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An Analogue of the Hartogs Lemma for R-Analytic Functions

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Abstract. The paper is devoted to the problem of R-analytic continuation of functions of several real variables which admit R-analytic continuation along parallel sections. We prove an analogue of the well-known Hartogs lemma for R-analytic functions.

Keywords: R-analytic functions, holomorphic functions, plurisubharmonic functions, pluripolar sets, Hartogs series.

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1. Introduction and preliminaries

In this paper we consider the R-analytic continuation of functions of several real variables that admit R-analytic continuation along parallel sections. Regarding to holomorphic functions, the first result in this direction is due to Hartogs [1]: if a holomorphic function $f('z, z_n)$ in the domain $'U \times \{|z_n| < r\} \subset \mathbb{C}^n_{'z} \times \mathbb{C}_{z_n}$, where $'z = (z_1, z_2, \ldots, z_{n-1})$, for each fixed $'z \in ('U)$ by z_n extends holomorphically to the disk $|z_n| < R$, $R > r > 0$, then it is holomorphic with respect to all variables in the domain $'U \times \{|z_n| < R\}$.

The following Forelli's theorem [2] is also directly related to Hartogs theorem: if f is infinitely smooth at a point $0 \in \mathbb{C}^n, f \in C^\infty\{0\}$, and the restrictions $f|_l$ are holomorphic in the disc $U(0,1) = l \bigcap B(0,1)$ for all complex lines $l \ni 0$, then f can be holomorphically extended to the ball $B(0,1) \subset \mathbb{C}^n$.

In a recent paper [3] A. Sadullaev proved the following analogue of Forelli's theorem for R-analytic functions.

Theorem 1. *Let a function $f(x)$, $x = (x_1, x_2, \ldots, x_n)$ be smooth in some neighborhood of the origin $0 \in \mathbb{R}^n$, $f(x) \in C^\infty\{0\}$ and let for any real line $l : x = \lambda t$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in S(0,1) \subset$*

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$\mathbb{R}^n$, $t \in \mathbb{R}$ *is a parameter, the restriction* $f|_l = f(\lambda t)$ *is real-analytic (R-analytic) in the interval* $t \in (-1, 1)$. *Then there is a closed pluripolar set* $S \subset B(0,1)$ *such that* $f(x)$ *is R-analytic in* $B(0,1) \backslash S$, *where* $B(0,1) \subset \mathbb{R}^n$ *is the unit ball and* $S(0,1) = \partial B(0,1)$ *is the unit sphere.*

Note that the well-known terminology is used here, a set $S \subset \mathbb{R}^n_x$ is called pluripolar if it is pluripolar in the ambient complex space $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z$, $z = x + iy$. An example of a function $f(x_1, x_2) = \dfrac{x_1^{k+1}}{(x_2-1)^2 + x_1^2}$ shows that exact analogues of Forelli's Theorem and Hartogs' Theorem for R-analytic functions are not true. The function $f(x_1, x_2)$ is real-analytic in the domain $\mathbb{R} \times \left\{|x_2| < \frac{1}{2}\right\}$, the restriction $f\left(x_1^0, x_2\right)$ is real-analytic on the whole line $\mathbb{R}$. However, f is not real-analytic at the point $(0, 1)$.

The main result of this work is

Theorem 2. *Let a function* $f(x) = f('x, x_n)$ *satisfy the following conditions:*

1) The function $f('x, x_n)$ *is R-analtic in a polycylinder* $U = ('U) \times \{|x_n| < r_n\}$, $r_n > 0$, *where* $'x = (x_1, x_2, \ldots, x_{n-1})$ *and*

$$'U = \left\{'x \in \mathbb{R}^{n-1} : |x_1| < r_1, |x_2| < r_2, \ldots, |x_{n-1}| < r_{n-1}\right\} =$$
$$= \left\{'x \in \mathbb{R}^{n-1} : -r_1 < x_1 < r_1, -r_2 < x_2 < r_2, \ldots, -r_{n-1} < x_{n-1} < r_{n-1}\right\}.$$

2) For each fixed $('x^0) \in ('U)$ *the function* $f\left('x^0, x_n\right)$ *that is R-analytic in the interval* $|x_n| < r_n$, *R-analytically continues into a larger interval* $|x_n| < R_n$, $R_n > r_n$.

