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# On Some Dynamics in Conceptual Spaces

**Piotr Lisowski**[1] · **Roman Urban**[2]

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**Abstract**
In this text, we consider Gärdenfors' conceptual spaces that are separable Hilbert spaces. In particular, the results we obtained apply to finite-dimensional Euclidean spaces. Our main contribution can be formulated as a combination of the theory of opinion dynamics with the theory of conceptual spaces. This combination, in turn, leads us to propose a new model for the time evolution of conceptual spaces. To achieve this goal, we propose some extension of the multidimensional opinion dynamics model of Parsegov, Proskurnikov, Tempo and Friedkin to opinions with values in Hilbert spaces.

**Keywords** Conceptual space · Prototypes · Opinion dynamics · Friedkin–Johnsen model · Consensus

**Mathematics Subject Classification** 91D30 · 93D50 · 05C82

## 1 Introduction

The main result of this work is to propose a model to describe the dynamics of conceptual spaces. It seems to us that the approach we proposed in the construction of our model, namely the use of opinion dynamics theory, is very interesting in that it leaves a lot of freedom for the construction of other models in the future.

Our second main result, interesting in itself and somewhat independent of the first one, is a generalization of the multidimensional model of opinion dynamics given in the article by Parsegov et al. (2017), which is itself a multidimensional extension of the classical Friedkin–Johnsen model (Friedkin, 1998; Friedkin & Johansen, 1999, 2011).

✉ Roman Urban
roman.urban@math.uni.wroc.pl

Piotr Lisowski
piotr.lisowski@cs.uni.wroc.pl

[1] Institute of Computer Science, Wroclaw University, ul. Joliot-Curie 15, 50-383 Wroclaw, Poland

[2] Institute of Mathematics, Wroclaw University, Plac Grunwaldzki 2/4, 50-384 Wroclaw, Poland

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Our opinion dynamics model admits not only multidimensional opinions (i.e., vectors in $\mathbb{R}^n$) but also models with opinions in a separable Hilbert space (e.g., $\ell^2(\mathbb{N})$ - see Sect. 2). This generalization allows us to model complex cognitive processes in our paper.

Although we formulate our results and theorems in terms of infinite-dimensional separable Hilbert spaces, we encourage the reader who is not interested in such far-reaching generalizations to assume that Hilbert space $\mathcal{H}$ is simply $\mathbb{R}^n$, for some $n \in \mathbb{N}$.

Both the theory of opinion dynamics and the theory of conceptual spaces have developed extremely rapidly in recent years and have attracted the interest of many researchers: mathematicians, philosophers (in formulation of the notion of conceptual spaces), and engineers on the one hand and scientists in the social sciences on the other. This interest in conceptual spaces is due to the fact that both theories have extremely wide applications to other sciences. To name a few examples, which are far from fully describing the capabilities of both theories, information theory, social sciences such as sociology, political science, economics, psychology, cognitive science, theory of cognition, and learning theory (see a recent survey on opinion dynamics by Noorazar (2020) and also (Noorazar et al., 2020), a tutorial consisting of two parts by Proskurnikov and Tempo (2017, 2018) a survey by Mossel and Tamuz (Mossel & Tamuz, 2017), see also (Stamoulas & Rathinam, 2018; Aydoğdu et al., 2017); for conceptual spaces see works of Gärdenfors (2000, 2017); Kaipainen et al. (2019); Zenker and Gärdenfors (2015) and literature cited therein).

At this point, we must emphasize that, as far as we know, this is the first time that a combination of conceptual space theory and opinion dynamics has been proposed.

### 1.1 Conceptual Spaces

In his pioneering book (Gärdenfors, 2000) Gärdenfors introduces theory for knowledge representation in geometric terms. It models how humans think, reason and comprehend a knowledge. Basically it is widely assumed that some metric space $\mathcal{S}$ (usually finite-dimensional Euclidean space) represents all that can be comprehended by a human being. Every dimension (or finite set of dimensions) of $\mathcal{S}$ should be understood as some measurable (or observable), numeric (or symbolic) information (or property) and is called a quality. Pairwise, disjoint regions in $\mathcal{S}$ are called concepts (or conceptual fields). This notions originate from the prototype theory of Kamp and Partee (1995). Following (Kamp & Partee, 1995) and linguists' language the most representative subset of a concept is called its prototype. The idea of prototypes comes from Rosch's 1973 work (Rosch, 1973). Typically, in most of the literature the prototype is considered as a point but this is not a necessary assumption, and there are many applications showing that single-point prototypes are too limiting.

