Research Article

Josef Diblík*

# Bounded solutions to systems of fractional discrete equations

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**Abstract:** The article is concerned with systems of fractional discrete equations

$$\Delta^\alpha x(n+1) = F_n(n, x(n), x(n-1), \dots, x(n_0)), \quad n = n_0, n_0+1, \dots,$$

where $n_0 \in \mathbb{Z}$, $n$ is an independent variable, $\Delta^\alpha$ is an $\alpha$-order fractional difference, $\alpha \in \mathbb{R}$, $F_n : \{n\} \times \mathbb{R}^{n-n_0+1} \to \mathbb{R}^s$, $s \geqslant 1$ is a fixed integer, and $x : \{n_0, n_0+1, \dots\} \to \mathbb{R}^s$ is a dependent (unknown) variable. A retract principle is used to prove the existence of solutions with graphs remaining in a given domain for every $n \geqslant n_0$, which then serves as a basis for further proving the existence of bounded solutions to a linear nonhomogeneous system of discrete equations

$$\Delta^\alpha x(n+1) = A(n)x(n) + \delta(n), \quad n = n_0, n_0+1, \dots,$$

where $A(n)$ is a square matrix and $\delta(n)$ is a vector function. Illustrative examples accompany the statements derived, possible generalizations are discussed, and open problems for future research are formulated as well.

**Keywords:** fractional discrete difference, asymptotic behavior, system of fractional discrete equations, estimates of solutions

**MSC 2020:** 39A05, 39A06, 39A22

## 1 Introduction

The study of fractional discrete equations is intensive from various points of view. Many new results have been published on the properties of their solutions and applications in various fields. We refer, at least, to [1,7,17] dealing with the stability and decay rate of stable solutions, [2,6,13,15,16] considering controllability problems, [11,12] modeling electrical circuits by fractional equations, and to references therein. For completeness, rudiments of fractional differential equations can be found, e.g., in [14,18].

Let $n_0$ be an integer, $\mathbb{N}(n_0) := \{n_0, n_0+1, \dots\}$, and let $x = (x_1, \dots, x_s)^T$, $x : \mathbb{N}(n_0) \to \mathbb{R}^s$ be a given function, where $s$ is a fixed positive integer. Let $\alpha \in \mathbb{R}$. The concept of an $\alpha$-order fractional difference $\Delta^\alpha x(n)$ of a function $x$, used in the article, is defined (we refer, e.g., to [6,12,15]) by the formula

$$\Delta^\alpha x(n) := \sum_{j=0}^{n-n_0} C_\alpha(j)x(n-j), \quad n \in \mathbb{N}(n_0), \tag{1}$$

where

$$C_\alpha(0) := 1, \quad C_\alpha(j) := (-1)^j \binom{\alpha}{j}, \quad \binom{\alpha}{j} := \frac{1}{j!}\alpha(\alpha-1)\dots(\alpha-(j-1)), \quad j > 1.$$

* **Corresponding author: Josef Diblík,** Brno University of Technology, CEITEC - Central European Institute of Technology, Brno 612 00, Czech Republic, e-mail: josef.diblik@ceitec.vutbr.cz, diblik@vut.cz

The present article considers systems of fractional nonlinear discrete equations

$$\Delta^{\alpha}x(n+1) = F_n(n, x(n), x(n-1), \dots, x(n_0)), \quad n \in \mathbb{N}(n_0), \tag{2}$$

where $\alpha \in \mathbb{R}$ is a fixed number and $F_n : \{n\} \times \mathbb{R}^{n-n_0+1} \to \mathbb{R}^s$, $F_n = (F_n^1, \dots, F_n^s)^T$. The vector function on the right-hand side of (2) is denoted by $F_n$ to clearly reflect the fact that, for every fixed $n \in \mathbb{N}(n_0)$, it can depend on $n - n_0 + 1$ of possible arguments $x(n_0), x(n_0+1), \dots, x(n)$. Such a notation is used throughout the article. A function $x^* : \mathbb{N}(n_0) \to \mathbb{R}^s$ is called a solution to (2) if equation (2) with $x := x^*$ is satisfied for every $n \in \mathbb{N}(n_0)$.

Analyzing the definition of fractional discrete difference (1), we see that, for an $n^* \in \mathbb{N}(n_0)$, the difference $\Delta^{\alpha}x(n^*+1)$ uses the current value $x(n^*+1)$ as well as all the preceding ones $x(n^*), x(n^*-1), \dots, x(n_0)$. This accounts for the fact that initial problem for equation (2),

$$x(n_0) = x_0 \tag{3}$$

determines a unique solution $x : \mathbb{N}(n_0) \to \mathbb{R}^s$ to system (2) within the meaning of the above definition of solution. This solution, as can easily be seen, depends continuously on the initial value $x_0$ if, for every fixed $n \in \mathbb{N}(n_0)$, the function $F_n(n, Y_n, Y_{n-1}, \dots, Y_{n_0})$ continuously depends on $Y_{n_0}, Y_{n_0+1}, \dots, Y_n$, where $Y_p = (Y_p^1, \dots, Y_p^s)^T$, $p = n_0, n_0+1, \dots, n$. This is tacitly assumed throughout the article.

Considered is also a particular linear case of system (2),

$$\Delta^{\alpha}x(n+1) = A(n)x(n) + \delta(n), \tag{4}$$

where $n \in \mathbb{N}(n_0)$, $A(n) = \{a_{ij}(n)\}_{i,j=1}^{s}$, $i, j = 1, \dots, s$ is a matrix, $a_{ij} : \mathbb{N}(n_0) \to \mathbb{R}$, and $\delta = (\delta_1, \dots, \delta_s)^T$: $\mathbb{N}(n_0) \to \mathbb{R}^s$ is a vector function.

## 1.1 Problem formulation

This article follows the following usual conventions: the inequality $v < w$ between two $s$-dimensional vectors $v = (v_1, \dots, v_s)^T$ and $w = (w_1, \dots, w_s)^T$ is equivalent with inequalities $v_i < w_i$, $i = 1, \dots, s$ and the inequality $v \leqslant w$ is equivalent with inequalities $v_i \leqslant w_i$, $i = 1, \dots, s$.

Let vectors $b = (b_1, \dots, b_s)^T$, $c = (c_1, \dots, c_s)^T$ where $b, c : \mathbb{N}(n_0) \to \mathbb{R}^s$ be given. Throughout the article, assume

$$b(n) < c(n), \quad n \in \mathbb{N}(n_0). \tag{5}$$

For a fixed $n \in \mathbb{N}(n_0)$, define a domain

$$\mathcal{D}_n := \left\{ \left( Y_n, Y_{n-1}, \dots, Y_{n_0} \right) : b(i) < Y_i < c(i),\ i = n_0, \dots, n \right\}.$$

We say that a solution $x$ to (2) is $(b, c)$-bounded if, for every $n \in \mathbb{N}(n_0)$,

$$(x(n), x(n-1), \dots, x(n_0)) \in \mathcal{D}_n. \tag{6}$$

The article gives sufficient conditions for the right-hand sides of systems (2) and (4) to guarantee the existence of at least one $(b, c)$-bounded solution $x$ to these systems. Note that (6) can be rewritten in an equivalent form $b(n) < x(n) < c(n)$, $n \in \mathbb{N}(n_0)$, or, using coordinates, as

$$b_i(n) < x_i(n) < c_i(n), \quad n \in \mathbb{N}(n_0), \quad i = 1, \dots, s. \tag{7}$$

## 1.2 Article structure

This article is structured as follows. Known formulas, used in the article, are presented in Section 1.3. In Section 2, an auxiliary difference system of Volterra type is investigated. As the main result of the article, Theorem 1 gives conditions sufficient for the existence of $(b, c)$-bounded solutions. System (2) can be transformed into a system of Volterra type. This is done in Section 3 where Theorem 1 is applied to a transformed system, and result is formulated in Theorem 2 on the existence of a $(b, c)$-bounded solution to fractional discrete system (2) with relevant examples considered. Next, Theorems 3–6 and Corollary 1 in Section 4, use the last result to give conditions for the existence of a $(b, c)$-bounded solution to system (4) if $c = -b = \text{const}$ and if $\alpha$ is arbitrary. An illustrative example is considered, documenting the impact of an $\alpha$-order of fractional discrete derivative on the restrictions guaranteeing the existence of bounded solutions. Concluding remarks and comments together with some open problems are listed in Section 5.

