity in this case is a periodic function of time:

$$\omega(t) = \widetilde{\omega} - \frac{\varepsilon}{\Omega}\cos\Omega t, \quad \widetilde{\omega} = \omega_0 + \frac{\varepsilon}{\Omega}, \tag{3.9}$$

where ω_0 is the initial value of the angular velocity and the angle φ is defined by the expression

$$\varphi(t) = \varphi_0 + \widetilde{\omega}t - \frac{\varepsilon}{\Omega^2}\sin\Omega t. \tag{3.10}$$

In view of (3.10), Eq. (2.10) takes the form

$$\dot{v} = i\Gamma v + \delta\exp\big(i\varphi_0\big)\exp\Big(\widetilde{\omega}t - \frac{\varepsilon}{\Omega^2}\sin\Omega t\Big)\sin\Omega t, \tag{3.11}$$

and its solution can be represented as

$$v(t) = v_0\exp\left(i\Gamma t\right) + \delta\exp\left(i\left(\Gamma t + \varphi_0\right)\right)\int\limits_0^t \exp\left(i\left(\widetilde{\omega} - \Gamma\right)s\right)\exp\Big(-i\frac{\varepsilon}{\Omega^2}\sin\Omega s\Big)\sin\Omega s ds. \tag{3.12}$$

Let us expand in a Fourier series the periodic function $\exp\Big(-i\frac{\varepsilon}{\Omega^2}\sin\Omega s\Big)\sin\Omega s$, which appears in the solution (3.12), and perform term-by-term integration:

$$v(t) = v_0\exp\left(i\Gamma t\right) - i\delta\exp\left(i\varphi_0\right)\sum_{n=-\infty}^{\infty}\sigma_n\frac{\exp\left(i\left(n\Omega + \widetilde{\omega}\right)t\right) - \exp(i\Gamma t)}{n\Omega + \widetilde{\omega} - \Gamma}, \tag{3.13}$$

where the Fourier coefficients σ_n are expressed in terms of Bessel functions of the first kind [35]:

$$\sigma_n = \frac{i}{2}(-\operatorname{sign}\varepsilon)^{n+1}\left(J_{n+1}\left(\frac{|\varepsilon|}{\Omega^2}\right) - J_{n-1}\left(\frac{|\varepsilon|}{\Omega^2}\right)\right).$$

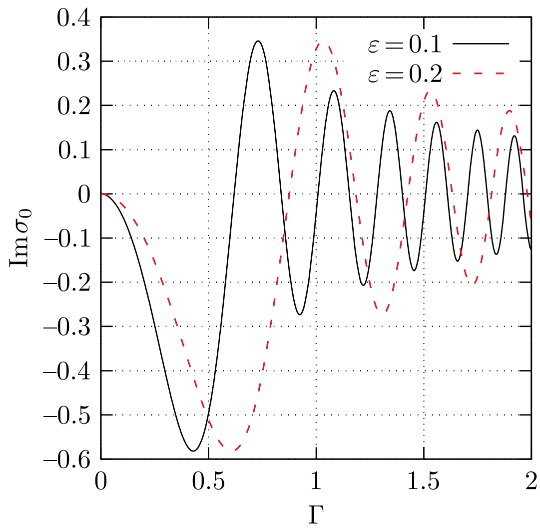


Fig. 5. Dependences $\sigma_0(\Gamma)$ for $\varepsilon = 0.1$ and $\varepsilon = 0.2$.

If $n\Omega + \widetilde{\omega} - \Gamma \neq 0$, then the series written on the right-hand side of (3.13) converges and is a bounded function of time. The relations

$$\Omega = \begin{cases} \dfrac{\varepsilon}{\Gamma - \omega_0}, & n = 0, \\[2ex] \dfrac{\Gamma - \omega_0 \pm \sqrt{(\Gamma - \omega_0)^2 - 4n\varepsilon}}{2n}, & n \neq 0 \end{cases} \tag{3.14}$$

are resonant frequencies at which the translational velocity components can undergo oscillations with increasing amplitude. In this case, the corresponding Fourier expansion coefficient must be different from zero. Figure 5 shows the dependences of the Fourier coefficient $\sigma_0(\Gamma)$ in the resonant case with $\omega_0 = 0$ and $\Omega = \dfrac{\varepsilon}{\Gamma}$ for $\varepsilon = 0.1$ and $\varepsilon = 0.2$. It is seen from Fig. 5 that σ_0 is mainly nonzero.