The problem of classifying objects into classes is one of the most natural cognitive processes. It is performed by human beings upon the objects of their surrounding reality constantly. The next process is the naming of the encountered objects. Hence the importance of classification. Specifically, having a finite set of prototypes in a conceptual space $\mathcal{S}$ it is important to partition the space $\mathcal{S}$ into domains determined by these prototypes. Usually such a partition is obtained using Voronoi diagrams (Okabe et al., 2011).

Even though conceptual space theory originates in cognitive science it has found many applications even in some seemingly exotic fields as music theory (Gärdenfors, 1988) and study of the concept of metaphor (Gärdenfors, 1996). So it seems strange that no one so far (to our knowledge) considered it in a dynamical context.

The main objective of our work is to explore conceptual spaces from a dynamic point of view. It is very natural to consider dynamics in conceptual spaces. Concepts change over time. As an example, we can mention linguistics, where conceptual spaces are used, among other things, to study the semantics of lexemes. Lexemes change their meaning in time, which is a commonly noticed phenomenon.

To some extent, the dynamics of conceptual spaces have already been studied by their creator, Gärdenfors, along with colleagues in papers (Masterton et al., 2017; Gärdenfors & Zenker, 2013). They modeled physical theories (classical mechanics, quantum mechanics, general theory of relativity) by means of conceptual spaces. They justified the possibility of equating phase spaces with conceptual spaces. This made it possible to determine the similarity between the theories. Our approach to dynamics is quite different and is used to model a different phenomenon.

To achieve our goal, that is, to give a satisfactory model of the dynamics of conceptual spaces we use a recent model built by Parsegov et al. (2017) concerning the opinion dynamics in a group consisting of a finite number of individuals.

### 1.2 Opinion Dynamics

Let's assume that we have a group of individuals and that these individuals exchange arguments on a particular topic of conversation, and then they may modify their opinion to take into account what they have learned. At the end of the discussion, the group will be characterized by what is known as a consensus of opinion or coexistence of opinions (fragmentation). There are many approaches to describing the problem of opinion formation. The resulting models, in turn, are characterized by varying degrees of complexity (see, for example, Liggett (1997), Bernardo et al. (2024), Anderson et al. (2020), Castellano et al. (2009)).

### 1.3 Our Opinion Dynamics Model

Here we generalize the multivariate model of opinion dynamics given in the paper by Parsegov et al. (2017) which is a multidimensional extension of the classical Friedkin–Johnsen model (FJ-model) (Friedkin, 1998; Friedkin & Johansen, 1999, 2011).

Specifically, in Parsegov et al. (2017) the following model is considered. There is $n$ agents. Each agent at time $k$ has its opinion $x_j(k) \in \mathbb{R}^m$, $j = 1, \dots, n$ on $m$ different topics. Let $x(k) = \big(x_1(k), \dots, x_n(k)\big)^T$ be the state at time $k$. We assume that there is some interdependence between $m$ topics which is described by the correlation matrix $C \in M_{m\times m}(\mathbb{R})$. In each step, agents exchange their views on $m$ topics. The exchange is synchronous but not necessarily symmetric. The description of the structure of agents' influence on each other is stored in the matrix $W \in M_{n\times n}(\mathbb{R})$. The element

$w_{i,j}$ specifies what influence the $j$-th agent has on the $i$-th agent. The last component of the model is the $\Lambda$ matrix, which reflects, let us call it, the stubbornness of the agents. In each step an agent takes into account not only the opinions of the agents it communicates with but also its initial opinion at time $k = 0$.

Finally, dynamics of the system is given by the following equation:

$$x(k+1) = \big((\Lambda W) \otimes C\big)x(k) + \big((I_n - \Lambda) \otimes I_m\big)x(0),$$

where $\otimes$ denotes the Kronecker product of matrices (Gantmacher, 1959; Horn & Johnson, 2013).

In our paper instead of $\mathbb{R}^n$ we admit an arbitrary separable Hilbert space $\mathcal{H}$. This requires a non-trivial Schechter's result on the spectrum of the tensor product of linear operators acting on Hilbert spaces (Schechter, 1969).

### 1.4 Our Conceptual Space Dynamic Model

The second goal is to apply the obtained opinion dynamics model in the theory of conceptual spaces.

In our work we want to look at conceptual spaces from a dynamic point of view.

The model we consider in our work consists of $n$ agents (or actors), except for Corollary 4.18, where we consider an infinite network that is, a countably infinite number of agents.[1] Each agent has some idea of $m$ given concepts (or notions) that are encoded as elements of the conceptual space $\mathcal{H}$. We can think that there are $m$ elements of the conceptual space $p_i^1(k), \ldots, p_i^m(k) \in \mathcal{H}$ associated with the $i$-th agent, here $k \geq 0$ is a discrete time. These elements are called prototypes of concepts. It is convenient to think that in fact each actor has its own conceptual space $\mathcal{H}_i = \mathcal{H}$ (although sometimes it is useful not to distinguish them). In our work, the conceptual space is a separable Hilbert space $\mathcal{H}$.