### 1.3 Some formulas

In transforming sums, we will assume, as customary, that $\sum_{j=p}^{q}\omega(j) = 0$ whenever the integers $p, q$ satisfy $q < p$ with $\omega$ being a given expression. We will keep to this convention even in the event of a sum being given explicitly as $\omega(p) + \dots + \omega(q)$. To make expressions more succinct, below we show a list of formulas with references used in the sequel:

- For formula

$$\sum_{p=0}^{m}(-1)^p\binom{\alpha}{p} = (-1)^m\binom{\alpha-1}{m}, \quad \alpha \in \mathbb{R},\ m \in \mathbb{N}(0) \tag{8}$$

  we refer, e.g., to [1, formula (5)] and [10, p. 165].
- For formula

$$\lim_{\ell\to\infty}(-1)^{\ell+1}\binom{\beta}{\ell+1}\Gamma(-\beta)\ell^{1+\beta} = 1, \quad \beta \in \mathbb{R}\setminus\mathbb{Z} \tag{9}$$

  we refer, e.g., to [1, formula (6)].
- For the statements: if $\alpha \in (0, 1)$, then

$$(-1)^n\binom{-\alpha}{n}, \quad n \in \mathbb{N}(0) \quad \text{is a decreasing sequence of positive terms} \tag{10}$$

  and

$$\frac{\delta_1}{n^{1-\alpha}} < (-1)^n\binom{-\alpha}{n} < \frac{\delta_2}{n^{1-\alpha}}, \quad n \in \mathbb{N}(1) \tag{11}$$

  for some $0 < \delta_1 < \delta_2$, we refer to [1, Lemma 1].

## 2 Volterra-type discrete systems

In this section, we consider Volterra-type discrete systems

$$x(n+1) = G_n(n, x(n), x(n-1), \dots, x(n_0)), \quad n \in \mathbb{N}(n_0), \tag{12}$$

where $G_n : \{n\} \times \mathbb{R}^{n-n_0+1} \to \mathbb{R}^s$, $G_n = (G_n^1, \dots, G_n^s)^T$. In the sequel, it will be assumed that, if $n \in \mathbb{N}(n_0)$ is fixed, the function $G_n(n, X_n, X_{n-1}, \dots, X_{n_0})$ of $n - n_0 + 1$ variables $X_{n_0}, X_{n_0+1}, \dots, X_n$ is continuous provided that each coordinate $X_i$, $i = n_0, \dots, n$, varies within $\mathbb{R}$. For every fixed $n \in \mathbb{N}(n_0)$ and every fixed $\nu \in \{1, \dots, s\}$, define domains

$$\begin{aligned}\mathcal{D}_n^{b_\nu} := \{\big(Y_n, Y_{n-1}, \dots, Y_{n_0}\big) : Y_n^\nu = b_\nu(n), \quad & b(n) \leqslant Y_n \leqslant c(n),\\ & b(i) < Y_i < c(i), \quad i = n_0, n_0+1, \dots, n-1\},\end{aligned} \tag{13}$$

$$\mathcal{D}_n^{c_\nu} := \{\big(Y_n, Y_{n-1}, \dots, Y_{n_0}\big) : Y_n^\nu = c_\nu(n), \quad b(n) \leqslant Y_n \leqslant c(n), \quad b(i) < Y_i < c(i), \quad i = n_0, n_0+1, \dots, n-1\}. \tag{14}$$

The following theorem specifies sufficient conditions for the existence of at least one $(b, c)$-bounded solution to (12) (defined in much the same way as a $(b, c)$-bounded solution for system (2)).

**Theorem 1.** *Let vectors $b, c : \mathbb{N}(n_0) \to \mathbb{R}^s$ satisfying* (5) *be given. Let, for every fixed $n \in \mathbb{N}(n_0)$, $\nu \in \{1, \dots, s\}$,*

$$c_\nu(n+1) < G_n^\nu\big(n, Y_n, Y_{n-1}, \dots, Y_{n_0}\big) \tag{15}$$

*whenever $(Y_n, Y_{n-1}, \dots, Y_{n_0}) \in \mathcal{D}_n^{c_\nu}$, and*

$$b_\nu(n+1) > G_n^\nu\big(n, Y_n, Y_{n-1}, \dots, Y_{n_0}\big) \tag{16}$$

*whenever $(Y_n, Y_{n-1}, \dots, Y_{n_0}) \in \mathcal{D}_n^{b_\nu}$. Then, there exists a $(b, c)$-bounded solution of* (12).

The proof uses the concepts of retraction and retract. Referring, e.g., to [5,19], they are defined as follows.

**Definition 1.** Let $\mathcal{A} \subset \mathcal{B}$ be any two sets of a topological space and let $\pi : \mathcal{B} \to \mathcal{A}$ be a continuous mapping from $\mathcal{B}$ onto $\mathcal{A}$ preserving the position of all points in $\mathcal{A}$. Then, $\pi$ is said to be a retraction of $\mathcal{B}$ onto $\mathcal{A}$ and $\mathcal{A}$ is called a retract of $\mathcal{B}$.

**Proof.** Any possible $(b, c)$-bounded solution of equation (12) is defined by an initial value $(n_0, x(n_0))$, where $b(n_0) < x(n_0) < c(n_0)$, that is,

$$b_i(n_0) < x_i(n_0) < c_i(n_0), \quad i = 1, \dots, s. \tag{17}$$

Suppose that, on the contrary, there exists no $(b, c)$-bounded solution of (12). It means the following. For an arbitrary solution $x = x(n)$ of equation (12), fixed for this proof, defined by initial values (again arbitrary but fixed for this proof) satisfying (17), there exists a value $n = n^* \in \mathbb{N}(n_0)$ such that inequalities

$$b_i(n) < x_i(n) < c_i(n), \quad i = 1, \dots, s \tag{18}$$

hold for $n = n_0, n_0+1, \dots, n^*$, and there exists a value $i = i^* \in \{1, \dots, s\}$ such that the inequality

$$b_{i^*}(n^*+1) < x_{i^*}(n^*+1) < c_{i^*}(n^*+1) \tag{19}$$

does not hold. Subsequently, we show that this assumption leads to a contradiction. Let

$$\Omega(n) := \{(n, Y) : b(n) < Y < c(n)\}, \quad n \in \mathbb{N}(n_0).$$

We will prove that there exists a continuous mapping of the set $\overline{\Omega}(n_0)$ onto its boundary $\partial\Omega(n_0)$ such that all points of $\partial\Omega(n_0)$ preserve their position. That is, we prove that there exists a retraction of $\overline{\Omega}(n_0)$ onto $\partial\Omega(n_0)$ (in the proof we construct this retraction using auxiliary mappings $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M} = \mathcal{M}_2 \circ \mathcal{M}_1$). This statement is topologically equivalent with the existence of a retraction of an $s$-dimensional ball onto its boundary, which is an $(s-1)$-dimensional sphere. This is, however, not possible given the topological properties of the sets considered, we refer, e.g., to [5]. To show this, we need some technical preliminaries described below.

**Technical preliminaries**. Define, for $t \geqslant n_0$, continuous and piecewice-linear functions

$$b^*(t) := b(\lfloor t \rfloor) + (b(\lfloor t \rfloor + 1) - b(\lfloor t \rfloor))(t - \lfloor t \rfloor), \quad c^*(t) := c(\lfloor t \rfloor) + (c(\lfloor t \rfloor + 1) - c(\lfloor t \rfloor))(t - \lfloor t \rfloor),$$

where $\lfloor \cdot \rfloor$ is the floor function. Obviously, $c^*(\lfloor t \rfloor) = c(\lfloor t \rfloor)$, $b^*(\lfloor t \rfloor) = b(\lfloor t \rfloor)$. Inequality (5) implies

$$b^*(t) < c^*(t), \quad t \geqslant n_0. \tag{20}$$

Let us consider a domain

$$\mathcal{H} := \{(t, Y) : t \geqslant n_0, b^*(t) \leqslant Y \leqslant c^*(t)\}$$

and remark that each of its subdomains

$$\mathcal{H}(n) := \{(t, Y) : n \leqslant t \leqslant n+1, b^*(t) \leqslant Y \leqslant c^*(t)\}, \quad n \in \mathbb{N}(n_0)$$

is convex. Moreover, define sets

$$\begin{aligned} \operatorname{int}_Y\mathcal{H} &:= \{(t, Y) : t \geqslant n_0, b^*(t) < Y < c^*(t)\}, \\ \operatorname{int}_Y\mathcal{H}(n) &:= \{(t, Y) : n \leqslant t \leqslant n+1, b^*(t) < Y < c^*(t)\}, \\ \partial_Y\mathcal{H} &:= \{(t, Y) : t \geqslant n_0, b^*(t) \leqslant Y \leqslant c^*(t), (t, Y) \notin \operatorname{int}_Y\mathcal{H}\}, \\ \partial_Y\mathcal{H}(n) &:= \{(t, Y) : n \leqslant t \leqslant n+1, b^*(t) \leqslant Y \leqslant c^*(t), (t, Y) \notin \operatorname{int}_Y\mathcal{H}(n)\}. \end{aligned}$$

Let

$$\ell(x(n^*))(t) := x(n^*) + t(x(n^*+1) - x(n^*)), \quad t \in [0, 1].$$

By (18),

$$(n^*, \ell(x(n^*))(0)) = (n^*, x(n^*)) \in \Omega(n^*) \subset \operatorname{int}_Y\mathcal{H}(n^*).$$

**Analysis of the negation of inequality** (19). Since inequality (19) does not hold, one of the following relations $(\alpha)$–$(\delta)$:

$$\begin{aligned} &(\alpha) \quad x_{i^*}(n^*+1) > c_{i^*}(n^*+1), \\ &(\beta) \quad x_{i^*}(n^*+1) = c_{i^*}(n^*+1), \\ &(\gamma) \quad x_{i^*}(n^*+1) < b_{i^*}(n^*+1), \\ &(\delta) \quad x_{i^*}(n^*+1) = b_{i^*}(n^*+1) \end{aligned}$$

holds. In the case of $(\alpha)$, we have

$$(n^*+1, \ell(x(n^*))(1)) = (n^*+1, x(n^*+1)) \notin \overline{\Omega}(n^*+1),$$

that is,

$$(n^*+1, \ell(x(n^*))(1)) = (n^*+1, x(n^*+1)) \notin \mathcal{H}(n^*),$$

which means that a unique point

$$\left(t^*, x_{t^*}\right) \in \partial\mathcal{H}_Y(n^*), \tag{21}$$

where $t^* \in (n^*, n^*+1)$ exists as the intersection of the set $\partial\mathcal{H}_Y(n^*)$ and the line segment $(n^*+t, \ell(x(n^*))(t))$, $t \in [0, 1]$.