Then there exists a closed pluripolar set $'S \subset ('U)$ *such that the function* $f('x, x_n)$ *R-analytically with respect to all variables* $('x, x_n)$ *continues into the domain* $('U \times \{|x_n| < R_n\}) \backslash ('S \times \{|x_n| \geqslant r_n\})$.

The proof of Theorem 2 essentially uses the method of proving Theorem 1 proposed by A. Sadullaev, namely, the embedding of a real space $\mathbb{R}^n_x \subset \mathbb{C}^n_z$, $z = x + iy$, and the natural holomorphic continuation of R-analytic functions into $\mathbb{C}^n$, the holomorphic continuation of the Hartogs series and methods of pluripotential theory (see [4–5]).

Note that using the local transformation of the pencil of lines $l \ni 0$, into parallel ones, from Theorem 2 one can obtain a proof of Theorem 1.

Real analytic functions were also studied in the work of J. Sichak [6], where he proved that if the function $f(x)$ is smooth in a domain $D \subset \mathbb{R}^n f \in C^\infty(D)$ and for each real line $l : x = x^0 + \lambda t$, $x^0 \in D$, $\lambda \in \mathbb{R}^n$, $|\lambda| = 1$, $t \in \mathbb{R}$, the restriction $f|_\ell$ is R-analytic by t in some neighborhood of zero, then $f(x)$ is R-analytic in D.

2. Domain of holomorphy of Hartogs series

Let $U = ('U) \times U_n$ be a domain in $\mathbb{C}^{n-1}_{'z} \times \mathbb{C}_{z_n}$, where U_n is a disc centered at the point $z_n = 0$ and with a radius $\delta > 0$. If the function $f('z, z_n)$ is holomorphic in U, then it can be expanded in a Hartogs series:

$$f('z, z_n) = \sum_{k=0}^{\infty} c_k('z)\, z_n^k, \tag{1}$$

where, the coefficients $c_k('z)$ are holomorphic in $'U$ and determined by the formula

$$c_k('z) = \frac{1}{2\pi i} \int\limits_{|\xi|=\delta'} \frac{f('z, \xi)}{\xi^{k+1}} d\xi, \quad 0 < \delta' < \delta,\ k = 0,\, 1,\, 2,\, \ldots. \tag{2}$$

Then, it is known that if $R('z)$ is the radius of convergence of series (1), then the function $u^*('z) = -\ln R_*('z)$ is plurisubharmonic in $'U$, and the set $\{'z \in ('U) : R_*('z) < R('z)\}$ is pluripolar. Here $R_*('z) = \underline{\lim}_{'w \to 'z} R('w)$ is the lower regularization. Moreover, the series (1) converges uniformly on any compact subset $K \subset\subset ('U) \times \{|z_n| < R_*('z)\}$. The proof of this fact can be found, for example, in [7, 8].

The following lemma, which plays the key role in the proof of Theorem 2, is widely used in the theory of analytic continuation.

Lemma 1. *Let a function $f('z, z_n)$ be holomorphic in the domain $'U \times \{z_n \in \mathbb{C} : |z_n| < \delta\}$, $'U \subset \mathbb{C}^{n-1}$. If for each fixed $'z^0 \in ('U_0)$ from some non-pluripolar set $'U_0 \subset ('U)$ the function $f('z^0, z_n)$ of variable z_n, extends holomorphically to the larger disc $\{z_n \in \mathbb{C} : |z_n| < \Delta\}$, $\Delta \geqslant \delta > 0$, then the function $f('z, z_n)$ holomorphically extends to the domain $\left\{'z \in' U, |z_n| < \delta^{\omega^*('z,'U_0,'U)} \cdot \Delta^{1-\omega^*('z,'U_0,'U)}\right\}$, where $\omega^*('z,' U_0,' U)$ is the well-known plurisubharmonic measure of the set $'U_0$ with respect to the domain $'U$, that is defined by the following*

$$\omega^*('z,' U_0,' U) = \left(\sup\left\{u('z) \in psh('U) : u(z)|_{'U} < 1, u(z)|_{'U_0} \leqslant 0\right\}\right)^*.$$