Now we introduce discrete-time dynamics into the conceptual spaces $\mathcal{H}_i$.

The dynamics we propose is inspired by the multidimensional FJ-model taken from opinion dynamics theory (Parsegov et al., 2017). In order to transfer opinion dynamics model to our conceptual space we need a matrix $W \in \mathbb{R}^{n\times n}$ of social influences between agents, a diagonal matrix of prejudices $\Lambda \in \mathbb{R}^{n\times n}$. Recall that qualities (dimensions) of conceptual spaces are (usually) dependent. This dependance is described by the correlation operator which is a bounded linear operator $\mathcal{C} : \mathcal{H} \to \mathcal{H}$.

By observing the dynamics as time goes to infinity ($k \to \infty$), we can observe the process of agents learning from each other and changing their $p_i^j(k)$ which implies a change of Voronoi diagrams in each conceptual space $\mathcal{H}_i$.

In our work, we investigate what conditions allow us to say whether, given $W$, $\Lambda$ and $\mathcal{C}$, we are dealing with convergence of prototypes in conceptual spaces or, using geometric language, whether Voronoi diagrams are in some sense convergent.

[1] Further research in this area can be of potential importance when studying finite complex networks but composed of a very large number of agents.

### 1.5 Structure of the Paper

This paper consists of the Introduction and six sections.

In Sect. 3, we define the notion of conceptual space. The next section, Sect. 4, is devoted to recalling facts from opinion dynamics theory. In particular, the De Groot and Friedkin–Johnsen models are discussed. A generalization of these models to multivariate models, in the sense that a finite number of topics that are correlated with each other are discussed simultaneously, is presented. Next, in Sect. 4.3, our main results are considering opinion dynamics are presented. Specifically, a generalization of the multivariate model by Parsegov et al. (2017) to the case of infinitely many topics discussed simultaneously. That is, the dynamics of the system takes place in a separable Hilbert space. In the next Sect. 5, we show the relation of the infinite-dimensional model built in the previous section to the dynamics of conceptual spaces. There we present two examples of dynamics in conceptual spaces which are separable Hilbert spaces. One of the conceptual spaces considered is the space of $L^2$ functions on the 2-dimensional torus. Sections 4.3 and 5 show what we think is an important phenomenon of the analogy of certain models of opinion dynamics with dynamics in conceptual spaces.

Finally, in Sect. 5.1 we address the question of a measure of the distance of the model's stationary state from the consensus and in Sect. 6 we summarize our work and outline directions for possible research.

## 2 Preliminaries

Here we present some facts about linear algebra, matrix theory, and functional analysis in Hilbert spaces, for details see (Laub, 2005; Horn & Johnson, 2013; Young, 1988). Let $M_{m\times n}(\mathbb{F})$ denotes the set of all $m \times n$-matrices with entries from the field $\mathbb{F}$ (here $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$).

**Definition 2.1** The *Kronecker product* of $A = [a_{ij}] \in M_{m\times n}(\mathbb{R})$ and $B = [b_{ij}] \in M_{p\times q}(\mathbb{R})$ is denoted by $A \otimes B$ and is defined to be the block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn}B \end{pmatrix} \in M_{mp\times nq}(\mathbb{R}).$$

In general, $A \otimes B \neq B \otimes A$.

**Definition 2.2** An *inner product* (or *scalar product*) on a complex vector space $V$ is a mapping $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ such that, for all $x, y, z \in V$ and all $\lambda \in \mathbb{C}$,

(i) $\langle x, y\rangle = \overline{\langle y, x\rangle}$,
(ii) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$,
(iii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,
(iv) $\langle x, x\rangle > 0$ when $x \neq 0$.

**Definition 2.3** An *inner product space* is a pair $(V, \langle\cdot,\cdot\rangle)$, where $V$ is a complex vector space and $\langle\cdot,\cdot\rangle$ is an inner product on $V$.

A *Hilbert space* is an inner product space which is a complete metric space with respect to the metric induced by its inner product.