If $(\beta)$ holds, then, repeating the aforementioned reasonings, we can conclude that there exists a nonempty intersection of the set $\partial\mathcal{H}_Y(n^*)$ and the segment $(n^*+t, \ell(x(n^*))(t))$, $t \in [0, 1]$. This intersection contains exactly one point $(t^*, x_{t^*})$, corresponding to the value $t = 1$. Then, $t^* = n^*+1$ and, similarly to (21), we can conclude

$$\left(t^*, x_{t^*}\right) = \left(n^*+1, x_{n^*+1}\right) \in \partial\mathcal{H}_Y(n^*).$$

In much the same way we can proceed in the cases of $(\gamma)$ and $(\delta)$.

**Auxiliary mapping $\mathcal{M}_1$ and its continuity**. For every point $(n_0, x(n_0)) \in \overline{\Omega}(n_0)$, define a mapping

$$\mathcal{M}_1 : (n_0, x(n_0)) \to \begin{cases} (n_0, x(n_0)) & \text{if} \quad (n_0, x(n_0)) \in \partial\Omega(n_0), \\ \left(t^*, x_{t^*}\right) & \text{if } (n_0, x(n_0)) \in \Omega(n_0). \end{cases}$$

Prove its continuity. $(i)$ Assume $(n_0, x(n_0)) \in \Omega(n_0)$. Let the inequality $(\alpha)$ hold. Consider a sequence of points

$$\{(n_0, x_r)\}_{r=1}^{\infty}, \quad (n_0, x_r) \in \Omega(n_0), \quad r = 1, \dots, \quad \lim_{r\to\infty}(n_0, x_r) = (n_0, x(n_0)) \tag{22}$$

and assume that all points of this sequence lie in a sufficiently small neighborhood of $(n_0, x(n_0))$ in the space $\{(n_0, y)\}$ where $y \in \mathbb{R}^s$. Let $r = r^*$ be fixed. Since the solutions of equation (12) are continuously dependent on the initial values, every solution, defined by an initial value $(n_0, x_{r^*})$, lies in, on a finite interval, sufficiently close to the solution, defined by $(n_0, x(n_0))$. If, without loss of generality, we consider a suitable subsequence of $\{(n_0, x_r)\}_{r=1}^{\infty}$, this property implies the existence of a unique intersection $(t^*_{r^*}, x_{r^*t^*})$ where $t^*_{r^*} \in (n^*, n^*+1)$ of the set $\partial\mathcal{H}_Y(n^*)$ with the segment $(n^*+t, \ell(x_{r^*}(n^*))(t))$, $t \in [0,1]$ because $(n^*, x_{r^*}(n^*)) \in \Omega(n^*)$ and $(n^*+1, x_{r^*}(n^*+1)) \notin \Omega(n^*+1)$. Repeating the above construction for every fixed $r$ (considering again, if necessary, a suitable subsequence of the initial sequence), we define a sequence of points

$$\left(t^*_r, x_{rt^*}\right) \in \partial\mathcal{H}_Y(n^*), \quad t^*_r \in (n^*, n^*+1) \tag{23}$$

with the property

$$\lim_{r\to\infty}\left(t^*_r, x_{rt^*}\right) = \left(t^*, x_{t^*}\right) \in \partial\mathcal{H}_Y(n^*), \quad t^*_r \in (n^*, n^*+1). \tag{24}$$

In other words, (24) says that

$$\lim_{r\to\infty} \mathcal{M}_1(n_0, x_r) = \mathcal{M}_1(n_0, x(n_0)) = \left(t^*, x_{t^*}\right) \tag{25}$$

and the continuity of $\mathcal{M}_1$, if $(\alpha)$ holds, is proved.

Now, we will prove the continuity of $\mathcal{M}_1$ in the case of $(\beta)$. Consider the sequence of points $\{(n_0, x_r)\}_{r=1}^{\infty}$ defined by (22). In this case, the solutions defined by the terms of this sequence will basically behave in the same way as in the case of $(\alpha)$. It is, however, also possible that, for some subsequence $\{(n_0, x^*_r)\}_{r=1}^{\infty}$ of the terms, every initial condition $(n_0, x^*(n_0)) = (n_0, x^*_r)$ will generate a solution $x^*_r(n)$ such that

$$(n^*+1, x^*_r(n^*+1)) \in \Omega(n^*+1). \tag{26}$$

We can eliminate the sequences containing infinitely many terms defining solutions with property (26) and infinitely many terms defining solutions with property

$$(n^*+1, x^*_r(n^*+1)) \notin \Omega(n^*+1) \tag{27}$$

by splitting each of them (without loss of generality) into two subsequences with their terms having the same property – all solutions defined by such terms satisfy either (26) or (27). If property (27) holds, we can proceed in much the same way as in the case of $(\alpha)$. Then, for limits (24) and (25), we have

$$\lim_{r\to\infty}\left(t^*_r, x_{rt^*}\right) = \left(t^*, x_{t^*}\right) = \left(n^*+1, x_{t^*}\right)$$

and

$$\lim_{r\to\infty} \mathcal{M}_1(n_0, x_r) = \mathcal{M}_1(n_0, x(n_0)) = \left(t^*, x_{t^*}\right) = \left(n^*+1, x_{t^*}\right),$$

respectively. If (26) holds, then, by (15), $(n^*+2, x(n^*+2)) \notin \overline{\Omega}(n^*+2)$ and, due to the continuous dependence of solutions on the initial data, $(n^*+2, x^*_r(n^*+2)) \notin \overline{\Omega}(n^*+2)$ holds as well. Therefore, there exists a unique nonempty intersection of the set $\partial_Y\mathcal{H}(n^*+1)$ with the segment $(n^*+1+t, \ell(x^*_r(n^*+1))(t))$, $t \in [0,1]$. Denoting this point of intersection by $(t^{**}_r, x^*_{t^{**}})$ where $t^{**}_r \in (n^*+1, n^*+2)$, we have (similarly to (23) and (24)) $(t^{**}_r, x^*_{t^{**}}) \in \partial_Y\mathcal{H}(n^*+1)$, $\quad t^{**}_r \in (n^*+1, n^*+2)$, and

$$\lim_{r\to\infty}\left(t^{**}_r, x^*_{t^{**}}\right) = \left(t^*, x_{t^*}\right) = \left(n^*+1, x_{t^*}\right). \tag{28}$$

Properties (24) and (28) imply that the solutions defined by the sequence (22) satisfy (25), i.e., the mapping $\mathcal{M}_1$ is continuous.

The continuity of $\mathcal{M}_1$ for the cases of $(\gamma)$ and $(\delta)$ can be proved in much the same way.

