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CR-warped product submanifolds of a generalized complex space form

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Abstract: In this paper CR-warped product submanifolds of a generalized complex space form are studied and a characterizing inequality for existence of CR-warped product submanifolds is established. Moreover, some special cases are also discussed.

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1. Introduction

The notion of warped product of manifolds was introduced by Bishop and O'Neill (1965) in order to construct a large velocity of manifolds of negative curvature. The idea of warped product on submanifolds was introduced by Chen (2001). Basically, Chen considered warped product of the type $N_\perp \times_f N_T$ such that $N_\perp$ is totally real submanifold and N_T is holomorphic submanifold of a Kaehler manifold and found that these warped product are simply CR-product as defined in Chen (1981). Therefore, Chen considered the warped product of the type $N_T \times_f N_\perp$, and obtained an inequality for squared norm of second fundamental form, these types of warped product are called CR-warped product. Later on, the geometrical behavior of these type of submanifolds were studied by many researchers (c.f. Arslan, Ezentas, Mihai, & Murathan, 2005; Khan, Khan, & Uddin, 2009; Sahin, 2006).

In Al-Luhaibi, Al-Solamy, and Khan (2009) investigated CR-warped product submanifolds in the setting of nearly Kaehler manifolds and obtained some basic results and finally worked out an estimation for squared norm of second fundamental form if ambient manifold is generalized complex space form. These types of warped products are also studied in different settings of almost Hermitian manifolds (c.f. Al-Luhaibi et al., 2009; Faghfouri & Majidi, 2015; Khan & Jamal, 2010; Sahin, 2009).

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PUBLIC INTEREST STATEMENT

The warped product manifolds are the generalization of product manifolds and occur naturally. These types of manifolds have wide applications in Differential geometry, Relativity, Physics as well as in different branches of engineering. The present study predicts the geometric behavior of underlying warped product submanifolds. Further it is known that the warping function of a warped product manifolds is a solution of some partial differential equations and most of the physical phenomenons are described by partial differential equations. We hope that our study may find applications in Physics as well as in engineering.

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The contact version of warped product manifolds were also studied in different settings (see Ateken, 2011, 2013; Sular & Ozgur, 2012). Recently, we also studied semi-invariant warped product submanifolds and obtained inequality for squared norm of second fundamental form Al-Solamy and Ali Khan (2012). Moreover, Aitceken (c.f. Ateken, 2011, 2013) investigated an inequality for squared norm of second fundamental form which characterize the existence of contact CR-warped product submanifolds in the setting of Cosymplectic and Kenmotsu space forms. Motivated by Aitceken (2011, 2013) and Sular and Ozgur (2012) studied the contact CR-warped product in more general setting namely trans-Sasakian generalized Sasakian space forms and obtained an inequality for existence of CR-warped product submanifolds. After reviewing the literature, we realized that characterizing inequality for existence of CR-warped product submanifolds is not yet investigated in the setting of generalized complex space forms. In this paper we obtained an estimation of second fundamental form in terms of Hessian of *lnf*, where *f* is a warping function and finally obtained an characterizing inequality for existence of CR-warped product submanifolds of generalized complex space forms and in particular for complex space forms.

2. Preliminaries

Let $\bar{M}$ be an almost Hermitian manifold with almost complex structure J and a Hermitian metric g i.e.

$$J^2 = -I \text{ and } g(JU, JV) = g(U, V), \tag{2.1}$$

for all vector fields $U, V \in T\bar{M}$. If almost complex structure J is parallel with respect to the Levi-civita connection $\bar{\nabla}$ on $\bar{M}$ i.e. $\bar{\nabla}J = 0$, then $(\bar{M}, J, g)$ is called Kaehler manifold. There is a more general structure on $\bar{M}$, namely nearly Kaehler structure and characterized by the following equation

$$(\bar{\nabla}_U J)V + (\bar{\nabla}_V J)U = 0, \tag{2.2}$$

equivalently, (2.2) can also be written as

$$(\bar{\nabla}_U J)U = 0, \tag{2.3}$$

for all $U, V \in T\bar{M}$.