**Theorem 2.4** *Let $\mathcal{H}$ be a separable Hilbert space. Then $\mathcal{H}$ is isomorphic either to $\mathbb{C}^n$ for some $n \in \mathbb{N}$ or to $\ell^2$.*

**Definition 2.5** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_i$, $i = 1, 2$, respectively. A *tensor product of* $\mathcal{H}_1$*and* $\mathcal{H}_2$ is a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ and a mapping $(x, y) \mapsto x \otimes y$ of $\mathcal{H}_1 \times \mathcal{H}_2$ into $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that

$$\begin{aligned}(x_1 + x_2) \otimes y =& x_1 \otimes y + x_2 \otimes y\\ (\lambda x)\otimes =& \lambda(x \otimes y)\\ x \otimes (y_1 + y_2) =& x \otimes y_1 + x \otimes y_2\\ x \otimes (\lambda y) =& \lambda(x \otimes y)\end{aligned}$$

and

(1) the vectors $x \otimes y$ form a *total* subset of $\mathcal{H}_1 \otimes \mathcal{H}_2$, i.e. its closed linear span is equal to $\mathcal{H}_1 \otimes \mathcal{H}_2$,
(2) $\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle = \langle x_1, y_1\rangle_1 \langle x_2, y_2\rangle_2$.

A bounded linear operator between two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ is a continuous linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$. We will denote by $L(\mathcal{H}_1, \mathcal{H}_2)$ the space of all bounded linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. The space $L(\mathcal{H}_1, \mathcal{H}_2)$ is a Banach space for the operator norm

$$\|T\| = \sup\{\|Tx\|_2 : x \in \mathcal{H}_1,\ \|x\|_1 \le 1\},$$

where $\|x\|_i = \langle x, x\rangle_i^{1/2}$. When $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ we simply write $L(\mathcal{H})$ for $L(\mathcal{H}, \mathcal{H})$. If we take the composition of operators as multiplication, then $L(\mathcal{H})$ is a (usually non commutative) Banach algebra with unit (the identity operator). In particular, $\|ST\| \le \|S\|\|T\|$, for $S, T \in L(\mathcal{H})$.

**Definition 2.6** Let, for $i = 1, 2$, $T_i$ be a linear bounded operator on a Hilbert space $\mathcal{H}_i$. There exists a unique linear bounded operator $T$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ such that

$$T(x_1 \otimes x_2) = T_1x_1 \otimes T_2x_2$$

for all $x_1$ in $\mathcal{H}_1$ and $x_2$ in $\mathcal{H}_2$. This operator is called a *tensor product of operators* $T_1$*and* $T_2$ and is denoted by $T_1 \otimes T_2$.

**Definition 2.7** Let $T$ be in $L(\mathcal{H})$, the *spectrum* of $T$ is the set $\sigma(T)$ of complex numbers $\lambda$ such that $T - \lambda I$ is not invertible.

## 3 Conceptual Spaces

**Definition 3.1** A *conceptual space* $\mathcal{S}$ is a subspace of some metric space $(X, d)$, together with family of prototypes $\mathcal{P} = \{P_1, \ldots, P_k\}$, where every $P_i \subset \mathcal{S} \subset X$.

The space $\mathcal{S}$ is divided (using appropriate methods) into $k$ cells called *conceptual domains* or *concepts*. The prototype $P_i$ which belongs to $i$-th concept $C_i$ is thought of as "the most representative example" of $C_i$. Usually $P_i$ are singleton sets and such partitions are usually obtained using Voronoi diagrams (see (Okabe et al., 2011)).

**Definition 3.2** Let $(X, d)$ be a metric space. Let, for $i = 1, \ldots, n$ $P_i \subset X$. The Voronoi cell $C_i$ associated with $P_i$ is the set of all points in $X$ whose distance to $P_i$ is smaller than their distance to the other prototype $P_j$, where $j$ is any index different from $i$, i.e.

$$C_i = \{x \in X : \ d(x, P_i) < d(x, P_j) \text{ for all } j \neq i\},$$

where $d(x, A) = \inf\{d(x, a) \mid a \in A\}$ is the distance between the point $x$ and the subset $A \subset X$. The *Voronoi diagram* is the tuple of cells $(C_i)_{1 \leq i \leq n}$.

Usually conceptual space $X$ is of higher dimension, e.g. $\mathbb{R}^D$, $D \in \mathbb{N}$, in which each *dimension* corresponds to quality of a concept. Typically, the dimensions of conceptual space are not independent of each other. As mentioned in the introduction, ties are imposed on the dimensions describing the concept. This phenomenon is very nicely illustrated by an example from the work (Rickard, 2006). There is considered an example of the concept of *apple* described by dimensions corresponding to the following qualities: *red, green, yellow, brown, smooth, wrinkled.* It is clear that these dimensions are correlated with each other. It is more common to find a *wrinkled* apple among *brown* apples than among *green* ones. When describing a concept with certain dimensions, we can therefore speak of a covariance matrix between the dimensions. The covariance matrix $C$ can be constructed from observations using statistical methods (see (Rickard, 2006)).

