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# Self-Similar Ripples Through Math and Mind

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# Self-Similar Ripples Through Math and Mind

## Commentary on Marks-Tarlow's "A Fractal Epistemology for Transpersonal Psychology")

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### 1 Introduction

It is in fact not a surprise to me that someone, such as Terry Marks-Tarlow, is attempting to relate a field of pure mathematics with transpersonal psychology. One can understand the field of mathematics at large from a few different angles.

- At the level of rigor and proof.
- At the level of intuition and the level of *creation* or *creativity*. This is often a transpersonal phenomena; mathematicians create new mathematics from an intuitive understanding of the material, which itself comes from years of study and meditation. Moreover, the key insights in discovery often come "out of thin air."
- At the level of communication and metaphor.
- As a language for the physical laws and as a model of certain structures in the universe.

*A Fractal Epistemology for Transpersonal Psychology* seems to be issued as a combination of the second and third items above.

Because the creative element of pure mathematics is often an exercise in the transpersonal realm, one could in fact argue that the scope of the Marks-Tarlow's paper could and should be expanded. The idea of "the unreasonable effectiveness of mathematics in the natural sciences" is well-known. But it is also true that much of mathematics does not model the physical universe, and that many pure mathematicians are not interested in doing so. Thus much of pure mathematics does not manifest itself in the material plane. A mathematician is often driven by an aesthetic function, a sense of wonder, curiosity, or marvel, and an imagination with guides him or her through the infinite expanse of the transpersonal, mathematical realm. These ideas are considered once again in section 3 of this commentary.

At the same time, much of modern mathematics is *synergistic* work; many theorems require techniques from a diverse range of mathematical subjects to prove and thus communicate. Thus within the parts lie the whole – a rather apt (fractal) metaphor in this context. That is, in using one branch of modern mathematics as metaphor and model you will in fact be using several.

In this commentary my aim is to introduce a clear, sometimes technical, but simultaneously intuitive presentation of fractal geometry. In doing so I will then attempt to derive what I see to be the connections between transpersonal psychology and fractal geometry. This will both echo and reformulate Marks-Tarlow's synthesis. Rather than going line by line through Marks-Tarlow's paper and commenting on what is and what is not clear, correct, or needs further edit, I have taken this approach in hopes that it would lead to a greater understanding. I have not avoided using technical language in what follows. I feel that this is best. I have tried to derive the technicalities from an intuitive foundation. I have used Kenneth Falconer's (2003) *Fractal Geometry* as my major technical, mathematical source on this subject. A more readable and introductory book on this topic is Robert Devaney's (1990) *Chaos, Fractals, and Dynamics* [2]. However, if you feel that you do not want to go through all the mathematical details right away, then you can skip to section 3, and refer back to section 2 as needed. I must admit that pictures are invaluable in this branch of mathematics, especially at the intersection of complex dynamics and fractal geometry. Thus a second, more illustrated source (such as [2]) will certainly be of use.

### 2 Fractal Geometry

Fractal geometry as an emerging field and a fractal as a mathematical concept seems to have eluded a rigorous definition within the mathematical


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community, but essentially a fractal—call it *F*—possesses the following attributes [1]:

1. *F* has a fine structure, i.e. detail on arbitrarily small scales.
2. *F* is too irregular to be described in traditional geometrical language, both locally and globally.
3. Often *F* has some form of self-similarity, perhaps approximate or statistical.
4. Usually, the 'fractal dimension' of *F* (defined in some way) is greater than its topological dimension.
5. In most cases of interest *F* is defined in a very simple way, perhaps recursively.

A canonical example of a fractal is the Cantor set (briefly mentioned in Marks-Tarlow's paper as *Cantor dust*). This set is traditionally constructed via an iterative process which repeatedly deletes *middle third* intervals at each stage of iteration. Specifically, let $C_0 = [0, 1]$, and construct $C_{i+1}$ from $C_i$ recursively by removing the middle third from each interval in $C_i$. The Cantor set is then defined as $C := \cap_{i=1}^{\infty} C_i$. This construction is displayed below.

