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WEAK ESTIMATES FOR THE MAXIMAL AND RIESZ POTENTIAL OPERATORS ON NON-HOMOGENEOUS CENTRAL MORREY TYPE SPACES IN L^1 OVER METRIC MEASURE SPACES

Katsuo Matsuoka, Yoshihiro Mizuta and Tetsu Shimomura

Nihon University, College of Economics
1-3-2 Misaki-cho Kanda Chiyoda-ku Tokyo 101-8360, Japan; katsu.m@nihon-u.ac.jp

Hiroshima University, Graduate School of Science, Department of Mathematics
Higashi-Hiroshima 739-8521, Japan; yomizuta@hiroshima-u.ac.jp

Hiroshima University, Graduate School of Education, Department of Mathematics
Higashi-Hiroshima 739-8524, Japan; tshimo@hiroshima-u.ac.jp

Abstract. In a metric measure space (X, d, μ), our first aim in this paper is to discuss the weak estimates for the maximal and Riesz potential operators in the non-homogeneous central Morrey type space $M^{1,q,a}(X)$ (about $x_0 \in X$) of all measurable functions f on X satisfying

$$\|f\|_{M^{1,q,a}(X)} = \left(\int_1^\infty \left(r^{-a} \|f\|_{L^1(B(x_0,r))} \right)^q \frac{dr}{r} \right)^{1/q} < \infty$$

for $a \geq 0$ and $0 < q < \infty$; when $q = \infty$, we apply a necessary modification. To do this, we consider the family $WM^{\varphi,q,a}(X)$ of all measurable functions $f \in L^1_{\rm loc}(X)$ such that

$$\|f\|_{WM^{\varphi,q,a}(X)} = \sup_{\lambda>0} \lambda \left(\int_1^\infty \left(r^{-a} \varphi^{-1} \left(\int_{B(x_0,r)} \chi_{E_f(\lambda)}(x)\, d\mu(x) \right) \right)^q \frac{dr}{r} \right)^{1/q} < \infty,$$

where φ is a general function satisfying certain conditions and $\chi_{E_f(\lambda)}$ denotes the characteristic function of $E_f(\lambda) = \{x \in X \colon |f(x)| > \lambda\}$. In connection with $M^{1,q,a}(X)$, we treat the complementary space $N^{\infty,q,a}(X)$ of all measurable functions f on X satisfying

$$\|f\|_{N^{\infty,q,a}(X)} = \|f\|_{L^\infty(B(x_0,2))} + \left(\int_1^\infty \left(r^a \|f\|_{L^1(X \setminus B(x_0,r))} \right)^q \frac{dr}{r} \right)^{1/q} < \infty.$$

1. Introduction

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. The space $B^p(\mathbf{R}^n)$ given by Beurling [4] is a special case of Herz spaces $K_p^{\alpha,r}(\mathbf{R}^n)$ introduced by Herz [17]. As an extension of the space $B^p(\mathbf{R}^n)$, Alvarez, Guzmán-Partida and Lakey [3] introduced the non-homogeneous central Morrey space $B^{p,a}(\mathbf{R}^n)$. Fu, Lin and Lu [12] proved the boundedness of the Riesz potential operator I_α on $B^{p,a}(\mathbf{R}^n)$, where $-n/p \leq a < -\alpha$; see also [21].

We denote by (X, d, μ) a metric measure space, where d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ). For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball in X centered at x with radius r and let $d_X = \sup\{d(x, y) \colon x, y \in X\}$. We assume that

$$\mu(\{x\}) = 0$$

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for $x \in X$ and

(μ1) $0 < \mu(B(x,r)) < \infty$ for $x \in X$ and $r > 0$;
(μ2) μ is doubling, that is, there exists a constant $A_1 > 1$ such that

$$\mu(B(x,2r)) \le A_1 \mu(B(x,r)) \quad \text{for all } r > 0 \text{ and } x \in X.$$

For a fixed point $0 \in X$, write $|x| = d(0,|x|)$. For $0 < q \le \infty$ and $a \ge 0$, we consider the non-homogeneous central Morrey type space $M^{1,q,a}(X)$ consisting of all measurable functions f on X satisfying

$$\|f\|_{M^{1,q,a}(X)} = \left(\int_1^\infty \left(r^{-a} \|f\|_{L^1(B(0,r))} \right)^q \frac{dr}{r} \right)^{1/q} < \infty$$

when $q < \infty$ and

$$\|f\|_{M^{1,\infty,a}(X)} = \sup_{r>1} r^{-a} \|f\|_{L^1(B(0,r))} < \infty$$

when $q = \infty$. Note here that $M^{1,q,a}(X)$ is independent of 0 by the doubling condition on μ (see also Definition 2.1).

In connection with $M^{1,q,a}(X)$, let us consider the family $N^{\infty,q,a}(X)$ of all measurable functions f on X satisfying

$$\|f\|_{N^{\infty,q,a}(X)} = \|f\|_{L^\infty(B(0,2))} + \left(\int_1^\infty \left(r^{a} \|f\|_{L^\infty(X \setminus B(0,r))} \right)^q \frac{dr}{r} \right)^{1/q} < \infty$$

when $q < \infty$ and

$$\|f\|_{N^{\infty,\infty,a}(X)} = \|f\|_{L^\infty(B(0,2))} + \sup_{r>1} r^{a} \|f\|_{L^\infty(X \setminus B(0,r))} < \infty$$

when $q = \infty$.

There are several Morrey type spaces related to our non-homogeneous central Morrey type spaces; e.g. Morrey spaces by Adams–Xiao [1], local Morrey type spaces and complementary local Morrey type spaces by Burenkov and al. [6, 7, 8, 9, 15], grand and small Lebesgue spaces by Fiorenza–Karadzhov [11].

In harmonic analysis, the maximal operator is a classical tool when studying Sobolev functions and partial differential equations (see [5, 18, 26], etc.). It is well known that the maximal operator is weakly bounded in the Lebesgue space $L^1(\mathbf{R}^n)$ (see [26]) and the Riesz potential operator is weakly bounded in $L^1(\mathbf{R}^n)$ (see [16]). Recently, the first and second authors [20] studied the weak boundedness of the maximal and Riesz potential operators in $M^{1,q,a}(\mathbf{R}^n)$ when μ is the Lebesgue measure on $\mathbf{R}^n$, as an extension of [26, Theorem 1(b), Chapter 1] and [16, Proposition 3.19]. We know the weak boundedness of the maximal and Riesz potential operators in the Lebesgue space $L^1(X)$ (see [16, Theorems 2.2 and 3.22]).

Our first aim in this paper is to establish the weak boundedness of the maximal and Riesz potential operators in $M^{1,q,a}(X)$ (Theorems 3.3 and 4.6, Corollaries 3.4 and 4.7 below), as an extension of [20, Theorem 3.2], [26, Theorem 1(b), Chapter 1] and [16, Proposition 3.19, Theorems 2.2 and 3.22]. To do so, we consider the weak central Morrey type spaces $WM^{\varphi,q,a}(X)$ defined by a general function φ satisfying certain conditions (see Section 3 for the definition of $WM^{\varphi,q,a}(X)$). In connection with [19, Remark 3.7], we show the boundedness for the Riesz potential operator from $M^{1,q,a}(X)$ to $M^{p,q_1,a_1}(X)$ when $X = \mathbf{R}^n$ and $1 < p < 1^* = n/(n-\alpha)$ (Corollary 4.8).

Next, following Di Fratta–Fiorenza [10] and Gogatishvili–Mustafayev [13], we study the duality properties among our Morrey type spaces $M^{1,q,a}(X)$ and $N^{\infty,q',a}(X)$

(Theorem 5.1), which gives another characterization of Morrey spaces by Adams–Xiao [1] (see also [14, 23, 24]).

Throughout this paper, let C denote various positive constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 0$.

2. Non-homogeneous central Morrey type spaces

For $1 \le p < \infty$ and a (Borel) measurable set $E \subset X$, set

$$\|f\|_{L^p(E)} = \left(\int_E |f(x)|^p \, d\mu(x) \right)^{1/p};$$

when $p = \infty$, $\|\cdot\|_{L^\infty(E)}$ denotes the essential supremum on E.