Thus,

at frequencies of external action which are defined by (3.14)*, the velocity of the point O_b of the body undergoes oscillations with increasing amplitude.*

We note that in the case $\Gamma - \omega_0 = 0$ the expressions (3.14) take an especially simple form

$$\Omega = \pm\sqrt{-\frac{\varepsilon}{n}}. \tag{3.15}$$

Examples of phase trajectories of the system for $\varepsilon = 0.1$, $\omega_0 = 0.1$, $v_0 = 0$, $\varphi_0 = 0$, $\delta = 0.2 - 0.1i$ and different values of Ω, Γ are given in Fig. 6. It can be seen from (3.13) that, when $n\Omega + \widetilde{\omega} - \Gamma \neq 0$ and $\dfrac{\Gamma}{n\Omega + \widetilde{\omega}} \in \mathbb{Q}$, the phase trajectory on the plane (v_x, v_y) is a periodic curve (see Fig. 6a), otherwise it is quasi-periodic (see Fig. 6b). If the parameter Ω takes a value equal to one of the resonant frequencies (3.14), then the phase trajectory is a spiral curve (see Fig. 6c).

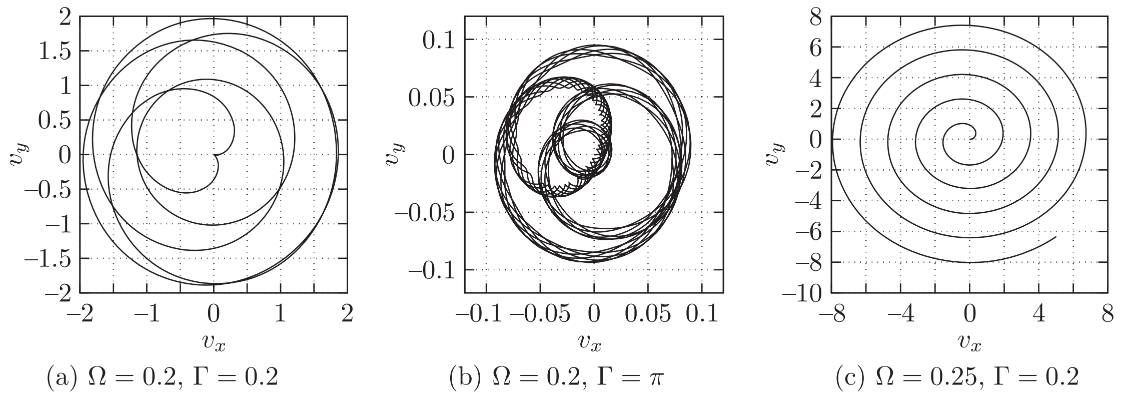


Fig. 6. Phase trajectories of the system for $\varepsilon = 0.1$, $\omega_0 = 0.1$, $v_0 = 0$, $\varphi_0 = 0$, $\delta = 0.2 - 0.1i$ and different values of Ω, Γ.

The trajectory of the point O_b of the body can be obtained by immediate integration of the expression (3.13):

$$z = x + iy = z_0 - \frac{iv_0}{\Gamma}\big(\exp(i\Gamma t) - 1\big) - \delta \exp(i\varphi_0) \sum_{n=-\infty}^{+\infty} \frac{\sigma_n}{n\Omega + \widetilde{\omega} - \Gamma} \times \\ \times \left(\frac{\exp(i(n\Omega + \widetilde{\omega})t) - 1}{n\Omega + \widetilde{\omega}} - \frac{\exp(i\Gamma t) - 1}{\Gamma} \right), \quad (3.16)$$

where z_0 is the initial position of the body.

When the conditions $n\Omega + \widetilde{\omega} - \Gamma \neq 0$ and $n\Omega + \widetilde{\omega} \neq 0$ are satisfied, the series written on the right-hand side of (3.16) converges, and the trajectory of the O_b of the body lies in some bounded region of the plane Oxy (see Fig. 7a). Relation (3.14) and the relations

$$\Omega = \begin{cases} -\dfrac{\varepsilon}{\omega_0}, & n = 0, \quad \omega_0 \neq 0, \\ \dfrac{-\omega_0 \pm \sqrt{\omega_0^2 - 4n\varepsilon}}{2n}, & n \neq 0 \end{cases} \quad (3.17)$$