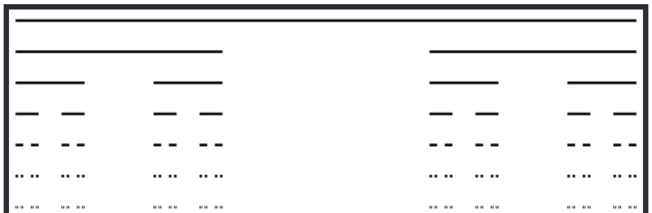

**Figure 1:** The Cantor Set *C*

You may in fact wonder why *C* is non-empty. This relies on a fundamental topological notion known as *compactness*. Such constructions can be made in higher dimensions as well, such as the *Sierpinski triangle*, which is obtained by repeatedly removing inverted equilateral triangles from an initial equilateral triangle. See figure 2. Features of the Cantor set include:

- The length of $C_n$ is $(2/3)^n$, and thus the length of the Cantor set is $n \lim_{n\to\infty}(2/3)^n = 0$.
- The Cantor set is uncountably infinite. Specifically it is in bijective correspondence with the real number line.
- The Cantor set is self-similar. The part of *C* in the interval [0, 1/3] and the part of *C* in the interval [2/3, 1], are both geometrically similar to *C*, scaled by a factor of 1/3. The parts of *C* in each of the four intervals of $C_2$ are similar to *C* but scaled by a factor of 1/9. The Cantor set contains copies of itself at many different scales. [1]
- The Cantor set has a 'fine structure'; that is, it contains detail at arbitrarily small scales. The more we enlarge the picture of the Cantor set, the more gaps become apparent to the eye. [1]
- Although *C* has an intricate detailed structure, the actual definition of *C* is straightforward. [1]
- *C* is obtained by a recursive procedure.
- The geometry of *C* is not easily described in classical terms: it is not the locus of the points that satisfy some simple geometric condition, nor is it the set of solutions of an simple equation. [1]
- It is awkward to describe the local geometry of *C* – near each of its points are a large number of other points, separated by gaps of varying lengths. [1]

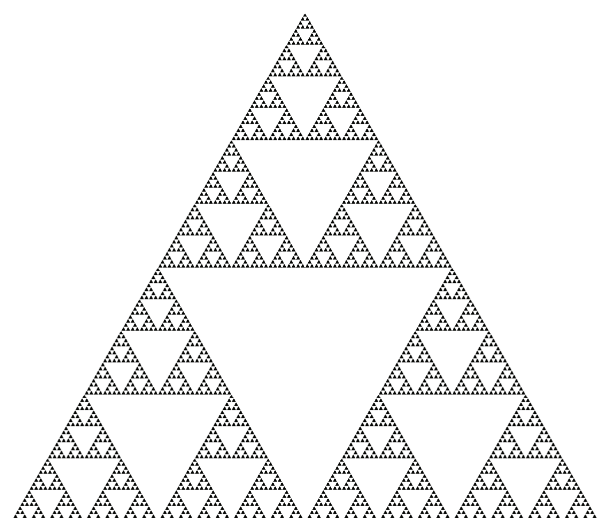

**Figure 2:** The Sierpinski triangle

- Topologically *C* contains no open interval (*a*, *b*), has no isolated points, every point of *C* is the limit of a sequence of other points in *C*, and *C* is a closed and bounded subset of the real line.
- Although *C* is large in the sense of cardinality – uncountably infinite – it has zero length. Thus it cannot be quantified by "usual" measures. [1]

I would like to expand on the first, second, and last items above as key insights in the development of fractal geometry. Essentially, what is at stake here is that our usual measuring sticks are not properly equipped to distinguish between various fractal sets. The Cantor set is uncountable, so cardinality alone is insufficient; if we measure the "length" of the Cantor set we obtain length zero. Objects such as the von Koch curve (figure 3) or a fractal coastline – examples of fractal curves – have infinite length but are of zero area. One is thus led to the notion of fractal dimension. Fractal dimension was popularized by Benoit Mandelbrot in his 1977 book *The Fractal Geometry of Nature*. However, a "proper" notion of fractal dimension predates Mandelbrot by over half a century, and was first introduced in 1919 by Felix Hausdorff. Now referred to as *Hausdorff dimension,* this notion provides a geometric, quantitative way to distinguish between fractal sets. However, there are many alternative definitions of fractal dimension, including upper and lower box dimension, correlation dimension, information dimension, Lyapunov dimension. It seems that on the most basic of fractals, these notions agree. However, many of these notions of dimension may not coincide on a variety of fractal constructions. Ultimately, definitions of a dimension are accepted based on experience and intuition [1].