A nearly Kaehler manifold $\bar{M}$ is Kaehler manifold if and only if Neijenhuis tensor of J vanish identically. Any four dimensional nearly Kaehler manifold is a Kaehler manifold. Six dimensional sphere S^6 is a typical example of a nearly Kaehler non Kaehler manifold. The complex structure J on S^6 is defined by vector cross product in the space of purely imaginary Cayley numbers. There is a more general class of almost Hermitian manifolds than nearly Kaehler manifold, this class is known as $RK-$manifolds. A generalized complex space form is an $RK-$manifold of constant holomorphic sectional curvature c and of constant α is denoted by $\bar{M}(c, \alpha)$. The sphere S^6 endowed with the standard nearly Kaehler structure is an example of a generalized complex space form which is not a complex space form. The curvature tensor $\bar{R}$ of generalized complex space form $\bar{M}(c, \alpha)$ is given by

$$(\bar{R}(U, V)W = \frac{c + 3\alpha}{4}[g(V, W)U - g(U, W)V] + \frac{c - \alpha}{4}[g(U, JW)JV - g(V, JW)JU + 2g(U, JV)JW]. \tag{2.4}$$

Let M be a submanifold of $\bar{M}$, then the induced Riemannian metric on M is denoted by the same symbol g and the induced connection on M is denoted by the symbol ∇. If $T\bar{M}$ and TM denote the tangent bundle on $\bar{M}$ and M respectively and $T^\perp M$, the normal bundle on M, then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.5}$$

$$\bar{\nabla}_U N = -A_N U + \nabla_U^\perp N, \tag{2.6}$$

for all $U, V \in TM$ and $N \in T^\perp M$ where $\nabla^\perp$ denotes the connection on the normal bundle $T^\perp M$. h and A_N are the second fundamental form and the shape operator of immersions of M in to $\bar{M}$. Corresponding to the normal vector field N they are related as

$$g(A_N U, V) = g(h(U, V), N). \tag{2.7}$$

The mean curvature vector H of M is given by

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$

where n is the dimension of M and $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame of vector fields on M. The squared norm of the second fundamental form is defined as

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.8}$$

A submanifold M of $\bar{M}$ is said to be a totally geodesic submanifold if $h(U, V) = 0$, for each $U, V \in TM$, and totally umbilical submanifold if $h(U, V) = g(U, V)H$.

For $U \in TM$ and $N \in T^\perp M$ we write

$$JU = PU + PU, \tag{2.9}$$

$$JN = tN + fN, \tag{2.10}$$

where PU and tN are tangential components of JU and JN respectively and FU and fN are the normal components of JU and JN.

The covariant differentiation of the tensors J, P, F, t and f are defined as respectively

$$(\bar{\nabla}_U J)V = \bar{\nabla}_U JV - J\bar{\nabla}_U V, \tag{2.11}$$

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{2.12}$$

$$(\bar{\nabla}_U F)V = \nabla_U^\perp FV - F\nabla_U V, \tag{2.13}$$

$$(\bar{\nabla}_U t)N = \nabla_U tN - t\nabla_U^\perp N, \tag{2.14}$$

$$(\bar{\nabla}_U f)N = \nabla_U^\perp fN - f\nabla_U^\perp N. \tag{2.15}$$

Furthermore, for any $U, V \in TM$, the tangential and normal parts of $(\bar{\nabla}_U J)V$ are denoted by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ i.e.

$$(\bar{\nabla}_U J)V = \mathcal{P}_U V + \mathcal{Q}_U V. \tag{2.16}$$

Moreover, it is easy to verify the following property

$$(\bar{\nabla}_U J)JV = -J(\bar{\nabla}_U)JV. \tag{2.17}$$

On using equations (2.5)–(2.13) and (2.16), we may obtain that

$$\mathcal{P}_U V = (\bar{\nabla}_U P)V - A_{FV}U - th(U, V), \tag{2.18}$$

$$Q_U V = (\bar{\nabla}_U F)V + h(U, TY) - fh(U, V). \tag{2.19}$$

Similarly, for $N \in T^\perp M$, denoting by $\mathcal{P}_U N$ and $\mathcal{Q}_U N$ respectively the tangential and normal parts of $(\bar{\nabla}_U J)N$, we find that

$$\mathcal{P}_U N = (\bar{\nabla}_U t)N + PA_N U - A_{fN} U, \tag{2.20}$$

$$\mathcal{Q}_U N = (\bar{\nabla}_U f)N + h(tN, U) + FA_N U. \tag{2.21}$$

On a submanifold M of a nearly Kaehler manifold by (2.2) and (2.16)

(a) $\mathcal{P}_U V + \mathcal{P}_V U = 0,$ (b) $\mathcal{Q}_U V + \mathcal{Q}_V U = 0$ $\tag{2.22}$

for any $U, V \in TM$.