Getting a little ahead of our narrative, let us mention here that in the theory of opinion dynamics, when considering a model in which agents discuss m dependent topics an analogous covariance matrix will naturally appear.

The space $X$ is usually endowed with a *(dis)similarity measure* $s = s(x, y)$, $x, y \in X$ which is a function of the metric $d$.

As a simple example let, for $x \neq y$,

$$s(x, y) = \frac{1}{d(x, y)} \text{ or } s(x, y) = e^{-d(x,y)}.$$

Then, a small value of $s(x, y)$ indicates a small degree of similarity between $x$ and $y$, whereas conversely a large value of $s(x, y)$ signifies strong similarity between objects (concepts) $x$ and $y$.

Let us now consider an example in which objects can be described by means of $D = 3$ real parameters. That is our conceptual space is now $\mathbb{R}^3$ and each point in $\mathbb{R}^3$

corresponds to a color concept. Each color can be defined by three parameters - namely, through the wavelength, saturation and hue. It is clear that, for example, the concept of 'red' is not only its prototype (one single point $x$ in 3-dimensional space) but also colors in the close vicinity of the prototype (i.e. some set in 3-dimensional space called a *conceptual domain* or *conceptual field*, which contains a point-prototype $x$). In this example, conceptual space will be the union of all concept domains corresponding to different colors.

From a linguistic point of view, concepts usually correspond to the grammatical category of a noun or verb if time is one of the dimensions of conceptual space. The qualities correspond to the adjective descriptions.

An important issue is the division - partition - of conceptual space. Let us consider the simplest example. Let Euclidean space $X = \mathbb{R}^D$ be a conceptual space. We have $n$ prototypes $p_1, \dots, p_n \in \mathbb{R}^D$. We want to know which concept fields (which subsets of $\mathbb{R}^D$) these prototypes designate. That is to say: if we are given an element $x \in \mathbb{R}^D$, to which concept field does it belong? This is a particular example of data classification (see (Gordon, 1999; Suthaharan, 2016; Aggarwal, 2015; Dougherty, 2013)). Applying Definition 3.2 with $P_i = \{p\}$ the conceptual field $C_i$ corresponding to the prototype $p_i$ is simply a set of all points of $\mathbb{R}^D$ whose $\ell^2$-distance to $p_i$ is less than to any other $p_j$ with $j \neq i$, i.e.

$$C_i = \{x \in \mathbb{R}^D : \ d(x, p_i) < d(x, p_j) \text{ for all } j \neq i\},$$

where $d(x, y) = \|x - y\|_{\ell^2} = \left(\sum_{i=1}^{D}(x_i - y_i)^2\right)^{\frac{1}{2}}$.

In this way we get a partition of the space $\mathbb{R}^D$ for polygonal regions, some of which may be unbounded (Okabe et al., 2011).

# 4 Opinion Dynamics

## 4.1 The One-Dimensional Friedkin–Johnsen Model

Let us first consider the one dimensional opinions held by $n$ agents or actors in the network. We denote these opinions as a vector $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$. By $x(k)$ we mean the distribution of opinions among agents at stage (or time) $k \in \mathbb{N}$.

**Definition 4.1** A *model* $M$ is a transformation of distribution of the opinions of $n$ agents at the stage $k$ to the distribution in the next stage,

$$x(k) \overset{M}{\mapsto} x(k+1).$$

**Definition 4.2** A fixed-point (or invariant distribution) of the model $M$, i.e. a distribution $x^\sharp$ such that $M(x^\sharp) = x^\sharp$ is called a *limit state* of the model $M$.

Consider the easiest case where opinions are just real numbers and agents can have an influence on each other. Let the non-negative coefficients or in other words weights

$0 \leqslant w_{ij} \leqslant 1$, $1 \leq i, j \leq n$ denote what influence agent $j$ has on agent $i$ or stating it symmetrically: how susceptible agent $i$ is to the arguments of agent $j$. We assume that $\sum_{1 \leqslant i \leqslant n} w_{ij} = 1$. Then the influence matrix $W = (w_{ij})_{i,j=1}^{n} \in M_{n \times n}(\mathbb{R})$ is row-stochastic.

The simplest model is the famous one due to Degroot (1974):

$$x(k+1) = Wx(k). \tag{4.1}$$

**Definition 4.3** According to Meyer and Plemmons (1977) we say that a matrix $A$ is called *semi-convergent* if there exists a matrix $A^{\sharp}$ such that

$$A^{\sharp} = \lim_{n \to \infty} A^n.$$

Clearly, a system (4.1) can only reach a limit state provided that the stochastic matrix of influences $W$ is semi-convergent.