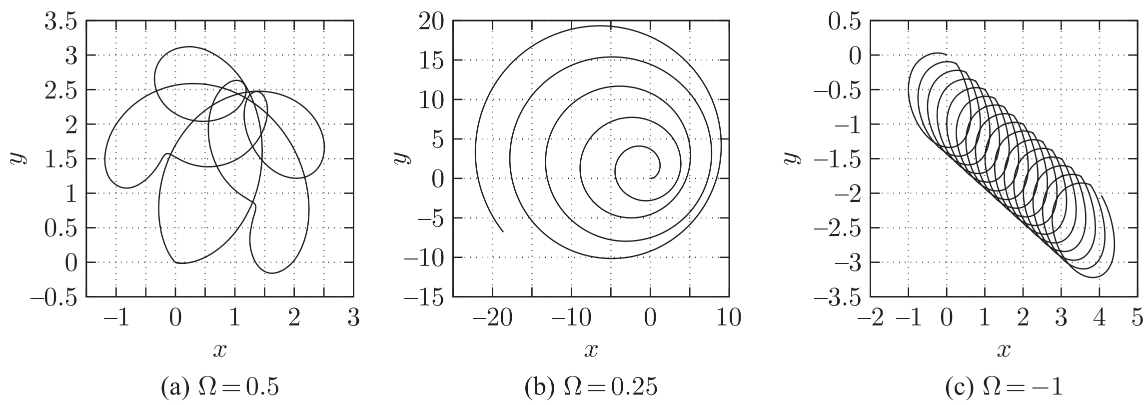


Fig. 7. Trajectories of the point O_b of the body for $\varepsilon = 0.1$, $\omega_0 = 0.1$, $\Gamma = 0.5$, $v_0 = 0$, $\varphi_0 = 0$, $\delta = 0.2 - 0.1i$, $v_0 = 0$, $z_0 = 0$ and different values of Ω.

define the resonant frequencies at which unbounded propulsion of the body is possible. We note that, when relations (3.14) are satisfied, the trajectory is a helical curve (see Fig. 7b), and when relations (3.17) are satisfied, the mean motion of point O_b is rectilinear (see Fig. 7c).

Thus,

the mean rectilinear motion of the system in an ideal fluid is possible due to the periodic force and periodic torque, which act with the frequency defined by (3.17).

4. Motion in a viscous fluid

4.1. Motion in the absence of external torque

Let us consider the motion of the body in the presence of dissipation ($\mu_1 \neq 0$, $\mu_2 \neq 0$) and in the absence of external torque ($\varepsilon = 0$). We show that in this case the projections v_x and v_y of the translational velocity of point O_b are bounded functions of time and the trajectory of point O_b lies in some bounded region of the plane Oxy.

The third equation of (2.7) has an exponentially decaying solution of the following form:

$$\omega(t) = \omega_0 \exp(-\mu_2 t), \tag{4.1}$$

where ω_0 is the initial value of the angular velocity and the angle φ changes as follows:

$$\varphi(t) = \varphi_0 + \frac{\omega_0}{\mu_2}\big(1 - \exp(-\mu_2 t)\big). \tag{4.2}$$

In view of (4.2), Eq. (2.10) then takes the form

$$\dot{v} = -\big(\mu_1 - i\Gamma\big)v + \delta \exp(i\widetilde{\varphi}) \exp\left(-\frac{i\omega_0}{\mu_2}\exp(-\mu_2 t)\right) \sin\Omega t, \quad \widetilde{\varphi} = \varphi_0 + \frac{\omega_0}{\mu_2}. \tag{4.3}$$

The solution of Eq. (4.3) can be represented as

$$\begin{gathered} v(t) = v_0 \exp\big(-(\mu_1 - i\Gamma)t\big) - \frac{i\delta}{2}\exp\left(-(\mu_1 - i\Gamma)t + i\widetilde{\varphi}\right) \times \\ \times \int\limits_0^t \Big(\exp\big(i\Omega s\big) - \exp\big(-i\Omega s\big)\Big) \exp\big((\mu_1 - i\Gamma)s\big) \exp\left(-\frac{i\omega_0}{\mu_2}\exp\left(-\mu_2 s\right)\right) ds. \end{gathered} \tag{4.4}$$

Let us expand the function $\exp\left(-\frac{i\omega_0}{\mu_2}\exp\left(-\mu_2 s\right)\right)$, which appears in (4.4), in a Taylor series and perform term-by-term integration:

$$\begin{gathered} v(t) = v_0 \exp\big(-(\mu_1 - i\Gamma)t\big) - \frac{i\delta}{2}\exp\left(i\widetilde{\varphi}\right) \times \\ \times \left(\sum_{n=0}^{\infty}\left(-\frac{i\omega_0}{\mu_2}\right)^n \frac{\exp\big(-(n\mu_2 - i\Omega)t\big) - \exp\big(-(\mu_1 - i\Gamma)t\big)}{n!(\mu_1 - n\mu_2 + i\left(\Omega - \Gamma\right))} - \right. \\ \left. - \sum_{n=0}^{\infty}\left(-\frac{i\omega_0}{\mu_2}\right)^n \frac{\exp\big(-(n\mu_2 + i\Omega)t\big) - \exp\big(-(\mu_1 - i\Gamma)t\big)}{n!(\mu_1 - n\mu_2 - i\left(\Omega + \Gamma\right))}\right). \end{gathered} \tag{4.5}$$

