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Semigroups in which the radical of every interior ideal is a subsemigroup

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Abstract. In this paper, we characterize when the radical $\sqrt{I}$ of every interior ideal I of a semigroup S is a subsemigroup of S. Also, the radical of every interior ideal (or right ideal or left ideal or quasi-ideal or ideal or bi-ideal or subsemigroup) of S is an interior ideal (or a right ideal or a left ideal or a quasi-ideal or an ideal or a bi-ideal) of S.

1. Introduction and Preliminaries

The theory of different types of ideals in semigroups was studied by several researchers such as: in 1952, Good and Hughes [2] introduced the notion of bi-ideals of semigroups. In 1956, Steinfeld [7] introduced the notion of quasi-ideals in semigroups. In 1976, Lajos [4] gave the concept of interior ideals of semigroups.

Let S be a semigroup and A, B be non-empty subsets of S. The *radical* $\sqrt{A}$ of A is defined by

$$\sqrt{A} = \{a \in S \mid a^n \in A \text{ for some positive integer } n\}.$$

For $a, b \in S$, the subsemigroup of S generated by $\{a, b\}$ is denoted by $\langle a, b\rangle$. A non-empty subset A of S is called a *left* (*right*) *ideal* of S if $SA \subseteq A(AS \subseteq A)$. If A is both a left and right ideal of S, then A is called an *ideal* of S. A non-empty subset Q of S is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$. A subsemigroup B of S is called a *bi-ideal* of S if $BSB \subseteq B$. A subsemigroup I of S is called an *interior ideal* of S if $SIS \subseteq I$. In 1992, Bogdanovic and Ciric [1] characterized semigroups in which the radical of every ideal (right ideal, bi-ideal, subsemigroup) is a subsemigroup (or ideal

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or bi-ideal or right ideal). Later, the case of quasi-ideals was considered in semigroups and ordered semigroups by Sanborisoot and Changphas in [5] and [6], respectively.

In this paper, we characterize when the radical $\sqrt{I}$ of every interior ideal I of a semigroup S is a subsemigroup of S. Also, the radical of every interior ideal (or right ideal or left ideal or quasi-ideal or ideal or bi-ideal or subsemigroup) of S is an interior ideal (or a right ideal or a left ideal or a quasi-ideal or an ideal or a bi-ideal) of S.

Let $\mathbb{N} = \{1, 2, 3, ...\}$ denote the set of all positive integers. Let S be a semigroup with identity and let $a, b \in S$, define

$$\begin{aligned}
a \mid b &\Longleftrightarrow b = xay \text{ for some } x, y \in S;\\
a \mid_r b &\Longleftrightarrow b = ax \text{ for some } x \in S;\\
a \mid_l b &\Longleftrightarrow b = ya \text{ for some } y \in S;\\
a \mid_t b &\Longleftrightarrow a \mid_r b \wedge a \mid_l b;\\
a \rightarrow b &\Longleftrightarrow a \mid b^n \text{ for some } n \in \mathbb{N}; \text{ and}\\
a \overset{h}{\rightarrow} b &\Longleftrightarrow a \mid_h b^n \text{ for some } n \in \mathbb{N} \text{ where } h \text{ is } r, l \text{ or } t.
\end{aligned}$$

2. Main Results

In general, the radical of a interior ideal of a semigroup with identity need not be a subsemigroup. The following theorem characterizes when the radical of every interior ideal of a semigroup with identity is a subsemigroup.

Theorem 2.1. *Let S be a semigroup with identity. Then the radical of every interior ideal of S is a subsemigroup of S if and only if*

$$\forall a, b \in S \quad \forall i, j \in \mathbb{N} \quad \exists n \in \mathbb{N} \quad [(ab)^n \in S\{a^i, b^j\}S].$$

Proof. Assume that the radical of every interior ideal of S is a subsemigroup of S. Let $a, b \in S$ and let $i, j \in \mathbb{N}$. Put $I = S\{a^i, b^j\}S$. Since

$$II = (S\{a^i, b^j\}S)(S\{a^i, b^j\}S) \subseteq S\{a^i, b^j\}S = I$$

and

$$SIS = S(S\{a^i, b^j\}S)S \subseteq S\{a^i, b^j\}S = I,$$

I is an interior ideal of S. Observe that $a, b \in \sqrt{I}$ because $a^i, b^j \in I$. By assumption, $\sqrt{I}$ is a subsemigroup of S. Thus, $ab \in \sqrt{I}$. Hence, $(ab)^n \in I = S\{a^i, b^j\}S$ for some $n \in \mathbb{N}$.