Definition 2.1. (Non-homogeneous central Morrey type spaces) Let $1 \le p \le \infty$, $0 < q \le \infty$ and $a \ge 0$. We define the non-homogeneous central Morrey type space $M^{p,q,a}(X)$ of all measurable functions f on X such that

$$\|f\|_{M^{p,q,a}(X)} = \left(\int_1^\infty \left(r^{-a} \|f\|_{L^p(B(0,r))} \right)^q \frac{dr}{r} \right)^{1/q} < \infty$$

when $0 < q < \infty$, and

$$\|f\|_{M^{p,\infty,a}(X)} = \sup_{r>1} r^{-a} \|f\|_{L^p(B(0,r))} < \infty$$

when $q = \infty$. Further we denote by $N^{p,q,a}(X)$ the family of all measurable functions f on X such that

$$\|f\|_{N^{p,q,a}(X)} = \|f\|_{L^p(B(0,2))} + \left(\int_1^\infty \left(r^{a} \|f\|_{L^p(X \setminus B(0,r))} \right)^q \frac{dr}{r} \right)^{1/q} < \infty$$

when $q < \infty$ and

$$\|f\|_{N^{p,\infty,a}(X)} = \|f\|_{L^p(B(0,2))} + \sup_{r>1} r^{a} \|f\|_{L^p(X \setminus B(0,r))} < \infty$$

when $q = \infty$.

It is easy to see that $M^{p,q,a}(X)$ is independent of $0 \in X$ by the doubling condition on μ since

$$\|f\|_{L^p(B(x_0,r))} \le \|f\|_{L^p(B(0,r+|x_0|))} \le \|f\|_{L^p(B(0,(1+|x_0|)r))}$$

for $x_0 \in X$ and $r > 1$.

Note that

(1) if $a = 0$ and $0 < q < \infty$, then $M^{p,q,a}(X) = \{0\}$;
(2) if $a = 0$, then $M^{p,\infty,a}(X) = N^{p,\infty,a}(X) = L^p(X)$;
(3) if $a > 0$, then $N^{p,\infty,a}(X) \subset L^p(X) \subset M^{p,\infty,a}(X)$.

Further,

(4) $N^{p,q,a}(X) \supset L_0^p(X)$,

where $L_0^p(X)$ denotes the family of functions in $L^p(X)$ with compact support in X.

For fundamental properties of our Morrey type spaces, we have the following lemmas (see [23, Lemma 2.2]).

Lemma 2.2. *Let $1 \le p \le \infty$ and $a \ge 0$. For $0 < q_1 < q_2 < \infty$,*

$$M^{p,q_1,a}(X) \subset M^{p,q_2,a}(X) \subset M^{p,\infty,a}(X)$$

and

$$N^{p,q_1,a}(X) \subset N^{p,q_2,a}(X) \subset N^{p,\infty,a}(X).$$

Lemma 2.3. *For $1 \le p \le \infty$, $0 < q < \infty$ and $a \ge 0$,*

$$\|f\|_{M^{p,q,a}(X)} \sim \left(\sum_{j=1}^{\infty} \left(2^{-aj} \|f\|_{L^p(B(0,2^j))}\right)^q \right)^{1/q}$$

and

$$\|f\|_{N^{p,q,a}(X)} \sim \|f\|_{L^p(B(0,4))} + \left(\sum_{j=1}^{\infty} \left(2^{aj} \|f\|_{L^p(X \setminus B(0,2^j))}\right)^q \right)^{1/q}.$$

3. Maximal functions

For a locally integrable function f on X, the maximal function of f is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y).$$

It is well known that the maximal operator $M\colon f \longrightarrow Mf$ is weakly bounded in $L^1(X)$, that is,

$$\mu(\{x\colon Mf(x) > \lambda\}) \le C\lambda^{-1} \int_X |f(y)|\, d\mu(y)$$

for all $\lambda > 0$ and f with $\|f\|_{L^1(X)} < \infty$ (see [16], etc.).

Consider a function $\varphi(r)$ satisfying the following conditions (φ1) and (φ2):

(φ1) $\varphi(0) = \lim_{r\to 0+} \varphi(r) = 0$;

(φ2) $\varphi(\cdot)$ is positive, doubling and convex in $(0,\infty)$.

Definition 3.1. (Weak central Morrey type spaces) Let $0 < q \le \infty$ and $a \ge 0$. We denote by $WM^{\varphi,q,a}(X)$ the family of all measurable functions f on X such that

$$\|f\|_{WM^{\varphi,q,a}(X)} = \sup_{\lambda>0} \lambda \left(\int_1^{\infty} \left(r^{-a} \varphi^{-1} \left(\int_{B(0,r)} \chi_{E_f(\lambda)}(x)\, d\mu(x) \right) \right)^q \frac{dr}{r} \right)^{1/q} < \infty$$

when $0 < q < \infty$ and

$$\|f\|_{WM^{\varphi,\infty,a}(X)} = \sup_{\lambda>0, r>1} \lambda r^{-a} \varphi^{-1} \left(\int_{B(0,r)} \chi_{E_f(\lambda)}(x)\, d\mu(x) \right) < \infty$$

when $q = \infty$, where χ_G denotes the characteristic function of a measurable set $G \subset X$ and

$$E_f(\lambda) = \{x \in X\colon |f(x)| > \lambda\}.$$

Note here that $WM^{\varphi,q,a}(X)$ is a linear space.

When $1 \le p \le \infty$ and $\varphi(r) = r^p$, $WM^{\varphi,q,a}(X)$ is denoted by $WM^{p,q,a}(X)$. For other examples of φ, see Examples 4.2 and 4.3 below.

Remark 3.2. It is seen that

$$\sup_{\lambda>0} \lambda \int_{E_f(\lambda)} (1+|x|)^{-a}\, d\mu(x) < \infty$$

if and only if $f \in WM^{1,1,a}(X)$. It is easy to see that if $\int_X (1+|x|)^{-a} |f(x)|\, d\mu(x) < \infty$, then $f \in WM^{1,1,a}(X)$.

In view of Almeida and Drihem [2], we know that the maximal operator M is bounded in both $M^{p,q,a}(\mathbf{R}^n)$ and $N^{p,q,a}(\mathbf{R}^n)$, when $1 < p < \infty$ and μ is the Lebesgue measure on $\mathbf{R}^n$. The case $p = 1$ is treated in the following, which is an extension of [20, Theorem 3.2].

Theorem 3.3. *Let $a \geq 0$. Suppose there is $C > 0$ such that*

($\mu 3$) $\displaystyle\int_r^\infty \mu(B(0,t))^{-1}\frac{dt}{t} \leq C\mu(B(0,r))^{-1}$ *when* $r > 1$;

($\mu 4$) $\displaystyle\int_1^r t^{-a}\mu(B(0,t))\frac{dt}{t} \leq Cr^{-a}\mu(B(0,r))$ *when* $r > 1$.

Then the maximal operator M is bounded from $M^{1,q,a}(X)$ to $WM^{1,q,a}(X)$.

Corollary 3.4. *Let μ be the Lebesgue measure on $\mathbf{R}^n$. If $0 \leq a < n$ and $0 < q \leq \infty$, then the maximal operator M is bounded from $M^{1,q,a}(\mathbf{R}^n)$ to $WM^{1,q,a}(\mathbf{R}^n)$.*

Theorem 3.5. *Let $a \geq 0$ and $0 < q \leq \infty$. When $a = 0$, suppose*

($\mu 5$) $\displaystyle\sup_{r>1}\, (\log r)\mu(B(0,r))^{-1} < \infty.$

If $q < \infty$, then there exist constants $A, C > 0$ such that

$$\sup_{\lambda > A}\int_1^\infty \left(r^a\lambda\int_{X\setminus B(0,r)}\chi_{E_{Mf}(\lambda)}(x)\,d\mu(x)\right)^q\frac{dr}{r} \leq C$$

when $\|f\|_{N^{1,q,a}(X)} \leq 1$; if $q = \infty$, then there exist constants $A, C > 0$ such that

$$\sup_{\lambda > A, r>1} r^a\lambda\int_{X\setminus B(0,r)}\chi_{E_{Mf}(\lambda)}(x)\,d\mu(x) \leq C$$

when $\|f\|_{N^{1,\infty,a}(X)} \leq 1$.

Remark 3.6. In Theorem 3.5, we need the restriction that $\lambda > A$ when $a > 0$. For showing this, take $\delta > a > 0$ and let us consider the function $f(y) = (1+|y|)^{-\delta-n}$. If μ is the Lebesgue measure on $\mathbf{R}^n$, then $f \in N^{1,q,a}(\mathbf{R}^n)$ and when $|x| > 1$

$$Mf(x) \geq \frac{1}{\mu(B(x,2|x|))}\int_{B(0,|x|)}(1+|y|)^{-\delta-n}\,d\mu(y) \geq c_1\frac{1}{\mu(B(x,2|x|))} = c_1|x|^{-n}.$$

Hence, this implies that for $\lambda = c_1 r^{-n}/2$

$$r^a\lambda\mu(\{x \in \mathbf{R}^n \setminus B(0,r)\colon Mf(x) > \lambda\}) \geq r^a\lambda\mu(\{x \in \mathbf{R}^n\colon r \leq |x| < 2^{1/n}r\}) = Cr^a,$$

which tends to ∞ as $r \to \infty$, when $a > 0$.