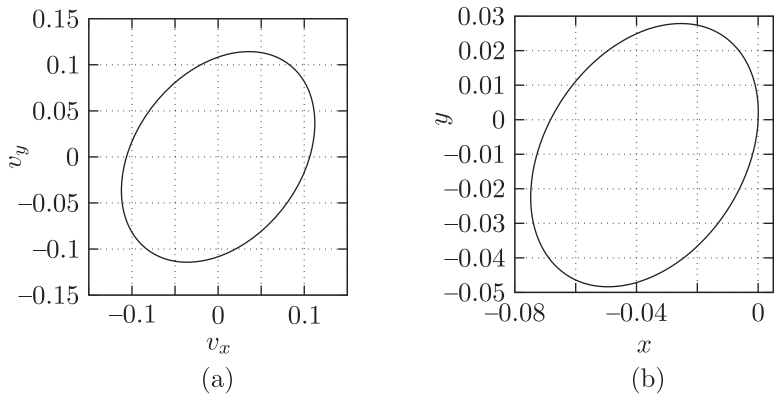


Fig. 8. Limit cycle (a) and the corresponding trajectory (b) for the parameter values $\mu_1 = 1$, $\mu_2 = 0.1$, $\Omega = 3$, $\Gamma = 3$, $\delta = 0.2 - 0.1i$, $\varphi_0 = 0$, $\omega_0 = 1$.

From the solution (4.5) one can find the following limit cycle as $t \to \infty$, which arises due to viscous dissipation (see Fig. 8a):

$$v_{lim}(t) = -\frac{i\delta}{2}\exp\left(i\widetilde{\varphi}\right)\left(\frac{\exp(i\Omega t)}{\mu_1 + i(\Omega - \Gamma)} - \frac{\exp(-i\Omega t)}{\mu_1 - i(\Omega + \Gamma)}\right). \tag{4.6}$$

Consider the possibility of a resonance arising in the system of interest. Let $\mu_1 - N\mu_2 + i(\Omega - \Gamma) = 0$ be satisfied for $n = N$. Then the corresponding term in the solution (4.5) takes the form

$$U_N = \left(-\frac{i\omega_0}{\mu_2}\right)^N \frac{t\exp\big(-(\mu_1 - i\Gamma)t\big)}{N!}. \tag{4.7}$$

It is seen from (4.7) that $\lim\limits_{t\to\infty} U_N = 0$. Thus,

in the presence of dissipation and in the absence of external torque the projections of the translational velocity of the point O_b of the body onto the axes of the fixed coordinate system are bounded functions of time.

Explicit integration of the expression (4.6) shows that the trajectory of O_b which corresponds to the limit cycle is a periodic curve (see Fig. 8b):

$$z_{lim} = x + iy = z_0 - \frac{\delta}{2\Omega}\exp\left(i\widetilde{\varphi}\right)\left(\frac{\exp(i\Omega t) - 1}{\mu_1 + i(\Omega - \Gamma)} + \frac{\exp(-i\Omega t) - 1}{\mu_1 - i(\Omega + \Gamma)}\right). \tag{4.8}$$

Thus,

in the system under consideration, unbounded propulsion in a viscous fluid only by means of the periodic external force is impossible.

4.2. Motion in the presence of external torque

Consider the motion of the system in the presence of dissipation ($\mu_1 \neq 0$, $\mu_2 \neq 0$) and external periodic torque ($\varepsilon \neq 0$). In this case, the solution of Eq. (2.10) can be constructed in the form of a multiple series.

The solution of the third equation of (2.7) can be written as

$$\omega(t) = \left(\omega_0 + \frac{\varepsilon\Omega}{\mu^2+\Omega^2}\right)\exp(-\mu_2 t) + \frac{\varepsilon\mu_2}{\mu_2^2+\Omega^2}\sin\Omega t - \frac{\varepsilon\Omega}{\mu_2^2+\Omega^2}\cos\Omega t, \tag{4.9}$$

and the angle φ will change as follows:

$$\varphi(t) = \widetilde{\varphi} - \alpha\exp(-\mu_2 t) - \sqrt{\beta^2+\gamma^2}\sin(\Omega t+\theta), \quad \widetilde{\varphi} = \varphi_0 + \alpha + \beta, \tag{4.10}$$

$$\alpha = \frac{1}{\mu_2}\left(\omega_0 + \frac{\varepsilon\Omega}{\mu^2+\Omega^2}\right), \quad \beta = \frac{\varepsilon\mu_2}{\Omega(\mu_2^2+\Omega^2)}, \quad \gamma = \frac{\varepsilon}{\mu_2^2+\Omega^2}, \quad \theta = \operatorname{arctg}\frac{\beta}{\gamma}.$$

