A global approach for the redefinition of higher-order flexibility and rigidity

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Abstract. The famous example of the double-Watt mechanism given by Connelly and Servatius raises some problems concerning the classical definitions of higher-order flexibility and rigidity, respectively, as they attest the cusp configuration of the mechanism a third-order rigidity, which conflicts with its continuous flexion. Some attempts were done to resolve the dilemma but they could not settle the problem. As cusp mechanisms demonstrate the basic shortcoming of any local mobility analysis using higher-order constraints, we present a global approach inspired by Sabitov's finite algorithm for testing the bendability of a polyhedron, which allows us (a) to compute iteratively configurations with a higher-order flexion and (b) to come up with a proper redefinition of higher-order flexibility and rigidity. The presented approach is demonstrated on several examples (double-Watt mechanisms and Tarnai's Leonardo structure). Moreover, we determine all configurations of a given 3-RPR manipulator with a third-order flexion and present a corresponding joint-bar framework of flexion order 23.

Key words: higher-order flexibility, higher-order rigidity, double-Watt mechanism, 3-RPR robot

1 Introduction

In this paper we give a redefinition of higher-order flexibility and rigidity of bar-joint frameworks. Such a framework $G(\mathscr{K})$ consists of a knot set

$$\mathscr{K} = \{X_1, \ldots, X_w\} \tag{1}$$

and a graph G on $\mathscr{K}$. A knot X_i corresponds a rotational/spherical joint (without clearance) in the case of a planar/spatial framework. An edge connecting two knots corresponds to a bar. We denote the number of edges by e.

By defining the combinatorial structure of the framework as well as the lengths of the bars, which are assumed to be non-zero, the intrinsic geometry of the framework is fixed. In general the assignment of the intrinsic metric does not uniquely determine the embedding of the framework into the Euclidean space $\mathbb{R}^d$, thus such a framework can have different incongruent realizations.

1.1 Algebraic approach to rigidity theory

The relation that two elements of the knot set are edge-connected can also be expressed algebraically. They are either quadratic constraints resulting from a squared distance of vertices (implied by an edge) or linear condition, in the case that one of the pin-joints gets an ideal-point. There are further linear conditions steaming from the elimination of isometries[1]. In total this results in a system of l algebraic equations $c_1 = 0, \ldots, c_l = 0$ in m unknowns $z_1, \ldots, z_m$, which constitute an algebraic variety $V(c_1, \ldots, c_l)$. Note that l equals $e+6$ in the spatial case and $e+3$ in the planar one. Moreover, m equals for the planar case $2w$ and for the spatial one $3w$.

If $V(c_1, \ldots, c_l)$ is positive-dimensional then the framework is flexible; otherwise rigid. The framework is called minimally rigid (isostatic) if the removal of any algebraic constraint (resulting from an edge) will make the framework flexible. In this case $m = l$ has to hold. Rigid frameworks, which are not isostatic, are called *overbraced* or *overconstrained* ($l > m$).

If $V(c_1, \ldots, c_l)$ is zero-dimensional, then each real solution corresponds to a realization $G(\mathbf{X})$ of the framework for $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_w)$. If there is exactly one real solution, then the framework is called globally rigid.

We can compute in a realization the tangent-hyperplane to each of the hypersurfaces $c_i = 0$ in $\mathbb{R}^m$ for $i = 1, \ldots, l$. Note that this is always possible as all hypersurfaces are either hyperplanes or regular hyperquadrics. The normal vectors of these tangent-hyperplanes constitute the columns of a $m \times l$ matrix $\mathbf{R}_{G(\mathbf{X})}$, which is also known as *rigidity matrix* of the realization $G(\mathbf{X})$; i.e.

$$\mathbf{R}_{G(\mathbf{X})} = \begin{pmatrix} \frac{\partial c_1}{\partial z_1} & \frac{\partial c_2}{\partial z_1} & \cdots & \frac{\partial c_l}{\partial z_1} \\ \frac{\partial c_1}{\partial z_2} & \frac{\partial c_2}{\partial z_2} & \cdots & \frac{\partial c_l}{\partial z_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial c_1}{\partial z_m} & \frac{\partial c_2}{\partial z_m} & \cdots & \frac{\partial c_l}{\partial z_m} \end{pmatrix}. \tag{2}$$

If its rank is m then the realization is infinitesimal rigid otherwise it is infinitesimal flexible; i.e. the hyperplanes have a positive-dimensional affine subspace in common. Therefore the intersection multiplicity of the l hypersurfaces is at least two in a shaky realization. Therefore a shaky configuration can also be seen as the limiting case where at least two realizations of a framework coincide [34, 40].

Clearly, by using the rank condition $rk(\mathbf{R}_{G(\mathbf{X})}) < m$ one can also characterize all shaky realizations $G(\mathbf{X})$ algebraically by the affine variety V_1. This so-called shakiness variety is the zero set of the ideal generated by the polynomials $p_1, \ldots, p_\mu$ which correspond to all $\mu := \binom{l}{l-m}$ minors of $\mathbf{R}_{G(\mathbf{X})}$ of order $m \times m$. Note that for minimally rigid framework $\mu = 1$ holds, where the infinitesimal flexibility is given by $p_1 : \det(\mathbf{R}_{G(\mathbf{X})}) = 0$.

[1] This are 6 linear constraints for $d = 3$ and 3 linear constraints for $d = 2$.

2 Review on higher-order flexibility and rigidity

A first paper on the higher-order flexion of surfaces was written by Rembs [29]. In contrast first results on higher-order rigidity of surfaces date back to Efimov [9]. An exhaustive treatment of higher-order flexion and rigidity of surfaces was done by Sabitov in [30], in which also a section is devoted to discrete structures. Connelly gave a definition of 2^{nd}-order flexibility and rigidity of frameworks in [7]. Tarnai wrote a paper [43] on the definition of higher-order infinitesimal mechanisms, which seems to be more problematic than that of a framework due to the existence of non-analytic kinematic pairs. According to Stachel [38] all these approaches to higher-order flexible frameworks can be unified to the so-called *classical* definition, which reads as follows:

Definition 1. A framework has a n^{th}-order flex if for each vertex $\mathbf{x}_i$ $(i = 1, \ldots, w)$ there is a polynomial function

$$\mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,1}t + \ldots + \mathbf{x}_{i,n}t^n \quad \text{with} \quad n > 0 \tag{3}$$

such that

1. the replacement of $\mathbf{x}_i$ by $\mathbf{x}_i'$ in the equation of the edge lengths gives stationary values of multiplicity $\geq n+1$ at $t = 0$;
2. the velocity vectors $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{w,1}$ do not originate from a rigid body motion (incl. standstill) of the complete framework; i.e. they are said to be non-trivial.

Remark 1. Tarnai's definition relies on the power-series expansion of the elongation of the bar in terms of the displacement, but his definition is equivalent to Definition 1 (cf. [43]). Following an idea of Koiter, one can replace the bar elongation by the strain energy of the bars, which also results in an equivalent definition (cf. [31]). Moreover, Kuznetsov [19] gave another definition of higher-flexibility, which relies on the Taylor expansion of the constrained equations of the framework. Without noticing it, exactly the same approach was used by Chen [6] to define the local mobility of a mechanism. It can be seen from [28], that the (identical) definitions of Kuznetsov and Chen are again equivalent with Definition 1. ◇

Based on the notion of n^{th}-order flex given in Definition 1 one can define n^{th}-order rigidity as follows [7, 8]:

Definition 2. A framework is n^{th}-order rigid if every n^{th}-order flex has $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{w,1}$ trivial as a 1^{st}-order flex.

Remark 2. Clearly, in the context of Definition 1 one is only interested in the flex with maximal n; i.e. the framework has to be rigid of order $(n+1)$ according to Definition 2. ◇

But the famous example of the double-Watt mechanism (cf. Fig. 1) given by Connelly and Servatius [8] raises some problems concerning these Definitions 1 and 2, as they attest this mechanism a 3^{rd}-order rigidity in a certain configuration, which

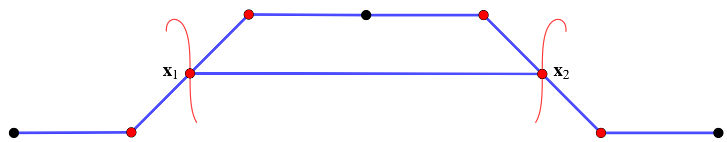

Fig. 1 Double-Watt mechanism of Connelly and Servatius in its cusp configuration; i.e. the mechanism has an instantaneous standstill. The dimensions of each Watt mechanism are as follows: the arms have length 1 and the coupler is of length $\sqrt{2}$. The midpoints $\mathbf{x}_1$ and $\mathbf{x}_2$ of both couplers are connected by a bar of length 3.

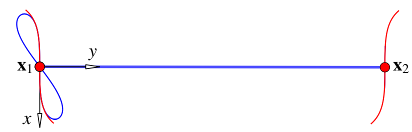

Fig. 2 Reduction of the double-Watt mechanism of Connelly and Servatius to a two-point guidance problem.

conflicts[2] with its continuous flexibility. This configuration corresponds to a cusp in the configuration space [8], which was also pointed out by Müller's study [26] of the mechanism from the perspective of kinematic singularities. Based on the latter work further examples of cusp mechanisms (even spatial ones) were given in [22].