We write $A(0,r) = B(0,2r)\setminus B(0,r)$ for $r > 0$. Let q' denote the Hölder conjugate of q, that is,

$$q' = \begin{cases} \infty & \text{when } q = 1, \\ q/(q-1) & \text{when } 1 < q < \infty, \\ 1 & \text{when } q = \infty. \end{cases}$$

For a proof of Theorem 3.3, we treat the Hardy type integral. Before doing so, it is convenient to note the following result due to Nakai [25, Lemma 7.1].

Lemma 3.7. *For $0 < q < \infty$ and a real a, consider*

($\mu qa;\infty$)**:** $\exists C > 0$ *s.t.* $\displaystyle\int_r^\infty (t^a\mu(B(0,t))^{-1})^q\frac{dt}{t} \leq C(r^a\mu(B(0,r))^{-1})^q$ *when* $r > 1$;

($\mu qa;0$)**:** $\exists C > 0$ *s.t.* $\displaystyle\int_1^r (t^{-a}\mu(B(0,t)))^q\frac{dt}{t} \leq C(r^{-a}\mu(B(0,r)))^q$ *when* $r > 1$.

Then

(1) (μ3) *implies* ($\mu q\varepsilon;\infty$) *for small* $\varepsilon > 0$;
(2) (μ4) *implies* ($\mu q(a+\varepsilon);0$) *for small* $\varepsilon > 0$.

Lemma 3.8. *For* $\beta \geq 0$, $1 < q < \infty$ *and* $a \geq 0$, *suppose* ($\mu 1\beta;\infty$) *and* ($\mu 1(a+\beta);0$) *hold. Then there exists a constant* $C > 0$ *such that*

$$\int_1^\infty \left(r^{-a-\beta}\mu(B(0,r)) \int_{X\setminus B(0,2r)} |f(y)||y|^\beta \mu(B(0,|y|))^{-1}\, d\mu(y)\right)^q \frac{dr}{r} \leq C$$

when $\|f\|_{M^{1,q,a}(X)} \leq 1$.

Proof. By Lemma 3.7, for $1 < q < \infty$ and $a \geq 0$, there exists $\varepsilon > 0$ such that

$$\left(\int_r^\infty \left(t^\beta \mu(B(0,t))^{-1}\right)^{(1-\varepsilon)q'} \frac{dt}{t}\right)^{1/q'} \leq C \left(r^\beta \mu(B(0,r))^{-1}\right)^{1-\varepsilon} \tag{3.1}$$

and

$$\int_1^r t^{-aq}\left(t^\beta \mu(B(0,t))^{-1}\right)^{-\varepsilon q} \frac{dt}{t} \leq C r^{-aq}\left(r^\beta \mu(B(0,r))^{-1}\right)^{-\varepsilon q} \tag{3.2}$$

when $r > 1$. Then we have by Fubini's theorem and Hölder's inequality

$$\begin{aligned}
J &= \int_1^\infty \left(r^{-a-\beta}\mu(B(0,r)) \int_{X\setminus B(0,2r)} |f(y)||y|^\beta \mu(B(0,|y|))^{-1}\, d\mu(y)\right)^q \frac{dr}{r} \\
&\leq C\int_1^\infty \left(r^{-a-\beta}\mu(B(0,r)) \int_{X\setminus B(0,2r)} |f(y)| \left(\int_{|y|/2}^{|y|} t^\beta \mu(B(0,t))^{-1}\frac{dt}{t}\right) d\mu(y)\right)^q \frac{dr}{r} \\
&= C\int_1^\infty \left(r^{-a-\beta}\mu(B(0,r)) \int_r^\infty t^\beta \mu(B(0,t))^{-1} \left(\int_{A(0,t)} |f(y)|\, d\mu(y)\right)\frac{dt}{t}\right)^q \frac{dr}{r} \\
&\leq C\int_1^\infty \left(r^{-a-\beta}\mu(B(0,r)) \left(\int_r^\infty \left(t^\beta \mu(B(0,t))^{-1}\right)^{(1-\varepsilon)q'} \frac{dt}{t}\right)^{1/q'} \right. \\
&\qquad \left. \cdot \left(\int_r^\infty \left(t^\beta \mu(B(0,t))^{-1}\right)^{\varepsilon q} \left(\int_{A(0,t)} |f(y)|\, d\mu(y)\right)^q \frac{dt}{t}\right)^{1/q}\right)^q \frac{dr}{r}.
\end{aligned}$$

From (3.1) we see that

$$\begin{aligned}
J &\leq C\int_1^\infty r^{-(a+\beta)q}\mu(B(0,r))^q \left(r^\beta \mu(B(0,r))^{-1}\right)^{(1-\varepsilon)q} \\
&\qquad \cdot \left(\int_r^\infty \left(t^\beta \mu(B(0,t))^{-1}\right)^{\varepsilon q} \left(\int_{A(0,t)} |f(y)|\, d\mu(y)\right)^q \frac{dt}{t}\right) \frac{dr}{r} \\
&= C\int_1^\infty \left(t^\beta \mu(B(0,t))^{-1}\right)^{\varepsilon q} \left(\int_{A(0,t)} |f(y)|\, d\mu(y)\right)^q \\
&\qquad \cdot \left(\int_1^t r^{-(a+\beta)q}\mu(B(0,r))^q \left(r^\beta \mu(B(0,r))^{-1}\right)^{(1-\varepsilon)q} \frac{dr}{r}\right)\frac{dt}{t} \\
&= C\int_1^\infty \left(t^\beta \mu(B(0,t))^{-1}\right)^{\varepsilon q} \left(\int_{A(0,t)} |f(y)|\, d\mu(y)\right)^q \\
&\qquad \cdot \left(\int_1^t r^{-aq}\left(r^\beta \mu(B(0,r))^{-1}\right)^{-\varepsilon q} \frac{dr}{r}\right)\frac{dt}{t},
\end{aligned}$$

so that (3.2) yields

$$J \le C \int_1^\infty \left(t^{-a} \int_{A(0,t)} |f(y)|\, d\mu(y) \right)^q \frac{dt}{t},$$

which proves the result. □

When $q = \infty$, Lemma 3.8 must be replaced by the following.

Lemma 3.9. *For $\beta \ge 0$ and $a \ge 0$, suppose $(\mu 1(a+\beta); \infty)$. Then there exists a constant $C > 0$ such that*

$$r^{-a-\beta} \mu(B(0,r)) \int_{X \setminus B(0,2r)} |f(y)||y|^\beta \mu(B(0,|y|))^{-1}\, d\mu(y) \le C$$

when $\|f\|_{M^{1,\infty,a}(X)} \le 1$.

Proof. As the proof of Lemma 3.8, we have

$$\begin{aligned} & r^{-a-\beta} \mu(B(0,r)) \int_{X \setminus B(0,2r)} |f(y)||y|^\beta \mu(B(0,|y|))^{-1}\, d\mu(y) \\ & \le C r^{-a-\beta} \mu(B(0,r)) \int_r^\infty t^\beta \mu(B(0,t))^{-1} \left(\int_{A(0,t)} |f(y)|\, d\mu(y) \right) \frac{dt}{t} \\ & \le C r^{-a-\beta} \mu(B(0,r)) \int_r^\infty t^{\beta+a} \mu(B(0,t))^{-1} \frac{dt}{t} \le C \end{aligned}$$

with the aid of $(\mu 1(a+\beta); \infty)$. □

Lemma 3.10. *Let $\beta \ge 0$. When $0 < q \le 1$ and $a \ge 0$, suppose $(\mu 1(a+\beta); 0)$. Then there exists a constant $C > 0$ such that*

$$\int_1^\infty \left(r^{-a-\beta} \mu(B(0,r)) \int_{X \setminus B(0,2r)} |f(y)||y|^\beta \mu(B(0,|y|))^{-1}\, d\mu(y) \right)^q \frac{dr}{r} \le C$$

when $\|f\|_{M^{1,q,a}(X)} \le 1$.