*(ii)* It remains to prove the continuity of $\mathcal{M}_1$ if an initial value $x = x(n_0)$ satisfies $(n_0, x) \in \partial\,\Omega(n_0)$. In this case, by assumptions (15) and (16), for every fixed $\nu \in \{1, \ldots, s\}$,

$$c_\nu(n_0+1) < G^\nu_{n_0}\big(n_0, Y_{n_0}\big), \tag{29}$$

whenever $(Y_{n_0}) \in \mathcal{D}^{c_\nu}_{n_0}(Y_{n_0})$ and

$$b_\nu(n_0+1) > G^\nu_{n_0}\big(n_0, Y_{n_0}\big), \tag{30}$$

whenever $(Y_{n_0}) \in \mathcal{D}^{b_\nu}_{n_0}(Y_{n_0})$. This means that the segment

$$(n_0+t, \ell(x(n_0))(t)) = (n_0+t, x(n_0) + t(x(n_0+1) - x(n_0))), \quad t \in [0,1]$$

intersects the set $\partial_Y\mathcal{H}(n_0)$ at a single point, namely, $(n_0, x(n_0))$ itself. Consider a sequence of points $\{(n_0, x_r)\}_{r=1}^\infty$ such that

$$(n_0, x_r) \in \overline{\Omega}(n_0),\ \lim_{r\to\infty}(n_0, x_r) = (n_0, x).$$

Then, in view of the continuous dependence of solutions of equation (12) on the initial values and by (29), (30), we can conclude that $(n_0+1, x_r(n_0+1)) \notin \overline{\Omega}(n_0+1)$ for all sufficiently large $r$ because $(n_0+1, x(n_0+1)) \notin \overline{\Omega}(n_0+1)$. The segment

$$(n_0+t, \ell(x_r(n_0))(t)) := (n_0+t, x_r(n_0) + t(x_r(n_0+1) - x_r(n_0))), \quad t \in [0,1]$$

intersects the set $\partial_Y\mathcal{H}(n_0)$ at a single point $(t_r^*, x_{t_r^*})$, $t_r^* \in [n_0, n_0+1)$ and, if for an $r$, $(n_0, x_r(n_0)) \in \partial\Omega(n_0)$, then this intersection point coincides with $(n_0, x_r(n_0))$ itself. Due to the convexity of $\mathcal{H}_Y(n_0)$ and the property of continuous dependence of solutions on the initial values, we conclude that $\lim_{r\to\infty}(t_r^*, x_{t_r^*}) = (n_0, x(n_0))$, which proves the continuity of $\mathcal{M}_1$ at points lying in $\partial\Omega(n_0)$.

**Auxiliary mapping $\mathcal{M}_2$ and its continuity**. In this section, we prove that the set $\partial\Omega(n_0)$ is a retract of the set $\partial_Y\mathcal{H}$. Let $(t_\mathcal{H}, x_\mathcal{H}) \in \partial_Y\mathcal{H}$ where $x_\mathcal{H} = (x_{\mathcal{H}1}, \dots, x_{\mathcal{H}s})^T$. This means that there exists an index $i \in \{1, \dots, s\}$ such that either $x_{\mathcal{H}i} = c_i^*(t_\mathcal{H})$ or $x_{\mathcal{H}i} = b_i^*(t_\mathcal{H})$. Consider the first variant only as the second one can be treated in much the same way. Then

$$(t_\mathcal{H}, x_\mathcal{H}) = (t_\mathcal{H}, x_{\mathcal{H}1}, \dots, x_{\mathcal{H},i-1}, c_i^*(t_\mathcal{H}), x_{\mathcal{H},i+1}, \dots, x_{\mathcal{H}s}).$$

Define a mapping $\mathcal{M}_2$ by the formula

$$M_2 : (t_\mathcal{H}, x_\mathcal{H}) \longrightarrow (n_0, x^*_\mathcal{H}) = (n_0, x^*_{\mathcal{H}1}, \dots, x^*_{\mathcal{H},i-1}, c_i(t_0), x^*_{\mathcal{H},i+1}, \dots, x^*_{\mathcal{H}s}) \in \partial\Omega(n_0),$$

where

$$x^*_{\mathcal{H}j} = b_j(t_0) + \frac{c_j(t_0) - b_j(t_0)}{c_j^*(t_H) - b_j^*(t_H)}(x_{\mathcal{H}j} - b_j^*(t_H)), \quad j = 1, 2, \dots, s. \tag{31}$$

Since $b_j^*(t_H) \leqslant x_{\mathcal{H}j} \leqslant c_j^*(t_H)$, $j = 1, 2, \dots, s$, using (20), we see that $b_j(t_0) \leqslant x^*_{\mathcal{H}j} \leqslant c_j(t_0)$, $j = 1, 2, \dots, s$ and the mapping $\mathcal{M}_2$ is well defined. The points of $\partial\Omega(n_0)$ are the fixed points of $\mathcal{M}_2$ and this mapping, as it is clear from (31), depends continuously on the coordinates of the point $(t_\mathcal{H}, x_\mathcal{H})$.

**Mapping $\mathcal{M} = \mathcal{M}_2 \circ \mathcal{M}_1$.** Define a mapping $\mathcal{M} = \mathcal{M}_2 \circ \mathcal{M}_1$ summarizing its properties. The domain of $\mathcal{M}_1$ is the set $\overline{\Omega}(n_0)$ and its values belong to $\partial_Y\mathcal{H}$. The domain of $\mathcal{M}_2$ is the set $\partial_Y\mathcal{H}$ and its range is the whole set $\partial\Omega(n_0)$. Therefore, the domain of $\mathcal{M}$ is the set $\overline{\Omega}(n_0)$ and its range is the whole set $\partial\Omega(n_0)$ because the points of the set $\partial\Omega(n_0)$ remain fixed. As both $\mathcal{M}_1$ and $\mathcal{M}_2$ are continuous, the resulting map $\mathcal{M}$ is continuous as well. Then, by Definition 1, the set $\mathcal{A} := \partial\Omega(n_0)$ is a retract of the set $\mathcal{B} := \overline{\Omega}(n_0)$ and the retraction is defined by $\pi := \mathcal{M}$. As mentioned above, this is impossible. Therefore, our assumption that there exists no $(b, c)$-bounded solution of (12) is false. This contradiction proves the theorem. □

## 3 $(b, c)$-bounded solutions to system (2)

The fractional discrete system (2) can be transformed into a (non-fractional) discrete system of Volterra type (12). From the definition of fractional derivative (1), we obtain

$$\begin{aligned}\Delta^{\alpha}x(n+1) &= \sum_{j=0}^{n+1-n_0} C_{\alpha}(j)x(n+1-j)\\ &= C_{\alpha}(0)x(n+1) + \sum_{j=1}^{n+1-n_0} C_{\alpha}(j)x(n+1-j)\\ &= x(n+1) + \sum_{j=1}^{n+1-n_0} C_{\alpha}(j)x(n+1-j), \quad n \in \mathbb{N}(n_0),\end{aligned}$$

and system (2) is equivalent with the following one:

$$x(n+1) = -\sum_{j=1}^{n+1-n_0} C_{\alpha}(j)x(n+1-j) + F_n(n, x(n), x(n-1), \dots, x(n_0)), \quad n \in \mathbb{N}(n_0). \tag{32}$$

**Theorem 2.** *Let functions $b, c : \mathbb{N}(n_0) \to \mathbb{R}^s$ satisfying* (5) *be given. Let, for every fixed $n \in \mathbb{N}(n_0)$ and $\nu \in \{1, \dots, s\}$,*

$$\alpha c_{\nu}(n) - \sum_{j=2}^{n-n_0+1} (-1)^j \binom{\alpha}{j} Y^{\nu}_{n-j+1} + F^{\nu}_n\big(n, Y_n, Y_{n-1}, \dots, Y_{n_0}\big) > c_{\nu}(n+1), \tag{33}$$

*whenever $(Y_n, Y_{n-1}, \dots, Y_{n_0}) \in \mathcal{D}^{c_\nu}_n$ and*

$$\alpha b_{\nu}(n) - \sum_{j=2}^{n-n_0+1} (-1)^j \binom{\alpha}{j} Y^{\nu}_{n-j+1} + F^{\nu}_n\big(n, Y_n, Y_{n-1}, \dots, Y_{n_0}\big) < b_{\nu}(n+1), \tag{34}$$

*whenever $(Y_n, Y_{n-1}, \dots, Y_{n_0}) \in \mathcal{D}^{b_\nu}_n$. Then, there exists a $(b, c)$-bounded solution of* (2).

**Proof.** The conclusion of the theorem is a consequence of Theorem 1 if, in accordance with (12), (32),

$$\begin{aligned}&G_n(n, x(n), x(n-1), \dots, x(n_0+1), x(n_0))\\ &\coloneqq -\sum_{j=1}^{n+1-n_0} C_{\alpha}(j)x(n+1-j) + F_n(n, x(n), x(n-1), \dots, x(n_0))\\ &= \alpha x(n) - \sum_{j=2}^{n+1-n_0} C_{\alpha}(j)x(n+1-j) + F_n(n, x(n), x(n-1), \dots, x(n_0)), \quad n \in \mathbb{N}(n_0).\end{aligned}$$ □

**Example 1.** Let $s = 1$, $n_0 = 0$ (since we consider a scalar equation, below we write $F_n$ rather than $F^1_n$, $x$ rather than $x_1$, etc.),

$$F_n(n, x(n), x(n-1), \dots, x(n_0)) \coloneqq a(n)x^3(n) + x(n)x(n-1)\dots x(0) + \cos n, \quad n \in \mathbb{N}(0),$$

$a : \mathbb{N}(0) \to \mathbb{R}$ and $\alpha = 1/2$. Then, system (2) reduces to the equation

$$\Delta^{1/2}x(n+1) = a(n)x^3(n) + x(n)x(n-1)\dots x(0) + \cos n, \quad n \in \mathbb{N}(0). \tag{35}$$