Example 1. In the following we present the analysis of the double-Watt mechanism according to the method presented by Stachel in [39]. With respect to the coordinate system displayed in Fig. 2 the coupler-curve of the point $\mathbf{x}_1$ is given by the algebraic equation

$$x^6+3x^4y^2+3x^2y^4+y^6+3x^4+6x^3y-2x^2y^2+6xy^3-5y^4-6xy+8y^2=0. \quad (4)$$

We are interested in the branch where the x-axis is the tangent to the inflection point. It can be parametrized locally by means of Puiseux series as:

$$\mathbf{x}_1=\begin{pmatrix}\tau_1\\ \frac{1}{2}\tau_1^3+\tau_1^5+\frac{9}{4}\tau_1^7+\frac{13}{2}\tau_1^9+\ldots\end{pmatrix}. \quad (5)$$

Clearly, the path of $\mathbf{x}_2$ is obtained by reflection on the x-axis and by translation along the vector $(0,3)^T$ yielding:

$$\mathbf{x}_2=\begin{pmatrix}\tau_2\\ 3-\frac{1}{2}\tau_2^3-\tau_2^5-\frac{9}{4}\tau_2^7-\frac{13}{2}\tau_2^9-\ldots\end{pmatrix}. \quad (6)$$

Thus we end up with a two-point guidance problem, where the time dependence of τ_i is set up by

[2] One expects from a proper definition that an n^{th}-order rigidity implies rigidity (cf. [8]).

$$\tau_i = v_{i,1}t + v_{i,2}t^2 + v_{i,3}t^3 + \ldots. \tag{7}$$

Now the $v_{i,j}$ have to be adjusted in order to fulfill

$$F := \|\mathbf{x}_2(\tau_2) - \mathbf{x}_1(\tau_1)\|^2 - 3^2 = o(t^n) \tag{8}$$

for a n^{th}-order flexibility at $t = 0$. We substitute Eq. (7) into Eq. (8) and consider the coefficients f_i of t^i in the resulting expression. We get $f_1 = 0$ and $f_2 = (v_{1,1} - v_{2,1})^2$. Setting $v_{2,1} = v_{1,1}$ we get $f_3 = -6v_{1,1}^3$. This means with $v_{1,1} \neq 0$ it is only flexible of 2^{nd}-order implying 3^{rd}-order rigidity. ⋄

Two attempts are known to the author to resolve the dilemma (cf. Footnote 2): Gaspar and Tarnai [11] suggested to use fractional exponents which corresponds to the replacement of Eq. (3) by

$$\mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,1}t + \mathbf{x}_{i,\frac{3}{2}}t^{\frac{3}{2}} + \mathbf{x}_{i,2}t^2 + \mathbf{x}_{i,\frac{5}{2}}t^{\frac{5}{2}} \ldots + \mathbf{x}_{i,n}t^n, \tag{9}$$

where $\mathbf{x}_{1,1}, \ldots, \mathbf{x}_{w,1}$ is non-trivial. This solved the particular problem for the cusp configuration of the double-Watt mechanism but not the parametrization problem according to [44], where it is also written that "*a very promising approach was presented recently by [38]*".

This approach of Stachel follows the more general notation of (k,n)-flexibility suggested by Sabitov [30] which replaces Eq. (3) by

$$\mathbf{x}_i' := \mathbf{x}_i + \mathbf{x}_{i,k}t^k + \ldots + \mathbf{x}_{i,n}t^n \quad \text{with} \quad n \geq k > 0 \tag{10}$$

where $\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{w,k}$ is non-trivial. In addition Eq. (10) has to represent an irreducible flex; this means that Eq. (10) does not result from a polynomial parameter substitution of a lower-order flex.

Example 2. Continuation of the double-Watt mechanism: According to the notation of Eq. (10) the double-Watt mechanism in the cusp configuration is $(1,2)$-flexible but not $(1,3)$-flexible (cf. Example 1). Therefore we set $v_{1,1}$ and continue Example 1 by considering $f_4 = (v_{1,2} - v_{2,2})^2$. We set $v_{2,2} = v_{1,2}$ and get $f_5 = 0$. Moreover, for f_6 we obtain the expression $-6v_{1,2}^3 + v_{1,3}^2 - 2v_{1,3}v_{2,3} + v_{2,3}^2$, which can be solved for[3] $v_{2,3} = v_{1,3} \pm \sqrt{6v_{1,2}^3}$ showing $(2,6)$-flexibility. Moreover, we can proceed in this way (i.e. solving $f_i = 0$ for $v_{2,i-3}$ for $i > 6$) implying $(2,\infty)$-flexibility.

We only have to check that the $(2,\infty)$-flexibility was not obtained by the $(1,2)$-flexibility by a polynomial parameter substitution of the form

$$t = \bar{t}^p(a_0 + a_1\bar{t} + a_2\bar{t}^2 + \ldots) \tag{11}$$

with $a_0 \neq 0$ and $p > 1$. For $p = 2$ we get $\overline{f}_1 = \overline{f}_2 = \overline{f}_3 = 0$. $\overline{f}_4 = a_0^2(v_{1,1} - v_{2,1})^2$ implies $v_{2,1} = v_{1,1}$. Then $\overline{f}_5 = 0$ and $\overline{f}_6 = -6a_0^3v_{1,1}^3$. Therefore the substitution turns

[3] Note that the $\pm$ sign corresponds to the two ways out of the cusp configurations.

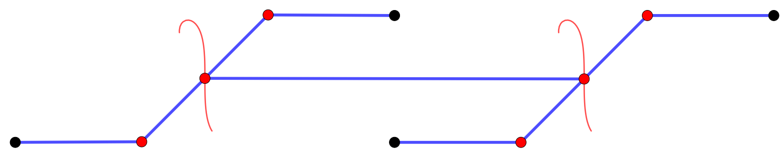

Fig. 3 Double-Watt mechanism of Stachel in a branching configuration; i.e. it corresponds to a double point in the configuration space.

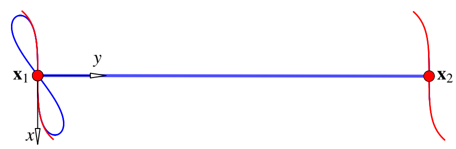

Fig. 4 Reduction of Stachel's double-Watt mechanism to a two-point guidance problem.

the $(1,2)$-flexibility into a reducible $(2,5)$-flexibility. As a consequence the $(2,\infty)$-flexibility has to be an irreducible flex. $\diamond$

Remark 3. Note that the substitution of Eq. (11) into Eq. (9) for $p=2$ yields the $(2,\infty)$-flexibility of Stachel discussed in the last example. Therefore, Stachel's approach also includes the one of Gaspar and Tarnai [11]. $\diamond$

Stachel's proposal was only presented within the Tensegrity Workshop in 2007 [38], but remained unpublished so far. According to Stachel [41], the reason for this is the example of another double-Watt mechanism, which is extended by a Kempe-mechanism (cf. Fig. 5), presented in [39], as no unique flexion order can be identified with his proposed definition. Therefore the problem is not yet settled.

Example 3. Stachel's double-Watt framework: In the following we also give this example of Stachel where the second Watt-mechanism is just a translation of the first one (see Fig. 3) by the vector $(0,3)^T$. Thus we get for the path of $\mathbf{x}_2$ the following parametrization

$$\mathbf{x}_2 = \begin{pmatrix} \tau_2 \\ 3+\frac{1}{2}\tau_2^3+\tau_2^5+\frac{9}{4}\tau_2^7+\frac{13}{2}\tau_2^9+\ldots \end{pmatrix} \tag{12}$$

for the interpretation as a two-point guidance problem, which is illustrated in Fig. 4. In this case the two-point guidance is in a branching configuration; i.e. it corresponds to a double point in the configuration space.

Then Stachel extended his double-Watt linkage by a Kempe-mechanism for the generation of the straight line motion of the midpoint $\mathbf{x}_3$ of $\mathbf{x}_1$ and $\mathbf{x}_2$ (see Fig. 5). In contrast, we only use a point guidance[4] to restrict the location of $\mathbf{x}_3$ on the line $y=\frac{3}{2}$ (cf. Fig. 6); i.e.

[4] This can also be interpreted in the terms of bar-joint framework, where the corresponding pin-joint is the ideal point of the y-axis.

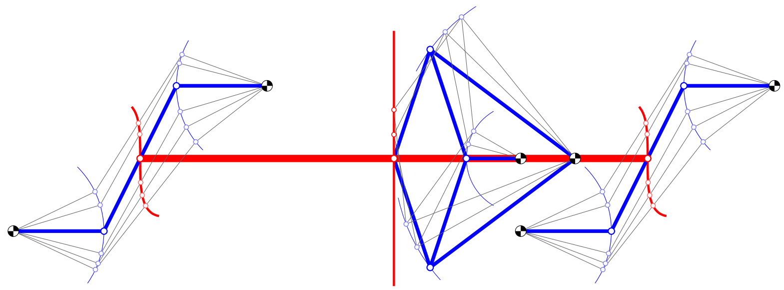

Fig. 5 Stachel's double-Watt mechanism extended by a Kempe-mechanism (Figure by courtesy of Hellmuth Stachel [39]).