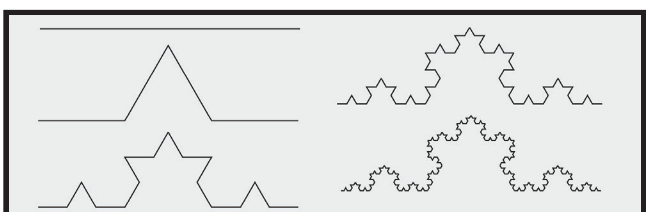
**Figure 3:** Constructing the von Koch curve

### 2.1 Fractal Dimension

I will now give a brief mathematical overview of fractal dimension. In considering a square and a rectangle, we are considering a two and a three dimensional object, respectively. In a square, there exist two independent modes of movement, or two perpendicular directions. Within the cube a third independent direction has been added. At the same time, the area of a square with side length *a* is *a*2; the area of a cube with side length *a* is *a*3. Evidently these areas respectively obey a *power law* in the dimension. We also notice that the dimension of the square is that of the plane, and the dimension of the cube is that of 3-space. That is, dimension obeys a certain scale-invariance.

In general when determining the "area" of a complex set (such as a fractal), one first *approximates* or *covers* the set by a sequence of simplified sets which have a well defined area (cf. *measure theory*). The most basic of fractals, however, such as the Cantor set or the Sierpinski Triangle, may be realized as a union of (covered, approximated by) self-similar, scaled subsets. The relationship between the number of such self-similar, scaled subsets required to cover the fractal set and the scaling factor will determine the dimension.

Let's first see how this relationship works in such standard geometric objects as the unit square and unit cube. Fix $n > 1$. We may realize the unit square as a union of $n^2$ smaller squares each of side length $1/n$ – the scaling factor is $1/n$. We may realize the unit cube as a union of $n^3$ smaller cubes each of side length $1/n$ – the scaling factor is again $1/n$. Thus, in both cases we derive the formula $n = (1/m)^d$, where $n$ = number of self-similar pieces, $m$ = scaling factor, and $d$ = dimension. Said another way, the dimension satisfies the formula $d = -\log n / \log m$. The dimension of the square is $2 = -\log n^2 / \log(1/n)$; the dimension of the cube is $3 = -\log n^3 / \log(1/n)$.

We can thus apply this method and formula to fractals which retain a constant self-similarity at each iteration, such as the Cantor set or the Sierpinski triangle. The Cantor set may be regarded as comprising four copies of itself scaled by a factor of 1/9. Thus, we might say that the dimension of the Cantor set is $-\log 4 / \log(1/9) = \log 2 / \log 3$.

In general, the Cantor set may be regarded as comprising $2^n$ copies of itself scaled by a factor of $1/3^n$; in all cases, however, we arrive at the same dimension: $-\log 2^n / \log(1/3^n) = \log 2 / \log 3$. The Sierpinski triangle may be regarded as comprising three copies of itself scaled by a factor of 1/2; or $3^n$ copies of itself scaled by a factor of $1/2^n$. In any case we arrive at a dimension of log 3/ log 2. See figure 4.

This construction is known as the *similarity dimension*. Due to the intricacy of most fractal constructions, we cannot hope for this method to work in general. We must consider approximations of our general set by a wider array of simplified sets. This leads to a fundamental notion known as 'measurement at scale δ.' Consider the Cantor set. This set may be *exactly* approximated at scales $\delta = 1/3^n$ for all $n \geq 1$: the parts of $C$ in the respective intervals of length $1/3^n$ are similar to the Cantor set but scaled by a factor of $1/3^n$, and so $C$ comprises the union of these sets. The smaller δ is, the finer the approximation. In dealing with more intricate sets, or fractals, we are interested in approximations as the scale $\delta \to 0$.