By Theorem 2 we show that if

$$a(n) > 4 - 4(-1)^{n+1}\binom{-1/2}{n+1} + \frac{1}{2^{n-2}} + 8|\cos n|, \quad n \in \mathbb{N}(0), \tag{36}$$

then there exists a solution $x(n)$ of (35) satisfying $|x(n)| < 1/2$, $n \in \mathbb{N}(0)$. Now we will specify the necessary computations in detail. Set $c(n) = -b(n) = 1/2$, $n \in \mathbb{N}(0)$. Let us verify inequality (33). If $(Y_n, Y_{n-1}, \dots, Y_0) \in \mathcal{D}^{1/2}_n$, then $Y_n = 1/2$ and $|Y_i| < 1/2$, $i = 0, \dots, n-1$. Moreover,

$$(-1)^j\binom{\alpha}{j} = (-1)^j\binom{1/2}{j} < 0, \quad j = 2, \dots, n+1.$$

Using formula (8), the left-hand side of (33) can be estimated as follows:

$$
\begin{aligned}
\mathcal{L} &:= \alpha c(n) - \sum_{j=2}^{n+1}(-1)^j\binom{\alpha}{j}Y_{n-j+1} + a(n)Y_n^3(n) + Y_nY_{n-1}\dots Y_0 + \cos n\\
&> \frac{1}{4} + \frac{1}{2}\sum_{j=2}^{n+1}(-1)^j\binom{1/2}{j} + a(n)\frac{1}{8} - \frac{1}{2^{n+1}} + \cos n\\
&= \frac{1}{4} + \frac{1}{2}\sum_{j=0}^{n+1}(-1)^j\binom{1/2}{j} - \frac{1}{2}\sum_{j=0}^{1}(-1)^j\binom{1/2}{j} + a(n)\frac{1}{8} - \frac{1}{2^{n+1}} + \cos n\\
&= \frac{1}{4} + \frac{1}{2}(-1)^{n+1}\binom{-1/2}{n+1} - \frac{1}{4} + a(n)\frac{1}{8} - \frac{1}{2^{n+1}} + \cos n\\
&= \frac{1}{2}(-1)^{n+1}\binom{-1/2}{n+1} + a(n)\frac{1}{8} - \frac{1}{2^{n+1}} + \cos n.
\end{aligned}
$$

Now, inequality (33) will hold if $\mathcal{L} > 1/2$, i.e., if

$$a(n) > 4 + 4(-1)^n\binom{-1/2}{n+1} + \frac{1}{2^{n-2}} - 8\cos n, \quad n \in \mathbb{N}(0).$$

Considering, in much the same way, inequality (34), we obtain inequality

$$a(n) > 4 + 4(-1)^n\binom{-1/2}{n+1} + \frac{(-1)^{n+1}}{2^{n-2}} + 8\cos n, \quad n \in \mathbb{N}(0).$$

Now the last two inequalities will hold if (36) holds. Theorem 2 is applicable and the above solution exists. To finish the example, let us note that, by formula (10), the sequence

$$(-1)^{n+1}\binom{-1/2}{n+1}$$

is positive and decreasing and, by (11), converges to zero.

**Example 2.** (Fractional Bernoulli-type equation) Consider equation (35) where the term $x(n)x(n-1)\dots x(0)$ is omitted, i.e., consider the following equation:

$$\Delta^{1/2}x(n+1) = a(n)x^3(n) + \cos n, \quad n \in \mathbb{N}(0). \tag{37}$$

By analogy with ordinary differential equations, this equation can be called a fractional Bernoulli-type equation. Proceeding in a much the same way as in Example 1 and omitting similar computations, we can prove that there exists a solution $x(n)$ of (37) satisfying $|x(n)| < 1/2$, $n \in \mathbb{N}(0)$ if

$$a(n) > 4 - 4(-1)^{n+1}\binom{-1/2}{n+1} + 8|\cos n|, \quad n \in \mathbb{N}(0).$$

# 4 Bounded solutions to linear system (4)

In this section, we prove the existence of a solution $x = (x_1, \dots, x_s)^T$ to linear nonhomogeneous systems of fractional discrete equations (4) satisfying

$$|x_i(n)| < M, \quad n \in \mathbb{N}(n_0), \quad i = 1, \dots, s, \tag{38}$$

where $M$ is a constant. Below, all values $\alpha \in \mathbb{R}$ are covered by four cases: Section 4.1 considers the case of $\alpha \in (r-1, r)$ for a fixed integer $r \in \{1, 2, \dots\}$, Section 4.2 deals with the case of $\alpha$ being a positive integer, the case of $\alpha < 0$ is treated in Section 4.3 and 4.4 considers the case of $\alpha = 0$. All the following proofs are based on Theorem 2 as system (4) is a particular case of (2) if

$$F_n(n, x(n), x(n-1), \dots, x(n_0)) := A(n)x(n) + \delta(n), \quad n \in \mathbb{N}(n_0),$$

and, consequently, system (4) becomes a Volterra-type system (32),

$$x(n+1) = -\sum_{j=1}^{n+1-n_0} (-1)^j \binom{\alpha}{j} x(n+1-j) + A(n)x(n) + \delta(n), \quad n \in \mathbb{N}(n_0).$$

This system can be written in the form:

$$x(n+1) = (\alpha I + A(n))x(n) - \sum_{j=2}^{n+1-n_0} (-1)^j \binom{\alpha}{j} x(n+1-j) + \delta(n), \quad n \in \mathbb{N}(n_0)$$

where $I$ is an $s \times s$ unit matrix. Defining $b, c : \mathbb{N}(n_0) \to \mathbb{R}^s$ as constant vectors with coordinates $b_\nu(n) = -M$, $c_\nu(n) = M$, $\nu = 1, \dots, s$, $n \in \mathbb{N}(n_0)$, by (13) and (14), we specify

$$\mathcal{D}_n^{b_\nu} = \mathcal{D}_n^{-M} = \{(Y_n, Y_{n-1}, \dots, Y_{n_0}) : Y_n^\nu = -M, -M \leqslant Y_n \leqslant M, -M < Y_i < M, \ i = n_0, n_0+1, \dots, n-1\},$$
$$\mathcal{D}_n^{c_\nu} = \mathcal{D}_n^{M} = \{(Y_n, Y_{n-1}, \dots, Y_{n_0}) : Y_n^\nu = M, -M \leqslant Y_n \leqslant M, -M < Y_i < M, \ i = n_0, n_0+1, \dots, n-1\}.$$

Now, we define an auxiliary expression used in the following proofs

$$\begin{aligned} S_\nu &:= -\sum_{j=2}^{n-n_0+1} (-1)^j \binom{\alpha}{j} Y_{n-j+1}^\nu \\ &= \sum_{j=2}^{n-n_0+1} (-1)^{j-1} \binom{\alpha}{j} Y_{n-j+1}^\nu \\ &= -\binom{\alpha}{2} Y_{n-1}^\nu + (-1)^2 \binom{\alpha}{3} Y_{n-2}^\nu + \dots + (-1)^{n-n_0} \binom{\alpha}{n-n_0+1} Y_{n_0}^\nu. \end{aligned} \tag{39}$$

Then, inequality (33) can be written as

$$\alpha M + S_\nu + a_{\nu\nu}(n)M + \sum_{j=1, j\neq \nu}^{s} a_{\nu j}(n) Y_n^j + \delta_\nu(n) > M, \tag{40}$$

and inequality (34) as

$$-\alpha M + S_\nu - a_{\nu\nu}(n)M + \sum_{j=1, j\neq \nu}^{s} a_{\nu j}(n) Y_n^j + \delta_\nu(n) < -M, \tag{41}$$

where, referring to definitions of $\mathcal{D}_n^M$, $\mathcal{D}_n^{-M}$,

$$-M < Y_{n-j+1}^\nu < M, \quad j = 2, \dots, n-n_0+1. \tag{42}$$

## 4.1 The case of $\alpha \in (r-1, r)$, $r \in \{1, 2, \dots\}$

Being the most interesting one, this case is covered by the following theorem.

**Theorem 3.** *Let $r \in \{1, 2, \dots\}$ be fixed and let $\alpha \in (r-1, r)$. Let $M$ be a positive constant such that:*

(*i*) *For every $n_0 \leqslant n \leqslant n_0 + r - 1$ and $i \in \{1, \dots, s\}$,*

$$a_{ii}(n) > \sum_{j=1, j\neq i}^{s} |a_{ij}(n)| + \frac{1}{M}|\delta_i(n)| + (1-\alpha) + \binom{\alpha}{2} + \binom{\alpha}{3} + \dots + \binom{\alpha}{n-n_0} + \binom{\alpha}{n-n_0+1}. \tag{43}$$

(*ii*) *For every $n > n_0 + r - 1$ and $i \in \{1, \dots, s\}$,*

$$a_{ii}(n) > \sum_{j=1, j\neq i}^{s} |a_{ij}(n)| + \frac{1}{M}|\delta_i(n)| + 2(1-\alpha) + 2\left[\binom{\alpha}{2} + \binom{\alpha}{4} + \dots + \binom{\alpha}{r-1}\right] - (-1)^{n-n_0+1}\binom{\alpha-1}{n-n_0+1} \tag{44}$$

*whenever $r$ is odd.*

(*iii*) *For every* $n > n_0 + r - 1$ *and* $i \in \{1, \dots, s\}$,

$$a_{ii}(n) > \sum_{j=1, j\neq i}^{s} |a_{ij}(n)| + \frac{1}{M}|\delta_i(n)| + 2\left[\binom{\alpha}{3} + \binom{\alpha}{5} + \dots + \binom{\alpha}{r-1}\right] + (-1)^{n-n_0+1}\binom{\alpha - 1}{n - n_0 + 1} \tag{45}$$

*whenever* $r$ *is even.*

*Then, there exists a solution* $x : \mathbb{N}(n_0) \to \mathbb{R}^s$ *of equation* (4) *satisfying inequality* (38).