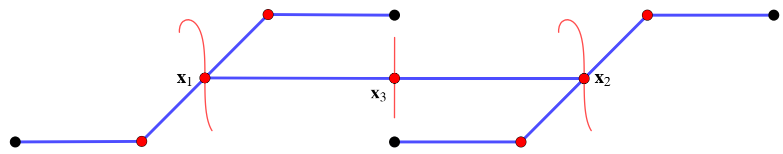

Fig. 6 Stachel's double-Watt mechanism extended by the guidance of the midpoint $\mathbf{x}_3$ of $\mathbf{x}_1$ and $\mathbf{x}_2$ along a straight line.

$$\mathbf{x}_3 = \begin{pmatrix} \tau_3 \\ \frac{3}{2} \end{pmatrix}. \tag{13}$$

For a n^{th}-order flexibility at $t = 0$ still Eq. (8) has to hold as well as the affine combination

$$\begin{pmatrix} G \\ H \end{pmatrix} := \mathbf{x}_1(\tau_1) + \mathbf{x}_2(\tau_2) - 2\mathbf{x}_3(\tau_3) = \mathbf{o}(t^n). \tag{14}$$

We substitute Eq. (7) into Eqs. (8) and (14) and consider the coefficients f_i, g_i and h_i of t^i in the resulting expressions. It can easily be seen that $g_i = v_{1,i} + v_{2,i} - 2v_{3,i}$ holds for all $i = 1,2,\ldots$, thus we set

$$v_{3,i} = \tfrac{v_{1,i}+v_{2,i}}{2}. \tag{15}$$

Moreover we get $f_1 = h_1 = h_2 = 0$ and $f_2 = (v_{1,1} - v_{2,1})^2$. We set $v_{2,1} = v_{1,1}$ and obtain $f_3 = 0$ and $h_3 = v_{1,1}^3$. Therefore this results in a $(1,2)$-flexibility.

Now we consider the case $v_{1,1} = 0$: Then we get $f_4 = (v_{1,2} - v_{2,2})^2$. Thus we set $v_{2,2} = v_{1,2}$ and get $h_4 = f_5 = h_5 = 0$. Moreover we obtain $f_6 = (v_{1,3} - v_{2,3})^2$, implying $v_{2,3} = v_{1,3}$ and $h_6 = v_{1,2}^3$. The latter shows a $(2,5)$-flexibility.

Now we can set $v_{1,2} = 0$ and proceed this procedure yielding the following sequence of flexion orders $(k, 3k-1)$ for $k = 1,2,\ldots$. According to Stachel the question remained open which is the correct order, as all the obtained ones are irreducible. This can be seen as follows:

The conditions for a $(1,2)$-flex which are

$$v_{3,1} = \tfrac{v_{1,1}+v_{2,1}}{2}, \quad v_{3,2} = \tfrac{v_{1,2}+v_{2,2}}{2}, \quad v_{2,1} = v_{1,1} \tag{16}$$

imply under the polynomial parameter substitution of Eq. (11) a reducible $(p, 3p-1)$-flexibility. Let's do this explicitly for $p = 2$. Then we get:

$$\overline{v}_{i,2} = a_0 v_{i,1} \tag{17}$$
$$\overline{v}_{i,3} = a_1 v_{i,1} \tag{18}$$
$$\overline{v}_{i,4} = a_0^2 v_{i,2} + a_2 v_{i,1} \tag{19}$$
$$\overline{v}_{i,5} = 2a_0 a_1 v_{i,2} + a_3 v_{i,1}. \tag{20}$$

Therefore the conditions for the $(2,5)$-flexibility, which are

$$\overline{v}_{1,2} - \overline{v}_{2,2} = 0, \quad \overline{v}_{1,3} - \overline{v}_{2,3} = 0, \quad \overline{v}_{1,j} + \overline{v}_{2,j} - 2\overline{v}_{3,j} = 0 \tag{21}$$

for $j = 2,3,4,5$ are fulfilled identically under Eq. (16). But $\overline{v}_{1,4}$ and $\overline{v}_{2,4}$ are in a certain relation as only a_2 can act as a free parameter, which in general has not to be the case. This shows the irreducibly of Stachel's $(2,5)$-flexibility. The same argument can be done also for the higher flexion orders in Stachel's sequence $(k, 3k-1)$. ⋄

Remark 4. Note that flexibility of 1^{st}-order is invariant under projectivities [47, 52] but this does not hold for higher-orders (even not for affine transformations). ⋄

Remark 5. Note that it is well known (cf. [30, page 232] and [2]) that there exists for each geometric structure an upper bound n^* such that the n^*-order flexibility results in a continuous flexion. ⋄

2.1 Structures studied with respect to higher-order flexibility

Wohlhart [49] followed Kuznetsov's approach (using a kinematic interpretation of the power-expansion in terms of velocity, acceleration, jerk, and so forth) for the study of higher-order flexible planar and spatial parallel manipulators of Stewart–Gough type. A deeper geometric study of these planar mechanisms was done by Stachel in [35], who also studied higher-order flexibility of bipartite planar frameworks [36] as well as octahedra [37]. Open and closed spatial serial chains where studied in [6, 26, 51]. Kuznetsov [19] and Tarnai [43] demonstrate their theoretical considerations only on basis of some simple planar linkages, where the so-called Leonardo structure [44] has to be pointed out as in this way frameworks with a $(2^\lambda - 1)$-order flex (according to Def. 1) for arbitrary $\lambda \in \mathbb{N}$ can be constructed (cf. Fig. 7). Local rigidity analysis of origami structures up to the 2^{nd}-order were done by He and Guest [13]. A characterization for 2^{nd}-order flexibility of quad-surfaces with planar faces was given by Schief et al. [33]. Finally, Tachi [42] capped rigid-foldable tubes with 2^{nd}-order flexible structures.

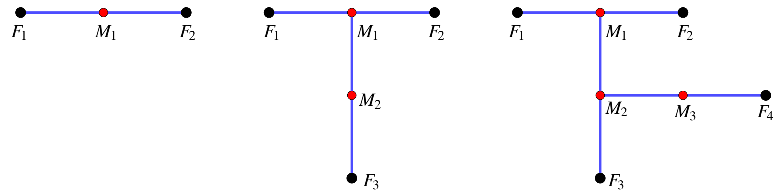

Fig. 7 Leonardo structure for $\lambda = 1$ (left), $\lambda = 2$ (center) and $\lambda = 3$ (right).

Remark 6. One should not forget about the work of Walter Wunderlich, who studied the geometry of several shaky structures and sometimes pointed out special ones with a higher-order flexibility (see overview article [34]). $\diamond$

Due to Remark 5 the idea of higher-order flexibility can also be used to compute over-constrained mechanisms. Based on the approach of Kuznetsov this method was stressed by Wohlhart [50] to determine a special class of Stewart–Gough platforms with self-motions and by Bartkowiak and Woernle [4, 5] as well as Milenkovic [25] for the design of overconstrained single-loop mechanisms. In contrast, Rameau and Serre [28] focused on different computational methods of this problem. From the computation point of view also the work [45] should be mentioned, where numeric algebraic geometry is used to test locally a so-called *high-multiplicity infinitesimal degree of freedom* by means of Macaulay matrices.

3 Redefinition of a higher-order flexibility and rigidity

According to Müller [26] the above mentioned examples with cusps in the configuration space (cusp mechanisms) demonstrate the basic shortcoming of any local mobility analysis using higher-order constraints. Therefore we present a global approach, which is also inspired by an idea of Sabitov like Stachel's approach; namely by his finite algorithm for testing the bendability of a polyhedron [30, page 231]. This can be formulated as follows:

Let us consider the configuration-set $\mathscr{S}$ of all frameworks having the same connectivity but only differ in their intrinsic metric. Note that $\mathscr{S}$ is only a subset of $\mathbb{R}^m$ (due to the fact that edges are not allowed to have zero length). In the case of 1^{st}-order flexibility each vertex $\mathbf{x}_i$ ($i = 1, \ldots, w$) can be associated with a velocity vector $\mathbf{x}_{i,1}$ such that the edge lengths do not change instantaneously, where the set of velocity vectors is not allowed to originate from a rigid body motion (incl. standstill); i.e. no trivial 1^{st}-order flex. The subset $\mathscr{S}_1 \subset \mathscr{S}$ of 1^{st}-order flexible configurations corresponds to the already mentioned shakiness variety V_1 in $\mathbb{R}^m$. The sets $\mathscr{S}_j$ with $j > 1$ are defined recursively as follows: If in a point of $\mathscr{S}_{j-1}$ a non-trivial 1^{st}-order flex exists, which is tangential to V_{j-1} then this point belongs to the set $\mathscr{S}_j$ thus we get a hierarchical structure of flexibility of higher-order. A configuration is called n^{th}-order flexible if it belongs to $\mathscr{S}_n$ but not to $\mathscr{S}_{n+1}$.

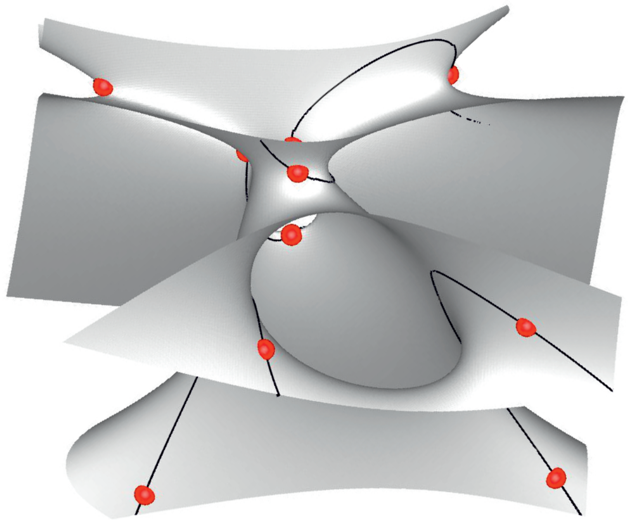

Fig. 8 Illustration of the surface $\mathscr{S}_1$ (gray), the curve $\mathscr{S}_2$ (black) and the discrete set $\mathscr{S}_3$ of points (red) for the configurations of the 3-RPR manipulator discussed in Example 8.