At this point it is required to tell how such a state can be reached. We can define two notion of convergence for a model.

**Definition 4.4** We say that the model $M$ is *convergent* if

$$\forall x(0)\ \exists x^{\sharp} \text{ such that } \lim_{k \to \infty} x(k) = x^{\sharp} \implies x^{\sharp} \overset{M}{\mapsto} x^{\sharp}.$$

In other words for a given initial opinion distribution the model iteration over it reaches a limit.

It is obvious that convergence of the DeGroot model is equivalent to the semi-convergence of its matrix of influences $W$, as was mentioned earlier.

First meaningful extension of above is the Friedkin–Johnsen model (the FJ-model, for short), see (Friedkin, 1998; Friedkin & Johansen, 1999, 2011). Now we also need actors' vulnerability to social influence given as diagonal matrix $\Lambda = (\lambda_{ij})$ where $0 \leqslant \lambda_{ij} \leqslant 1$. Then the updating equation has the following form:

$$x(k+1) = \Lambda W x(k) + (I - \Lambda)x(0). \tag{4.2}$$

A matrix $\Lambda$ ought to be interpreted as *stubbornness* of agents. So at every step of iteration an agent not only takes for consideration neighboring agents' opinions but also his own initial opinion. It is clear that the DeGroot model is a special case of the FJ-model when $\Lambda = I$. Sufficient conditions for convergence obtained in Frasca et al. (2013); Friedkin and Johansen (1999) are going to be presented in the next section.

**Definition 4.5** With the matrix $W$ we associate the finite graph $\mathcal{G}(W) = (V, \mathcal{E}(W))$. The set of nodes $V = \{1, \ldots, n\}$ of this graph is in one-to-one correspondence with the agents and the edges stand for the inter-personal influences, that is $(i, j) \in \mathcal{E}(W)$ if and only if $w_{ij} > 0$. A positive self-influence weight $w_{ii} > 0$ corresponds to the self-loop $(i, i)$. We call $\mathcal{G} = \mathcal{G}(W)$ the *interaction graph* of the social network. A sequence of nodes $i = i_0 \mapsto i_1 \mapsto \ldots \mapsto i_r = i'$ such that $(i_j, i_{j+1}) \in \mathcal{E}(W)$, $j = 0, \ldots, r-1$, is called a *walk* from $i$ to $i'$.

### 4.2 The Multidimensional Friedkin–Johnsen Model

Parsegov et al. (2017) gave a multidimensional extension of the FJ-model dealing with interdependent topics of discussion.

Let's assume that the opinion concerns $m$ topics. That is, a given opinion $x$ is represented by an $m$-dimensional vector $x \in \mathbb{R}^m$. We will now be interested in the dynamics of the opinions of $n$ agents at discrete time $k$. At each time $k \in \mathbb{N}$ there is an interaction between agents. As a result of this interaction, the agents can change their opinions. More formally, at time $k$ we have $n$ vectors $x_1(k), \ldots, x_n(k) \in \mathbb{R}^m$, where the $j$-th coordinate ($1 \leq j \leq m$) of the $i$-th vector $x_i^j(k)$ denotes the opinion of the $i$-th agent regarding the $j$-th subject at time $k$.

When dealing with interdependent discussion topics, we need to equip the model with information about the level of entanglement or the dependencies between topics. This information is given by the matrix $C \in M_{m\times m}(\mathbb{R})$. The following equations generalize the model given by (4.2) to the situation of interdependence of opinions

$$x_i(k+1) = \lambda_{ii} C \sum_{j=1}^{n} w_{ij} x_j(k) + (1 - \lambda_{ii}) x_i(0), \qquad i = 1, \ldots, n. \tag{4.3}$$

To write this system of equations in a compact form, we combine all $x_i(k)$, $i = 1, \ldots, n$, into a single column vector $x(k) \in \mathbb{R}^{nm}$, i.e.

$$x(k) = \left(x_1^1(k), \ldots, x_1^m(k), x_2^1(k), \ldots, x_2^m(k), \ldots, x_n^1(k), \ldots, x_n^m(k)\right)^T.$$

Introducing the Kronecker multiplication operator $\otimes$ (see Appendix) we get

$$x(k+1) = \big((\Lambda W) \otimes C\big) x(k) + \big((I_n - \Lambda) \otimes I_m\big)\, x(0). \tag{4.4}$$

***Remark 4.6*** It is worth mentioning that the FJ-model and its multidimensional extension were originally designed to operate on opinions. Vector-valued opinion is supposed to be multiple opinions on certain potentially interdependent topics. The conceptual space perspective is, in a sense opposite. We have one object of interest that we want to describe with its (also potentially dependent) attributes.