Conversely, assume that for all a, b in S and i, j in $\mathbb{N}$, there exists $n \in \mathbb{N}$ such that $(ab)^n \in S\{a^i, b^j\}S$. Let I be an interior ideal of S, and let $a, b \in \sqrt{I}$. Then $a^i \in I$ and $b^j \in I$ for some $i, j \in \mathbb{N}$. By assumption, there exists $n \in \mathbb{N}$ such that $(ab)^n \in S\{a^i, b^j\}S$. Thus, $ab \in \sqrt{I}$, because $(ab)^n \in S\{a^i, b^j\}S \subseteq SIS \subseteq I$. Hence, $\sqrt{I}$ is a subsemigroup of S. □

Example 2.2. Let $S = \{a, b, c, d, e\}$ be a semigroup ([3]) with the multiplication:

$\cdot$	a	b	c	d	e
a	e	b	a	d	e
b	b	b	b	b	b
c	a	b	c	d	e
d	d	b	d	d	d
e	e	b	e	d	e

The interior ideals of S are $\{b\}, \{b, d\}, \{b, d, e\}, \{a, b, d, e\}$ and S. We have $\sqrt{\{b\}} = \{b\}$, $\sqrt{\{b, d\}} = \{b, d\}$, $\sqrt{\{b, d, e\}} = \{a, b, d, e\}$, $\sqrt{\{a, b, d, e\}} = \{a, b, d, e\}$ and $\sqrt{S} = S$. Then the radical of every interior ideal of S is a subsemigroup of S.

Theorem 2.3. *Let S be a semigroup with identity. The radical of every interior ideal of S is a right ideal of S if and only if*

$$a^k \to ab \quad \text{for all } a, b \in S \text{ and } k \in \mathbb{N}.$$

Proof. Assume that the radical of every interior ideal of S is a right ideal of S. Let $a, b \in S$ and $k \in \mathbb{N}$. Put $I = Sa^kS$. Next, we claim that I is an interior ideal of S. Consider

$$II = (Sa^kS)(Sa^kS) \subseteq Sa^kS = I \quad \text{and} \quad SIS = S(Sa^kS)S \subseteq Sa^kS = I.$$

Then I is an interior ideal of S and $a \in \sqrt{I}$. By assumption, $\sqrt{I}$ is a right ideal of S. Thus, $ab \in \sqrt{I}S \subseteq \sqrt{I}$. Hence, $(ab)^n \in I$ for some $n \in \mathbb{N}$. Therefore, $a^k \to ab$.

Conversely, assume that $a^k \to ab$ for all $a, b \in S$ and $k \in \mathbb{N}$. Let I be an interior ideal of S. Let $a \in \sqrt{I}$ and $b \in S$. Then $a^k \in I$ for some $k \in \mathbb{N}$. Since $a^k \to ab$, we obtain that $(ab)^n \in Sa^kS \subseteq SIS \subseteq I$ for some $n \in \mathbb{N}$. Hence, $ab \in \sqrt{I}$. Therefore, $\sqrt{I}$ is a right ideal of S. □

Similar to Theorem 2.3, we have the following theorem.

Theorem 2.4. *Let S be a semigroup with identity. The radical of every interior ideal of S is a left ideal of S if and only if*

$$a^k \to ba \quad \text{for all } a, b \in S \text{ and } k \in \mathbb{N}.$$

Theorem 2.5. *Let S be a semigroup with identity. Then the radical of every interior ideal of S is a quasi-ideal of S if and only if*

$$\forall a,b,c \in S \ \ [a \mid_r c \wedge b \mid_l c \Rightarrow \forall i,j \in \mathbb{N} \ [a^i \rightarrow c \vee b^j \rightarrow c]].$$

Proof. Assume that the radical of every interior ideal of S is a quasi-ideal of S. Let $a,b,c \in S$ such that $a \mid_r c$ and $b \mid_l c$. Then $c = au$ and $c = vb$ for some $u,v \in S$. Let $i,j \in \mathbb{N}$. Put $I = S\{a^i, b^j\}S$. Thus I is an interior ideal of S and $a,b \in \sqrt{I}$. By assumption, $\sqrt{I}$ is a quasi-ideal of S. Since $c = au$ and $c = vb$, $c \in \sqrt{I}S \cap S\sqrt{I} \subseteq \sqrt{I}$. Thus, $c^n \in S\{a^i, b^j\}S$ for some $n \in \mathbb{N}$. Hence, $a^i \rightarrow c$ or $b^j \rightarrow c$.