**Proof.** The case of (*i*). Verify that (40) holds for every fixed $n_0 \leqslant n \leqslant n_0 + r - 1$ and $\nu \in \{1, \dots, s\}$. Because $\alpha \in (r-1, r)$, $n_0 \leqslant n \leqslant n_0 + r - 1$, we have

$$\binom{\alpha}{j} > 0 \quad \text{if } j = 2, \dots, n - n_0 + 1 \tag{46}$$

and, therefore,

$$S_\nu > -M\left[\binom{\alpha}{2} + \binom{\alpha}{3} + \dots + \binom{\alpha}{n - n_0 + 1}\right].$$

Then, inequality (40) (and, consequently, inequality (33)) for the indicated values of $n$ will hold if

$$\alpha M - M\left[\binom{\alpha}{2} + \binom{\alpha}{3} + \dots + \binom{\alpha}{n - n_0 + 1}\right] + a_{\nu\nu}(n)M - \sum_{j=1, j\neq \nu}^{s} |a_{\nu j}(n)|M + \delta_\nu(n) > M.$$

This inequality is equivalent with inequality (43). The verification of inequality (34) leads, as it follows from (41), to the same inequality where the term $\delta_\nu(n)$ is replaced by $-\delta_\nu(n)$. Therefore, inequality (43) is sufficient for the validity of both inequalities (33) and (34) if $n_0 \leqslant n \leqslant n_0 + r - 1$.

The case of (*ii*). Assume $n > n_0 + r - 1$, $\nu \in \{1, \dots, s\}$. Then, $S_\nu$ can be written in the form

$$S_\nu = -\binom{\alpha}{2}Y^\nu_{n-1} + (-1)^2\binom{\alpha}{3}Y^\nu_{n-2} + \dots + (-1)^{r-1}\binom{\alpha}{r}Y^\nu_{n-(r-1)} + (-1)^r\binom{\alpha}{r+1}Y^\nu_{n-r} + \dots + (-1)^{n-n_0}\binom{\alpha}{n-n_0+1}Y^\nu_{n_0}.$$

Since $r$ is odd, we have

$$\binom{\alpha}{j} > 0 \quad \text{if } j = 2, \dots, r, \quad (-1)^j\binom{\alpha}{j} < 0 \quad \text{if } j = r+1, \dots, n - n_0 + 1$$

and $S_\nu$ can be estimated as follows:

$$\begin{aligned} S_\nu > {} & M\left[-\binom{\alpha}{2} - \binom{\alpha}{3} - \dots - \binom{\alpha}{r-1}\right] + M\left[(-1)^r\binom{\alpha}{r} + (-1)^{r+1}\binom{\alpha}{r+1} + \dots + (-1)^{n-n_0+1}\binom{\alpha}{n-n_0+1}\right] \\ = {} & M\left[-\binom{\alpha}{2} - \binom{\alpha}{3} - \dots - \binom{\alpha}{r-1}\right] + M\left[(-1)^0\binom{\alpha}{0} + \dots + (-1)^{r-1}\binom{\alpha}{r-1} + (-1)^r\binom{\alpha}{r}\right. \\ & \left. + (-1)^{r+1}\binom{\alpha}{r+1} + \dots + (-1)^{n-n_0+1}\binom{\alpha}{n-n_0+1}\right] \\ & - M\left[(-1)^0\binom{\alpha}{0} + (-1)^1\binom{\alpha}{1} + (-1)^2\binom{\alpha}{2} + \dots + (-1)^{r-1}\binom{\alpha}{r-1}\right]. \end{aligned} \tag{47}$$

Now we apply formula (8) to the middle expression and add up the first and the third expressions. We derive

$$S_\nu > M(\alpha - 1) - 2M\left[\binom{\alpha}{2} + \binom{\alpha}{4} + \dots + \binom{\alpha}{r-1}\right] + (-1)^{n-n_0+1}\binom{\alpha-1}{n-n_0+1}M. \tag{48}$$

Then, inequality (40) (and, consequently, inequality (33)) will hold for the indicated values of $n$ if

$$\alpha M + M(\alpha-1) - 2M\left[\binom{\alpha}{2} + \binom{\alpha}{4} + \cdots + \binom{\alpha}{r-1}\right] + (-1)^{n-n_0+1}\binom{\alpha-1}{n-n_0+1}M + a_{\nu\nu}(n)M + \sum_{j=1, j\neq\nu}^{s} a_{\nu j}(n)Y_n^j$$
$$+ \delta_\nu(n) > M.$$

It follows from (41) that, verifying inequality (34), we obtain the same inequality only with the term $\delta_\nu(n)$ replaced by $-\delta_\nu(n)$. Therefore, inequality (44) suffices to prove the validity of both inequalities.

The case of (*iii*). Since $r$ is even, we have

$$\binom{\alpha}{j} > 0 \quad \text{if } j = 2, \ldots, r, \quad (-1)^j\binom{\alpha}{j} > 0 \quad \text{if } j = r+1, \ldots, n-n_0+1$$

and $S_\nu$ can be estimated as follows (compare with (47)):

$$\begin{aligned}
S_\nu &> M\left[-\binom{\alpha}{2} - \binom{\alpha}{3} - \cdots - \binom{\alpha}{r-1}\right] - M\left[(-1)^r\binom{\alpha}{r} + (-1)^{r+1}\binom{\alpha}{r+1} + \cdots + (-1)^{n-n_0+1}\binom{\alpha}{n-n_0+1}\right] \\
&= M\left[-\binom{\alpha}{2} - \binom{\alpha}{3} - \cdots - \binom{\alpha}{r-1}\right] \\
&\quad - M\left[(-1)^0\binom{\alpha}{0} + \cdots + (-1)^{r-1}\binom{\alpha}{r-1} + (-1)^r\binom{\alpha}{r} + (-1)^{r+1}\binom{\alpha}{r+1} + \cdots + (-1)^{n-n_0+1}\binom{\alpha}{n-n_0+1}\right] \\
&\quad + M\left[(-1)^0\binom{\alpha}{0} + (-1)^1\binom{\alpha}{1} + (-1)^2\binom{\alpha}{2} + \cdots + (-1)^{r-1}\binom{\alpha}{r-1}\right].
\end{aligned}$$

Using formula (8) and proceeding as above, we derive (as in (48))

$$S_\nu > -M(\alpha-1) - 2M\left[\binom{\alpha}{3} + \binom{\alpha}{5} + \cdots + \binom{\alpha}{r-1}\right] - (-1)^{n-n_0+1}\binom{\alpha-1}{n-n_0+1}M.$$

Thus, inequality (40) (and, consequently, inequality (33)) for the indicated values of $n$ will hold if

$$\alpha M - M(\alpha-1) - 2M\left[\binom{\alpha}{3} + \binom{\alpha}{5} + \cdots + \binom{\alpha}{r-1}\right] - (-1)^{n-n_0+1}\binom{\alpha-1}{n-n_0+1}M + a_{\nu\nu}(n)M + \sum_{j=1, j\neq\nu}^{s} a_{\nu j}(n)Y_n^j$$
$$+ \delta_\nu(n) > M.$$

It follows from (41) that, verifying inequality (34), we obtain the same inequality only with the term $\delta_\nu(n)$ replaced by $-\delta_\nu(n)$. Therefore, inequality (45) suffices to prove the validity of both inequalities. Theorem 2 is applicable and the hypothesis of Theorem 3 is its consequence. Formula (38) is a consequence of (7). □

**Remark 1.** From the formula (9) we obtain, setting $\beta + 1 = \alpha > 0$, $\ell = n - n_0$, that the limits of the last terms in formulas (44) and (45) are zeroes since

$$\lim_{n\to\infty} (-1)^{n-n_0+1}\binom{\alpha-1}{n-n_0+1} = 0.$$

Moreover, these last terms are negative as it is easy to show that

$$-(-1)^{n-n_0+1}\binom{\alpha-1}{n-n_0+1} < 0, \tag{49}$$

in the case of inequality (44) and

$$(-1)^{n-n_0+1}\binom{\alpha-1}{n-n_0+1} < 0 \tag{50}$$

in the case of inequality (45).

The following corollary to Theorem 3 provides us with a set of conditions that, although less accurate, are simpler to verify.