***Remark 4.7*** In the theory of opinion dynamics proposed in the work of Parsegov et al. (2017), the dimensions of the opinion vectors of individual agents are dependent on each other as they discuss dependent topics. This is the reason for introducing the $C$ matrix into the model. Parsegov *et al.* call this matrix a *multi-issues dependence structure* matrix (MiDS matrix). Consequently, in applying the theory to practical cases, the problem of MiDS matrix estimation arises. Two convex optimization methods have been proposed in Parsegov et al. (2017). One is based on a recursive equation describing the dynamics of the agent system, and the other is based on the limit form of this equation. We refer the interested reader for details to [(Parsegov et al., 2017), Sect. VIII].

***Remark 4.8*** Ye et al. (2020) consider models similar to those in the article (Parsegov et al., 2017), but considered in continuous time. In their models also $C$-matrix appears to play a similar role. In the context considered by by Ye *et al.* this matrix is referred to as a *logical matrix*.

## 4.3 Opinion Dynamics in a Separable Hilbert Space

The theory of conceptual spaces is mainly developed in metric spaces (Gärdenfors, 2000). However, there is often a need to consider spaces with additional structures. For example, in some applications (see e.g. [(Derrac & Schockaert, 2015), p. 72], the notion of an angle between the points of the space is necessary, and thus spaces with a scalar product play an important role. So Hilbert spaces are natural in this context.

Our main contribution to the theory of conceptual spaces is to define the dynamics of conceptual spaces that are separable Hilbert spaces. As we noted in Remark 4.6, there is a connection between multidimensional opinions on various interrelated topics on the one hand and concepts and their interrelated attributes on the other.

Following this correspondence our first aim is to extend the multidimensional FJ-model from Parsegov et al. (2017) to the case of a separable Hilbert space.

Let $\mathcal{H}$ be a separable Hilbert space. The state space is the Cartesian product $\mathcal{H}^n$, $n \in \mathbb{N}$. We denote the state of opinions at time $k$ as $x(k) = (x_1(k), \ldots, x_n(k))^T \in \mathcal{H}^n$. In this setting $\mathcal{C} : \mathcal{H} \to \mathcal{H}$ is a linear operator. Now the Kronecker product $\otimes$ has to be replaced by a tensor product (Aubin, 2000). We can think of an operation $\Lambda W \otimes \mathcal{C}$, where $\Lambda$ is again a 'stubbornness' matrix, in two equivalent ways:

(a) either as it would be a matrix of linear operators written as:

$$\Lambda W \otimes \mathcal{C} = \begin{bmatrix} \lambda_{11} w_{11} \mathcal{C} & \cdots & \lambda_{11} w_{1n} \mathcal{C} \\ \vdots & \ddots & \vdots \\ \lambda_{nn} w_{n1} \mathcal{C} & \cdots & \lambda_{nn} w_{nn} \mathcal{C} \end{bmatrix}$$

(b) or a linear operator obtained as a tensor product of two operators $\Lambda W$ and $\mathcal{C}$ acting on two Hilbert spaces $\mathcal{H}$ and $\mathbb{R}^n$, respectively. Thus $\Lambda W \otimes \mathcal{C}$ acts on $\mathbb{R}^n \otimes \mathcal{H}$ in a usual way:

$$(\Lambda W \otimes \mathcal{C})(x \otimes y) = (\Lambda W x) \otimes (\mathcal{C} y).$$

Formally, the equation describing the dynamics in $\mathcal{H}^n$ does not differ from the equation (4.4) and takes the following form:

$$x(k+1) = \big((\Lambda W) \otimes \mathcal{C}\big) x(k) + \big((I_n - \Lambda) \otimes \mathcal{I}\big) x(0), \tag{4.5}$$

where $\mathcal{I}$ is the identity operator defined in $\mathcal{H}$. In order to use notation (b) we need to decompose $x(k)$ as an element in the tensor product space $\mathbb{R}^n \otimes \mathcal{H}$. Let $\{e_1, \ldots, e_n\}$ be the standard basis in $\mathbb{R}^n$. Because $\mathcal{H}$ is separable we know that there is a countable orthonormal basis $\{h_1, h_2, \ldots\}$ of $\mathcal{H}$. Then $\{e_i \otimes h_j\}$ forms the basis of $\mathbb{R}^n \otimes \mathcal{H}$ in which we can write