Now we have the following properties of $\mathcal{P}$ and $\mathcal{Q}$, which can be verified very easily

(P_1) (i) $\mathcal{P}_{U+V} W = \mathcal{P}_U W + \mathcal{P}_V W$ (ii) $\mathcal{Q}_{U+V} W = \mathcal{Q}_U W + \mathcal{Q}_Y Z,$
(P_2) (i) $\mathcal{P}_U(V + W) = \mathcal{P}_U V + \mathcal{P}_U W$ (ii) $\mathcal{Q}_U(V + W) = \mathcal{Q}_U V + \mathcal{Q}_U W,$
(P_3) (i) $g(\mathcal{P}_U V, W) = -g(V, \mathcal{P}_U W)$ (ii) $g(\mathcal{Q}_U V, N) = -g(V, \mathcal{P}_U N),$

for all $U, V, W \in TM$ and $N \in T^\perp M$.

Let $\bar{M}$ be an almost Hermitian manifold with an almost complex structure J and Hermitian metric g and let M be a submanifold of $\bar{M}$, M is said to be CR-submanifold if there exist two orthogonal complementary distributions D and $D^\perp$ such that D is holomorphic distribution i.e., $JD \subseteq D$ and $D^\perp$ is totally real distribution i.e., $JD^\perp \subseteq T^\perp M$.

If μ is the invariant subspace of the normal bundle $T^\perp M$, then in the case of CR-submanifold, the normal bundle $T^\perp M$ can be decomposed as follows

$$T^\perp M = \mu \oplus JD^\perp. \tag{2.23}$$

A CR-submanifold M is called CR-product if the distribution D and $D^\perp$ are parallel on M. In this case M is foliated by the leaves of these distributions. In general, if N_1 and N_2 are Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively, then the product manifold $(N_1 \times N_2, g)$ is a Riemannian manifold with Riemannian metric g defined as

$$g(U, V) = g_1(d\pi_1 U, d\pi_1 V) + g_2(d\pi_2 U, d\pi_2 V), \tag{2.24}$$

where π_1 and π_2 are the projection maps of M onto N_1 and N_2, respectively, and $d\pi_1$ and $d\pi_2$ are their differentials.

As a generalization of the product manifold and in particular of CR-product submanifold, one can consider warped product of manifolds which are defined as follows.

Definition 2.1   Let (B, g_B) and (C, g_C) be two Riemannian manifolds with Riemannian metric g_B and g_C respectively and ψ be a positive differentiable function on B. The warped product of B and C is the Riemannian manifold $(B \times C, g)$, where

$$g = g_B + \psi^2 g_C.$$

For a warped product manifold $N_1 \times_\psi N_2$, we denote by D_1 and D_2 the distributions defined by the vectors tangent to the leaves and fibers respectively. In other words, D_1 is obtained by the tangent

vectors of N_1 via the horizontal lift and D_2 is obtained by the tangent vectors of N_2 via vertical lift. In case of CR-warped product submanifolds D_1 and D_2 are replaced by D and $D^\perp$ respectively.

The warped product manifold $(B \times C, g)$ is denoted by $B \times_\psi C$. If U is the tangent vector field to $M = B \times_\psi C$ at (p, q) then

$$\|U\|^2 = \|d\pi_1 U\|^2 + \psi^2(p)\|d\pi_2 U\|^2. \tag{2.25}$$

Bishop and O'Neill (1965) proved the following

THEOREM 2.1 *Let* $M = B \times_\psi C$ *be warped product manifolds. If X, Y* $\in$ *TB and V, W* $\in$ *TC then*

(i) $\nabla_X Y \in TB$
(ii) $\nabla_X V = \nabla_V X = \left(\frac{X\psi}{\psi}\right)V,$
(iii) $\nabla_V W = \frac{-g(V, W)}{\psi}\nabla\psi.$

From above Theorem, for the warped product $M = B \times_f C$ it is easy to conclude that

$$\nabla_X V = \nabla_V X = (U ln\psi)V, \tag{2.26}$$

for any $X \in TB$ and $V \in TC$.
$grad\psi$ is the gradient of ψ and is defined as

$$g(grad\psi, U) = U\psi, \tag{2.27}$$

for all $U \in TM$.

COROLLARY *2.1   On a warped product manifold* $M = N_1 \times_\psi N_2$, *the following statements hold*

(i) N_1 *is totally geodesic in M*
(ii) N_2 *is totally umbilical in M.*

In what follows, $N_\perp$ and N_T will denote a totally real and holomorphic submanifold respectively of an almost Hermitian manifold $\bar{M}$.

A warped product manifold is said to be trivial if its warping function *f* is constant. More generally, a trivial warped product manifold $M = N_1 \times N_2$ is a Riemannian product $N_1 \times N_2^\psi$ where N_2^ψ is the manifold with the Riemannian metric $\psi^2 g_2$ which is homothetic to the original metric g_2 of N_2. For example, a trivial CR-warped product is CR-product.