Conversely, assume that for all $a,b,c \in S$ such that

$$a \mid_r c \wedge b \mid_l c \Rightarrow \forall i,j \in \mathbb{N} \ [a^i \rightarrow c \vee b^j \rightarrow c].$$

Let I be an interior ideal of S. To show that $\sqrt{I}S \cap S\sqrt{I} \subseteq \sqrt{I}$, we let $x \in \sqrt{I}S \cap S\sqrt{I}$. Then $x = au$ and $x = vb$ for some $u,v \in S$ and $a,b \in \sqrt{I}$. Since $a,b \in \sqrt{I}$, $a^i, b^j \in I$ for some $i,j \in \mathbb{N}$. By assumption, there exists $n \in \mathbb{N}$ such that $x^n \in S\{a^i, b^j\}S \subseteq SIS \subseteq I$. Thus, $x \in \sqrt{I}$. Hence, $\sqrt{I}$ is a quasi-ideal of S. □

The next theorem follows from Theorem 2.3 and 2.4.

Theorem 2.6. *Let S be a semigroup with identity. The radical of every interior ideal of S is an ideal of S if and only if*

$$a^k \rightarrow ab \ \ \text{and } a^k \rightarrow ba \text{ for all } a,b \in S \text{ and } k \in \mathbb{N}.$$

Theorem 2.7. *Let S be a semigroup with identity. The radical of every interior ideal of S is a bi-ideal of S if and only if*

(1) $\forall a,b \in S \ \forall i,j \in \mathbb{N} \ \ [a^i \rightarrow ab \vee b^j \rightarrow ab]$,

(2) $\forall a,b,c \in S \ \forall i,j \in \mathbb{N} \ \ [a^i \rightarrow abc \vee c^j \rightarrow abc]$.

Proof. Assume that the radical of every interior ideal of S is a bi-ideal of S. To show that (1) holds, we let $a,b \in S$, and let $i,j \in \mathbb{N}$. Put $I = S\{a^i, b^j\}S$. Then I is an interior ideal of S and $a,b \in \sqrt{I}$. By assumption, $\sqrt{I}$ is a bi-ideal of S. Thus, $ab \in \sqrt{I}\sqrt{I} \subseteq \sqrt{I}$. Hence, $(ab)^n \in I = S\{a^i, b^j\}S$ for some $n \in \mathbb{N}$, and so $a^i \rightarrow ab$ or $b^j \rightarrow ab$. Next, to show that (2) holds, we let $a,b,c \in S$, and $i,j \in \mathbb{N}$. Put $I = S\{a^i, c^j\}S$. Then I is an interior ideal of S and $a,c \in \sqrt{I}$. By assumption, $\sqrt{I}$ is a bi-ideal of S. Thus,

$abc \in \sqrt{I}S\sqrt{I} \subseteq \sqrt{I}$. Hence, $(abc)^n \in I = S\{a^i, b^j\}S$ for some $n \in \mathbb{N}$, and so $a^i \to abc$ or $c^j \to abc$.

Conversely, assume that (1) and (2) hold. Let I be an interior ideal of S. We will show that $\sqrt{I}$ is a bi-ideal of S. First, let $a, b \in \sqrt{I}$. Then $a^i \in I$ and $b^j \in I$ for some $i, j \in \mathbb{N}$. By (1), $(ab)^n \in S\{a^i, b^j\}S$ for some $n \in \mathbb{N}$. Thus, $(ab)^n \in S\{a^i, b^j\}S \subseteq SIS \subseteq I$. Hence, $ab \in \sqrt{I}$ and so $\sqrt{I}$ is a subsemigroup of S. Next, let $a, c \in \sqrt{I}$ and $b \in S$. Since $a, c \in \sqrt{I}$, $a^i \in I$ and $c^j \in I$ for some $i, j \in \mathbb{N}$. By (2), $(abc)^n \in S\{a^i, c^j\}S \subseteq SIS \subseteq I$. Hence, $abc \in \sqrt{I}$. Therefore, $\sqrt{I}$ is a bi-ideal of S. □

Theorem 2.8. *Let S be a semigroup with identity. The radical of every right ideal of S is an interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \ [a^i \xrightarrow{r} ab \vee b^j \xrightarrow{r} ab]$,

(2) $\forall a, b, \in S\ [a \mid b \Rightarrow \forall k \in \mathbb{N}\ \ [a^k \xrightarrow{r} b]]$.