We proceed with a discussion of this approach:

- This approach goes along with a recent result of Alexandrov [3] for smooth surfaces, who was able to show that a 1^{st}-order flex tangential to V_1 can be extended to a 2^{nd}-order flex.
- Sabitov assumed that all the appearing sets $\mathscr{S}, \mathscr{S}_1, \mathscr{S}_2, \ldots$ are manifolds and submanifolds, respectively. In general the varieties $V_1, V_2, \ldots$ contain singular points, which correspond mostly to the interesting configurations in the study of higher-order flexibility.
- An analogous assumption has to be done by Alexandrov [3] in the smooth setting mentioned above, namely the restriction to regular points of V_1.

This means that this approach gives a proper definition of n^{th}-order flexibility for configurations that correspond to points of $\mathbb{R}^m$ which are regular with respect to each of the varieties $V_1, V_2, \ldots, V_n$.

Lemma 1. *Every regular point of V_1 has to have a single non-trivial instantaneous flexion.*

Proof. Let recall that V_1 is the zero set of the ideal generated by all minors $p_1, \ldots, p_\mu$ of $\mathbf{R}_{G(\mathbf{X})}$ of order $m \times m$.

Let p_j equals the $\det(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m)$ where the $\mathbf{r}_i$'s denote pairwise distinct columns of the rigidity matrix $\mathbf{R}_{G(\mathbf{X})}$ given in Eq. (2).

Now the entries of the gradient of p_j, which is given by

$$\nabla p_j = \left(\frac{\partial p_j}{\partial z_1}, \frac{\partial p_j}{\partial z_2}, \ldots, \frac{\partial p_j}{\partial z_m} \right), \tag{22}$$

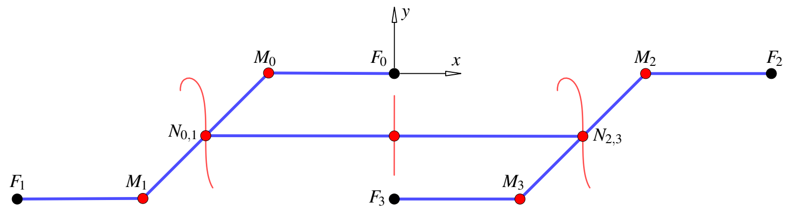

Fig. 9 Stachel's double-Watt mechanism extended by the guidance of the midpoint $\mathbf{x}_3$ of $\mathbf{x}_1$ and $\mathbf{x}_2$ along a straight line.

can be computed due to the following product rule for determinants [1, page 626]:

$$\frac{\partial p_j}{\partial z_i} = \det(\frac{\partial \mathbf{r}_1}{\partial z_i}, \mathbf{r}_2, \ldots, \mathbf{r}_m) + \det(\mathbf{r}_1, \frac{\partial \mathbf{r}_2}{\partial z_i}, \ldots, \mathbf{r}_m) + \ldots + \det(\mathbf{r}_1, \mathbf{r}_2, \ldots, \frac{\partial \mathbf{r}_m}{\partial z_i}). \quad (23)$$

This already shows that for points of V_1 with $rk(\mathbf{R}_{G(\mathbf{X})}) < m-1$ all these gradients $\nabla p_{1,j}$ are zero vectors, as all summands of Eq. (23) are zero. As a consequence, these points have to be singular ones of V_1. □

Remark 7. Lemma 1 explains Husty's observation for 3-RPR mechanisms given in [14]; namely "*the surprising property that it* (singularity surface) *has a singularity itself at the point which corresponds to the pose with two dof local mobility*."

Moreover, this lemma also gives another reasoning for the note in [12] that the transverse rigidity test always fails if more than one non-trivial infinitesimal flex exist, as in this case the corresponding point on V_1 has to be a singular one. ◇

Example 4. Let us consider Stachel's extended double-Watt framework introduced in Example 3. For setting up the algebraic equations we use the following coordinatization according to Fig. 9:

$$F_0 = (0,0)^T, \quad F_1 = (-3,-1)^T, \quad F_2 = (3,0)^T, \quad F_3 = (0,-1)^T, \quad (24)$$

for the points pinned to the base and

$$M_0 = (a_0, b_0)^T, \quad M_1 = (a_1, b_1)^T, \quad M_2 = (a_2, b_2)^T, \quad M_3 = (a_3, b_3)^T, \quad (25)$$

for the moving points. Then the mechanism is determined by the following set of eight equations:

$$\|F_i - M_i\|^2 = 1 \qquad \text{for} \quad i = 0, \ldots, 3 \quad (26)$$

$$\|M_j - M_{j+1}\|^2 = 2 \qquad \text{for} \quad j = 0, 2 \quad (27)$$

$$\|N_{0,1} - N_{2,3}\|^2 = 9 \quad (28)$$

$$a_0 + a_1 + a_2 + a_3 = 0 \quad (29)$$

with $N_{0,1} = \frac{M_0+M_1}{2}$ and $N_{2,3} = \frac{M_2+M_3}{2}$. Note that Eq. (29) corresponds with the straight line motion of the midpoint of $N_{0,1}$ and $N_{2,3}$. Direct computations show that the rank of the (8×8)-rigidity matrix $\mathbf{R}_{G(\mathbf{X})}$ in the configuration $\mathbf{X}$ given by

$$(a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3) = (-1, 0, -2, -1, 2, 0, 1, -1) \tag{30}$$

equals 6. This confirms that $\mathbf{X}$ is a singular point of the variety V_1. ⋄

Beside Lemma 1 we also have to keep in mind that according to Remark 5 a certain flexibility order n^* exists which implies flexibility of order ∞. With this ingredients we can prove the following alternative characterization:

Theorem 1. *The n^{th}-order flexibility with $n < n^*$ of a configuration which corresponds to a regular point of each variety $V_1, V_2, \ldots, V_n$ is equivalent with the fact that the configuration is a framework realization of multiplicity $n+1$.*

Proof. V_1 is the set of points determined by the constraint equations $c_1, \ldots, c_l$ with multiplicity of at least two. In a general point $\mathbf{X}$ of V_1 the intersection multiplicity with respect to $V_1 \cap c_1 \cap \ldots \cap c_l$ is 1. For increasing it a necessary and sufficient condition is that the tangent spaces have a positive-dimensional subspace in common. This is exactly the condition that in $\mathbf{X}$ an instantaneous flexion exists, which is tangential to V_1. Due to Lemma 1 the common subspace has to be 1-dimensional. Therefore by construction a generic element of the variety V_2 has to have multiplicity 3 with respect to $V(c_1, \ldots, c_l)$.

This line of argumentation can be iterated until we reach the set V_{n^*}, which consists of points having multiplicity ∞. Thus points of $V_n \setminus V_{n+1}$ with $n < n^*$ have to correspond with framework realizations of multiplicity $n+1$. □

A redefinition can be based on this property as it can also be extended to singular points of the varieties $V_1, V_2, \ldots$ which are not covered by Sabitov's algorithm.

Definition 3. If a configuration does not belong to a continuous flexion of the framework then we define its order of flexibility by the number of coinciding framework realizations minus 1.

Remark 8. This definition follows the way Wunderlich (cf. Remark 6) studied infinitesimal flexibility; namely as the limiting case where two realizations of a framework coincide (cf. Stachel [40]). ⋄

Based on Definition 3 we can also give a redefinition of higher-order rigidity as follows:

Definition 4. Is a configuration n^{th}-order flexible according to Definition 3 then it is $(n+1)$-rigid.

3.1 Computational Aspects

As we now have obtained proper redefinitions of higher-order flexibility and rigidity, we remain with the problem of how to compute the number of coinciding realizations. For that we have to calculate the intersection multiplicity of the hypersurfaces

$c_1,\dots,c_l$ in the considered configuration $\mathbf{X}$. For the determination of the flexion order we suggest the following 3-step algorithm:

1. According to the Lasker–Noether theorem every algebraic set is the union of a finite number of uniquely defined algebraic sets known as irreducible components. They can be computed with an irredundant primary decomposition[5] algorithm (see e.g. [21]).
2. Then one has to test if the given realization is contained in a irreducible composition of dimension 1 or higher. If this is the case the configuration $\mathbf{X}$ is assigned with the flexion order ∞ (in accordance with Fulton's properties [10] of an intersection number). If this is not the case then we identify all zero-dimensional primary ideals $I_1,\dots,I_s$ containing $\mathbf{X}$.
3. We compute the intersection multiplicity q_i of $\mathbf{X}$ with respect to each primary ideal I_i for $i=1,\dots,s$. Then the intersection multiplicity of $\mathbf{X}$ with respect to the hypersurfaces $c_1,\dots,c_l$ equals the sum $q_1+\dots+q_s$.