Let *M* be a *m*–dimensional Riemannian manifold with Riemannian metric *g* and let $\{e_1, \ldots, e_m\}$ be an orthogonal basis of *TM*. For a smooth function ψ on *M* the Hessian of *f* are defined as

$$H^\psi(U, V) = UV\psi - (\nabla_U V)\psi = g(\nabla_U grad\psi, V), \tag{2.28}$$

for any $U, V \in TM$. The Laplacian of *f* is defined by

$$\Delta\psi = \sum_{i=1}^{m}\{(\nabla_{e_i} e_i)\psi - e_i e_i\psi\} = -\sum_{i=1}^{m} g(\nabla_{e_i} gradf, e_i). \tag{2.29}$$

It is eviedent from the above two equations that Laplacian is the negative of the Hessian. Moreover from the integration theory on manifolds, for a compact orientable Riemannian manifold *M* without boundary, we have

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$$\int\limits_M \Delta\psi\, dV = 0,$$
(2.30)

where dV is the volume element of M (O' Neill, 1983).

3. CR-warped product submanifolds

In this section we consider warped product of the type $M = N_T \times_\psi N_\perp$ in a nearly Kaehler manifold, where N_T is holomorphic submanifold and $N_\perp$ is totally real submanifold, these warped product submanifolds are called CR-warped product submanifolds. N_T and $N_\perp$ are the integral submanifolds of the distributions D and $D^\perp$.

Estimation of squared norm of second fundamental form for CR-warped product submnaifolds in the setting of almost Hermitian manifolds has been worked out by many authors (see Al-Luhaibi et al., 2009; Chen, 2003; Khan et al., 2009). Our aim in this paper is to obtain a characterizing inequality for squared norm of second fundamental form for CR-warped product submanifolds in the setting of generalized complex space form.

Now, we obtain some basic results in the following Lemma.

LEMMA 1   Let $M = N_T \times_\psi N_\perp$ be a CR-warped product submanifold of a nearly kaehler manifold $\bar{M}$, then

(i) $g(h(JU, W), JW) = U\ln\psi\|W\|^2,$

(ii) $g(h(U, W), JW) = -JU\ln\psi\|W\|^2,$

(iii) $g(Jh(U, W), \mathcal{P}_W U) = -\|\mathcal{P}_W U\|^2,$

(iv) $g(h(JU, W) - Jh(U, W), \mathcal{Q}_W U) = \|\mathcal{Q}_W U\|^2,$

for any $U \in TN_T$ and $W \in TN_\perp$.

Proof   Assume that M is a CR-warped product submanifold of a nearly Kaehler manifold. From (2.5) and (2.16), we can write

$$g(h(JU, W), JW) = g(\mathcal{Q}_W U, JW) + g(\nabla_W U, W),$$

using part (ii) of P_3 and (2.26), above equation gives

$$g(h(JU, W), JW) = -g(U, \mathcal{P}_W JW) + U\ln\psi\|W\|^2.$$

By use of (2.16) and (2.17) in above equation we get

$$g(h(JU, W), JW) = U\ln\psi\|W\|^2,$$

replacing U by JU, we can find part (ii).

From Gauss formula and (2.16), we have

$$g(Jh(U, W), \mathcal{P}_W U) = g(\bar{\nabla}_W JU - (\bar{\nabla}_W J)U, \mathcal{P}_W U).$$

Using (2.5), (2.26), (2.11) and (2.16), the above equation yields

$$g(Jh(U, W), \mathcal{P}_W U) = JU\ln\psi\, g(W, \mathcal{P}_W U) - \|\mathcal{P}_W U)\|^2.$$

By use of part (i) of P_3 and some easy calculations, the second term of above equation becomes zero and we get the required result.

Applying Gauss formula, (2.16) and (2.26), we obatin

$$g(h(JU, W), Q_W U) = \|Q_W U\|^2 + Uln\psi g(W, Q_W U)) + g(Jh(U, W), Q_W U),$$

by making use of part (ii) of P_3, (2.16) and (2.3), the second term on right hand side becomes zero and we get part (iv) of Lemma.

LEMMA 2   *For a CR-warped product submanifold $M = N_T \times_\psi N_\perp$ of a nearly Kaehler manifold $\bar{M}$, we have*

$$g(h(JU, W), Q_W U) = -(Uln\psi)^2 \|W\|^2 - g(h(JU, W), Jh(U, W)) + \|h(JU, W)\|^2,$$

for any $U \in TN_T$ and $W \in TN_\perp$.