Proof. Assume that the radical of every right ideal of S is an interior ideal of S. To show that (1) holds, let $a, b \in S$ and $i, j \in \mathbb{N}$. Put $I = \{a^i, b^j\}S$. Then I is a right ideal of S and $a, b \in \sqrt{I}$. By assumption, $\sqrt{I}$ is an interior ideal of S. Thus, $ab \in \sqrt{I}\sqrt{I} \subseteq \sqrt{I}$ Hence, $(ab)^n \in I = \{a^i, b^j\}S$ for some $n \in \mathbb{N}$ and so $a^i \xrightarrow{r} ab$ or $b^j \xrightarrow{r} ab$. Next, to show that (2) holds, we let $a, b \in S$ such that $a \mid b$. Then $b = xay$ for some $x, y \in S$. Let $k \in \mathbb{N}$. Put $I = a^kS$. Then I is a right ideal of S and $a \in \sqrt{I}$. By assumption, $\sqrt{I}$ is an interior ideal of S. Since $b = xay \in S\sqrt{I}S \subseteq \sqrt{I}$, $b^n \in I = a^kS$ for some $n \in \mathbb{N}$. Hence, $a^k \xrightarrow{r} b$.

Conversely, assume that (1) and (2) hold. Let I be a right ideal of S. We will show that $\sqrt{I}$ is an interior ideal of S. First, let $a, b \in \sqrt{I}$. Then $a^i, b^j \in I$ for some $i, j \in \mathbb{N}$. By (1), there exists $n \in \mathbb{N}$ such that $(ab)^n \in \{a^i, b^j\}S \subseteq IS \subseteq I$. Thus, $ab \in \sqrt{I}$. Hence, $\sqrt{I}$ is a subsemigoup of S. Next, let $x \in S\sqrt{I}S$. Then $x = yaz$ for some $y, z \in S$ and $a \in \sqrt{I}$. Since $a \in \sqrt{I}$, then there exists $k \in \mathbb{N}$ such that $a^k \in I$. By (2), there exists $n \in \mathbb{N}$ such that $x^n \in a^kS \subseteq IS \subseteq I$. Thus, $x \in \sqrt{I}$. Hence, $\sqrt{I}$ is an interior ideal of S. □

By Theorem 2.8, we have the following theorem.

Theorem 2.9. *Let S be a semigroup with identity. The radical of every left ideal of S is an interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \ [a^i \xrightarrow{l} ab \vee b^j \xrightarrow{l} ab]$,

(2) $\forall a, b \in S\ \ [a \mid b \Rightarrow \forall k \in \mathbb{N}\ [a^k \xrightarrow{l} b]]$.

Theorem 2.10. *Let S be a semigroup with identity. The radical of every quasi-ideal of S is an interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ \ [(ab)^n \in \{a^i, b^j\}S \cap S\{a^i, b^j\}]$,

(2) $\forall a, b, c \in S\ \ [a \mid b \Rightarrow \forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [b^n \in a^k S \cap S a^k]]$.

Proof. Assume that the radical of every quasi-ideal of S is an interior ideal of S. To show that (1) holds, we let $a, b \in S$ and $i, j \in \mathbb{N}$. Put $I = \{a^i, b^j\}S \cap S\{a^i, b^j\}$. Then I is a quasi-ideal of S and $a, b \in \sqrt{I}$. By assumption, $\sqrt{I}$ is an interior ideal of S. Thus, $ab \in \sqrt{I}\sqrt{I} \subseteq \sqrt{I}$. Hence, $(ab)^n \in I = \{a^i, b^j\}S \cap S\{a^i, b^j\}$ for some $n \in \mathbb{N}$. Next, to show that (2) holds, we let $a, b, c \in S$ such that $a \mid b$. Then $b = xay$ for some $x, y \in S$. Let $k \in \mathbb{N}$. Put $I = a^k S \cap S a^k$. Then I is a quasi-ideal of S and $a \in \sqrt{I}$. By assumption, $\sqrt{I}$ is an interior ideal of S. Thus, $b = xay \in S\sqrt{I}S \subseteq \sqrt{I}$. Hence, $b^n \in I = a^k S \cap S a^k$ for some $n \in \mathbb{N}$.