Indeed, if we expand the function $f('z, z_n)$ in a Hartogs series of the form (1) in the domain $'U \times \{z_n \in \mathbb{C} : |z_n| < \delta\}$, then the function $u('z) = -\ln R_*('z)$ is plurisubharmonic in the domain $'U$ and by the conditions of the lemma $u('z)|_{'U} \leqslant -\ln \delta$, $u(z)|_{'U_0} \leqslant -\ln \Delta$. According to the theorem on two constants (see [9], p. 103), we obtain the inequality

$$u('z) \leqslant (1 - \omega^*('z,' U_0,' U)) \cdot (-\ln \Delta) + \omega^*('z,' U_0,' U) \cdot (-\ln \delta).$$

Hence it follows that

$$\ln R_*('z) \geqslant (1 - \omega^*('z,' U_0,' U)) \cdot \ln \Delta + \omega^*('z,' U_0,' U) \cdot \ln \delta,$$

or $R_*(z) \geqslant \delta^{\omega^*('z,'U_0,'U)} \cdot \Delta^{1-\omega^*('z,'U_0,'U)}$. Thus in accordance with above mentioned, the function $f('z, z_n)$ extends holomorphically to the domain

$$'U \times \{|z_n| < R_*('z)\} \supset ('U) \times \left\{|z_n| < \delta^{\omega^*('z,'U_0,'U)} \cdot \Delta^{1-\omega^*('z,'U_0,'U)}\right\}.$$

3. Proof of the main result

Without loss of generality we assume that for each fixed $'x \in ('U)$ the function $f('x, x_n)$ is R-analytic in the interval $(-R_n - \varepsilon, R_n + \varepsilon)$, $\varepsilon > 0$. The proof of the theorem will be implemented in several steps.

Step 1. We embed the real space $\mathbb{R}^n_x$ into the complex space $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z$, $z = x + iy$. Then, by definition of R-analyticity of a function $f('x, x_n)$, there exists a domain $\hat{U} \subset \mathbb{C}^n$, $\hat{U} \supset U$ and a holomorphic function $F(z) = F('z, z_n) \in O\left(\hat{U}\right)$ such that $F('z, z_n)|_U = f('x, x_n)$.

It follows that from the conditions of the theorem the function $F(z) = F('z, z_n)$ satisfies the following conditions:

1) $F(z) \in O\left(\hat{U}\right)$.

2) For each fixed $'z = ('x) \in ('U)$ the function $F('x, z_n)$ of the variable z_n, can be extended holomorphically into the ellipse of type $E_j : \dfrac{(Re z_n)^2}{R_n^2} + j^2(Im z_n)^2 < 1$, $j \in \mathbb{N}$, such that $E_j \supset\supset \{|x_n| \leqslant R_n\}$ $\forall j \in \mathbb{N}$.

We put $'\hat{U} = \hat{U} \cap \mathbb{C}^{n-1}_{'z}$ and fix a subdomain $'\hat{V} \subset\subset ('\hat{U})$ such that $'V = ('\hat{V}) \cap ('U) \neq \emptyset$. Then there is a circle $\{|z_n| < \sigma\}$, $\sigma > 0$, such that $'\hat{V} \times \{|z_n| < \sigma\} \subset \hat{U}$, i.e. the function $F(z) = F('z, z_n)$ is holomorphic with respect to the $('z, z_n)$ in $'\hat{V} \times \{|z_n| < \sigma\}$. We fix the number $j \in \mathbb{N}$ and denote by $'V_j$ the set of points $'x$ from $'V = ('\hat{V}) \cap ('U)$ for which the function $F('x, z_n)$ of variable z_n extends holomorphically into the ellipse E_j, i.e.

$$'V_j = \{'x \in ('V) : F('x, z_n) \in O(E_j)\}$$

It is obvious that

$$V_j \subset V_{j+1} \ \forall j \in \mathbb{N}$$

and

$$\bigcup_{j=1}^{\infty} ('V_j) = {'V}.$$

Step 2. Since an open non-empty subset $'V \subset \mathbb{R}^{n-1}$ is not pluripolar in $\mathbb{C}^{n-1}$, then there exists a number $j_0 \in \mathbb{N}$ such that for all $j > j_0$ the sets $'V_j \subset ('V)$ will be non-pluripolar in $\mathbb{C}^n$.