Proof   In view of part (ii) of P_3 and (2.16), we have

$$g(h(JU, W), Q_W U) = -g(\mathcal{P}_W h(JU, W), U) = -g((\bar{\nabla}_W J)h(JU, W), U),$$

By use of (2.11) in above equation, we have

$$g(h(JU, W), Q_W U) = g(Jh(JU, W), \bar{\nabla}_W U) - g(\bar{\nabla}_W h(JU, W), JU).$$

On making use of (2.5), (2.26) and part (i) of Lemma 3.1, above equation reduced to

$$g(h(JU, W), Q_W U) = -(Uln\psi)^2 \|W\|^2 - g(h(JU, W), Jh(U, W)) + \|h(JU, W)\|^2,$$

which is the required result.

THEOREM 3.1   *Let $M = N_T \times_\psi N_\perp$ be a CR-warped product submanifold of a nearly Kaehler manifold $\bar{M}$, then*

$$\|h(U, W)\|^2 + \|h(JU, W)\|^2 = 2g(h(JU, W), Jh(U, W)) + (Uln\psi)^2 \|W\|^2$$
$$+ (JUln\psi)^2 \|W\|^2 + \|\mathcal{P}_W U\|^2 + \|Q_W U\|^2, \tag{3.1}$$

for any $U \in TN_T$ and $W \in TN_\perp$.

Proof   Let M be a CR-warped product submanifold of a nearly Kaehler manifold, then by Gauss formula we have

$$g(h(JU, W), Jh(U, W)) = g(Jh(U, W), \bar{\nabla}_W JU - \nabla_W JU),$$

By use of (2.11), (2.16), parts (ii) and (iii) of Lemma 3.1, above equation yiels

$$\|h(U, W)\|^2 = g(h(JU, W), Jh(U, W)) + \|\mathcal{P}_W U\|^2$$
$$- g(Jh(U, W), Q_W U) + (JUln\psi)^2 \|W\|^2. \tag{3.2}$$

Now, further calculating the third term on right hand side as follows

$$g(Jh(U, W), Q_W U) = g(J\bar{\nabla}_U W - J\nabla_W U, Q_W U),$$

In view of (2.26), the above equation becomes

$$g(Jh(U, W), Q_W U) = g(J\bar{\nabla}_W U, Q_W U) - Uln\psi g(JW, Q_W U),$$

After using part (ii) of P_3, (2.16) and (2.17), the second term on right hand side of above equation becomes zero. Further, on applying (2.5), (2.16) and (2.17), we get

$$g(Jh(U, W), Q_W U) = -\|Q_W U\|^2 + g(h(JU, W), Q_W U).$$

(3.3)

By use of Lemma 3.2 and (3.3) in (3.2), we get (3.1), this completes the proof.

Now we will prove the following theorem for CR-warped product submnaifolds of a generalized complex space form.

THEOREM 3.2 Let $M = N_T \times_f N_\perp$ be a CR-warped product submanifold of a generalized complex space form $\bar{M}(c, \alpha)$, then we have

$$\|h(U, W)\|^2 + \|h(JU, W)\|^2 = H^{ln\psi}(U, U) + H^{ln\psi}(JU, JU) + \frac{c - \alpha}{2}$$
$$+ 2\|\mathcal{P}_W U\|^2 + 2\|Q_W U\|^2 + (Uln\psi)^2 g(W, W) + (JUln\psi)^2 g(W, W),$$

(3.4)

for any $U \in TN_T$ and $W \in TN_\perp$.

Proof Suppose M be a CR-warped product submanifolds of a generalized complex space form then by (2.4), we have

$$\bar{R}(U, JU, W, JW) = -\frac{c - \alpha}{2} g(U, U) g(W, W).$$

(3.5)

On the other hand by Coddazi equation

$$\bar{R}(U, JU, W, JW) = Xg(h(JU, W), JW) - g(\bar{\nabla}_U JW, h(JU, W)) - g(h(\nabla_U JU, W), JW)$$
$$- g(h(\nabla_U W, JU), JW) - JUg(h(U, W), JW) + g(h(U, W), \bar{\nabla}_{JU} JW)$$
$$+ g(h(\nabla_{JU} U, W), JW) + g(h(\nabla_{JU} W, U), JW),$$

on using (2.5), (2.26), decomposition (2.16), parts (i) and (ii) of Lemma 3.1, above equation takes the form

$$\bar{R}(U, JU, W, JW) = U(Uln\psi)g(W, W) + 2(Uln\psi)^2 g(W, W) - g(h(JU, W), Q_U W)$$
$$- g(h(JU, W), Jh(U, W)) - Uln\psi g(h(JU, W), JW)$$
$$- g(h(\nabla_U JU, W), JW) - (Uln\psi)^2 g(W, W)$$
$$+ JU(JUln\psi)g(W, W) + 2JUln\psi g(\nabla_{JU} W, W)$$
$$+ g(h(U, W), Q_{JU} W) + g(h(U, W), Jh(JU, W))$$
$$+ JUln\psi g(h(U, W), JW) + g(h(\nabla_{JU} U, W), JW)$$
$$- (JUln\psi)^2 g(W, W).$$