In view of (4.10), Eq. (2.10) takes the following form:

$$\dot{v} = \big(-\mu_1 + i\Gamma\big)v + \delta\exp\big(i\widetilde{\varphi}\big)\exp\big(-i\alpha\exp(-\mu_2 t)\big)\exp\big(-i\sqrt{\beta^2+\gamma^2}\sin(\Omega s+\theta)\big)\sin\Omega t. \tag{4.11}$$

The solution of the equation can be represented in the form of the following quadrature:

$$\begin{aligned} v(t) = {}& v_0\exp\big(-(\mu_1 - i\Gamma)t\big) + \delta\exp(i\widetilde{\varphi})\exp\big(-(\mu_1 - i\Gamma)t\big)\times \\ &\times\int\limits_0^t \sin\Omega s\exp\big((\mu_1 - i\Gamma)s\big)\exp\left(-i\alpha\exp(-\mu_2 s)\right)\exp\left(-i\sqrt{\beta^2+\gamma^2}\sin(\Omega s+\theta)\right)ds. \end{aligned} \tag{4.12}$$

To calculate the integral written on the right-hand side of (4.12), we expand the function $\exp\left(-i\alpha\exp(-\mu_2 s)\right)$ in a Taylor series and the function $\exp\left(-i\sqrt{\beta^2+\gamma^2}\sin(\Omega s+\theta)\right)$ in a Fourier series. Performing term-by-term integration, we obtain the solution of Eq. (4.11) in the form of a multiple series:

$$\begin{aligned} v(t) = {}& v_0\exp\big(-(\mu_1 - i\Gamma)t\big) - \frac{i\delta}{2}\exp(i\widetilde{\varphi})\times \\ &\times\left(\sum_{n=-\infty}^{\infty}\sum_{p=0}^{\infty}\frac{(-i\alpha)^p\sigma_n}{p!}\frac{\exp(-(p\mu_2 - i\Omega(n+1))t) - \exp(-(\mu_1 - i\Gamma)t)}{\mu_1 - p\mu_2 + i(\Omega(n+1)-\Gamma)} - \right. \\ &\left. - \sum_{n=-\infty}^{\infty}\sum_{p=0}^{\infty}\frac{(-i\alpha)^p\sigma_n}{p!}\frac{\exp(-(p\mu_2 - i\Omega(n-1))t) - \exp(-(\mu_1 - i\Gamma)t)}{\mu_1 - p\mu_2 + i(\Omega(n-1)-\Gamma)}\right), \end{aligned} \tag{4.13}$$

where the Fourier coefficients σ_n are expressed in terms of Bessel functions of the first kind [35]:

$$\sigma_n = \exp(in(\theta+\pi))J_n\Big(\sqrt{\beta^2+\gamma^2}\Big). \tag{4.14}$$

From the expression (4.13) one can find the limit cycle (see Fig. 9a), which arises in the system as $t\to\infty$ due to viscous dissipation:

$$v_{lim}(t) = -\frac{i\delta}{2}\exp(i\widetilde{\varphi})\sum_{n=-\infty}^{\infty}\sigma_n\left(\frac{\exp(i\Omega(n+1)t)}{\mu_1 + i(\Omega(n+1)-\Gamma)} - \frac{\exp(i\Omega(n-1)t)}{\mu_1 + i(\Omega(n-1)-\Gamma)}\right). \tag{4.15}$$

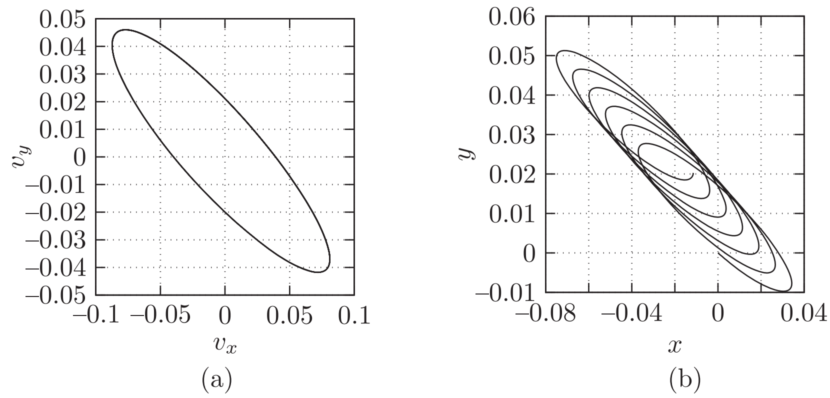


Fig. 9. Limit cycle (a) and the corresponding trajectory (b) for the parameter values $\mu_1 = 0.05$, $\mu_2 = 0.02$, $\varepsilon = 0.1$ $\Omega = 2.5$, $\Gamma = 0.5$, $\delta = 0.2 - 0.1i$, $\varphi_0 = 0$, $\omega_0 = 0.4$.