Remark 9. According to Definition 3 the flexion order equals $q_1+\dots+q_s-1$, but if one is interested in a more detailed analysis of the configuration and its flexion order, then one should have a look at the sequence $(q_1,\dots,q_s)$. It is well known that the irredundant primary decomposition has not to be unique; but in our case we are save as we assumed that all primary ideals containing $\mathbf{X}$ are zero-dimensional. Therefore they have to correspond to minimal prime ideals and not to embedded ones, which are causing non-uniqueness (cf. [21]). ⋄

In the following we sketch a possibility for the computation of q_i. Let us assume that the zero-dimensional primary ideal I_i is generated by polynomials $g_1,\dots,g_\gamma$. We distinguish the following two cases:

a) If $\gamma=m$; i.e. I_i is a complete intersection, then we can use theoretically the U-resultant method (see [24, § 18], [46, § 83] or [18]), which works as follows: One adds the so-called U-polynomial

$$g_0=u_0+u_1z_1+\dots+u_mz_m \tag{31}$$

to the set $g_1,\dots,g_m$ and eliminates $z_1,\dots,z_m$ by means of Macaulay resultant [23]. This results in a homogeneous polynomial $R(g_0,\dots,g_m)$ where the degree equals the product of the degrees of $g_1,\dots,g_m$. Moreover, $R(g_0,\dots,g_m)$ factorizes into powers of f linear factors

$$\prod_{j=1}^{f}\left(\zeta_{j,0}u_0+\zeta_{j,1}u_1+\dots+\zeta_{j,m}u_m\right)^{q_j}. \tag{32}$$

Then the jth common point of $g_1,\dots,g_m$ has multiplicity q_j and its coordinates are given by $z_i=\zeta_{j,i}/\zeta_{j,0}$ for $i=1,\dots,m$.

b) If $\gamma>m$ one can use a generalization of the U-resultant method given by Lazard [20] to end up with an expression of the form given in Eq. (32).

[5] The prime decomposition is not valid as it does not preserve the intersection multiplicity.

Let us demonstrate the above algorithm for the already mentioned Leonardo structure [44].

Example 5. According to Tarnai [43] these frameworks with a $(2^\lambda - 1)$-order flex can be generated by an iterative procedure. In the following we demonstrate this for $\lambda = 1$, $\lambda = 2$ and $\lambda = 3$ (cf. Fig. 7), using the following coordinatization:

$$F_1 = (-1,0)^T, \quad F_2 = (1,0)^T, \quad F_3 = (0,-2)^T, \quad F_4 = (2,-1)^T, \tag{33}$$

for the points pinned to the base and

$$M_1 = (a,b)^T, \quad M_2 = (c,d)^T, \quad M_3 = (e,f)^T, \tag{34}$$

for the moving points.

- $\lambda = 1$: In this case one has to solve the two equations $\|M_1 - F_i\|^2 = 1$ for $i = 1,2$, which read after homogenizing with h as:

$$a^2 + 2ah + b^2 = 0, \quad a^2 - 2ah + b^2 = 0. \tag{35}$$

  The primary decomposition of the ideal spanned by these two equations yields the two primary ideals $I_1^1 = \langle a, b^2 \rangle$ and $I_2^1 = \langle h, a^2 + b^2 \rangle$. Only I_1^1, which is zero-dimensional, contains the considered configuration $\mathbf{X}$ having homogeneous coordinates $(h:a:b) = (1:0:0)$. Computation of the U-resultant (with MACAULAY2) yields u_0^2, which shows that the configuration has multiplicity 2 and therefore a 1^{st}-order flexion.
- $\lambda = 2$: In addition to Eq. (35) one has to consider the two conditions $\|M_2 - M_1\|^2 = 1$ and $\|M_2 - F_3\|^2 = 1$, which read after homogenizing with h as:

$$a^2 - 2ac + b^2 - 2bd + c^2 + d^2 - h^2 = 0, \quad c^2 + d^2 + 4dh + 3h^2 = 0. \tag{36}$$

  The primary decomposition (operated by MAPLE 2022) of the ideal spanned by Eqs. (35-36) yields the following primary ideals

$$\begin{aligned}
I_1^2 &= \langle a, b^2, b - 2d - 2h, bh + c^2 \rangle, \\
I_2^2 &= \langle h, b^2 + a^2, d^2 + c^2, bc - ad, bd + ca \rangle, \\
I_3^2 &= \langle a^3, h^2, ah, a^2 b, hb, b^2 + a^2, ad - bc - 2ch, ac + bd + 2dh, c^2 + d^2 - 2ac - 2bd \rangle, \\
I_4^2 &= \langle a^4, c^5, h^6, ah, a^3 c^4, a^3 bc, hbc^2, b^2 + a^2, bh^3 + c^2 h^2, 2cdh^2 - bch^2 + 2ch^3, \\
&\quad 2dh^3 - bh^3 + 2h^4, c^2 + d^2 + 4dh + 3h^2, a^2 d - abc - 2bh^2 + 4dh^2 + 4h^3, \\
&\quad ac + bd + 2dh + 2h^2, acd - bc^2 - 3bh^2 - 2c^2 h + 2dh^2 + 2h^3 \rangle.
\end{aligned} \tag{37}$$

  Again only I_1^2, which is zero-dimensional, contains the considered configuration $\mathbf{X}$ having homogeneous coordinates $(h:a:b:c:d) = (1:0:0:0:-1)$. Computation of the U-resultant (with MACAULAY2) yields $2^4(u_0 - u_4)^4$. This validates the 3^{rd}-order flexion.

- $\lambda = 3$: In addition to Eqs. (35) and (36) one has to consider the two conditions $\|M_3 - M_2\|^2 = 1$ and $\|M_3 - F_4\|^2 = 1$, which read after homogenizing with h as:

$$c^2 - 2ce + d^2 - 2df + e^2 + f^2 - h^2, \quad e^2 - 4eh + f^2 + 2fh + 4h^2. \tag{38}$$

The primary decomposition (operated by MAPLE 2022) of the ideal spanned by Eqs. (35,36,38) contains only[6] one primary ideal with $h \neq 0$, which reads as:

$$I_1^3 = \langle a, b, c^2, c - 2e + 2h, 2e - c + 2d, e^2 + 2ef + f^2 - 2ce - cf \rangle. \tag{39}$$

But this cannot be correct as the U-resultant (with MACAULAY2) yields $2^8(u_0 - u_4 + u_5 - u_6)^4$, which shows only a 4-fold realization at the considered configuration $\mathbf{X}$ having homogeneous coordinates $(h : a : b : c : d : e : f) = (1 : 0 : 0 : 0 : -1 : 1 : -1)$.
We did a recheck following the idea of [48] by slightly perturbating the system of equations. Then it can easily be seen that there are 8 solutions[7] in the neighborhood of $\mathbf{X}$.
This shows up a problem of the `PrimaryDecomposition` command in MAPLE 2022. In order to correct I_1^3 of Eq. (39) one has to replace b by b^2 (as this is the case in I_1^1 and I_1^2). Then the U-resultant (operated with MACAULAY2) yields the expected expression $2^{16}(u_0 - u_4 + u_5 - u_6)^8$. ⋄

Example 6. Continuation of Example 4: The primary decomposition (operated by MAPLE 2022) of the ideal spanned by Eq. (26–29) yields only one zero-dimensional primary ideal I containing the configuration $\mathbf{X}$ of Eq. (30); namely

$$\begin{aligned} I = \langle &(1+a_0)^2, (a_2-2)^2, (3-a_2)^2 + b_2^2 - 1, (a_0+3)a_1 + 5 + (b_0+1)b_1, \\ &(a_3-3)a_2 + 5 + (b_2+1)b_3, a_1 + a_0 + a_3 + a_2, a_0^2 + b_0^2 - 1, \\ &(6-2a_3)a_2 + a_3^2 - 2b_2b_3 + b_3^2 - 10, a_1^2 - 2a_0a_1 - 2b_0b_1 + b_1^2 - 1, \\ &(a_0 - a_2 - a_3 - 3)a_1 + (3 - a_0 + a_3)a_2 + (b_0 - b_2 - b_3 - 1)b_1 + \\ &(b_2 - b_0 - 1)b_3 - a_0a_3 - b_0b_2 - 26\rangle. \end{aligned} \tag{40}$$

As this ideal has more than eight generators, we cannot apply the U-resultant method as done in Example 5. As we are not aware of any implementation of the generalized U-resultant method of Lazard [20], we proceeded as follows: The ideal I of Eq. (40) only has the solution $\mathbf{X}$ and we determined its multiplicity by the MAPLE command `NumberOfSolutions`[8] of the `PolynomialIdeals` package, which yields 6.

As one cannot trust for sure the `PrimaryDecomposition` command in MAPLE 2022 as demonstrated in Example 5, we did again a recheck by the per-

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6 The other primary ideals with $h = 0$ are not given due to their length.