(3.6)

From part (i) of Lemma 3.1 and (2.7)

$$g(A_{JW} W, JU) = Uln\psi g(W, W).$$

Since $\nabla_U U \in TN_T$, then we can replace U by $\nabla_U U$ as

$$g(A_{JW} W, J\nabla_U U) = \nabla_U Uln\psi g(W, W).$$

Applying Gauss formula, we find

$$g(A_{JW} W, J\bar{\nabla}_U U - Jh(U, U)) = \nabla_U Uln\psi g(W, W).$$

(3.7)

On using (2.11), (2.3) and (2.26) in the following equation

$$g(h(U, U), JW) = -g(J\bar{\nabla}_U U, W),$$

one can conclude that $h(U, U) \in \mu$, using this fact in (3.7), we get

$$g(A_{JW} W, J\bar{\nabla}_U U) = \nabla_U ln\psi g(W, W).$$

By use of (2.5) and (2.3), the above expression reduced to

$$g(A_{JW} W, \nabla_U JU) = \nabla_U ln\psi g(W, W),$$

or

$$g(h(\nabla_U JU, W), JW) = \nabla_U U ln\psi g(W, W). \tag{3.8}$$

Similarly, we can get

$$g(h(\nabla_{JU} U, W), JW) = \nabla_{JU} JU ln\psi g(W, W). \tag{3.9}$$

Moreover, using (2.22)(b) as follows

$$- g(h(JU, W), \mathcal{Q}_U W) + g(h(U, W), \mathcal{Q}_{JU} W) = g(h(JU, W), \mathcal{Q}_W U)$$
$$- g(h(U, W), \mathcal{Q}_W JU),$$

or

$$- g(h(JU, W), \mathcal{Q}_U W) + g(h(U, W), \mathcal{Q}_{JU} W) = g(h(JU, W), \mathcal{Q}_W U)$$
$$- g(h(U, W), (\bar{\nabla}_W J)JU),$$

using (2.17), it is easy to see that

$$- g(h(JU, W), \mathcal{Q}_U W) + g(h(U, W), \mathcal{Q}_{JU} W) = g(h(JU, W) - Jh(U, W), \mathcal{Q}_W U)$$
$$- g(Jh(U, W), \mathcal{P}_W U). \tag{3.10}$$

By making use of (3.8), (3.9), (3.10) in (3.6), we get

$$\bar{R}(U, JU, W, JW) = \{U(Uln\psi) + JU(JUln\psi) - (\nabla_U Uln\psi + \nabla_{JU} JUln\psi)\}\|W\|^2$$
$$- 2g(Jh(U, W), h(JU, W)) - g(Jh(U, W), \mathcal{P}_W U)$$
$$+ g(h(JU, W) - Jh(U, W), \mathcal{Q}_W U).$$

By parts (iii) and (iv) of Lemma 3.1, and (2.28), we have

$$\bar{R}(U, JU, W, JW) = H^{ln\psi}(U, U) + H^{ln\psi}(JU, JU) + \|\mathcal{P}_W U\|^2$$
$$+ \|\mathcal{Q}_W U\|^2 - 2g(Jh(U, W), h(JU, W)). \tag{3.11}$$

Applying (3.1) and (3.5) in (3.11), we get

$$-\frac{c-\alpha}{2} = H^{ln\psi}(U, U) + H^{ln\psi}(JU, JU) + 2\|\mathcal{P}_W U\|^2 + +2\|\mathcal{Q}_W U\|^2$$
$$- \|h(U, W)\|^2 - \|h(JU, W\|^2 + (Uln\psi)^2\|W\|^2 + (JUln\psi)^2\|W\|^2,$$

which is the required result.

Note 1. If for the CR-warped product submanifolds $M = N_T \times_f N_\perp$ the ambient manifold is complex space form $\bar{M}(c)$, then $\alpha = 0$ and $\mathcal{P} = \mathcal{Q} = 0$.