The equation of the trajectory (see Fig. 9b) of the point O_b corresponding to the limit cycle (4.15) can be found by integrating the expression (4.15):

$$z_{lim} = z_0 - \frac{\delta}{2}\exp\big(i\widetilde{\varphi}\big)\left(\sum_{n=-\infty}^{-2} A_n + \sum_{n=0}^{\infty} A_n - \sum_{n=-\infty}^{0} B_n - \sum_{n=2}^{\infty} B_n\right) + \langle v_{lim}\rangle t, \tag{4.16}$$

$$\langle v_{lim}\rangle = \frac{i\delta\exp(i\widetilde{\varphi})\cos\theta \cdot J_1\Big(-\sqrt{\beta^2+\gamma^2}\Big)}{\mu_1 - i\Gamma}, \tag{4.17}$$

$$A_n = \frac{\sigma_n(\exp(i\Omega(n+1)t) - 1)}{(\mu_1 + i(\Omega(n+1) - \Gamma))\Omega(n+1)}, \quad B_n = \frac{\sigma_n(\exp(i\Omega(n-1)t) - 1)}{(\mu_1 + i(\Omega(n-1) - \Gamma))\Omega(n-1)}.$$

It is seen from (4.16) that the motion occurs with the mean velocity $\langle v_{lim}\rangle$ defined by (4.17).

Thus,

the propulsion of the circular cylinder in a viscous fluid is possible due to periodic external force and torque.

5. Conclusion

Explicit integration of the equations of motion has shown that, in the absence of dissipation and under a suitable choice of the frequency of external action, the system considered may exhibit a resonance due to which the translational velocity components will undergo oscillations with increasing amplitude. In the absence of external torque the system has only one value of resonant frequency, and in the presence of external torque the system possesses a spectrum of resonant frequencies. In the nonresonant case, mean rectilinear motion of the system is possible.

In the presence of dissipation, no resonance arises, and the translational velocity components and the angular velocity of the system are always bounded functions of time. Moreover, mean rectilinear motion turns out to be possible only in the presence of external torque.

To conclude, we note that it is also of interest to consider the motion of an elliptic body in an ideal and viscous fluid by means of external periodic force and torque.

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References

[1] Bizyaev, I. A., Borisov, A. V., and Kuznetsov, S. P., The Chaplygin Sleigh with Friction Moving due to Periodic Oscillations of an Internal Mass, *Nonlinear Dyn.,* 2019, vol. 95, no. 1, pp. 699–714.

[2] Bizyaev, I. A., Borisov, A. V., and Kuznetsov, S. P., Chaplygin Sleigh with Periodically Oscillating Internal Mass, *Europhys. Lett.,* 2017, vol. 119, no. 6, 60008, 7 pp.

[3] Bizyaev, I. A., Borisov, A. V., and Mamaev, I. S., The Chaplygin Sleigh with Parametric Excitation: Chaotic Dynamics and Nonholonomic Acceleration, *Regul. Chaotic Dyn.,* 2017, vol. 22, no. 8, pp. 955–975.

[4] Borisov, A. V., Kozlov, V. V., and Mamaev, I. S., Asymptotic Stability and Associated Problems of Dynamics of Falling Rigid Body, *Regul. Chaotic Dyn.,* 2007, vol. 12, no. 5, pp. 531–565.

[5] Borisov, A. V., Kuznetsov, S. P., Mamaev, I. S., and Tenenev, V. A., Describing the Motion of a Body with an Elliptical Cross Section in a Viscous Uncompressible Fluid by Model Equations Reconstructed from Data Processing, *Tech. Phys. Lett.,* 2016, vol. 42, no. 9, pp. 886–890; see also: *Pis'ma Zh. Tekh. Fiz.,* 2016, vol. 42, no. 17, pp. 9–19.

[6] Borisov, A. V. and Kuznetsov, S. P., Comparing Dynamics Initiated by an Attached Oscillating Particle for the Nonholonomic Model of a Chaplygin Sleigh and for a Model with Strong Transverse and Weak Longitudinal Viscous Friction Applied at a Fixed Point on the Body, *Regul. Chaotic Dyn.,* 2018, vol. 23, nos. 7–8, pp. 803–820.

[7] Borisov, A. V. and Mamaev, I. S., *Rigid Body Dynamics: Hamiltonian Methods, Integrability, Chaos,* Izhevsk: R&C Dynamics, Institute of Computer Science, 2005 (Russian).

[8] Borisov, A. V. and Mamaev, I. S., On the Motion of a Heavy Rigid Body in an Ideal Fluid with Circulation, *Chaos,* 2006, vol. 16, no. 1, 013118, 7 pp.