Proof. For $0 < q \le 1$ and $a \ge 0$, we have by (3.2) with $\varepsilon = 0$

$$\begin{aligned} & \int_1^\infty \left(r^{-a-\beta} \mu(B(0,r)) \int_{X \setminus B(0,2r)} |f(y)||y|^\beta \mu(B(0,|y|))^{-1}\, d\mu(y) \right)^q \frac{dr}{r} \\ & \le C \sum_{j=1}^\infty \left(2^{j(-a-\beta)} \mu(B(0,2^j)) \left(\sum_{k \ge j} \mu(B(0,2^k))^{-1} \int_{A(0,2^k)} |f(y)||y|^\beta\, d\mu(y) \right) \right)^q \\ & \le C \sum_{j=1}^\infty (2^{j(-a-\beta)} \mu(B(0,2^j)))^q \sum_{k \ge j} (2^{k\beta} \mu(B(0,2^k))^{-1})^q \left(\int_{A(0,2^k)} |f(y)|\, d\mu(y) \right)^q \\ & \le C \sum_{k=1}^\infty (2^{k\beta} \mu(B(0,2^k))^{-1})^q \left(\int_{A(0,2^k)} |f(y)|\, d\mu(y) \right)^q \sum_{j=1}^k (2^{j(-a-\beta)} \mu(B(0,2^j)))^q \\ & \le C \sum_{k=1}^\infty (2^{k\beta} \mu(B(0,2^k))^{-1})^q \left(\int_{A(0,2^k)} |f(y)|\, d\mu(y) \right)^q (2^{k(-a-\beta)} \mu(B(0,2^k)))^q \\ & \le C \sum_{k=1}^\infty \left(2^{-ak} \int_{B(0,2^k)} |f(y)|\, d\mu(y) \right)^q \le C, \end{aligned}$$

which proves the result. □

Proof of Theorem 3.3. We show only the case when $1 < q < \infty$, because the remaining case is similarly obtained; when $0 < q \le 1$ and $q = \infty$, we can use Lemma 3.10 and Lemma 3.9, respectively, instead of Lemma 3.8.

Let f be a measurable function on X such that $\|f\|_{M^{1,q,a}(X)} \le 1$. For $r > 1$, write $f = f\chi_{B(0,2r)} + f\chi_{X\setminus B(0,2r)} = f_1 + f_2$. Note here that if $x \in B(0,r)$ and $t \le r$, then $B(0,2r)^c \cap B(x,t) = \emptyset$. Hence,

$$\begin{aligned} Mf_2(x) &\le C \sup_{t>r} \frac{1}{\mu(B(0,t))} \int_{B(0,2t)\setminus B(0,2r)} |f(y)|\, d\mu(y) \\ &\le C \int_{X\setminus B(0,2r)} |f(y)|\mu(B(0,|y|))^{-1}\, d\mu(y) \end{aligned}$$

for $x \in B(0,r)$. Let $\lambda > 0$. Since $\{x \in B(0,r) \colon Mf(x) > \lambda\} \subset \{x \in B(0,r) \colon Mf_1(x) > \lambda/2\} \cup \{x \in B(0,r) \colon Mf_2(x) > \lambda/2\}$, we have

$$\begin{aligned} &\mu(\{x \in B(0,r) \colon Mf(x) > \lambda\}) \\ &\le \mu(\{x \in B(0,r) \colon Mf_1(x) > \lambda/2\}) + \mu(\{x \in B(0,r) \colon Mf_2(x) > \lambda/2\}) \\ &\le C\lambda^{-1} \int_{B(0,2r)} |f(y)|\, d\mu(y) + C\mu(B(0,r))\lambda^{-1} \int_{X\setminus B(0,2r)} |f(y)|\mu(B(0,|y|))^{-1} d\mu(y), \end{aligned}$$

so that

$$\begin{aligned} &r^{-a}\lambda\mu(\{x \in B(0,r) \colon Mf(x) > \lambda\}) \\ &\le Cr^{-a} \int_{B(0,2r)} |f(y)|\, d\mu(y) + Cr^{-a}\mu(B(0,r)) \int_{X\setminus B(0,2r)} |f(y)|\mu(B(0,|y|))^{-1}\, d\mu(y). \end{aligned}$$

Now we find from Lemma 3.8 with $\beta = 0$ that

$$\int_1^\infty \left(r^{-a}\mu(B(0,r)) \int_{X\setminus B(0,2r)} |f(y)|\mu(B(0,|y|))^{-1}\, d\mu(y) \right)^q \frac{dr}{r} \le C,$$

which is a consequence of the lemma. □

Proof of Theorem 3.5. We show only the case when $1 < q < \infty$, as the proof of Theorem 3.3. Let f be a measurable function on X such that $\|f\|_{N^{1,q,a}(X)} \le 1$. For $r > 1$, write $f = f\chi_{B(0,r/2)} + f\chi_{X\setminus B(0,r/2)} = f_1 + f_2$. Note here that

$$\begin{aligned} &Mf_1(x) \le C\mu(B(0,r))^{-1} \int_{B(0,r/2)} |f(y)|\, d\mu(y) \\ &\le C\mu(B(0,r))^{-1} \left(\int_{B(0,1)} |f(y)|\, d\mu(y) + \sum_{\{j:2^{-j}r>1\}} \int_{B(0,2^{-j}r)\setminus B(0,2^{-j-1}r)} |f(y)|\, d\mu(y) \right) \\ &\le C\mu(B(0,r))^{-1} \left(\int_{B(0,1)} |f(y)|\, d\mu(y) + \sum_{\{j:2^{-j}r>1\}} (2^{-j-1}r)^{-a} \right) \\ &\le C\mu(B(0,r))^{-1} \left(1 + \int_1^r t^{-a}\, \frac{dt}{t} \right) \le A \end{aligned}$$

for $x \in X \setminus B(0,r)$ and $r > 1$, with the aid of ($\mu 5$) when $a = 0$. Hence if $\lambda > 2A$, then

$$\begin{aligned}
&\mu(\{x \in X \setminus B(0,r)\colon Mf(x) > \lambda\}) \\
&\le \mu(\{x \in X \setminus B(0,r)\colon Mf_1(x) > \lambda/2\}) + \mu(\{x \in X \setminus B(0,r)\colon Mf_2(x) > \lambda/2\}) \\
&\le \mu(\{x \in X \setminus B(0,r)\colon Mf_2(x) > \lambda/2\}) \\
&\le C\lambda^{-1} \int_{X\setminus B(0,r/2)} |f(y)|\, d\mu(y).
\end{aligned}$$

Therefore

$$r^a \lambda \mu(\{x \in X \setminus B(0,r)\colon Mf(x) > \lambda\}) \le C r^a \int_{X\setminus B(0,r/2)} |f(y)|\, d\mu(y)$$

when $\|f\|_{N^{1,q,a}(X)} \le 1$. Thus this theorem is proved. $\square$

4. Riesz potentials

For $\alpha > 0$, we define the Riesz potential $I_\alpha f$ of order α of a measurable function f on X by

$$I_\alpha f(x) = \int_X \frac{|x-y|^\alpha}{\mu(B(x,|x-y|))} f(y)\, d\mu(y).$$

Lemma 4.1. [22, Theorem 1.1, Chap. 2] *If*

$$\int_X \frac{(1+|y|)^\alpha}{\mu(B(0,1+|y|))} |f(y)|\, d\mu(y) < \infty,$$

then the Riesz potential $I_\alpha f$ is finite a.e. and locally integrable on X.

Proof. For $R > 0$, write

$$\begin{aligned}
I_\alpha |f|(x) &= \int_{B(0,2R)} \frac{|x-y|^\alpha}{\mu(B(x,|x-y|))} |f(y)|\, d\mu(y) \\
&\quad + \int_{X\setminus B(0,2R)} \frac{|x-y|^\alpha}{\mu(B(x,|x-y|))} |f(y)|\, d\mu(y) \\
&= I_1(x) + I_2(x).
\end{aligned}$$

First, for $y \in B(0,2R)$, we see from ($\mu 1$) and ($\mu 2$) that

$$\begin{aligned}
\int_{B(0,R)} \frac{|x-y|^\alpha}{\mu(B(x,|x-y|))}\, d\mu(x) &\le \sum_{\{j\colon 2^{-j} < 3R\}} \int_{A(y,2^{-j})} \frac{|x-y|^\alpha}{\mu(B(y,|x-y|))}\, d\mu(x) \\
&\le C \sum_{\{j\colon 2^{-j} < 3R\}} \frac{2^{-j\alpha}}{\mu(B(y,2^{-j}))} \mu(B(y,2^{-j+1})) \\
&\le C R^\alpha,
\end{aligned}$$

so that

$$\begin{aligned}
\int_{B(0,R)} I_1(x)\, d\mu(x) &= \int_{B(0,2R)} |f(y)| \left(\int_{B(0,R)} \frac{|x-y|^\alpha}{\mu(B(x,|x-y|))}\, d\mu(x) \right) d\mu(y) \\
&\le C R^\alpha \int_{B(0,2R)} |f(y)|\, d\mu(y).
\end{aligned}$$

Moreover, if $x \in B(0,R)$, then

$$I_2(x) \le C \int_{X \setminus B(0,2R)} \frac{|y|^\alpha}{\mu(B(0,|y|))} |f(y)|\, d\mu(y) < \infty$$

since $|y|/2 \le |x-y| \le 3|y|/2$ and $B(0,|y|) \subset B(x, 5|x-y|)$. Thus the present lemma is proved. □

For $x \in X$ and $t > 0$, let

$$\mu^{-1}(x,t) = \sup_{\{r>0:\, \mu(B(x,r))<t\}} r.$$

Note that if $\mu(B(x,\cdot))$ is continuous on $(0,\infty)$ for all $x \in X$, then

$$\mu(B(x, \mu^{-1}(x,t))) = t.$$

Consider a function $\varphi(x,r)$ satisfying $(\varphi1)$–$(\varphi2)$ and the following conditions $(\varphi3)$–$(\varphi5)$:

$(\varphi3)$ there exists a constant $C_1 > 0$ such that

$$\varphi(x, r[\mu^{-1}(x,r^{-1})]^\alpha) \le C_1 r;$$

$(\varphi4)$ there exists $C_2 > 0$ such that

$$r \le C_2 \varphi(x, r[\mu^{-1}(x,r)]^{-\alpha});$$

$(\varphi5)$ there exists $C_3 > 0$ such that

$$\varphi(t)^{-1} \le C_3 \varphi(t^{-1}),$$

where

$$\varphi(t) = \inf_{x \in X} \varphi(x,t)$$

for $t \ge 0$.