In view of Note 3.1 we have the following Corollary.

COROLLARY 3.1   Let $M = N_T \times_\psi N_\perp$ be a CR-warped product submanifold of a complex space form $\bar{M}(c)$, then we have

$$\|h(U, W)\|^2 + \|h(JU, W)\|^2 = H^{ln\psi}(U, U) + H^{ln\psi}(JU, JU) + \frac{c}{2}$$
$$+ \{(Uln\psi)^2 + (JUln\psi)^2\}g(W, W).$$

Let $\{e_1, e_2, \ldots, e_p, Je_1, Je_2, \ldots, Je_p, e^1, e^2, \ldots, e^q\}$ be an orthonormal frame of TM such that $\{e_1, e_2, \ldots, e_p, Je_1, Je_2, \ldots, Je_p\}$ are tangential to TN_T and $\{e^1, e^2, \ldots, e^q\}$ are tangential to $TN_\perp$. Moreover assume that $\{Je^1, Je^2, \ldots, Je^q\}$ and $\{N_1, N_2, \ldots, N_{2r}\}$ are tangential to $JN_\perp$ and μ respectively.

Finally, we will prove the main theorem.

THEOREM 3.3 *Let $M = N_T \times_\psi N_\perp$ be a compact orientable CR-warped product submanifold of a generalized complex space form $\tilde{M}(c, \alpha)$. Then M is CR-product if the following inequality holds*

$$\|h_\mu(D, D^\perp)\|^2 \geq \frac{c - \alpha}{2}p.q + 2\|\mathcal{P}_{D^\perp}D\|^2 + 2\|\mathcal{Q}_{D^\perp}D\|^2,$$

where $2p$ and q are the real dimensions of N_T and $N_\perp$.

Proof By the definition of Laplacian of $ln\psi$

$$-\Delta ln\psi = \sum_{i=1}^{p} g(\nabla_{e_i} grad ln\psi, e_i) + \sum_{i=1}^{p} g(\nabla_{Je_i} grad ln\psi, Je_i)$$
$$+ \sum_{j=1}^{q} g(\nabla_{e^j} grad ln\psi, e^j).$$

Here we consider ∇ be a Levi-Civita connection, N_T is totally geodesic in M. Hence $grad ln\psi \in TN_T$, then by Equations (2.26) and (2.27) the previous equation reduced to

$$-\Delta ln\psi = \sum_{i=1}^{p} g(\nabla_{e_i} grad ln\psi, e_i) + \sum_{i=1}^{p} g(\nabla_{Je_i} grad ln\psi, Je_i)$$
$$+ \sum_{j=1}^{q} g(grad ln\psi, grad ln\psi)g(e^j, e^j),$$

or

$$-\Delta ln\psi = \sum_{i=1}^{p} \{H^{ln\psi}(e_i, e_i) + H^{ln\psi}(Je_i, Je_i)\} + q\|grad ln\psi\|^2.$$

$$(3.12)$$

In (3.4) using local frame of vector fields on N_T and $N_\perp$ and summing both sides over $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$, we have

$$\sum_{i=1}^{p}\sum_{j=1}^{q}(\|h(e_i, e^j)\|^2 + \|h(Je_i, e^j)\|^2) = \sum_{i=1}^{p}\{H^{ln\psi}(e_i, e_i) + H^{ln\psi}(Je_i, Je_i)\}q$$
$$+ \frac{c - \alpha}{2}p.q + q\sum_{i=1}^{p}\{(e_i ln\psi)^2 + (Je_i ln\psi)^2\}\|W\|^2$$
$$+ 2\sum_{i=1}^{p}\sum_{j=1}^{q}(\|\mathcal{P}_{e^j}e_i\|^2 + \|\mathcal{Q}_{e^j}e_i\|^2).$$

$$(3.13)$$

Furthermore, we can write the second fundamental form h as follows

$$h(e_i, e^j) = \sum_{k=1}^{q} g(h(e_i, e^j), Je^k)Je^k + \sum_{t=1}^{2r} g(h(e_i, e^j), N_t)N_t,$$

for each $1 \leq i \leq p$ and $1 \leq j \leq q$. Taking the inner product of the above equation with $h(e_i, e^j)$ we get

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \|h(e_i, e^j)\|^2 = \sum_{i=1}^{p} \sum_{j,k=1}^{q} g(h(e_i, e^j), Je^k)^2 + \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{t=1}^{2r} g(h(e_i, e^j), N_t)^2,$$

then making use of part (ii) of Lemma 3.1, the last equation takes the form

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \|h(e_i, e^j)\|^2 = q \sum_{i=1}^{p} (Je_i ln\psi)^2 + \sum_{i=1}^{p} \sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2. \tag{3.14}$$