**Corollary 1.** *Let $r \in \{1, 2, \dots\}$ be fixed and let $\alpha \in (r-1, r)$. If there exists a positive constant $M$ such that, for every $n \in \mathbb{N}(n_0)$ and $i \in \{1, \dots, s\}$,*

$$a_{ii}(n) > \sum_{j=1, j\neq i}^{s} |a_{ij}(n)| + \frac{1}{M}|\delta_i(n)| + 2(1-\alpha) + 2\left[\binom{\alpha}{2} + \binom{\alpha}{4} + \dots + \binom{\alpha}{r-1}\right], \tag{51}$$

*whenever $r$ is odd and*

$$a_{ii}(n) > \sum_{j=1, j\neq i}^{s} |a_{ij}(n)| + \frac{1}{M}|\delta_i(n)| + 2\left[\binom{\alpha}{3} + \binom{\alpha}{5} + \dots + \binom{\alpha}{r-1}\right], \tag{52}$$

*whenever $r$ is even, then there exists a solution $x : \mathbb{N}(n_0) \to \mathbb{R}^s$ of equation* (4) *satisfying inequalities* (38).

**Proof.** If inequalities (51) and (52) hold, then, due to properties (49) and (50), so do inequalities (44) and (45). Moreover, we show that if inequalities (51) and (52) hold, then so does inequality (43). The reason is that, for the values of $n$ indicated in (43), by formula (46), we have

$$(1-\alpha) + \binom{\alpha}{2} + \binom{\alpha}{3} + \dots + \binom{\alpha}{n-n_0} + \binom{\alpha}{n-n_0+1} \leqslant (1-\alpha) + \binom{\alpha}{2} + \binom{\alpha}{3} + \dots + \binom{\alpha}{r-1} + \binom{\alpha}{r},$$

and it is sufficient, if (51) holds, to prove inequality

$$(1-\alpha) + \binom{\alpha}{2} + \binom{\alpha}{3} + \dots + \binom{\alpha}{r-1} + \binom{\alpha}{r} \leqslant 2(1-\alpha) + 2\left[\binom{\alpha}{2} + \binom{\alpha}{4} + \dots + \binom{\alpha}{r-1}\right]$$

or, by formula (8),

$$0 \leqslant \binom{\alpha}{0} - \binom{\alpha}{1} + \binom{\alpha}{2} - \binom{\alpha}{3} + \dots + \binom{\alpha}{r-1} - \binom{\alpha}{r} = \sum_{p=0}^{r} (-1)^p \binom{\alpha}{p} = (-1)^r \binom{\alpha-1}{r}. \tag{53}$$

Inequality (53) holds because, as $r$ is odd, we have

$$(-1)^r \binom{\alpha-1}{r} > 0.$$

If (52) holds, it is sufficient to prove

$$(1-\alpha) + \binom{\alpha}{2} + \binom{\alpha}{3} + \dots + \binom{\alpha}{r-1} + \binom{\alpha}{r} \leqslant 2\left[\binom{\alpha}{3} + \binom{\alpha}{5} + \dots + \binom{\alpha}{r-1}\right]$$

or, by formula (8),

$$0 \leqslant -\left[\binom{\alpha}{0} - \binom{\alpha}{1} + \binom{\alpha}{2} - \binom{\alpha}{3} + \dots - \binom{\alpha}{r-1} + \binom{\alpha}{r}\right] = -\sum_{p=0}^{r} (-1)^p \binom{\alpha}{p} = -(-1)^r \binom{\alpha-1}{r}. \tag{54}$$

Inequality (54) holds because, as $r$ is even, we have

$$-(-1)^r \binom{\alpha-1}{r} > 0. \qquad \square$$

**Example 3.** Let $s = 2$, $n_0 = 0$ and let system (4) be specified as

$$\Delta^\alpha x_1(n+1) = a_{11}(n)x_1(n) + a_{12}(n)x_2(n) + \delta_1(n), \tag{55}$$

$$\Delta^\alpha x_2(n+1) = a_{21}(n)x_1(n) + a_{22}(n)x_2(n) + \delta_2(n), \tag{56}$$

where $n \in \mathbb{N}(0)$. Set $M = 1$. Inequalities (51) and (52) will hold and, by Corollary 1, systems (55) and (56) will have a solution $x(n) = (x_1(n), x_2(n))^T$ such that $|x_j(n)| < 1$, $j = 1, 2$, $n \in \mathbb{N}(0)$ if, for $i = 1, 2$ and

$$\begin{aligned}
\alpha = 1/2 \in (0,1): &\quad a_{ii}(n) > |a_{i,3-i}(n)| + |\delta_i(n)| + 1,\\
\alpha = 3/2 \in (1,2): &\quad a_{ii}(n) > |a_{i,3-i}(n)| + |\delta_i(n)|,\\
\alpha = 5/2 \in (2,3): &\quad a_{ii}(n) > |a_{i,3-i}(n)| + |\delta_i(n)| + 3/4,\\
\alpha = 7/2 \in (3,4): &\quad a_{ii}(n) > |a_{i,3-i}(n)| + |\delta_i(n)| + 35/8.
\end{aligned}$$

It is easy to see that the higher $\alpha > 1$ is the more restricted the inequalities are for $a_{ii}(n)$, $i = 1, 2$.

## 4.2 The case of $\alpha$ being a positive integer

Let $\alpha$ be a fixed integer $\alpha \in \{1, 2, \ldots\}$. To prove the existence of a solution to linear nonhomogeneous systems of fractional discrete equations (4) satisfying (38), we apply Theorem 2. Let us verify inequalities (40) and (41). In the case considered,

$$\binom{\alpha}{j} > 0 \quad \text{if } j = 2, \ldots, \alpha, \;\; (-1)^{j-1}\binom{\alpha}{j} = 0 \quad \text{if } j = \alpha + 1, \ldots. \tag{57}$$

Assume $n + 1 - n_0 \leqslant \alpha$. Then, referring to (39), and proceeding in much the same way as in the proof of Theorem 3, using (57), we obtain

$$S_\nu > -M\left[\binom{\alpha}{2} + \binom{\alpha}{3} + \cdots + \binom{\alpha}{n - n_0 + 1}\right].$$

Inequalities (40) and (41) (and, consequently, inequalities (33) and (34)) will hold for $n \leqslant n_0 - 1 + \alpha$ if

$$\alpha M - M\left[\binom{\alpha}{2} + \binom{\alpha}{3} + \cdots + \binom{\alpha}{n - n_0 + 1}\right] + a_{\nu\nu}(n)M - M \sum_{j=1, j\neq\nu}^{s} |a_{\nu j}(n)| - |\delta_\nu(n)| > M. \tag{58}$$

Similarly, assuming $n + 1 - n_0 > \alpha$, by (39) and (57), we obtain

$$S_\nu > -M\left[\binom{\alpha}{2} + \binom{\alpha}{3} + \cdots + \binom{\alpha}{\alpha - 1} + \binom{\alpha}{\alpha}\right].$$

Then, inequalities (40) and (41) (and, consequently, inequalities (33) and (34)), will be satisfied if

$$\alpha M - M\left[\binom{\alpha}{2} + \binom{\alpha}{3} + \cdots + \binom{\alpha}{\alpha - 1} + \binom{\alpha}{\alpha}\right] + a_{\nu\nu}(n)M - M \sum_{j=1, j\neq\nu}^{s} |a_{\nu j}(n)| - |\delta_\nu(n)| > M. \tag{59}$$

If (58) and (59) hold, then so do inequalities (40) and (41), and consequently, inequalities (33) and (34) in Theorem 2. Then the following theorem holds.

**Theorem 4.** *Let $\alpha$ be a positive integer. If there exists a positive constant $M$ such that, for every $n_0 \leqslant n \leqslant n_0 + \alpha - 1$ and $i \in \{1, \ldots, s\}$,*

$$a_{ii}(n) > \sum_{j=1, j\neq i}^{s} |a_{ij}(n)| + \frac{1}{M}|\delta_i(n)| + (1 - \alpha) + \left[\binom{\alpha}{2} + \binom{\alpha}{3} + \cdots + \binom{\alpha}{n - n_0} + \binom{\alpha}{n - n_0 + 1}\right],$$

*and for every $n > n_0 + \alpha - 1$ and $i \in \{1, \ldots, s\}$,*

$$a_{ii}(n) > \sum_{j=1, j\neq i}^{s} |a_{ij}(n)| + \frac{1}{M}|\delta_i(n)| + (1 - \alpha) + \left[\binom{\alpha}{2} + \binom{\alpha}{3} + \cdots + \binom{\alpha}{\alpha - 1} + \binom{\alpha}{\alpha}\right],$$

*then there exists a solution $x : \mathbb{N}(n_0) \to \mathbb{R}^s$ of equation* (4) *satisfying* (38).