Let us fix $j \in \mathbb{N}$, $j > j_0$ and let the function $w = g_j(z_n)$ conformally maps the ellipse E_j into the unit circle $\{|w| < 1\}$, $g_j(0) = 0$. Since the function $F('z, z_n)$ is holomorphic in the neighborhood $'\hat{V} \times \{|z_n| < \sigma\}$, the function $\Phi('z, w) = F('z, g_j^{-1}(w))$ is holomorphic in the domain $'\hat{V} \times g_j^{-1}(\{|z_n| < \sigma\})$. Since $g_j(0) = 0$. there is a number $\delta_j > 0$ such that $('\hat{V}) \times \{|w| < \delta_j\} \subset ('\hat{V}) \times g_j^{-1}(\{|z_n| < \sigma\})$, i.e. the function $\Phi('z, w)$ is holomorphic in the domain $'\hat{V} \times \{|w| < \delta_j\}$. In addition, for each fixed variable $'z = ('x) \in ('V_j)$, the function $\Phi('x, w)$ of the variable w extends holomorphically to the circle $\{|w| < 1\}$.

By Lemma 1, where $\delta = \delta_j$, $\Delta = 1$, the function $\Phi('z, w)$ is holomorphic in the domain

$$\left\{'z \in' \hat{V}, |z_n| < \delta_j^{\omega^*(\backslash z, 'V_j, '\hat{V})}\right\}.$$

Thus, if we substitute into $\Phi('z, w)$ the value $w = g_j(z_n)$, then we obtain that the function $F('z, z_n)$ extends holomorphically to the domain

$$G_j = \left\{('z, z_n) \in \mathbb{C}^n : ('z) \in ('\hat{V}), |g_j(z_n)| < \delta_j^{\omega^*('z, 'V_j, '\hat{V})}\right\} \quad (3)$$

Note that if the point $'x \in ('V_j)$ is pluriregular, i.e. $\omega^*('x, 'V_j, '\hat{V}) = 0$, then, according to (3), the ellipse $\{'x\} \times \{|g_j(z_n)| < 1\} \subset G_j$. Consequently, the domain G_j contains some neighborhood of the segment $\{'x\} \times [-R_n, R_n]$.

Step 3. By the construction of the domain G_j, F can be extended holomorphically to the domain $G_{'V} = \bigcup_{j=j_0}^{\infty} G_j$ as well. Let us denote by P_j the set of irregular points $'x \in ('V_j)$ and by $P_{'V} = \bigcup_{j=j_0}^{\infty} P_j$ the union of these sets $P_{'V} \subset ('V)$. It is a pluripolar set in $\mathbb{C}^{n-1}_{'z}$. For each fixed point $'z = ('x) \in ('V) \backslash P_{'V}$, the union $G_{'V} = \bigcup_{j=j_0}^{\infty} G_j$ contains a neighborhood of the segment $\{'x\} \times [-R_n, R_n]$.

Step 4. We take a sequence of domains $'\hat{V}_k \subset\subset' \hat{V}_{k+1} \subset\subset' \hat{U} : \bigcup_{k=1}^{\infty} ('\hat{V}_k) =' \hat{U}$ and put $P = \bigcup_{k=1}^{\infty} P_{'V_k}$. Then $P \subset ('U)$ is pluripolar set in $\mathbb{C}^{n-1}_{'z}$. According to Step 3, the function F extends holomorphically to the domain $G = \bigcup_{k=1}^{\infty} G_{'V_k}$, and for each fixed point $'z = ('x) \in ('U) \backslash P$ the union $G = \bigcup_{k=1}^{\infty} G_{'V_k}$ contains a neighborhood of the segment $\{'x\} \times [-R_n, R_n]$. Therefore, for such points the given function $f('x, x_n)$ is R-analytic in the set of variables in the neighborhood of the segment $\{'x\} \times [-R_n, R_n]$.

We note that the complement $S = ['U \times \{|x_n| < R_n\}] \setminus [G \bigcap \mathbb{R}^n]$ is a closed pluripolar set, $S \subset P \times \{|x_n| \geqslant r_n\}$, and the function $f('x, x_n)$ is R-analytically extended to $['U \times \{|x_n| < R_n\}] \setminus S$. The theorem is proved. □

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Об аналоге леммы Гартогса для R-аналитических функций

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Аннотация. Работа посвящена задачам R-аналитического продолжения функций многих действительных переменных, допускающих R-аналитическое продолжение на параллельные сечения. В ней доказывается аналог известной теоремы Гартогса для R-аналитических функций.

Ключевые слова: R-аналитические функции, голоморфные функции, плюрисубгармонические функции, плюриполярные множества, ряды Гартогса.