Example 4.2. (1) $\varphi(r) = r^p(\log(e+r))^q$ with $p \ge 1$ satisfies $(\varphi5)$ when $q \ge 0$.

(2) $\varphi_c(r) = r^p(\log(c+r))^q$ with $p > 1$ is convex when c is sufficiently large, relatively to q, and $\varphi_c \sim \varphi_e$ when $c > e$.

(3) $\varphi(r) = r^{p_1} + r^{p_2}$ with $1 \le p_1 \le p_2$ satisfies $(\varphi5)$.

Example 4.3. Let $\mu(B(x,r)) = r^a(\log(e+r^{-1}))^b$ with $a > \alpha$ and $b \ge 0$. Then $\mu^{-1}(x,r) \sim r^{1/a}(\log(e+r^{-1}))^{-b/a}$. Considering a function

$$\varphi(r) = r^{a/(a-\alpha)}(\log(e+r))^{b\alpha/(a-\alpha)},$$

we find

$$\begin{aligned}\varphi(x, r[\mu^{-1}(x,r^{-1})]^\alpha) &\sim [r[\mu^{-1}(x,r^{-1})]^\alpha]^{a/(a-\alpha)}(\log(e+r))^{b\alpha/(a-\alpha)} \\ &\sim r(\log(e+r))^{-(b/a)\alpha a/(a-\alpha)}(\log(e+r))^{b\alpha/(a-\alpha)} = r,\end{aligned}$$

which shows $(\varphi3)$. Similarly,

$$\begin{aligned}\varphi(x, r[\mu^{-1}(x,r)]^{-\alpha}) &\sim [r[\mu^{-1}(x,r)]^{-\alpha}]^{a/(a-\alpha)}(\log(e+r))^{b\alpha/(a-\alpha)} \\ &\sim r(\log(e+r^{-1}))^{b\alpha/(a-\alpha)}(\log(e+r))^{b\alpha/(a-\alpha)},\end{aligned}$$

which shows $(\varphi4)$.

Lemma 4.4. *Assume that φ satisfies $(\varphi1)$–$(\varphi5)$. Suppose*

$(\mu6)$ *$r^\alpha(\mu(B(x,r)))^{-1}$ is uniformly almost decreasing on $(1,\infty)$ for all $x \in X$.*

Then the inequality

$$\varphi^{-1}(\mu(\{x \in X\colon I_\alpha|f|(x) > \lambda\})) \le C\lambda^{-1}\int_X |f(y)|\,d\mu(y)$$

holds for all measurable functions f on X and $\lambda > 0$ with a constant $C > 0$ independent of f and λ.

Proof. Let f be a nonnegative measurable function on X such that

$$\int_X f(y)\,d\mu(y) = 1.$$

If $x \in X$ and $t > 0$, then we have

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,t)} \frac{|x-y|^\alpha}{\mu(B(x,|x-y|))} f(y)\,d\mu(y) + \int_{X\setminus B(x,t)} \frac{|x-y|^\alpha}{\mu(B(x,|x-y|))} f(y)\,d\mu(y) \\ &\le Ct^\alpha Mf(x) + C\frac{t^\alpha}{\mu(B(x,t))}\int_{X\setminus B(x,t)} f(y)\,d\mu(y) \\ &\le Ct^\alpha Mf(x) + C\frac{t^\alpha}{\mu(B(x,t))}, \end{aligned}$$

with the aid of (μ6). Here, taking $t = \mu^{-1}(x, 1/Mf(x))$, we find

$$I_\alpha f(x) \le CMf(x)[\mu^{-1}(B(x,1/Mf(x)))]^\alpha,$$

so that by (φ2) and (φ3)

$$\varphi(I_\alpha f(x)) \le \varphi(x, CMf(x)\ [\mu^{-1}(B(x,1/Mf(x)))]^\alpha) \le CMf(x).$$

Therefore we obtain for $\lambda > 0$,

$$\begin{aligned} \mu(\{x \in X\colon I_\alpha f(x) > \lambda/2\}) &\le \mu(\{x \in X\colon Mf(x) > C\varphi(\lambda/2)\}) \\ &\le C\varphi(\lambda/2)^{-1}\int_X f(y)\,d\mu(y) = C\varphi(\lambda/2)^{-1}, \end{aligned}$$

which gives by (φ5)

$$\varphi^{-1}(\mu(\{x \in X\colon I_\alpha f(x) > \lambda/2\})) \le C\lambda^{-1},$$

as required. □

Remark 4.5. Note that ($\mu1\alpha;\infty$) implies (μ6).

Theorem 4.6. *For $a \ge 0$, suppose ($\mu1\alpha;\infty$) and ($\mu1(a+\alpha);0$) hold. Then the Riesz potential operator $I_\alpha\colon f \longrightarrow I_\alpha f$ is bounded from $M^{1,q,a}(X)$ to $WM^{\varphi,q,a}(X)$.*

Proof. Suppose $1 < q < \infty$, as before. Let f be a nonnegative measurable function on X such that $\|f\|_{M^{1,q,a}(X)} \le 1$. For $r > 1$, write $f = f\chi_{B(0,2r)} + f\chi_{X\setminus B(0,2r)} = f_1 + f_2$.

By Lemma 4.4, we have

$$\varphi^{-1}(\mu(\{x \in B(0,r)\colon I_\alpha f_1(x) > \lambda/2\})) \le C\lambda^{-1}\int_{B(0,2r)} f(y)\,d\mu(y),$$

which gives

$$\left(\int_1^\infty \left(r^{-a}\lambda\varphi^{-1}(\mu(\{x\in B(0,r)\colon I_\alpha f_1(x)>\lambda/2\}))\right)^q \frac{dr}{r}\right)^{1/q}$$
$$\le C\left(\int_1^\infty \left(r^{-a}\int_{B(0,2r)} f(y)\,d\mu(y)\right)^q \frac{dr}{r}\right)^{1/q}.$$

Further, for $x\in B(0,r)$

$$I_\alpha f_2(x)\le C\int_{X\setminus B(0,2r)} \frac{|y|^\alpha}{\mu(B(0,|y|))} f(y)\,d\mu(y),$$

Therefore, we obtain

$$\begin{aligned}&\varphi^{-1}(\mu(\{x\in B(0,r)\colon I_\alpha f_2(x)>\lambda/2\}))\\ &\le C\lambda^{-1}\varphi^{-1}(\mu(B(0,r)))\int_{X\setminus B(0,2r)} \frac{|y|^\alpha}{\mu(B(0,|y|))} f(y)\,d\mu(y)\\ &\le C\lambda^{-1}r^{-\alpha}\mu(B(0,r))\int_{X\setminus B(0,2r)} \frac{|y|^\alpha}{\mu(B(0,|y|))} f(y)\,d\mu(y)\end{aligned}$$

since $\varphi^{-1}(x,r)\le Cr[\mu^{-1}(x,r)]^{-\alpha}$ by $(\varphi 4)$. Hence Lemma 3.8 yields

$$\begin{aligned}&\left(\int_1^\infty \left(r^{-a}\lambda\varphi^{-1}(\mu(\{x\in B(0,r)\colon I_\alpha f_2(x)>\lambda/2\}))\right)^q \frac{dr}{r}\right)^{1/q}\\ &\le C\left(\int_1^\infty \left(r^{-a-\alpha}\mu(B(0,r))\int_{X\setminus B(0,2r)} \frac{|y|^\alpha}{\mu(B(0,|y|))} f(y)\,d\mu(y)\right)^q \frac{dr}{r}\right)^{1/q}\\ &\le C\left(\int_1^\infty \left(r^{-a}\int_{B(0,r)} f(y)\,d\mu(y)\right)^q \frac{dr}{r}\right)^{1/q}.\end{aligned}$$

Thus, Theorem 4.6 is proved. □

Corollary 4.7. *Let $a\ge 0$ and $0<q\le\infty$. Let μ be the Lebesgue measure on $\mathbf{R}^n$. If $n-\alpha>a$, then the Riesz potential operator I_α is bounded from $M^{1,q,a}(\mathbf{R}^n)$ to $WM^{1^*,q,a}(\mathbf{R}^n)$.*

In connection with [19, Remark 3.7], we show the following result.