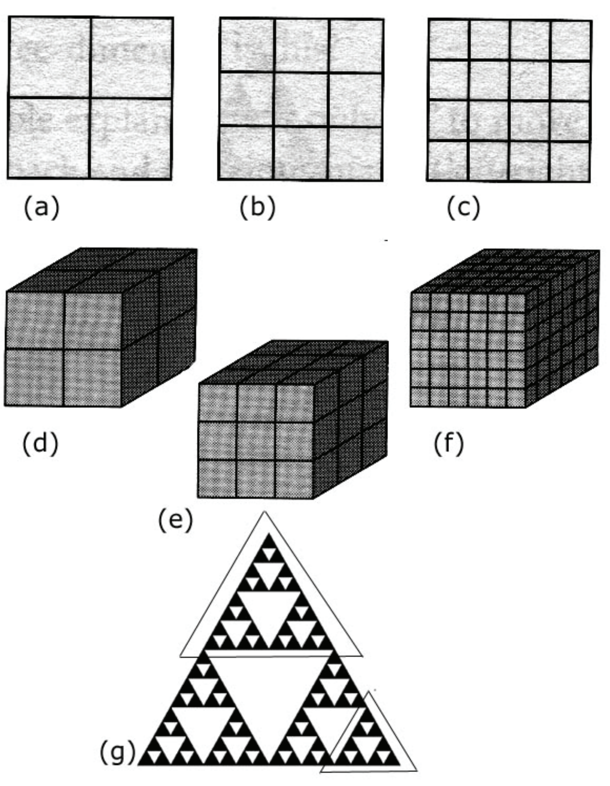


**Figure 4:** Self-similar coverings

Let $U$ be any non-empty subset of $n$-dimensional Euclidean space $\mathbf{R}^n$. We define the diameter of U by $|U| := \sup\{|x - y| : x,y \in U\}$. In words, the diameter of $U$ is the maximum distance (least upper bound) between any two points in $U$. Now let $F$ be any bounded, non-empty subset of $\mathbf{R}^n$. Denote by $N_\delta(F)$ the smallest number of sets of diameter at most δ which can cover $F$ . *A dimension of F is then determined by the power law (if any) obeyed by* $N_\delta(F)$ *as* $\delta \to 0$ [1]. That is,

$$N_\delta(F) \sim c\delta^{-s}$$

for constants $c, s$. Take the Cantor set, for example. For $\delta = 1/3^n$, we have $N_\delta(C) = 2^n$. Then

$$N_\delta(C) = 2^n = (1/3^n)^{-\log 2/\log 3} = \delta^{-\log 2/\log 3}.$$

Thus $s = \log 2/\log 3$, and so we expect $s$ to be a measure of the dimension as $\delta \to 0$. With this in mind, we define the *box dimension* of $F$ by

$$dim_B(F) := \lim \underline{\log N_\delta(F)},$$

assuming the limit exists. For example, we have shown that the box dimension of the Cantor set is log 2/ log 3, which agrees with its *similarity dimension* calculated above.

### 2.2 Fractals and Dynamics

I have introduced the notion of a fractal *in isolation*. It is just as important, if not more, to consider how fractals arise within dynamical systems. Dynamical systems is a mathematical study of processes in motion. Planetary motion, the dynamics of the weather and the stock market, or the motion of a simple pendulum are all examples. Processes can be viewed in discrete time, continuous time, and through the lens of complex variables. This is mathematically represented either by a function $f : D \to D$, $D \subset \mathbf{R}n$ or $D \subset \mathbf{C}$, and its *iterates* or *orbits*,

$$\{f^k(x)\}_{k=0}^{\infty} = \{x, f(x), f(f(x)), f(f(f(x))), \ldots\}, \quad (1)$$

or by a differential equation

$$x = \frac{dx}{dt} = f(x), \quad (2)$$

along with its family of solution curves $x(t)$, each of which is defined uniquely for a given initial condition

$x(0)$. (Assuming some technical conditions on the function $f$, such as 'smoothness'.) Note that in (1), $f^0(x) = x$, $f^1(x) = f(x)$, $f^2(x) = f(f(x))$, etc. In both cases, we can let time approach infinity and inquire as to the end behavior of the system; this is indeed an interesting problem because one is often interested in predicting the future of the system (will it rain tomorrow?). Roughly speaking, we define an *attractor* (*repeller*) of the system to be a set to which all nearby orbits/ solution curves converge (diverge) (in time, defined in both the discrete and continuous case). *This set can manifest itself as a fractal; moreover, when it does so the dynamics of the system is often chaotic*. What is perhaps most interesting, however, is that *very simple systems, even systems depending on only one variable, may behave just as unpredictably as the stock market, just as wildly as a turbulent waterfall, and just as violently as a hurricane* [2]. This is, in essence, the notion of chaos.