7 This number can additionally be verified by the `IntersectionMultiplicity` command implemented in MAPLE 2022.

8 In MAPLE 2022 there is no documentation on how the command `NumberOfSolutions` works.

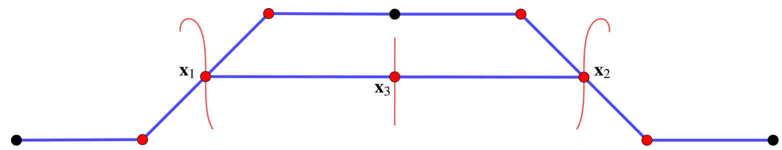

Fig. 10 Double-Watt mechanism of Connelly and Servatius extended by the guidance of the midpoint $\mathbf{x}_3$ of $\mathbf{x}_1$ and $\mathbf{x}_2$ along a straight line.

tubation approach of [48], which confirms multiplicity 6. According to Definition 3 this implies a flexion of order 5. ◇

Example 7. We can also force the midpoint $\mathbf{x}_3$ of $\mathbf{x}_1$ and $\mathbf{x}_2$ of the original double-Watt mechanism of Connelly and Servatius (cf. Example 1) to run on a vertical line. In analogy to Example 3 one can use Stachel's approach, which yields the sequence of flexion orders $(k, 3k-1)$ for odd k and $(k, 3k+\frac{k}{2}-1)$ for even k.

Similar considerations as in Example 6 show, that in this case seven solutions[9] coincide, yielding flexion order 6. ◇

Remark 10. The flexion order of the frameworks discussed in Examples 6 and 7 can be raised from 5 and 6 to 9 and 14, respectively, by modifying the dimension of the used Watt-linkage in a way that the coupler is vertical (i.e. the coupler is tangential to the considered branch) but the arms remain horizontal in the considered configuration. ◇

Clearly as the algorithm given in Section 3.1 is based on symbolic methods from computer algebra, we are faced with computational limits. But beside this problem the flexion order of a given configuration can be computed with the presented tools in all cases from the pure theoretical point of view. Another problem is the computation of configurations with a higher-order flexion, which is discussed in the next section.

4 Computing 3-RPR configurations with a higher-order flexion

In this section we demonstrate, how the idea of Sabitov's finite algorithm for testing the bendability of a polyhedron [30, page 231] can be used to compute iteratively configurations with a higher-order flexion. We do this exemplarily for a planar 3-RPR manipulator consisting of a moving triangle which is connected by three legs to the fixed base. The legs are jointed to the platform and the base by rotational (R)

[9] A slight perturbation of the system of equations shows that seven solutions converge against the given configuration. Note that in this case the `PrimaryDecomposition` command in MAPLE 2022 does not work as the resulting solution is only sixfold and also the `IntersectionMultiplicity` command fails for all possible $8! = 40320$ permutations.

joints and the corresponding anchor points are denoted by m_i and M_i, respectively, for $i = 1,2,3$. The length r_i of the legs can actively be controlled by prismatic (P) joints.

Our choice of the example was motivated by the following statement of Husty [15] that 3^{rd}-order flexibility "*can be reached by any design because the three necessary conditions could be imposed on the input parameters only. Unfortunately neither the conditions nor the number of corresponding poses are known*". We will clarify this in Section 4.1.

Note that we can interpret the triangular base and platform either as (a) triangular plates or (b) triangular bar structures. In case (a) the 3-RPR manipulator can be seen as a pin-jointed bar-plate framework and in case (b) as a classical bar-joint framework. In the following Subsections 4.1 and 4.2 we distinguish these two interpretations as they will effect the discussion of configurations with a higher-order flexion. But let us start with some review on this topic.

As already mentioned in Section 2.1 Wohlhart [49] followed a kinematic version of Kuznetsov's approach for the study of higher-order flexible 3-RPRs (interpreted as bar-plate frameworks). Stachel studied the geometry of higher-order flexible 3-RPRs (interpreted as bar-joint frameworks) in [35], where he has shown the following result for a configuration of flexion order $(1,n)$:

If one disconnects the leg M_im_i from the platform, then the trajectory of the point m_i under the resulting four bar motion has n^{th}-order contact with the circle centered in M_i having radius r_i.[10]

Moreover, this result implies that in this configuration $(n+1)$ realizations coincide, which also goes along with our redefinition given in Definition 3. Based on this characterization Husty [15] has given an approach for the computation of 3-RPR configurations (interpreted as bar-plate frameworks) with flexion order 5, which has to be done carefully as it can also yield pseudo-solutions[11].

4.1 Bar-plate framework

Let us start with the computation of V_1 for these mechanisms, which can be done in several ways. For the problem at hand we stress an approach of Husty and Gosselin [16], which is recapped next:

The coordinates $(a_i,b_i)^T$ of a point m_i of the moving platform with respect to the moving frame can be transformed into coordinates of the fixed frame using the so-called Blaschke-Grünwald parameters $(q_0 : q_1 : q_2 : q_3)$. They can be seen as

[10] According to [35, Lem. 1] a corresponding result also holds for Stewart–Gough platforms, which goes along with the definition of an "*order of a configuration*" given by Sarkissyan and Parikyan [32] in 1990 (see also Wohlhart [49, page 1116]).

[11] Note that the example illustrated in Fig. 8 of [15] does not show a 5^{th}-order flexion, as it is not a sixfold solution of the direct kinematics problem. The direct kinematic splits up into a fourfold solution and a twofold one. Therefore the two corresponding configurations are flexible of order 3 and 1, respectively.

homogeneous coordinates of points of a projective 3-dimensional space P^3. It is well known, that there is a bijection between points of this space sliced along the line $q_0 = q_1 = 0$ and the planar motion group SE(2). The slicing has to be done to ensure that the 4-tuple $(q_0 : q_1 : q_2 : q_3)$ can be normalized by $c_4 = 0$ with

$$c_4 := q_0^2 + q_1^2 - 1. \tag{41}$$

If this normalization condition holds the above mentioned transformation reads as follows:

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} q_0^2 - q_1^2 & -2q_0q_1 \\ 2q_0q_1 & q_0^2 - q_1^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 2q_1q_2 + 2q_0q_3 \\ 2q_1q_3 - 2q_0q_3 \end{pmatrix}. \tag{42}$$

Using these Blaschke-Grünwald parameters the condition that a point m_i is located on a circle with radius r_i around the fixed point M_i with coordinates $(A_i, B_i)^T$ with respect to the fixed frame, can be written as $c_i = 0$ with:

$$\begin{aligned} c_i := & 2A_ia_iq_1^2 - 2A_ia_iq_0^2 + 4A_ib_iq_0q_1 - 4B_ia_iq_0q_1 - 2B_ib_iq_0^2 + 2B_ib_iq_1^2 + \\ & a_i^2q_0^2 + a_i^2q_1^2 + b_i^2q_0^2 + b_i^2q_1^2 - 4A_iq_0q_3 - 4A_iq_1q_2 + 4B_iq_0q_2 - 4B_iq_1q_3 + \\ & 4a_iq_0q_3 - 4a_iq_1q_2 - 4b_iq_0q_2 - 4b_iq_1q_3 + A_i^2 + B_i^2 + 4q_2^2 + 4q_3^2 - r_i^2 \end{aligned} \tag{43}$$

The information of the leg lengths r_i complete the intrinsic metric of the framework. Then its realizations[12] $G(\mathbf{X})$ are obtained as the solutions of the four algebraic equations $c_1 = c_2 = c_3 = c_4 = 0$. It is well-known that there can only exist six solutions thus a 6^{th}-order flex (according to Definition 3) implies a continuous flexion; i.e. $n^* = 6$.

Now we are looking for poses of the platform yielding an infinitesimal flexibility of the framework. As described in Section 1.1, these configurations are characterized by the fact that the determinant of the rigidity matrix $\mathbf{R}_{G(\mathbf{X})}$ vanishes, which is given by

$$\mathbf{R}_{G(\mathbf{X})} = (\nabla c_1, \nabla c_2, \nabla c_3, \nabla c_4) = \begin{pmatrix} \frac{\partial c_1}{\partial q_0} & \frac{\partial c_2}{\partial q_0} & \frac{\partial c_3}{\partial q_0} & \frac{\partial c_4}{\partial q_0} \\ \frac{\partial c_1}{\partial q_1} & \frac{\partial c_2}{\partial q_1} & \frac{\partial c_3}{\partial q_1} & \frac{\partial c_4}{\partial q_1} \\ \frac{\partial c_1}{\partial q_2} & \frac{\partial c_2}{\partial q_2} & \frac{\partial c_3}{\partial q_2} & \frac{\partial c_4}{\partial q_2} \\ \frac{\partial c_1}{\partial q_3} & \frac{\partial c_2}{\partial q_3} & \frac{\partial c_3}{\partial q_3} & \frac{\partial c_4}{\partial q_3} \end{pmatrix} \tag{44}$$

according to Eq. (2). Then the shakiness variety V_1 equals the zero set of $s := \det\left(\mathbf{R}_{G(\mathbf{X})}\right)$. According to [17] this variety has only singularities for some special designs beside the singularities resulting from the parametrization, which equal the line $q_0 = q_1 = 0$. Therefore in the generic case each point of V_1 sliced along the line $q_0 = q_1 = 0$ is a regular one. Thus according to Lemma 1 the tangent planes to $c_1, \ldots, c_4$ have a line in common. The orthogonality of this line to ∇s is equivalent to the condition

$$rk(\nabla c_1, \nabla c_2, \nabla c_3, \nabla c_4, \nabla s) = 3 \tag{45}$$

which implies the four conditions $s_1 = s_2 = s_3 = s_4 = 0$ with:

[12] In this context the realizations are also known as solutions of the direct kinematics problem.