Similarly, we can get

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \|h(Je_i, e^j)\|^2 = q \sum_{i=1}^{p} (e_i ln\psi)^2 + \sum_{i=1}^{p} \sum_{j=1}^{q} \|h_\mu(Je_i, e^j)\|^2. \tag{3.15}$$

By use of (3.12), (3.14) and (3.15) in (3.13), we have

$$\sum_{i=1}^{p} \sum_{j=1}^{q} (\|h_\mu(e_i, e^j)\|^2 + \|h_\mu(Je_i, e^j)\|^2) = (-\Delta ln\psi - q\|gradln\psi\|^2)q + \frac{c-\alpha}{2}p.q$$

$$+ 2 \sum_{i=1}^{p} \sum_{j=1}^{q} (\|\mathcal{P}_{e^j}e_i\|^2 + \|\mathcal{Q}_{e^j}e_i\|^2). \tag{3.16}$$

Now, we use the following notations

$$\|h_\mu(D, D^\perp)\|^2 = \sum_{i=1}^{p} \sum_{j=1}^{q} (\|h_\mu(e_i, e^j)\|^2 + \|h_\mu(Je_i, e^j)\|^2),$$

$$\|\mathcal{P}_{D^\perp}D\|^2 + \|\mathcal{Q}_{D^\perp}D\|^2 = \sum_{i=1}^{p} \sum_{j=1}^{q} (\|\mathcal{P}_{e^j}e_i\|^2 + \|\mathcal{Q}_{e^j}e_i\|^2).$$

In view of above notations, (3.16) can be written as

$$-q\Delta ln\psi = \|h_\mu(D, D^\perp)\|^2 + q^2\|gradln\psi\|^2 - \frac{c-\alpha}{2}p.q$$

$$- 2\|\mathcal{P}_{D^\perp}D\|^2 - 2\|\mathcal{Q}_{D^\perp}D\|^2. \tag{3.17}$$

From (2.30), we can conclude that

$$\int_{M} \{ \|h_\mu(D, D^\perp)\|^2 + q^2\|gradln\psi\|^2 - \frac{c-\alpha}{2}p.q$$

$$- 2\|\mathcal{P}_{D^\perp}D\|^2 - 2\|\mathcal{Q}_{D^\perp}D\|^2) \} dV = 0. \tag{3.18}$$

Here, if

$$\|h_\mu(D, D^\perp)\|^2 \geq \frac{c-\alpha}{2}p.q + 2\|\mathcal{P}_{D^\perp}D\|^2 + 2\|\mathcal{Q}_{D^\perp}D\|^2,$$

(3.18) and above inequality implies that $\|gradln\psi\| = 0$ i.e. ψ is constant, since $q \neq 0$, which proves the Theorem completely.

Now, we have some consequences of above findings as follows.

COROLLARY 3.2 *Let $M = N_T \times_f N_\perp$ be a compact orientable CR-warped product submanifold of a complex space form $\bar{M}(c)$. Then M is CR-product if the following inequality holds*

$$\|h_\mu(D, D^\perp)\|^2 \geq \frac{c \cdot p \cdot q}{2},$$

where $2p$ and q are the real dimensions of N_T and $N_\perp$.

COROLLARY 3.3 *Let $M = N_T \times_f N_\perp$ be a compact orientable CR-warped product submanifold of a generalized complex space form $\bar{M}(c, \alpha)$. Then M is CR-product if and only if*

$$\|h_\mu(D, D^\perp)\|^2 = \frac{c - \alpha}{2}p.q + 2\|\mathcal{P}_{D^\perp}D\|^2 + 2\|\mathcal{Q}_{D^\perp}D\|^2. \tag{3.19}$$

Proof Assume that M is compact CR-warped product submanifold of a generalized complex space form $\bar{M}(c, \alpha)$ satisfying (3.19), then from (3.18) ψ is constant i.e. M is CR-product.

Conversely, if M is CR-product, i.e. f is constant, then from part (i) of Lemma 3.1 one can conclude $h(U, W) \in \mu$ for all $U \in TN_T$ and $W \in TN_\perp$. Hence, it is easy to see that (3.19) holds.

COROLLARY 3.4 *Let $M = N_T \times_f N_\perp$ be a compact orientable CR-warped product submanifold of a complex space form $\bar{M}(c)$. Then M is CR-product if and only if*

$$\|h_\mu(D, D^\perp)\|^2 = \frac{c.p.q}{2}.$$

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