Conversely, assume that (1) and (2) hold. Let I be a quasi-ideal of S. We will show that $\sqrt{I}$ is an interior ideal of S. First, let $a, b \in \sqrt{I}$. Then $a^i, b^j \in I$ for some $a^i, b^j \in \mathbb{N}$. By (1), $(ab)^n \in \{a^i, b^j\}S \cap S\{a^i, b^j\} \subseteq IS \cap SI \subseteq I$ for some $n \in \mathbb{N}$. Thus, $ab \in \sqrt{I}$. Hence, $\sqrt{I}$ is a subsemigroup of S. Next, let $x \in S\sqrt{I}S$. Then $x = yaz$ for some $x, y \in S$ and $a \in \sqrt{I}$. Since $a \in \sqrt{I}$, then there exists $k \in \mathbb{N}$ such that $a^k \in I$. By (2), there exists $n \in \mathbb{N}$ such that $x^n \in a^k S \cap S a^k \subseteq IS \cap SI \subseteq I$. Thus, $x \in \sqrt{I}$. Hence, $\sqrt{I}$ is an interior ideal of S. □

Theorem 2.11. *Let S be a semigroup with identity. The radical of every ideal of S is an interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \ [a^i \to ab \vee b^j \to ab]$,

(2)] $\forall a, b \in S\ \ [a \mid b \Rightarrow \forall k \in \mathbb{N}\ [a^k \to b]]$.

Proof. Assume that the radical of every ideal of S is an interior ideal of S. To show that (1) holds, we let $a, b \in S$ and $i, j \in \mathbb{N}$. Put $I = S\{a^i, b^j\}S$. Then I is an ideal of S and $a, b \in \sqrt{I}$. By assumption, $\sqrt{I}$ is an interior ideal of S. Thus, $ab \in \sqrt{I}\sqrt{I} \subseteq \sqrt{I}$. Hence, $(ab)^n \in I = S\{a^i, b^j\}S$ for some $n \in \mathbb{N}$, and so $a^i \to ab$ or $b^j \to ab$. Next, to show that (2) holds, we let $a, b \in S$ such that $a \mid b$. Then $b = xay$ for some $x, y \in S$. Let $k \in \mathbb{N}$. Put $I = S a^k S$. Then I is an ideal of S and $a \in \sqrt{I}$. By assumption, $\sqrt{I}$ is an interior ideal of S. Since $b = xay \in S\sqrt{I}S \subseteq \sqrt{I}$, thus $b^n \in I = S a^k S$ for some $n \in \mathbb{N}$. Hence, $a^k \to b$.

Conversely, assume that (1) and (2) hold. Let I be an ideal of S. We will show that $\sqrt{I}$ is an interior ideal of S. First, let $a, b \in \sqrt{I}$. Then

$a^i, b^j \in I$ for some $i, j \in \mathbb{N}$. By (1), $(ab)^n \in S\{a^i, b^j\}S \subseteq SIS \subseteq IS \subseteq I$ for some $n \in \mathbb{N}$. Thus, $ab \in \sqrt{I}$. Hence, $\sqrt{I}$ is a subsemigroup of S. Next, let $x \in S\sqrt{I}S$. Then $x = yaz$ for some $y, z \in S$ and $a \in \sqrt{I}$. Since $a \in \sqrt{I}$, then there exists $k \in \mathbb{N}$ such that $a^k \in I$. By (2), there exists $n \in \mathbb{N}$ such that $x^n \in Sa^kS \subseteq SIS \subseteq IS \subseteq I$. Thus, $x \in \sqrt{I}$. Hence, $\sqrt{I}$ is an interior ideal of S. □

Theorem 2.12. *Let S be a semigroup with identity. The radical of every bi-ideal of S is an interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \ \exists n \in \mathbb{N}\ [(ab)^n \in \{a^i, b^j\}S\{a^i, b^j\}]$,

(2) $\forall a, b, c \in S\ \ [a \mid b \Rightarrow \forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [b^n \in a^kSa^k]]$.

Proof. Assume that the radical of every bi-ideal of S is an interior ideal of S. To show that (1) holds, we let $a, b \in S$ and $i, j \in \mathbb{N}$. Put $I = \{a^i, b^j\}S\{a^i, b^j\}$. Then I is a bi-ideal of S and $a, b \in \sqrt{I}$. By assumption, $\sqrt{I}$ is an interior ideal of S. Thus, $ab \in \sqrt{I}\sqrt{I} \subseteq \sqrt{I}$. Hence, $(ab)^n \in I = \{a^i, b^j\}S\{a^i, b^j\}$ for some $n \in \mathbb{N}$. Next, to show that (2) holds, we let $a, b, c \in S$ such that $a \mid b$. Then $b = xay$ for some $x, y \in S$. Let $k \in \mathbb{N}$. Put $I = a^kSa^k$. Then I is a bi-ideal of S and $a \in \sqrt{I}$. By assumption, $\sqrt{I}$ is an interior ideal of S. So, we obtain that $b = xay \in \sqrt{I}S\sqrt{I} \subseteq \sqrt{I}$. Hence, $b^n \in I = a^kSa^k$ for some $n \in \mathbb{N}$.