[9] Borisov, A. V., Mamaev, I. S., and Vetchanin, E. V., Dynamics of a Smooth Profile in a Medium with Friction in the Presence of Parametric Excitation, *Regul. Chaotic Dyn.,* 2018, vol. 23, no. 4, pp. 480–502.

[10] Borisov, A. V., Mamaev, I. S., and Vetchanin, E. V., Self-Propulsion of a Smooth Body in a Viscous Fluid under Periodic Oscillations of a Rotor and Circulation, *Regul. Chaotic Dyn.,* 2018, vol. 23, nos. 7–8, pp. 850–874.

[11] Borisov, A. V., Vetchanin, E. V., and Kilin, A. A., Control Using Rotors of the Motion of a Triaxial Ellipsoid in a Fluid, *Math. Notes,* 2017, vol. 102, nos. 3–4, pp. 455–464; see also: *Mat. Zametki,* 2017, vol. 102, no. 4, pp. 503–513.

[12] Brendelev, V. N., On the Realization of Constraints in Nonholonomic Mechanics, *J. Appl. Math. Mech.,* 1981, vol. 45, no. 3, pp. 351–355; see also: *Prikl. Mat. Mekh.,* 1981, vol. 45, no. 3, pp. 481–487.

[13] Chaplygin, S. A., On the Action of a Plane-Parallel Air Flow upon a Cylindrical Wing Moving within It, in *The Selected Works on Wing Theory of Sergei A. Chaplygin,* San Francisco: Garbell Research Foundation, 1956, pp. 42–72.

[14] Chaplygin, S. A., Von dem Drucke eines planparallelen Stromes auf untergetauchte Körper (zur Theorie der Aeroplane), *Mat. Sb.,* 1911, vol. 28, no. 1, pp. 120–166 (Russian).

[15] Childress, S., Spagnolie, S. E., and Tokieda, T., A Bug on a Raft: Recoil Locomotion in a Viscous Fluid, *J. Fluid Mech.,* 2011, vol. 669, pp. 527–556.

[16] Eldering, J., Realizing Nonholonomic Dynamics as Limit of Friction Forces, *Regul. Chaotic Dyn.,* 2016, vol. 21, no. 4, pp. 390–409.

[17] Karapetyan, A. V., On Realizing Nonholonomic Constraints by Viscous Friction Forces and Celtic Stones Stability, *J. Appl. Math. Mech.,* 1981, vol. 45, no. 1, pp. 30–36; see also: *Prikl. Mat. Mekh.,* 1981, vol. 45, no. 1, pp. 42–51.

[18] Karavaev, Yu. L., Kilin, A. A., and Klekovkin, A. V., Experimental Investigations of the Controlled Motion of a Screwless Underwater Robot, *Regul. Chaotic Dyn.,* 2016, vol. 21, nos. 7–8, pp. 918–926.

[19] Kilin, A. A., Klenov, A. I., and Tenenev, V. A., Controlling the Movement of the Body Using Internal Masses in a Viscous Liquid, *Computer Research and Modeling,* 2018, vol. 10, no. 4, pp. 445–460 (Russian).

[20] Kirchhoff, G., *Vorlesungen über mathematische Physik: Vol. 1. Mechanik,* Leipzig: Teubner, 1876.

[21] Klenov, A. I. and Kilin, A. A., Influence of Vortex Structures on the Controlled Motion of an Above-Water Screwless Robot, *Regul. Chaotic Dyn.,* 2016, vol. 21, nos. 7–8, pp. 927–938.

[22] Korotkin, A. I., *Added Masses of Ship Structures,* Fluid Mech. Appl., vol. 88, Dordrecht: Springer, 2009.

[23] Kozlov, V. V., On the Problem of Fall of a Rigid Body in a Resisting Medium, *Mosc. Univ. Mech. Bull.,* 1990, vol. 45, no. 1, pp. 30–36; see also: *Vestn. Mosk. Univ. Ser. 1. Mat. Mekh.,* 1990, no. 1, pp. 79–86.

[24] Kozlov, V. V., Realization of Nonintegrable Constraints in Classical Mechanics, *Sov. Phys. Dokl.,* 1983, vol. 28, pp. 735–737; see also: *Dokl. Akad. Nauk SSSR,* 1983, vol. 272, no. 3, pp. 550–554.

[25] Kozlov, V. V. and Ramodanov, S. M., The Motion of a Variable Body in an Ideal Fluid, *J. Appl. Math. Mech.,* 2001, vol. 65, no. 4, pp. 579–587; see also: *Prikl. Mat. Mekh.,* 2001, vol. 65, no. 4, pp. 592–601.

[26] Kozlov, V. V. and Onishchenko, D. A., The Motion in a Perfect Fluid of a Body Containing a Moving Point Mass, *J. Appl. Math. Mech.,* 2003, vol. 67, no. 4, pp. 553–564; see also: *Prikl. Mat. Mekh.,* 2003, vol. 67, no. 4, pp. 620–633.