$$s_1 := \det(\nabla c_2, \nabla c_3, \nabla c_4, \nabla s), \qquad s_2 := \det(\nabla c_1, \nabla c_3, \nabla c_4, \nabla s), \tag{46}$$

$$s_3 := \det(\nabla c_1, \nabla c_2, \nabla c_4, \nabla s), \qquad s_4 := \det(\nabla c_1, \nabla c_2, \nabla c_3, \nabla s). \tag{47}$$

Then V_2 is the zero set of the ideal

$$I_2 = \langle s, s_1, s_2, s_3, s_4 \rangle. \tag{48}$$

Iteration of the above procedure yields the conditions $s_{1,i} = s_{2,i} = s_{3,i} = s_{4,i} = 0$ with:

$$s_{1,i} := \det(\nabla c_2, \nabla c_3, \nabla c_4, \nabla s_i), \qquad s_{2,i} := \det(\nabla c_1, \nabla c_3, \nabla c_4, \nabla s_i), \tag{49}$$

$$s_{3,i} := \det(\nabla c_1, \nabla c_2, \nabla c_4, \nabla s_i), \qquad s_{4,i} := \det(\nabla c_1, \nabla c_2, \nabla c_3, \nabla s_i), \tag{50}$$

for $i = 1, \ldots, 4$. Then V_3 is the zero set of the ideal

$$I_3 = \langle s, s_1, s_2, s_3, s_4, s_{1,1}, \ldots, s_{4,1}, s_{1,2}, \ldots, s_{4,2}, s_{1,3}, \ldots, s_{4,3}, s_{1,4}, \ldots, s_{4,4} \rangle. \tag{51}$$

In addition the singular points of V_2 have to considered separately. As V_2 is a curve in P^3 a singularity corresponds to the case

$$rk(\nabla s, \nabla s_1, \nabla s_2, \nabla s_3, \nabla s_4) = 1. \tag{52}$$

In the following we apply this procedure to a concrete example.

Example 8. The geometry of the platform and base is given by:

$$\begin{aligned} &A_1 = 0, \quad B_1 = 0, \quad A_2 = 3, \quad B_2 = 0, \quad A_3 = 1, \quad B_3 = 3, \\ &a_1 = 0, \quad b_1 = 0, \quad a_2 = 1, \quad b_2 = 0, \quad a_3 = 2, \quad b_3 = 1. \end{aligned} \tag{53}$$

For this values we obtain

$$\begin{aligned} s = &5q_0^3q_2 - 13q_0^2q_1q_2 - 4q_0^2q_1q_3 + 5q_0^2q_2^2 + 7q_0q_1^2q_2 + 11q_0q_1^2q_3 - \\ &6q_0q_1q_2^2 - 6q_0q_1q_3^2 + 10q_1^3q_3 - 5q_1^2q_3^2. \end{aligned} \tag{54}$$

In the next step we consider the ideal I_2 given in Eq. (48). By means of Hilbert dimension it can be verified that V_2 is a curve in P^3. Moreover, the degree of V_2 is 18. But V_2 splits up into a curve g of degree 14 and the line $q_0 = q_1 = 0$ of multiplicity 4, which can be seen as follows: We add the expression $(q_0^2 + q_1^2)u - 1$ to the ideal I_2 and eliminate the unknown u (also known as the Rabinowitsch trick). The variety of the resulting elimination ideal is only of degree 14. Moreover, it can be checked that only $(q_0 : q_1) = (0 : 0)$ fulfill the equations $s = s_1 = s_2 = s_3 = s_4 = 0$ but not $(q_0 : q_1) = (1 : \pm I)$. Due to the slicing of P^3 along $q_0 = q_1 = 0$ we can restrict to the curve g of degree 14. It can easily be checked that g does not contain any singular points by applying the criterion of Eq. (52).

In the last step we consider the ideal I_3 given in Eq. (51). V_3 again contains the line $q_0 = q_1 = 0$ with multiplicity 3. We can get rid of this line in the same way as done in the case of V_2 (Rabinowitsch trick). The elimination ideal then yields 32

Table 1 All real RPR-configurations with a 3^{rd}-order flex for the geometry given in Eq. (53)

#	q_0	q_1	q_2	q_3	Fig. 11
1	0.612011087187	0.790849182309	−1.605503824990	0.608460800603	(a)
2	−0.335887854729	0.941901984839	−0.360610831902	2.136077449950	(b)
3	0.933493296982	0.358594847269	−0.518343596625	0.387989956333	(c)
4	−0.351833124675	0.936062739553	1.595410958064	1.897762719666	(d)
5	0.926572314644	0.376116665058	0.064697675622	−0.063224689351	(e)
6	−0.985793710397	0.167960592226	0.665253728293	2.206010002417	(f)
7	−0.388425191626	0.921480260510	0.083061189759	0.100978528116	(g)
8	−0.430899664574	0.902399844342	1.635384670001	−0.158191823892	(h)
9	0.700957636960	0.713202910248	1.476082504043	0.619974829761	(i)
10	−0.981604898439	0.190923606082	0.557314730844	−1.086046127580	(j)

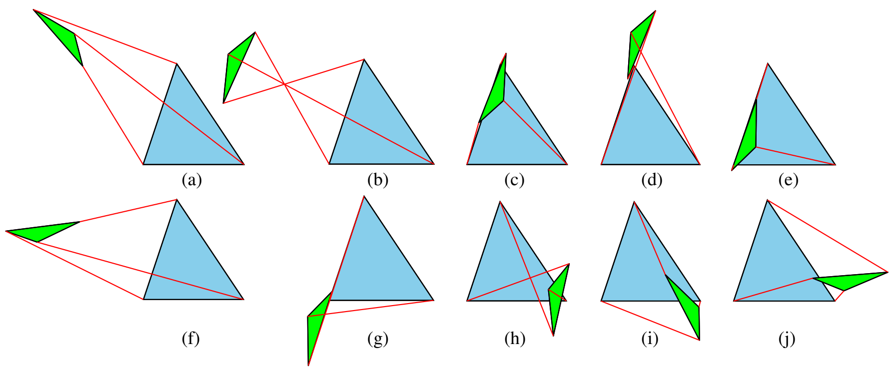

Fig. 11 Visualization of the 10 configuration with a 3^{rd}-order flex given in Table 1.

solutions. We can even eliminate q_0 and q_3 from the set of equations generating I_3 to end up with the polynomial of degree 32, which is given in the Appendix. By setting $q_1 = 1$ we can easily check that it has 10 real solutions, which are given in Table 1. Moreover, the corresponding configurations are illustrated in Fig. 11. In addition, the shakiness variety V_1, the curve g and the 10 configurations are illustrated in Fig. 8 for $q_0 = 1$.

Finally it should be noted, that it remains unclear if examples with 32 real solutions exist. ⋄

4.2 Bar-joint framework

For the interpretation as bar-joint framework there exists 24 realizations, as the platform triangle as well as the base triangle can flip. But this does not imply that $n^* = 24$ holds true for all cases, as the following study will show.

As we assumed in Section 1 that bar lengths are always non-zero we can assume a rescaling of the framework such that the bar between M_1 and M_2 has length one. Then the pin-joints can be coordinatized as follows with respect to the fixed frame:

$$M_1 = (0,0)^T, \quad M_2 = (1,0)^T, \quad M_3 = (A_3,B_3)^T, \quad m_j = (a_j,b_j)^T, \tag{55}$$

for $j = 1,2,3$. If the remaining 8 bar lengths are known they imply 8 distance equations $c_1,\ldots,c_8$. The solutions of this set of equations correspond to realizations of this isostatic bar-joint framework. Then we can compute the (8×8) rigidity matrix according to Eq. (2). Again the shakiness variety V_1 is given as the zero set of $\det\left(\mathbf{R}_{G(\mathbf{X})}\right)$ which splits up into the following three factors $s_1s_2s_3$ with:

$$\begin{aligned}
s_1 =& B_3,\\
s_2 =& a_1b_2 - a_1b_3 - a_2b_1 + a_2b_3 + a_3b_1 - a_3b_2,\\
s_3 =& A_3a_1b_2b_3 - A_3a_2b_1b_3 - B_3a_1a_3b_2 + B_3a_2a_3b_1 -\\
& A_3b_1b_2 + A_3b_1b_3 + B_3a_1b_2 - B_3a_3b_1 - a_1b_2b_3 + a_3b_1b_2
\end{aligned} \tag{56}$$

Their geometric interpretation is that for $s_1 = 0$ (resp. $s_2 = 0$) the base (resp. platform) degenerates into a line[13]. For $s_3 = 0$ the three legs belong to a pencil of lines. Let us denote the varieties $s_i = 0$ by S_i for $i = 1,2,3$. Now we can easily identify the following regions of V_1 where different values for n^* hold true:

$$S_1 \setminus (S_2 \cup S_3) \qquad n^* = 2 \tag{57}$$
$$S_2 \setminus (S_1 \cup S_3) \qquad n^* = 2 \tag{58}$$
$$S_3 \setminus (S_1 \cup S_2) \qquad n^* = 6 \tag{59}$$
$$(S_1 \cap S_2) \setminus S_3 \qquad n^* = 4 \tag{60}$$
$$(S_1 \cap S_3) \setminus S_2 \qquad n^* = 12 \tag{61}$$
$$(S_2 \cap S_3) \setminus S_1 \qquad n^* = 12 \tag{62}$$
$$S_1 \cap S_2 \cap S_3 \qquad n^* = 24 \tag{63}$$

Remark 11. We are aware of the fact that no point on $S_1 \setminus (S_2 \cup S_3)$ or $S_2 \setminus (S_1 \cup S_3)$ can reach a higher flexion order than 1, as a triangle does not allow an isometric deformation according to the side-side-side theorem. Therefore $n^* = 2$ of Eqs. (57) and (58) as well as $n^* = 4$ of Eq. (60) can never be reached and are only of theoretical nature. ◇

In the following we give the construction of configurations with the highest possible flexion order. Let us assume that the platform and the base triangles degenerate into lines l and L, respectively. A necessary condition for a configuration of flexion order 23, is that l and L coincide. If this would not be the case one can reflect the configuration on one of these lines to get another realization, which contradicts the assumption that all 24 realization coincide.