Conversely, assume that (1) and (2) hold. Let I be a bi-ideal of S. We will show that $\sqrt{I}$ is an interior ideal of S. First, let $a, b \in \sqrt{I}$. Then $a^i, b^j \in I$ for some $i, j \in \mathbb{N}$. By (1), $(ab)^n \in \{a^i, b^j\}S\{a^i, b^j\} \subseteq ISI \subseteq I$ for some $n \in \mathbb{N}$. Thus, $(ab) \in \sqrt{I}$. Hence, $\sqrt{I}$ is a subsemigroup of S. Next, let $x \in S\sqrt{I}S$. Then $x = yaz$ for some $y, z \in S$ and $a \in \sqrt{I}$. Since $a \in \sqrt{I}$, then there exists $k \in \mathbb{N}$ such that $a^k \in I$. By (2), there exists $n \in \mathbb{N}$ such that $x^n \in a^kSa^k \subseteq ISI \subseteq I$. Thus, $x \in \sqrt{I}$. Hence, $\sqrt{I}$ is an interior ideal of S. □

Theorem 2.13. *Let S be a semigroup with identity. The radical of every subsemigroup of S is an interior ideal of S if and only if*

(1) $\forall a, b \in S\ \forall i, j \in \mathbb{N}\ \ \exists n \in \mathbb{N}\ [(ab)^n \in \langle a^i, b^j\rangle]$,

(2) $\forall a, b \in S\ \ [a \mid b \Rightarrow \forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ [b^n \in \langle a^k\rangle]]$.

Proof. Assume that the radical of every subsemigroup of S is an interior ideal of S. To show that (1) holds, let $a, b \in S$ and $i, j \in \mathbb{N}$. Put $I = \langle a^i, b^j\rangle$. By assumption, $\sqrt{I}$ is an interior ideal of S and $a, b \in \sqrt{I}$. Thus, $ab \in \sqrt{I}\sqrt{I} \subseteq \sqrt{I}$. Hence, $(ab)^n \in I = \langle a^i, b^j\rangle$ for some $n \in \mathbb{N}$. This shows that (1) holds. Next, to show that (2) holds, let $a, b \in S$ such that $a \mid b$. Then $b = xay$ for some $x, y \in S$. Let $k \in \mathbb{N}$. Put $I = \langle a^k \rangle$. By assumption, $\sqrt{I}$ is an interior ideal of S and $a \in \sqrt{I}$. Since $b = xay$, then $b = xay \in S\sqrt{I}S \subseteq \sqrt{I}$. Thus, $b^n \in I = \langle a^k \rangle$ for some $n \in \mathbb{N}$.

Conversely, assume that (1) and (2) hold. Let I be a subsemigroup of S. We will show that $\sqrt{I}$ is an interior ideal of S. First, let $a, b \in \sqrt{I}$. Then $a^i, b^j \in I$ for some $i, j \in \mathbb{N}$. By (1), $(ab)^n \in \langle a^i, b^j \rangle$ for some $n \in \mathbb{N}$. Since $\langle a^i, b^j \rangle \subseteq I$, then $(ab)^n \in I$ for some $n \in \mathbb{N}$. Thus, $ab \in \sqrt{I}$. Hence, $\sqrt{I}$ is a subsemigroup of S. Next, let $x \in S\sqrt{I}S$. Then $x = yaz$ for some $y, z \in S$ and $a \in \sqrt{I}$. Since $a \in \sqrt{I}$, $a^k \in I$ for some $k \in \mathbb{N}$. By (2), $x^n \in \langle a^k \rangle \subseteq I$ for some $n \in \mathbb{N}$. Thus, $x \in \sqrt{I}$. Hence, $\sqrt{I}$ is an interior ideal of S. □

Finally, we obtain the following theorem.