Consider first a discrete system, given by a continuous function $f : D \to D$ and its orbits $\{f^k(x)\}_{k=0}^{\infty}$. For example, if $f(x) = \cos x$ then $f^k(x) \to 0.739 \ldots$ as $k \to \infty$ for every initial value $x \in \mathbf{R}$. We consider three cases in general.

1. $f^k(x)$ converges to a *fixed point* $w$, i.e. a point $w \in D$ such that $f(w) = w$. Convergence means that $|f^k(x) - w| \to 0$ as $k \to \infty$. A fixed point $w$ is *stable* or *unstable* according to whether $|f'(w)| < 1$ or $> 1$. Stable fixed points attract nearby orbits, unstable fixed points repel them.
2. $f^k(x)$ converges to a *periodic orbit*, i.e. a set of the form

   $$\{v, f(v), f^2(v), \ldots, f^{p-1}(v)\}$$

   where $p$ is the minimal integer such that $f^p(v) = v$. Convergence in this case means $|f^k(x) - f^k(v)| \to 0$ as $k \to \infty$. A periodic point $v$ of period $p$ is stable or unstable according to whether $|(f^p)'(v)| < 1$ or $> 1$. Stable periodic points attract nearby orbits, unstable periodic points repel them.
3. $f^k(x)$ may appear to move about at random, but always staying close to certain set, which may be a fractal.

A common example of a discrete dynamical system exhibiting chaotic dynamics is given by the logistic map $f\lambda : \mathbf{R} \to \mathbf{R}$, defined by $f_\lambda(x) = \lambda x(1 - x)$, where $\lambda$ is a positive constant. We outline the dynamics of $f_\lambda$ for various $\lambda$.

- $0 < \lambda \leq 1$. Then $f_\lambda$ maps the unit interval to itself, and so restricting our attention here we find that 0 is an attractive fixed point with $f^k_\lambda(x) \to 0$ for all $x \in [0, 1]$.
- $1 < \lambda < 3$. $f_\lambda$ has an unstable fixed point at 0 and a stable fixed point at $1 - 1/\lambda$. We have that $f^k(x) \to 1 - 1/\lambda$ for all $x \in (0, 1)$.
- As $\lambda$ increases through the value $\lambda_1 = 3$, the fixed point at $1–1/\lambda$ becomes unstable and almost every point in (0, 1) attracts to a new stable orbit of period 2.
- When $\lambda$ increases through $\lambda_2 = 1+\sqrt{6}$ the period-2 orbit becomes unstable and a new stable orbit of period 4 is born.
- This behavior continues as $\lambda$ increases further, with a stable orbit of period $2^q$ appearing at $\lambda = \lambda_q$ ; this orbit attracts all but countably many points of (0, 1). [1]
- As $q \to \infty$, $\lambda \to \lambda\infty \approx 3.570$, the period doubling occurs more frequently and *we obtain a sequence of attracting (periodic) orbits approximating a Cantor set. When $\lambda = \lambda_\infty$ the attractor is a Cantor-like set.* The Hausdorff dimension of this set has been approximated to be 0.538 . . . .
- $\lambda\infty < \lambda < 4$. Several types of behavior occur. There exist a set $K$ of positive measure such that for $\lambda \in K$, $f_\lambda$ has chaotic behavior.
- The behavior of the logistic map is universal in the sense that it is qualitatively the same as any family of transformations of the form $f(x) = \lambda f(x)$, provided that $f$ has a single maximum at a point $c$ with $f''(c) < 0$.

The values of $\lambda$ which mark a change in the qualitative behavior of the dynamics (iterates) of $f_\lambda$ are known as *bifurcations*. Thus above for the logistic map such values include $\lambda 1 = 3$, $\lambda 2 = 1 + \sqrt{6}$, . . . . This sequence lists a series of period-doubling bifurcations leading to chaos. We summarize this behavior in figure 5, known as the bifurcation diagram for $f_\lambda$. Each value along the horizontal axis represents a value of $\lambda$, above which is plotted the long term behavior of the iterates $f^k(x)$ for suitable $x$ (as described above).