Corollary 4.8. *Let $a\ge 0$ and $0<q\le\infty$. Let μ be the Lebesgue measure on $\mathbf{R}^n$. If $0<a<n/1^*$, $a_1>a+n(1/p-1/1^*)$ $(>a1^*/p)$ and $0<q<q_1<\infty$, then the Riesz potential operator I_α is bounded from $M^{1,q,a}(\mathbf{R}^n)$ to $M^{p,q_1,a_1}(\mathbf{R}^n)$ when $1<p<1^*$.*

Proof. Suppose $1<q<\infty$, as before. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{M^{1,q,a}(\mathbf{R}^n)}\le 1$. For $r>1$, write $f=f\chi_{B(0,2r)}+f\chi_{\mathbf{R}^n\setminus B(0,2r)}=f_1+f_2$.

By Lemma 4.4, we have

$$\mu(\{x\in B(0,r)\colon I_\alpha f_1(x)>\lambda/2\})\le C\left(\lambda^{-1}\int_{B(0,2r)} f(y)\,dy\right)^{1^*}.$$

If $1 < p_1 < p < 1^*$, then

$$\begin{aligned}\int_{B(0,r)} (I_\alpha f_1(x))^p\,dx &\le \int_0^\infty \mu(\{x \in B(0,r)\colon I_\alpha f_1(x) > \lambda\})\,d\lambda^p \\ &\le C\int_0^1 \left(\int_{B(0,r)} \left(\frac{I_\alpha f_1(x)}{\lambda}\right)^{p_1} dx\right) d\lambda^p \\ &\quad + C\int_1^\infty \left(\lambda^{-1}\int_{B(0,2r)} f(y)\,dx\right)^{1^*} d\lambda^p.\end{aligned}$$

Here note from Minkowski's inequality that

$$\begin{aligned}\left(\int_{B(0,r)} (I_\alpha f_1(x))^{p_1}\,dx\right)^{1/p_1} &\le \int f_1(y)\left(\int_{B(0,r)} |x-y|^{(\alpha-n)p_1}\,dx\right)^{1/p_1} dy \\ &\le Cr^{\alpha-n+n/p_1}\int f_1(y)\,dy\end{aligned}$$

so that

$$\begin{aligned}&\left(\int_{B(0,r)} (I_\alpha f_1(x))^p\,dx\right)^{1/p} \\ &\le C\left(r^{\alpha-n+n/p_1}\int_{B(0,2r)} f(y)\,dy\right)^{p_1/p} + C\left(\int_{B(0,2r)} f(y)\,dy\right)^{1^*/p}.\end{aligned}$$

Hence

$$\begin{aligned}&\left(\int_1^\infty \left(r^{-a_1}\|I_\alpha f_1\|_{L^p(B(0,r))}\right)^{q_1}\frac{dr}{r}\right)^{1/q_1} \\ &\le C\left(\int_1^\infty \left(r^{-a_1p/p_1+\alpha-n+n/p_1}\int_{B(0,2r)} f(y)\,dy\right)^{p_1q_1/p}\frac{dr}{r}\right)^{1/q_1} \\ &\quad + C\left(\int_1^\infty \left(r^{-a_1p/1^*}\int_{B(0,2r)} f(y)\,dy\right)^{q_1 1^*/p}\frac{dr}{r}\right)^{1/q_1} \\ &= CI_1 + CJ_1.\end{aligned}$$

We have by Hardy's inequality (cf. Stein [26, Appendices B.3])

$$\begin{aligned}I_1^{p/p_1} &\le C\left(\int_1^\infty \left(r^{-a_1p/p_1+\alpha-n+n/p_1}\int_{B(0,2r)} f(y)\,dy\right)^q\frac{dr}{r}\right)^{1/q} \\ &\le C\left(\int_1^\infty \left(r^{-a}\int_{B(0,2r)} f(y)\,dy\right)^q\frac{dr}{r}\right)^{1/q} \le C\end{aligned}$$

when $q_1p_1/p > q$ and $-a_1p/p_1 + n/p_1 - n/1^* < -a$. Similarly,

$$\begin{aligned}J_1^{p/1^*} &\le C\left(\int_1^\infty \left(r^{-a_1p/1^*}\int_{B(0,2r)} f(y)\,dy\right)^q\frac{dr}{r}\right)^{1/q} \\ &\le C\left(\int_1^\infty \left(r^{-a}\int_{B(0,2r)} f(y)\,dy\right)^q\frac{dr}{r}\right)^{1/q} \le C\end{aligned}$$

when $1^*q_1/p > q$ and $-a_1p/1^* < -a$.

Further, for $x \in B(0,r)$

$$I_\alpha f_2(x) \le C \int_{\mathbf{R}^n \setminus B(0,2r)} \frac{|y|^\alpha}{\mu(B(0,|y|))} f(y)dy.$$

Therefore, if

$$\lambda/2 < C \int_{\mathbf{R}^n \setminus B(0,2r)} \frac{|y|^\alpha}{\mu(B(0,|y|))} f(y)dy,$$

then

$$\begin{aligned}
&\mu(\{x \in B(0,r) \colon I_\alpha f_2(x) > \lambda/2\}) \\
&\le C\mu(B(0,r)) \left(\lambda^{-1} \int_{\mathbf{R}^n \setminus B(0,2r)} \frac{|y|^\alpha}{\mu(B(0,|y|))} f(y)\,dy \right)^{1^*},
\end{aligned}$$

since t^{1^*} is an increasing function. If $1 < p_1 < p < 1^*$, then

$$\begin{aligned}
\int_{B(0,r)} (I_\alpha f_2(x))^p dx &= \int_0^\infty \mu(\{x \in B(0,r) \colon I_\alpha f_2(x) > \lambda\}) d\lambda^p \\
&\le C \int_0^1 \left(\int_{B(0,r)} \left(\frac{I_\alpha f_2(x)}{\lambda} \right)^{p_1} dx \right) d\lambda^p \\
&\quad + Cr^n \int_1^\infty \left(\lambda^{-1} \int_{\mathbf{R}^n \setminus B(0,2r)} |y|^{\alpha-n} f(y)\,dy \right)^{1^*} d\lambda^p \\
&\le C \int_{B(0,r)} (I_\alpha f_2(x))^{p_1}\, dx \\
&\quad + Cr^n \left(\int_{\mathbf{R}^n \setminus B(0,2r)} |y|^{\alpha-n} f(y)\,dy \right)^{1^*}.
\end{aligned}$$

Here note that

$$\begin{aligned}
\left(\int_{B(0,r)} (I_\alpha f_2(x))^{p_1}\, dx \right)^{1/p_1} &\le Cr^{n/p_1} \int |y|^{\alpha-n} f_2(y)\,dy \\
&\le Cr^{n/p_1} \int \left(\int_{|y|/2}^{|y|} t^{\alpha-n} \frac{dt}{t} \right) f_2(y)\,dy \\
&\le Cr^{n/p_1} \int_r^\infty t^{\alpha-n} \left(\int_{A(t)} f_2(y)\,dy \right) \frac{dt}{t} \\
&\le Cr^{\varepsilon+\alpha-n+n/p_1} \left(\int_r^\infty \left(t^{-\varepsilon} \int_{A(t)} f_2(y)\,dy \right)^q \frac{dt}{t} \right)^{1/q}
\end{aligned}$$

for $\alpha - n + \varepsilon = \varepsilon - n/1^* < 0$, so that

$$\begin{aligned}
&\left(\int_{B(0,r)} (I_\alpha f_2(x))^p\, dx \right)^{1/p} \\
&\le Cr^{(\varepsilon+\alpha-n+n/p_1)p_1/p} \left(\int_r^\infty \left(t^{-\varepsilon} \int_{A(t)} f_2(y)\,dy \right)^q \frac{dt}{t} \right)^{p_1/(pq)} \\
&\quad + Cr^{n/p} \left(r^{(\varepsilon+n-\alpha)q} \int_r^\infty \left(t^{-\varepsilon} \int_{A(t)} f_2(y)\,dy \right)^q \frac{dt}{t} \right)^{1^*/(pq)}.
\end{aligned}$$