Theorem 2.14. *Let S be a semigroup with identity. Then the following conditions are equivalent:*

(1) *the radical of every interior ideal of S is an interior ideal of S,*

(2) *for any $a, b \in S$,*

 (2.1) $\forall i, j \in \mathbb{N}\ \exists n \in \mathbb{N}\ \ [(ab)^n \in S\{a^i, b^j\}S]$,

 (2.2) $a \mid b \Rightarrow \forall k \in \mathbb{N}\ \ \exists n \in \mathbb{N}\ [b^n \in Sa^kS]$,

(3) *for any $a, b \in S$, $\sqrt{SaS}$ and $\sqrt{S\{a, b\}S}$ are interior ideals of S,*

(4) *for any $a, b \in S$,*

 (4.1) *there exists $n \in \mathbb{N}$ such that $(ab)^n \in S\{a^2, b^2\}S$,*

 (4.2) *if $a \mid b$, then there exists $n \in \mathbb{N}$ such that $b^n \in Sa^2S$,*

(5) *for any $a, b \in S$ and $k \in \mathbb{N}$,*

 (5.1) *there exists $n \in \mathbb{N}$ such that $(ab)^n \in S\{a^k, b^k\}S$,*

 (5.2) *if $a \mid b$, then there exists $n \in \mathbb{N}$ such that $b^n \in Sa^kS$.*

Proof. First, we will show that (1) ⇔ (2). Assume that the radical of every interior ideal of S is an interior ideal of S. To show that (2.1) holds, we let $a, b \in S$ and $i, j \in \mathbb{N}$. Put $I = S\{a^i, b^j\}S$. Then I is an interior ideal of S and $a, b \in \sqrt{I}$. By assumption, $\sqrt{I}$ is an interior ideal of S. Thus, $ab \in \sqrt{I}\sqrt{I} \subseteq \sqrt{I}$. Hence, $(ab)^n \in I = S\{a^i, b^j\}S$ for some $n \in \mathbb{N}$. Next, to show that (2.2) holds, we let $a, b \in S$ such that $a \mid b$. Then $b = xay$ for some $x, y \in S$. Let $k \in \mathbb{N}$. Put $I = Sa^kS$. Then I is an interior ideal of S and $a \in \sqrt{I}$. By assumption, $\sqrt{I}$ is an interior ideal of S. So, we obtain that $b = xay \in S\sqrt{I}S \subseteq \sqrt{I}$. Hence, $b^n \in I = Sa^kS$ for some $n \in \mathbb{N}$.

Conversely, assume that (2.1) and (2.2) hold. Let I be an interior ideal of S. To show that $\sqrt{I}$ is an interior ideal of S, let $a, b \in \sqrt{I}$. Then $a^i, b^j \in I$ for some $i, j \in \mathbb{N}$. By (2.1), $(ab)^n \in S\{a^i, b^j\}S \subseteq SIS \subseteq I$ for some $n \in \mathbb{N}$. Thus, $ab \in \sqrt{I}$. This shows that $\sqrt{I}$ is a subsemigroup of S. Next, to show that $S\sqrt{I}S \subseteq \sqrt{I}$, let $x \in S\sqrt{I}S$. We have $x = yaz$ for some $y, z \in S$ and $a \in \sqrt{I}$. Since $a \in \sqrt{I}$, there exists $k \in \mathbb{N}$ such that $a^k \in I$. By (2.2), there exists $n \in \mathbb{N}$ such that $x^n \in Sa^kS \subseteq SIS \subseteq I$. Thus, $x \in \sqrt{I}$. Therefore, $\sqrt{I}$ is an interior ideal of S.

(1) $\Rightarrow$ (3). Assume (1) holds, and let $a, b \in S$. Since SaS and $S\{a, b\}S$ are interior ideals of S and (1), thus $\sqrt{SaS}$ and $\sqrt{S\{a, b\}S}$ are interior ideals of S.

(3) $\Rightarrow$ (4). Assume (3) holds, and let $a, b \in S$. Then obviously, $a, b \in \sqrt{S\{a^2, b^2\}S}$. By (3), $\sqrt{S\{a^2, b^2\}S}$ is an interior ideal of S. Thus,

$$ab \in (\sqrt{S\{a^2, b^2\}S})(\sqrt{S\{a^2, b^2\}S}) \subseteq \sqrt{S\{a^2, b^2\}S}.$$

Hence, $(ab)^n \in S\{a^2, b^2\}S$ for some $n \in \mathbb{N}$. This shows that (4.1) holds. Next, let $a, b \in S$ such that $a \mid b$. Then $b = xay$ for some $x, y \in S$. Obviously, $a \in \sqrt{Sa^2S}$. By (3), $\sqrt{Sa^2S}$ is an interior ideal of S. Since $b = xay$, thus $b = xay \in S(\sqrt{Sa^2S})S \subseteq \sqrt{Sa^2S}$. Hence, $b^n \in Sa^2S$ for some $n \in \mathbb{N}$. Therefore, (4.2) holds.