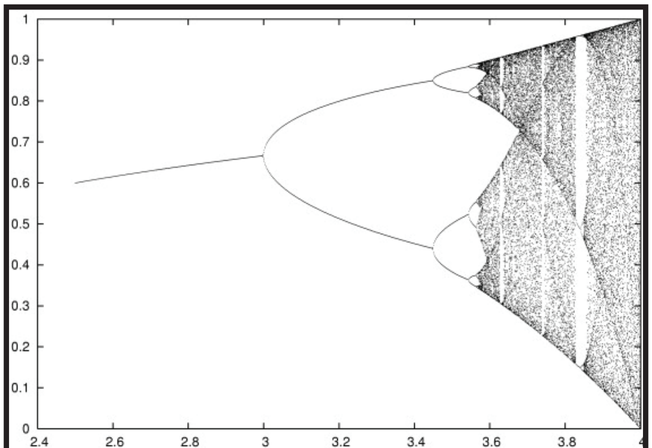


**Figure 5:** The bifurcation diagram of the logistic map $f\lambda(x) = \lambda x(1 - x)$

Finally, it will be useful to remark how fractals arise within the realm of complex dynamics. Let $f : \mathbf{C} \to \mathbf{C}$ be a polynomial of degree $n \geq 2$ with complex coefficients. We are again interested in orbits $\{f^k(z)\}_{k=0}^{\infty}$ of various $z \in \mathbf{C}$. In particular we are interested in the *boundary zone* between orbits which *diverge*

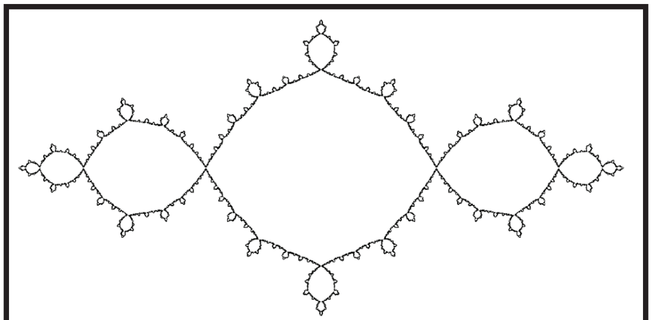

**Figure 6:** The Julia set of $f(z) = z^2 - 1$

*to* $\infty$ and orbits which do not. This "boundary" is known as the *Julia set of f*. Specifically, we define the *filled in Julia set* of $f$ by

$$K(f) := \{z \in \mathbf{C} : f^k(z) \not\to \infty\}.$$

The Julia set of $f$, denoted $J(f)$, is then the (topological) boundary of $K(f)$. That is, for every point $z \in J(f)$, there exists a small disk $D_z(r)$ of radius $r > 0$, and points $w$ and $v$ in $D_z(r)$ with $f^k(w) \not\to \infty$ and $f(v) \to \infty$.

The Julia set of the function $f(z) = z^2$ is the unit circle. Points inside this circle iterate to 0, while points outside the circle iterate to $\infty$. However, if $f$ is perturbed ever so slightly to $f_c(z) = z^2 + c$, then the possible structures of the Julia set become diverse, often fractal.

The pictures that arise in these examples are majestic to say the least. See, for example, [2] for an introduction to this topic.

We remark that the *Mandelbrot set M* is defined to be the collection of all $c \in \mathbf{C}$ for which $f_c^k(0) \not\to \infty$:

$$M = \{c \in \mathbf{C} : f^k(0) \not\to \infty\}.$$

The Mandelbrot set is, in words, the collection of all $c$ values (in $\mathbf{C}$) for which the Julia set of $f_c(z) = z^2 + c$ does not break into pieces (i.e. is connected). The Mandelbrot set may be viewed as a picture book of the various Julia sets that arise as $c$ varies in $f_c$. This is because each $c$ value corresponds to a specific connected Julia set. This is displayed in figure 7.

The Mandelbrot set is a highly complicated figure with a main cardioid region to which a series of circular 'buds' are attached. In addition fine 'hairs' grow out from the buds carrying within miniature copies of the entire Mandelbrot set. The Mandelbrot set is connected (topologically) and its boundary has Hausdorff dimension 2. The Mandelbrot set is pictured in figures 8, 9, and 10.