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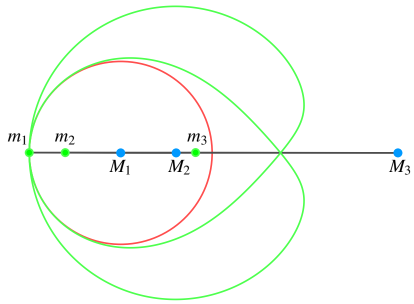
[13] A triangle and its mirrored version can only coincide if it degenerates into a line.



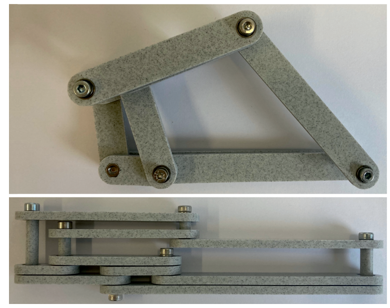


Fig. 12 (left) Visualization of the bar-joint framework with a flexion of order 23. The green curve shows the coupler curve of the point m_1, which results by giving away the first leg. The red curve is a circle with radius r_1 and midpoint M_1. The circle intersects the green coupler curve in m_1 with multiplicity 6. (right) Model of the bar-joint framework with flexion order 23, which is constructed using multiple layers (bottom). The model allows a large flexion as illustrated in the top.

Remark 12. Interestingly such a configuration is not only a singular point of V_1, as it is located in the intersection of S_1, S_2 and S_3 but already a singular point of S_3 according to [17]. ◇

Therefore a 23^{rd}-order flexible bar-joint framework follows from a 5^{th}-order flexible plate-bar framework, where all six anchor points are located on a line. This problem can be solved following the already mentioned approach of Husty [15]. In this way the following example was computed.

Example 9. The geometry of the base is given by

$$M_1 = (0,0)^T, \quad M_2 = (1,0)^T, \quad M_3 = (5,0)^T \tag{64}$$

with respect to the fixed system and the geometry of the platform is given by

$$m_1 = (0,0)^T, \quad m_2 = (\tfrac{1}{2} + \tfrac{2\sqrt{10}}{5} - \tfrac{\sqrt{120\sqrt{10}-255}}{10}, 0)^T, \quad m_3 = (3,0)^T \tag{65}$$

with respect to the moving frame. The information on the intrinsic metric of the framework is completed by the following lengths of the three legs:

$$r_1 = \tfrac{3}{2} + \tfrac{2\sqrt{10}}{5} - \tfrac{\sqrt{120\sqrt{10}-255}}{10}, \quad r_2 = 2, \quad r_3 = \tfrac{7}{2} + \tfrac{2\sqrt{10}}{5} - \tfrac{\sqrt{10}\sqrt{48\sqrt{10}-102}}{20}. \tag{66}$$

This configuration is illustrated in Fig. 12 where also pictures of a model can be seen, which was produced for validation of the higher-order flexion. ◇

Remark 13. It should be possible to determine the set of these frameworks with flexion order 23 in full generality (as only 4 unknowns are involved), which is dedicated to future research. ◇

5 Final remarks, open problems and future work

In the paper we presented a global approach for a proper redefinition of higher-order flexibility and rigidity. We only discussed planar frameworks, but the proposed algebraic method works for frameworks of any dimension. Especially, it is planned to apply the iterative procedure of Section 4 also to the spatial version of 3-RPR manipulators, which are Stewart–Gough platforms. Any such manipulator (interpreted as bar-body framework) has to have configurations with a 6^{th}-order flexion, whose detailed investigation is dedicated to future research. Furthermore we are interested in the highest possible flexion order[14] of Stewart–Gough configurations and their computation.

Let us close the paper with the following list of final remarks and open problems:

1. Note that the presented approach does not only work for bar-joint frameworks but it can be applied to any framework with algebraic joints; i.e. the relative position of two jointed rigid bodies can be described algebraically. But it remains open to extend it to frameworks with non-algebraic joints (cf. [43]).
2. With our approach we were able to give a proper redefinition of higher-order flexibility and rigidity, but the computation of the associated (k,n)-flex(es) in dependence of the time parameter t remains open and is dedicated to future research. We plan to solve this problem by means of tropical geometry and Puiseux series as this promising approach was already successfully used in [27] for analyzing the configuration space of mechanisms.
   Moreover, for this task we also want to generate further examples by following an idea of Stachel [41] using the two-point guidance method, where the points $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are in (higher-order) singularities of their paths at $t = 0$.
3. Our approach operates over $\mathbb{C}$ and does not take reality issues into account so far (for a local attempt see [45]). For example, a planar 4-bar mechanism where the bar lengths a,b,c,d fulfill the equation $a+b+c=d$ has a 1-dimensional set of configurations which are all complex with exception of one single configuration $\mathbf{X}$. Our algorithm would assign to $\mathbf{X}$ the flexion order ∞ but it is only shaky over $\mathbb{R}$. We suggest the following procedure for resolving this minor problem[15]. Namely, instead of just assigning the value ∞ as flexion order, we propose to consider the corresponding (k,n)-flex(es) mentioned in item 2 above. More precisely we are only interested in the degree $n^{\mathbb{R}} \leq n$ of the highest possible real flexion. The(se) number(s) can then be used to assign a real flexion order to the configuration.
   Note that the analysis of a framework configuration $\mathbf{X}$, which corresponds to an isolated real solution within a higher-dimensional complex configuration set,

[14] According to Definition 3 its upper bound is 39 but a configuration only depends on 30 unknowns (up to Euclidean motions), which can be adjusted. From that one might expect a maximal flexion order of 30.

[15] Note that this special case is circumvented by the formulation of Definition 3, as we assumed that the considered configuration does not belong to a continuous flexion of the framework (over $\mathbb{C}$).

has to be handled with special care, as in this case **X** can also arise as an embedded component in the complex solution set (cf. [21]). Then the irredundant primary decomposition proposed in the algorithm of Section 3.1 is not unique anymore (cf. Remark 9). The study of further examples in this context is dedicated to future research.

4. The algorithm presented in Section 3.1 for determining the intersection multiplicity requires global constructions (like primary decomposition and U-resultant method), but the multiplicity is a local property according to [18]. Therefore again one can think about using local methods (e.g. Serre's Tor formula) to determine this number. It remains open if these local methods can also detect a continuous flexion and if they work in all cases (like the presented global approach).

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Appendix: Polynomial of degree 32

$$
\begin{aligned}
&5169694889612648582969770 44q_1^{32} - 9280309213987777419484380570q_1^{31}q_2 + \\
&4352627023211727183455650 2073q_1^{30}q_2^2 - 45280692730479399589412412168q_1^{29}q_2^3 - \\
&71413409266992435779029661320q_1^{28}q_2^4 + 73378758260985908292649564 0512q_1^{27}q_2^5 - \\
&121605749941654633181633602 1712q_1^{26}q_2^6 + 11785250082683805084046729670 40q_1^{25}q_2^7 + \\
&30498385388148048358605431 5776q_1^{24}q_2^8 + 373067534199906557276943674880q_1^{23}q_2^9 - \\
&350686585730510814063735442 2016q_1^{22}q_2^{10} + 15154579062933804962148470312 96q_1^{21}q_2^{11} + \\
&2762451499211791028130610419712q_1^{20}q_2^{12} - 1507176820840441939654068420608q_1^{19}q_2^{13} - \\
&1140312255149192283851181674496q_1^{18}q_2^{14} + 3709177173793453321218277048 32q_1^{17}q_2^{15} + \\
&5403562343133463928666756874 24q_1^{16}q_2^{16} + 2186229830258054730458911211 52q_1^{15}q_2^{17} - \\
&5131297002972504583794199756 80q_1^{14}q_2^{18} - 1469983146306045877020183756 80q_1^{13}q_2^{19} + \\
&4532299499911891468096891781 12q_1^{12}q_2^{20} - 1326381457598636926294860759 04q_1^{11}q_2^{21} - \\
&1482409854476361709282824028 16q_1^{10}q_2^{22} + 1244258973315944101079049830 40q_1^{9}q_2^{23} - \\
&1238626973404818888381903667 2q_1^{8}q_2^{24} - 2704982109791373607741843046 4q_1^{7}q_2^{25} + \\
&1483141815808960467089699635 2q_1^{6}q_2^{26} - 1721669183596659665641930752q_1^{5}q_2^{27} - \\
&1309349875968694100160413696q_1^{4}q_2^{28} + 691975482131520534161129472q_1^{3}q_2^{29} - \\
&156210223994716269983039488q_1^{2}q_2^{30} + 18063680521134606070579200q_1q_2^{31} - \\
&874805860916262711853056q_2^{32} = 0
\end{aligned}
\tag{67}
$$