(4) $\Rightarrow$ (5). Assume (4) holds, and let $a, b \in S$. By (4.1), $(ab)^n \in S\{a^2, b^2\}S$ for some $n \in \mathbb{N}$. Since $S\{a^2, b^2\}S \subseteq S\{a, b\}S$, then $(ab)^n \in S\{a, b\}S$ for some $n \in \mathbb{N}$. Suppose that there exists $m \in \mathbb{N}$ where $k \in \mathbb{N}$ such that $(ab)^m \in S\{a^k, b^k\}S$. By (4.1), there exists $p \in \mathbb{N}$ such that $((ab)^m)^p \in S\{a^{2k}, b^{2k}\}S$. Thus,

$$((ab)^m)^p \in S\{a^{2k}, b^{2k}\}S = S\{a^{k+1}a^{k-1}, b^{k+1}b^{k-1}\}S \subseteq S\{a^{k+1}, b^{k+1}\}S.$$

Hence, $(ab)^{mp} = ((ab)^m)^p \in S\{a^{k+1}, b^{k+1}\}S$. This shows that (5.1) holds. Next, let $a, b \in S$ such that $a \mid b$. Then $b = xay$ for some $x, y \in S$. By (4.2), $b^n \in Sa^2S$ for some $n \in \mathbb{N}$. Thus, $b^n \in Sa^2S \subseteq SaS$. Hence, $b^n \in SaS$ for some $n \in \mathbb{N}$. Suppose that there exists $m \in \mathbb{N}$ where $k \in \mathbb{N}$ such that $b^m \in Sa^kS$. By (4.2), there exists $p \in \mathbb{N}$ such that $(b^m)^p \in Sa^{2k}S$. Thus, $(b^m)^p \in Sa^{2k}S = Sa^{k+1}a^{k-1}S \subseteq Sa^{k+1}S$. Hence, $b^{mp} = (b^m)^p \in Sa^{k+1}S$. This shows that (5.2) holds.

(5) $\Rightarrow$ (1). Assume (5) holds. Let I be an interior ideal of S. First, let $a, b \in \sqrt{I}$. Then there exist $i, j \in \mathbb{N}$ such that $a^i, b^j \in I$. By (5.1), there exists $n \in \mathbb{N}$ such that

$$(ab)^n \in S\{a^{i+j}, b^{i+j}\}S = S\{a^i a^j, b^i b^j\}S \subseteq S\{a^i, b^j\}S \subseteq SIS \subseteq I.$$

Thus, $ab \in \sqrt{I}$, and so $\sqrt{I}$ is a subsemigroup of S. Next, let $x \in S\sqrt{I}S$. Then $x = yaz$ for some $y, z \in S$ and $a \in \sqrt{I}$. Since $a \in \sqrt{I}$, then there exists $k \in \mathbb{N}$ such that $a^k \in I$. By (5.2), there exists $n \in \mathbb{N}$ such that $x^n \in Sa^kS$. Thus, $x^n \in Sa^kS \subseteq SIS \subseteq I$. Hence, $x \in \sqrt{I}$. Therefore, $\sqrt{I}$ is an interior ideal of S. □

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References

[1] **S. Bogdanović and M. Ciri c**, *Semigroups in which the radical of every ideal is a subsemigroup*, Zbornik rad. Fil. Nis. Ser. Mat., **6** (1992), 129 – 135.

[2] **R.A. Good and D.R. Hughes**, *Associated groups for a semigroup*, Bull. Amer. Math. Soc., **58** (1952), 624 – 625.

[3] **N. Kehayopulu and M. Tsingelis**, *On left regular and intra-regular ordered semigroup*, Math. Slovaca, **64** (2014), 1123 – 1134.

[4] **S. Lajos**, *(m, k, n)-ideals in semigroups. Notes on semigroups II*, Karl Marx Univ. Econ., Dept. Math. Budapest, **1** (1976), 12 – 19.

[5] **J. Sanborisoot and T. Changphas**, *Semigroups in which the radical of every quasi-ideal is a subsemigroup*, Quasigroups Related Syst., **28** (2020), 301-308.

[6] **J. Sanborisoot and T. Changphas**, *Ordered semigroups in which the radical of every quasi-ideal is a subsemigroup*, Int. J. Math. Comput. Sci., **16** (2021), 1385 – 1396.

[7] **O. Steinfeld**, *Über die Quasiideals von Halbgruppen*, (German). Publ. Math. Debrecen, **4** (1956), 262 – 275.

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