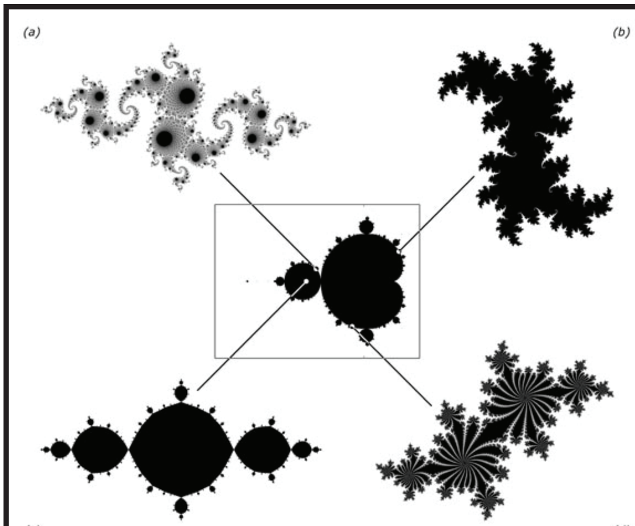


**Figure 7:** Julia sets for $c$ at different points in the Mandelbrot set

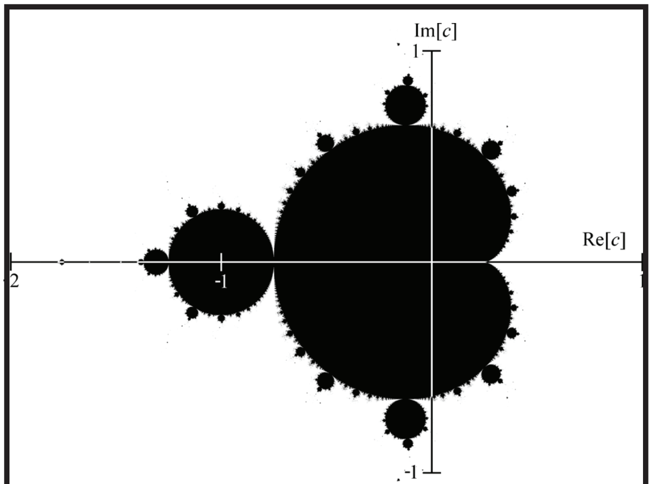


**Figure 8.** The Mandelbrot set

## 3 Fractal Geometry and Transpersonal Psychology

We began our overview of fractal geometry with the unit line segment [0, 1] and a simple iterative procedure. This iterative procedure quite literally *dissolved* the unit line segment until all that was left was