Hence

$$
\begin{aligned}
&\left(\int_1^\infty \left(r^{-a_1}\|I_\alpha f_2\|_{L^p(B(0,r))}\right)^{q_1} \frac{dr}{r}\right)^{1/q_1}\\
&\le C\left(\int_1^\infty\left(r^{(-a_1+\varepsilon p_1/p+n/p-(n-\alpha)p_1/p)pq/p_1}\int_r^\infty\left(t^{-\varepsilon}\int_{A(t)} f_2(y)\,dy\right)^q\frac{dt}{t}\right)^{p_1q_1/(pq)}\frac{dr}{r}\right)^{1/q_1}\\
&\quad + C\left(\int_1^\infty\left(r^{(-a_1+\varepsilon 1^*/p+n/p-(n-\alpha)1^*/p)pq/1^*}\int_r^\infty\left(t^{-\varepsilon}\int_{A(t)} f_2(y)\,dy\right)^q\frac{dt}{t}\right)^{1^*q_1/(pq)}\frac{dr}{r}\right)^{1/q_1}\\
&= CI_2 + CJ_2.
\end{aligned}
$$

We have by Hardy's inequality and Fubini's theorem

$$
\begin{aligned}
&I_2^{pq/p_1}\\
&\le \left(\int_1^\infty\left(r^{(-a_1+\varepsilon p_1/p+(n-\alpha)(1-p_1/p))pq/p_1}\int_r^\infty\left(t^{-\varepsilon}\int_{A(t)} f_2(y)\,dy\right)^q\frac{dt}{t}\right)^{q_1p_1/(pq)}\frac{dr}{r}\right)^{pq/(q_1p_1)}\\
&\le C\int_1^\infty \left(r^{(-a_1+\varepsilon p_1/p+n/p-(n-\alpha)p_1/p)pq/p_1}\left(t^{-\varepsilon}\int_{A(t)} f_2(y)\,dy\right)^q\frac{dt}{t}\right)\frac{dr}{r}\\
&\le C\int_1^\infty\left(\int_0^t r^{(-a+\varepsilon)q}\frac{dr}{r}\right)\left(t^{-\varepsilon}\int_{A(t)} f_2(y)\,dy\right)^q\frac{dt}{t}\\
&\le C\int_1^\infty\left(t^{-a}\int_{A(t)} f(y)\,dy\right)^q\frac{dt}{t}\le C
\end{aligned}
$$

when $q_1p_1/(pq) > 1$, $(-a_1 + \varepsilon p_1/p + n/p - (n-\alpha)p_1/p)p/p_1 < -a + \varepsilon$ and $0 < a < \varepsilon < n/1^*$. Similarly,

$$
\begin{aligned}
&J_2^{pq/1^*}\\
&\le \left(\int_1^\infty\left(r^{(-a_1+\varepsilon 1^*/p+n/p-(n-\alpha)1^*/p)pq/1^*}\int_r^\infty\left(t^{-\varepsilon}\int_{A(t)} f_2(y)\,dy\right)^q\frac{dt}{t}\right)^{1^*q_1/(pq)}\frac{dr}{r}\right)^{pq/(1^*q_1)}\\
&\le C\left(\int_1^\infty\left(r^{(-a_1+\varepsilon 1^*/p+n/p-(n-\alpha)1^*/p)pq/1^*}\int_r^\infty\left(t^{-\varepsilon}\int_{A(t)} f_2(y)dy\right)^q\frac{dt}{t}\right)\frac{dr}{r}\right)\\
&\le C\int_1^\infty\left(t^{-a}\int_{A(t)} f(y)dy\right)^q\frac{dt}{t}\le C
\end{aligned}
$$

when $1^*q_1/(pq) > 1$, $(-a_1 + \varepsilon 1^*/p + n/p - (n-\alpha)1^*/p)p/1^* < -a + \varepsilon$ and $0 < a < \varepsilon < n/1^*$. □

Lemma 4.9. *Suppose* $(\mu 1(a+\alpha);\infty)$. *If* $\mu(E) = \mu(B(x,r))$ *for* $x \in X$ *and* $r > 1$, *then*

$$
\int_E \frac{|x-y|^\alpha}{\mu(B(x,|x-y|))}\,d\mu(y) \le Cr^\alpha.
$$

Proof. By Remark 4.5, (μ1) and (μ2) , we have

$$\begin{aligned}
&\int_E \frac{|x-y|^\alpha}{\mu(B(x,|x-y|))}\, d\mu(y) \\
&= \int_{E\cap B(x,r)} \frac{|x-y|^\alpha}{\mu(B(x,|x-y|))}\, d\mu(y) + \int_{E\setminus B(x,r)} \frac{|x-y|^\alpha}{\mu(B(x,|x-y|))}\, d\mu(y) \\
&\le \int_{B(x,r)} \frac{|x-y|^\alpha}{\mu(B(x,|x-y|))}\, d\mu(y) + C\frac{r^\alpha}{\mu(B(x,r))}\int_{E\setminus B(x,r)} d\mu(y) \\
&\le Cr^\alpha + C\frac{r^\alpha}{\mu(B(x,r))}\mu(E) \le Cr^\alpha,
\end{aligned}$$

as required. □

Theorem 4.10. *Let $a \ge 0$. Suppose μ satisfies*

(1) (μ6) *when* $0 < q < \infty$;
(2) (μ6) *and*

(μ7) $\sup_{x\in\mathbf{R}^n, r>1} r^\alpha(\log r)(\mu(B(x,r)))^{-1} \le C$ *when* $q = \infty$.

If $q < \infty$, then there exist constants $A, C > 0$ such that

$$\sup_{\lambda>A}\int_1^\infty \left(r^a\lambda\varphi^{-1}(\mu(\{X\setminus B(0,r)\colon I_\alpha f(x) > \lambda\}))\right)^q \frac{dr}{r} \le C$$

when $\|f\|_{N^{1,q,a}(X)} \le 1$; if $q = \infty$, then there exist constants $A, C > 0$ such that

$$\sup_{\lambda>A, r>1} r^a\lambda\varphi^{-1}(\mu(\{X\setminus B(0,r)\colon I_\alpha f(x) > \lambda\})) \le C$$

when $\|f\|_{N^{1,q,a}(X)} \le 1$.

Proof. Suppose $1 < q < \infty$, as before. Let f be a nonnegative measurable function on X such that $\|f\|_{N^{1,q,a}(X)} \le 1$. For $r \ge 1$, write $f = f\chi_{B(0,r/2)} + f\chi_{X\setminus B(0,r/2)} = f_1 + f_2$.

If $x \in X\setminus B(0,r)$, then we have by (μ6)

$$\begin{aligned}
I_\alpha f_1(x) &\le C\frac{|x|^\alpha}{\mu(B(0,|x|))}\int_{B(0,r/2)} f(y)\, d\mu(y) \\
&\le C\frac{r^\alpha}{\mu(B(0,r))}\left(1 + \int_1^r t^{-a}\,\frac{dt}{t}\right) \le A,
\end{aligned}$$

since $\|f\|_{N^{1,q,a}(X)} \le 1$.

Set $E = \{x \in X\setminus B(0,r)\colon I_\alpha f(x) > \lambda\}$. If $\lambda > 2A$, then Lemma 4.4 gives

$$\begin{aligned}
\varphi^{-1}(E) &\le \varphi^{-1}(\mu(\{x\in X\setminus B(0,r)\colon I_\alpha f_2(x)) > \lambda/2\})) \\
&\le C\lambda^{-1}\int_X f_2(y)\, d\mu(y) \\
&= C\lambda^{-1}\int_{X\setminus B(0,r/2)} f(y)\, d\mu(y).
\end{aligned}$$

Hence

$$\left(\int_1^\infty \left(r^a\lambda\varphi^{-1}(\mu(\{x\in X\setminus B(0,r)\colon I_\alpha f_2(x)>\lambda/2\}))\right)^q\frac{dr}{r}\right)^{1/q}$$
$$\le C\left(\int_1^\infty\left(r^a\int_{X\setminus B(0,r/2)}f(y)\,d\mu(y)\right)^q\frac{dr}{r}\right)^{1/q},$$

which completes the proof. □

Corollary 4.11. *Let $a\ge 0$ and $0<q\le\infty$. Let μ be the Lebesgue measure on $\mathbf{R}^n$. If $n-\alpha>a$, then there exist constants $A,C>0$ such that*

$$\sup_{\lambda>A}\int_1^\infty\left(r^{-a}\lambda[\mu(\{\mathbf{R}^n\setminus B(0,r)\colon I_\alpha f(x)>\lambda\})]^{1/1^*}\right)^q\frac{dr}{r}\le C$$

when $\|f\|_{N^{1,q,a}(\mathbf{R}^n)}\le 1$.

5. The associate space of $M^{1,q,a}(X)$

Following Gogatishvili and Mustafayev [13, 14], we study the duality properties of $M^{1,q,a}(X)$ and $N^{\infty,q',a}(X)$ when $a\ge 0$; see also [23] and [24].