### 4.3 The case of $\alpha < 0$

If $\alpha < 0$, then

$$(-1)^j\binom{\alpha}{j} > 0 \quad j = 2, \dots. \tag{60}$$

Using (60) and formula (8), we estimate $S_\nu$ under assumption (42),

$$\begin{aligned} S_\nu &> -M\left[(-1)^2\binom{\alpha}{2} + (-1)^3\binom{\alpha}{3} + \dots + (-1)^{n-n_0+1}\binom{\alpha}{n-n_0+1}\right] \\ &= -M\left[(-1)^0\binom{\alpha}{0} + (-1)^1\binom{\alpha}{1} + (-1)^2\binom{\alpha}{2} + (-1)^3\binom{\alpha}{3} + \dots + (-1)^{n-n_0+1}\binom{\alpha}{n-n_0+1}\right] \\ &\quad + M\left[(-1)^0\binom{\alpha}{0} + (-1)^1\binom{\alpha}{1}\right] \\ &= -M(-1)^{n-n_0+1}\binom{\alpha-1}{n-n_0+1} - M(\alpha-1). \end{aligned}$$

Then, inequality (40) (as well as inequality (41)) will hold if

$$\alpha M - M(-1)^{n-n_0+1}\binom{\alpha-1}{n-n_0+1} - M(\alpha-1) + a_{\nu\nu}(n)M - M\sum_{j=1, j\neq\nu}^{s}|a_{\nu j}(n)| - |\delta_\nu(n)| > M.$$

Then, inequalities (33) and (34) in Theorem 2 hold as well, which proves the following theorem.

**Theorem 5.** *Let $\alpha < 0$. If there exists a positive constant $M$ such that, for every $n \in \mathbb{N}(n_0)$ and $i \in \{1, \dots, s\}$,*

$$a_{ii}(n) > \sum_{j=1, j\neq i}^{s}|a_{ij}(n)| + \frac{1}{M}|\delta_i(n)| + (-1)^{n-n_0+1}\binom{\alpha-1}{n-n_0+1},$$

*then there exists a solution $x : \mathbb{N}(n_0) \to \mathbb{R}^s$ of equation* (4) *satisfying inequality* (38).

### 4.4 The case of $\alpha = 0$

Let $\alpha = 0$. Then, $S_\nu = 0$, inequalities (40) and (41) will hold if

$$a_{\nu\nu}(n)M - M\sum_{j=1, j\neq\nu}^{s}|a_{\nu j}(n)| - |\delta_\nu(n)| > M,$$

so the following theorem holds.

**Theorem 6.** *Let $\alpha = 0$. If there exists a positive constant $M$ such that, for every $n \in \mathbb{N}(n_0)$ and $i \in \{1, \dots, s\}$,*

$$a_{ii}(n) > \sum_{j=1, j\neq i}^{s}|a_{ij}(n)| + \frac{1}{M}|\delta_i(n)| + 1,$$

*then there exists a solution $x : \mathbb{N}(n_0) \to \mathbb{R}^s$ of equation* (4) *satisfying inequality* (38).

## 5 Concluding remarks and open problems

In a scalar case, some particular results were announced on conferences [8,9]. The results of the article hold for an arbitrary value $\alpha \in \mathbb{R}$. This is a substantial difference against fractional-order ordinary differential equations where, usually, the cases of $\alpha \in (r-1, r)$ are investigated separately for $r = 1, 2, 3$, etc. Most investigations are related to the fixed values $r = 1$ and $r = 2$.

The usefulness of the method formulated in Theorem 1 was demonstrated by the proofs of the existence of bounded solutions to the linear system (4) satisfying inqualities (38). It is possible to consider various linear as well as nonlinear systems generalizing linear system (4). In particular, it is easy to consider the following generalization of (4):

$$\Delta^\alpha x(n+1) = A(n)x(n) + \delta_n(n, x(n), x(n-1), \dots, x(n_0)), \quad n \in \mathbb{N}(n_0),$$

where the vector $\delta_n = (\delta_n^1, \delta_n^2, \dots, \delta_n^s)^T$ depends on $n, x(n), x(n-1), \dots, x(n_0)$. Systems may also be considered with the $\alpha$-order fractional differences on the left-hand sides of equations being different for each equation. Below we formulate some open problems to suggest further investigations.

**Open problem 1.** Suppose that, for some values $\nu \in \{1, \dots, s\}$, inequalities (15) and (16) in Theorem 1 are replaced by reversed ones. Is it then possible to prove the existence of a $(b, c)$-bounded solution of Volterra-type discrete equation (12) or are some additional assumptions for its existence necessary? If the answer is affirmative, then further applications, such as an investigation of stability by Bohl-Perron theory, seem to be possible.

**Open problem 2.** Theorems 1–6 and Corollary 1 give sufficient conditions for the existence of $(b, c)$-bounded solutions to the systems considered. One of further avenues of investigation can be the construction of methods for determining the initial values $(n_0, x_1^*, \dots, x_s^*)$, $b_i(n_0) < x_i^* < c_i(n_0)$, $i = 1, \dots, s$ generating such solutions. For some first steps concerning non-delayed systems of difference equations, we refer to [3, 4] and references therein.

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## References

[1] P. T. Anh, A. Babiarz, A. Czornik, M. Niezabitowski, and S. Siegmund, *Asymptotic properties of discrete linear fractional equations*, Bull. Pol. Ac.: Tech. **67** (2019), no. 4, 749–759.

[2] A. Babiarz, and J. Klamka, *Controllability of fractional linear switched systems with delays for the fixed sequence*, 11th Asian Control Conference (ASCC) Gold Coast Convention Centre, Australia December 17–20, 2017, pp. 1894–1899.

[3] J. Baštinec, J. Diblík, and S. Pinelas, *Initial data generating bounded solutions of a system of two linear discrete equations*, AIP Conf. Proc. **2293** (2020), 340010–1–340010-4, DOI: https://doi.org/10.1063/5.0026616.

[4] J. Baštinec, J. Diblík, S. Pinelas, and J. Vala, *Determining the initial data generating solutions with prescribed behaviour of a triangular system of linear discrete equations*, Appl. Math. Comput. **425** (2022), 126533, 1–18, DOI: https://doi.org/10.1016/j.amc.2021.126533.

[5] K. Borsuk, *Theory of Retracts, Monografie Matematyczne*, vol. 44, Państwowe Wydawnictwo Naukowe, Warsaw, 1967.

[6] M. Busłowicz, *Controllability, reachability and minimum energy control of fractional discrete-time linear systems with multiple delays in state*, Bull. Polish Acad. Sci. Tech. Sci. **62** (2014), 233–239, DOI: https://doi.org/10.2478/bpasts-2014-0023.

[7] J. Čermák, T. Kisela, and L. Nechvátal, *Stability and asymptotic properties of a linear fractional difference equation*, Adv. Differ. Equ. **2012** (2012), no. 122, 1–14.

[8] J. Diblík, and J. Baštinec, *Bounded solutions of fractional discrete equations of positive non-integer orders*, AIP Conf. Proc. **2425** (2022), 270003–1–270003-4, DOI: https://doi.org/10.1063/5.0081310.

[9] J. Diblík and J. Baštinec, *Existence of solutions of discrete equations of Volterra type in prescribed domains and fractional discrete equations*, AIP Conf. Proc. **2293** (2020), 340007–1–340007-4, DOI: https://doi.org/10.1063/5.0026616.

[10] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd edition, Addison-Wesley Professional, 1994.

[11] T. Kaczorek, *An extension of the Cayley-Hamilton Theorem to different orders fractional linear systems and its application to electrical circuits*, IEEE Trans. Circuits and Systems-?II: Express Briefs **66** (2019), no. 7, 1169–1171.

[12] T. Kaczorek and K. Rogowski, *Fractional linear systems and electrical circuits*, Studies in Systems, Decision and Control, vol. 13, Springer, Cham, 2015, p. xii+254.

[13] T. Kaczorek and L. Sajewski, *Relationship between controllability and observability of standard and fractional different orders discrete-time linear system*, Fract. Calc. Appl. Anal. **22** (2019), no. 1, 158–169.

[14] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematical Studies, vol. 204, Elsevier Science Inc, New York, 2006.

[15] J. Klamka and M. Niezabitowski, *Controllability of the fractional discrete linear time-varying infinite-dimensional systems*, AIP Conf. Proc. **1738** (2015), 130004–1–130004-4, DOI: https://doi.org/10.1063/1.4951920.

[16] D. Mozyrska, E. Pawłuszewicz, and M. Wyrwas, *Local observability and controllability of nonlinear discrete-time fractional order systems based on their linearisation*, Int. J. Sys. Sci. **48** (2017), no. 4, 788–794.

[17] D. Mozyrska and M. Wyrwas, *Stability of discrete fractional linear systems with positive orders*, IFAC PapersOnLine **50** (2017), no. 1, 8115–8120, DOI: https://doi.org/10.1016/j.ifacol.2017.08.1250.

[18] I. Podlubny, *Fractional differential equations, an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in Science and Engineering, vol. 198, Academic Press, Inc., San Diego, CA, 1999.

[19] R. Srzednicki, *Ważewski method and Conley index*, Handbook of Differential Equations, Elsevier/North-Holland, Amsterdam, 2004, pp. 591–684.