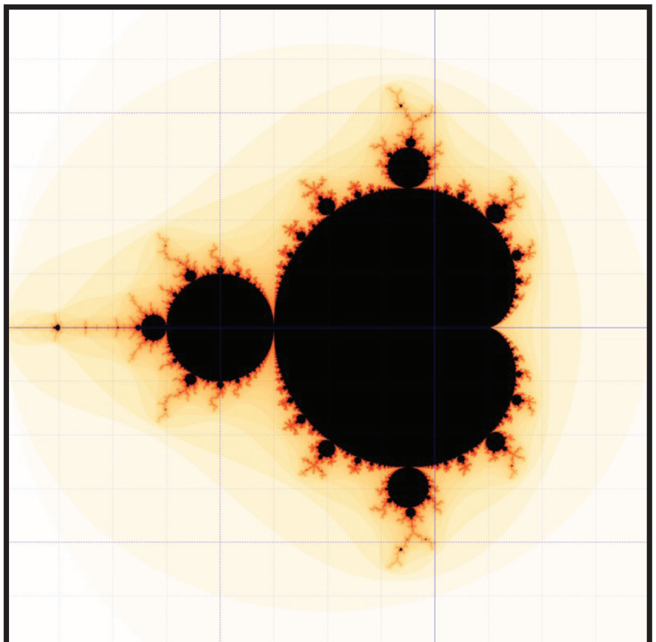

**Figure 9.** Seeing the 'hairs' of the Mandelbrot set

simultaneously non-trivial or non-empty and yet highly imaginative and detailed in nature. This was the Cantor set. This set resided between the ordinary zero-dimensional realm of discrete points and the one-dimensional realm of the real line. In fact, we attached to this set a dimension of log 2/ log 3, a number which quantifies how much space the set fills. This set was highly elusive in that the more we searched within the set, the more details we found—details which brought us right back where we started. Indeed, the Cantor set is self-similar on infinitely many scales. Both the concept of and the journey into the Cantor set can be seen or mirrored through a transpersonal and inner perspective. This requires poetic language, and so one might say: *The farther I reached upward, the tighter I grasped; the deeper I dove, the less structure I found, yet the more detail there was to get lost in. It was all so confusing to find myself, my struggles, my highs and lows mirrored back to me as if within me is you, within you is me, and together we are connected without by a web of profundity. And I realized that life was happening to me, dissolving my isolated conceptualizations, my isolating boundaries, forcing me ever deeper into myself—an ever elusive space, a self-similar dream which was simultaneously manifesting within and without.*

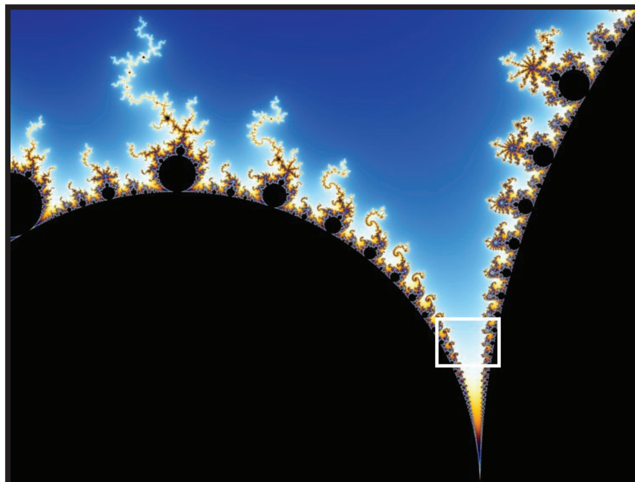

**Figure 10.** Glimpsing the Mandelbrot set within the Mandelbrot set

At the same time, we saw how fractals arose in the fate of a dynamical system. In a one-dimensional system, such as the logistic map, we saw that for certain values of $\lambda$ the long-term behavior of the system attracts to a fractal, Cantor-like set. That is, *in trying to predict the future we only found chaos.* And yet within this chaos exists an inherent order; the fractal attractor is highly self-similar. Abstractly, one might say that within such examples exists an inherent intelligence beyond the scope of our usual mode of inquiry. We may consider fractals in nature as well. A coastline exists at the boundary between land and sea. It is rather curious to remind ourselves that the sea is the great symbol of the divine. Thus the coastline—a natural fractal—existing at the boundary between land and sea, symbolically exists at the boundary between the physical and spiritual planes, between the concrete and the transpersonal. In complex dynamics, fractals exist as Julia sets, defined as the (topological) boundary between convergent and divergent orbits. Between the finite and the infinite. We thus have a physical boundary between land and sea, or a mathematical boundary between the finite and the infinite, both emanating from the same archetypal pool.

We thus see how mathematics, specifically the mathematics of fractal geometry, gives rise to metaphor which operates across different modes of reality. I would remark further that it is not fractal geometry that is giving rise to this phenomenon, and it is not just the mathematics alone that is of interest, but the larger concepts to which the mathematics points. The self-similar ripples that exist across the mathematical, physical, and inner worlds. It seems that it is not just that mathematics is excellent at modeling the physical world, but that the two worlds are archetypally *self-similar*, and further this self-similar ripple extends beyond a mere physical manifestation.

If I were to imagine myself manifesting at the three billionth stage of the Cantor set, I would be overwhelmed by a seemingly orderless detail. But if by chance I was one day able to step outside of my separate existence, and perceive the whole from a higher dimensional perspective, the majesty of a pattern would emerge. I was never separate to begin with, but a self-similar realization, caught in an illusion of low-dimensional separateness, unable to glimpse the forest through the trees.

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## About the Author


***Jonathan Root, PhD,*** received his doctorate in mathematics from Boston University in 2016 under the supervision of Mark Kon. He taught English in rural China during the 2016–2017 school year, and will continue his studies in the philosophy, cosmology, and consciousness program at the California Institute of Integrative Studies (CIIS) in San Francisco, California. Root's commentary aims to "introduce a clear, sometimes technical, but simultaneously intuitive presentation of fractal geometry" in order to derive his own understanding how transpersonal psychology connects with fractal geometry through self-similar ripples across different modes of reality.


## About the Journal


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