Let Y be a family of measurable functions on X with a norm $\|\cdot\|_Y$. Then the associate space Y' of Y is defined as the family of all measurable functions f on X such that

$$\|f\|_{Y'}=\sup_{g\in Y\colon \|g\|_Y\le 1}\int_X|f(x)g(x)|\,d\mu(x)<\infty.$$

Here we prove the following result.

Theorem 5.1. *For $a>0$ and $1\le q\le\infty$,*

$$\left(M^{1,q,a}(X)\right)'=N^{\infty,q',a}(X).$$

The proof will be done in a way similar to those of A. Gogatishvili and R. Ch. Mustafayev [13, 14]; but, in our case, we recall that

$$N^{\infty,q',a}(X)\subset L^\infty(X)\qquad\text{and}\qquad M^{1,q,a}(X)\supset L^1(X).$$

For a proof of Theorem 5.1, following [23] and [24], it is sufficient to prepare the following two lemmas.

Lemma 5.2. *Let $a>0$ and $1\le q\le\infty$. Then*

$$\int_X|f(x)g(x)|d\mu(x)\le C\|f\|_{M^{1,q,a}(X)}\|g\|_{N^{\infty,q',a}(X)}.$$

Proof. Let $1<q<\infty$, as before. Let f and g be measurable functions on X such that $\|f\|_{M^{1,q,a}(X)}\le 1$ and $\|g\|_{N^{\infty,q',a}(X)}\le 1$. We have by Fubini's theorem for

$\beta > 0$

$$\begin{aligned}
&\int_X |f(x)g(x)|\,d\mu(x)\\
&= \int_{B(0,1)} |f(x)g(x)|\,d\mu(x) + \int_{X\setminus B(0,1)} |f(x)g(x)| \left(\beta|x|^{-\beta}\int_0^{|x|} r^{\beta}\frac{dr}{r}\right) d\mu(x)\\
&= \int_{B(0,1)} |f(x)g(x)|\,d\mu(x) + \int_{X\setminus B(0,1)} |f(x)g(x)||x|^{-\beta}\,d\mu(x)\\
&\quad + \beta\int_1^{\infty}\left(\int_{X\setminus B(0,r)} |f(x)g(x)||x|^{-\beta} d\mu(x)\right) r^{\beta}\frac{dr}{r}.
\end{aligned}$$

Since

$$\int_1^{\infty}\left(\int_{X\setminus B(0,r)} |f(x)g(x)||x|^{-\beta}\,d\mu(x)\right) r^{\beta}\frac{dr}{r} \ge \int_{X\setminus B(0,1)} |f(x)g(x)||x|^{-\beta}\,d\mu(x),$$

we have

$$\begin{aligned}
\int_X |f(x)g(x)|\,d\mu(x) &\le \int_{B(0,1)} |f(x)g(x)|\,d\mu(x)\\
&\quad + C\int_1^{\infty}\left(\int_{X\setminus B(0,r)} |f(x)g(x)||x|^{-\beta}\,d\mu(x)\right) r^{\beta}\frac{dr}{r}\\
&= I + CJ.
\end{aligned}$$

Then we obtain by Hölder's inequality

$$I \le \|f\|_{L^1(B(0,1))}\|g\|_{L^{\infty}(B(0,1))} \le 1$$

and for $\beta > a > 0$

$$\begin{aligned}
J &\le \int_1^{\infty}\left(\|g\|_{L^{\infty}(X\setminus B(0,r))}\int_{X\setminus B(0,r)} |f(x)||x|^{-\beta}\,d\mu(x)\right) r^{\beta}\frac{dr}{r}\\
&\le \left(\int_1^{\infty}\left(\|g\|_{L^{\infty}(X\setminus B(0,r))}r^{a}\right)^{q'}\frac{dr}{r}\right)^{1/q'}\\
&\quad \cdot\left(\int_1^{\infty}\left(\int_{X\setminus B(0,r)} |f(x)||x|^{-\beta}\,d\mu(x)\right)^{q} r^{q(\beta-a)}\frac{dr}{r}\right)^{1/q}\\
&\le \left(\int_1^{\infty}\left(\int_{X\setminus B(0,r)} |f(x)||x|^{-\beta}\,d\mu(x)\right)^{q} r^{q(\beta-a)}\frac{dr}{r}\right)^{1/q}.
\end{aligned}$$

Here note from Fubini's theorem and Hölder's inequality that

$$\begin{aligned}
&\int_{X\setminus B(0,r)} |f(x)||x|^{-\beta}\,d\mu(x) = \beta\int_{X\setminus B(0,r)} |f(x)|\left(\int_{|x|}^{\infty} t^{-\beta}\frac{dt}{t}\right) d\mu(x)\\
&\le \beta\int_r^{\infty}\left(\int_{B(0,t)\setminus B(0,r)} |f(x)|\,d\mu(x)\right) t^{-\beta}\frac{dt}{t}\\
&\le \beta\left(\int_r^{\infty}\left(\int_{B(0,t)\setminus B(0,r)} |f(x)|\,dx\right)^{q} t^{(-\beta+\gamma)q}\frac{dt}{t}\right)^{1/q}\left(\int_r^{\infty} t^{-\gamma q'}\frac{dt}{t}\right)^{1/q'}\\
&\le C\left(\int_r^{\infty}\left(\int_{B(0,t)\setminus B(0,r)} |f(x)|\,d\mu(x)\right)^{q} t^{q(-\beta+\gamma)}\frac{dt}{t}\right)^{1/q} r^{-\gamma}
\end{aligned}$$

for $0 < \gamma < \beta - a$. Hence by Fubini's theorem

$$J \le C \left(\int_1^\infty \left(\int_r^\infty \left(\int_{B(0,t)\setminus B(0,r)} |f(x)|\, d\mu(x) \right)^q t^{q(-\beta+\gamma)} \frac{dt}{t} \right) r^{-\gamma q} r^{q(\beta-a)} \frac{dr}{r} \right)^{1/q}$$

$$\le C \left(\int_1^\infty \left(\int_{B(0,t)} |f(x)|\, d\mu(x) \right)^q t^{q(-\beta+\gamma)} \left(\int_0^t r^{-\gamma q} r^{q(\beta-a)} \frac{dr}{r} \right) \frac{dt}{t} \right)^{1/q}$$

$$\le C \left(\int_1^\infty \left(\int_{B(0,t)} |f(x)|\, d\mu(x) \right)^q t^{-aq} \frac{dt}{t} \right)^{1/q} \le C,$$

as required. □

Lemma 5.3. *For $a > 0$ and $1 \le q \le \infty$, set $Z = M^{1,q,a}(X)$. Then*

$$\|g\|_{N^{\infty,q',a}(X)} \le C \sup_{\{f\in Z:\, \|f\|_Z \le 1\}} \int_X |f(x)g(x)|\, d\mu(x).$$

Proof. Let g be a measurable function on X such that

$$\sup_{\{f\in Z:\, \|f\|_Z \le 1\}} \int_X |f(x)g(x)|\, d\mu(x) < \infty.$$

Let B be a ball of X. Then $f = f_1\chi_B \in Z$ for $f_1 \in L^1(X)$, so that

$$\int_B |f_1(x)g(x)|\, d\mu(x) < \infty,$$

which implies that $g \in L^\infty(B)$. Set

$$a_j = 2^{ja}\|g\|_{L^\infty(A_j)} \quad \text{and} \quad G_N = \left(\sum_{j<N} a_j^{q'} \right)^{1/q},$$

where $A_0 = B(0,1)$ and $A_j = B(0,2^j) \setminus B(0,2^{j-1})$ for $j \ge 1$. Consider

$$f = G_N^{-1} \sum_{j<N} 2^{ja} a_j^{q'-1} \chi_{B(x_j,r_j)}/\mu(B(x_j,r_j))$$

where $B(x_j,r_j) \subset A_j$. Then

$$\|f\|_Z^q \le C \sum_{j<N} \left(2^{-ja}\|f\|_{L^1(A_j)}\right)^q \le C G_N^{-q} \sum_{j<N} \left(a_j^{q'-1}\right)^q = C$$

and

$$\int_X |f(x)g(x)|\, d\mu(x) \ge C G_N^{-1} \sum_{j<N} 2^{ja} a_j^{q'-1} \left(\frac{1}{\mu(B(x_j,r_j))} \int_{B(x_j,r_j)} |g(x)|\, d\mu(x) \right),$$

which together with the Lebesgue density theorem gives

$$\int_X |f(x)g(x)|\, d\mu(x) \ge C G_N^{-1} \sum_{j<N} 2^{ja} a_j^{q'-1} \|g\|_{L^\infty(A_j)} = C G_N^{-1} \sum_{j<N} a_j^{q'}$$

$$= C \left(\sum_{j<N} a_j^{q'} \right)^{1/q'} = C\|g\|_{N^{\infty,q',a}(X)}.$$

This completes the proof. □

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