[27] Kutta, W. M., Auftriebskräfte in strömenden Flüssigkeiten, *Illustr. aeronaut. Mitteilungen,* 1902, vol. 6, pp. 133–135.

[28] Kuznetsov, S. P., Plate Falling in a Fluid: Regular and Chaotic Dynamics of Finite-Dimensional Models, *Regul. Chaotic Dyn.,* 2015, vol. 20, no. 3, pp. 345–382.

[29] Leonard, N. E., Periodic Forcing, Dynamics and Control of Underactuated Spacecraft and Underwater Vehicles, in *Proc. of the 34th IEEE Conf. on Decision and Control (New Orleans, La., Dec 1995),* pp. 3980–3985.

[30] Lighthill, M. J., On the Squirming Motion of Nearly Spherical Deformable Bodies through Liquids at Very Small Reynolds Numbers, *Comm. Pure Appl. Math.,* 1952, vol. 5, no. 2, pp. 109–118.

[31] Mamaev, I. S., Tenenev, V. A., and Vetchanin, E. V., Dynamics of a Body with a Sharp Edge in a Viscous Fluid, *Rus. J. Nonlin. Dyn.,* 2018, vol. 14, no. 4, pp. 473–494.

[32] Mamaev, I. S. and Vetchanin, E. V., The Self-Propulsion of a Foil with a Sharp Edge in a Viscous Fluid under the Action of a Periodically Oscillating Rotor, *Regul. Chaotic Dyn.,* 2018, vol. 23, nos. 7–8, pp. 875–886.

[33] Mason, R. J., Fluid Locomotion and Trajectory Planning for Shape-Changing Robots, *PhD Dissertation,* Pasadena, Calif.: California Institute of Technology, 2003, 264 pp.

[34] Michelin, S. and Llewellyn Smith, S. G., An Unsteady Point Vortex Method for Coupled Fluid-Solid Problems, *Theor. Comput. Fluid Dyn.,* 2009, vol. 23, no. 2, pp. 127–153.

[35] Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I., *Integrals and Series: Vol. 1. Elementary Functions,* New York: Gordon & Breach Sci. Publ., 1986.

[36] Ramodanov, S. M. and Tenenev, V. A., Motion of a Body with Variable Distribution of Mass in a Boundless Viscous Liquid, *Nelin. Dinam.,* 2011, vol. 7, no. 3, pp. 635–647 (Russian).

[37] Ramodanov, S. M., Tenenev, V. A., and Treschev, D. V., Self-Propulsion of a Body with Rigid Surface and Variable Coefficient of Lift in a Perfect Fluid, *Regul. Chaotic Dyn.,* 2012, vol. 17, no. 6, pp. 547–558.

[38] Tallapragada, Ph., A Swimming Robot with an Internal Rotor As a Nonholonomic System, in *Proc. of the American Control Conference (Chicago, Ill., USA, July 1–3, 2015),* pp. 657–662.

[39] Vetchanin, E. V. and Kilin, A. A., Controlled Motion of a Rigid Body with Internal Mechanisms in an Ideal Incompressible Fluid, *Proc. Steklov Inst. Math.,* 2016, vol. 295, pp. 302–332; see also: *Tr. Mat. Inst. Steklova,* 2016, vol. 295, pp. 321–351.

[40] Vetchanin, E. V. and Kilin, A. A., Control of Body Motion in an Ideal Fluid Using the Internal Mass and the Rotor in the Presence of Circulation around the Body, *J. Dyn. Control Syst.,* 2017, vol. 23, no. 2, pp. 435–458.

[41] Vetchanin, E. V. and Kilin, A. A., Control of the Motion of an Unbalanced Heavy Ellipsoid in an Ideal Fluid Using Rotors, *Nelin. Dinam.,* 2016, vol. 12, no. 4, pp. 663–674 (Russian).

[42] Vetchanin, E. V., Kilin, A. A., and Mamaev, I. S., Control of the Motion of a Helical Body in a Fluid Using Rotors, *Regul. Chaotic Dyn.,* 2016, vol. 21, nos. 7–8, pp. 874–884.

[43] Vetchanin, E. V. and Mamaev, I. S., Optimal Control of the Motion of a Helical Body in a Liquid Using Rotors, *Russ. J. Math. Phys.,* 2017, vol. 24, no. 3, pp. 399–411.

[44] Vetchanin, E. V., Mamaev, I. S., and Tenenev, V. A., The Self-Propulsion of a Body with Moving Internal Masses in a Viscous Fluid, *Regul. Chaotic Dyn.,* 2013, vol. 18, nos. 1–2, pp. 100–117.