$$x(k) = \left(x_1^1(k), x_1^2(k), \ldots, x_2^1(k), x_2^2, \ldots, x_n^1(k), x_n^2(k), \ldots\right)^T$$

and

$$\begin{aligned} x(k+1) &= \big(\Lambda W \otimes \mathcal{C}\big)x(k) + \big((I_n - \Lambda) \otimes \mathcal{I}\big)x(0) \\ &= \big(\Lambda W \otimes \mathcal{C}\big)\Big(\sum_{i=1}^{n}\sum_{j=1}^{\infty} x_i^j(k)(e_i \otimes h_j)\Big)+ \\ &\quad + \big((I_n - \Lambda) \otimes \mathcal{I}\big)\Big(\sum_{i=1}^{n}\sum_{j=1}^{\infty} x_i^j(0)(e_i \otimes h_j)\Big) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{\infty}\Big(x_i^j(k)\big((\Lambda W e_i) \otimes \mathcal{C}(h_j) + x_i^j(0)\big((e_i - \Lambda e_i) \otimes h_j\big)\big)\Big). \end{aligned}$$

### 4.4 Convergence

In examining the convergence for our model, we will follow almost the same procedure as can be found in Friedkin and Johansen (1999); Parsegov et al. (2017). We start with some useful definitions. Let $\mathcal{H}$ be a separable Hilbert space. Recall, that by $L(\mathcal{H})$ we denote the space of all bounded linear operators in $\mathcal{H}$.

**Definition 4.9** A sequence of operators $T_n \in L(\mathcal{H})$ *converges in norm* to $T$ if

$$\|T_n - T\|_{L(\mathcal{H})} \underset{n\to\infty}{\longrightarrow} 0,$$

where $\|\cdot\|_{L(\mathcal{H})}$ denotes the operator norm in the space $L(\mathcal{H})$.

**Definition 4.10** A sequence of operators $T_n \in L(\mathcal{H})$ *strongly converges* to $T$ if for every $x \in \mathcal{H}$,

$$\|T_n x - T x\| \underset{n\to\infty}{\longrightarrow} 0.$$

**Definition 4.11** We call the $i$-th agent *stubborn* if $\lambda_{ii} < 1$ and *totally stubborn* if $\lambda_{ii} = 0$. An agent that is neither stubborn nor influenced by a stubborn agent (connected to some stubborn agent by a walk in the interaction graph $\mathcal{G}(W)$) is called *oblivious*.

Notice that it is possible that there are agents that are neither stubborn nor oblivious. Notice also that the set of all oblivious agents obey (the Hilbert space extended) the DeGroot model dynamic. This can be easily seen when we decompose matrices $W$, $\Lambda$ and vector $x(k)$ simply by rearranging rows in such a way that all oblivious agents are at the bottom (compare with the paragraph before Dfinition 3 on page 4 in Parsegov et al. (2017)). Now

$$W = \begin{bmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & I \end{bmatrix}, \quad x(k) = \begin{bmatrix} \overline{x}(k) \\ \underline{x}(k) \end{bmatrix}.$$

We will denote the number of rows in the matrix $W_{11}$ by $\overline{n}$, and the number of rows of the matrix $W_{22}$ by $\underline{n}$.

The above observation allows us to formulate the following theorem.

**Theorem 4.12** *(Necessary conditions) If the model (4.5) contains oblivious agents then its convergence implies that*

- $\{\mathcal{C}^n\}_{n=1}^{\infty}$ *is strongly convergent to some* $\mathcal{C}^\sharp$ *and* $W_{22}$ *is semi-convergent*

*or*

- $\{\mathcal{C}^n\}_{n=1}^{\infty}$ *strongly converges to the null operator* $\mathcal{O}$.

***Proof*** Assume that the model converges. Then after row permutation, as explained earlier, iteration of bottom part of equations gives,

$$\begin{aligned} \underline{x}(k+1) &= \big(W_{22} \otimes \mathcal{C}\big)\underline{x}(k) \\ &= \big(W_{22} \otimes \mathcal{C}\big)^{k+1}\underline{x}(0) \\ &= W_{22}^{k+1} y(0) \otimes \mathcal{C}^{k+1} y'(0), \end{aligned}$$

where $y(k) \otimes y'(k)$ is a tensor space representation of $\underline{x}(k)$ in $\mathbb{R}^n \otimes \mathcal{H}$.
So the convergence of $\{\underline{x}(k)\}_{k=0}^{\infty}$ implies the strong convergence of $\{\mathcal{C}^k\}_{k=1}^{\infty}$ and the semi-convergence of $W_{22}$. There is also possibility of the convergence $\{\mathcal{C}^k\}_{k=1}^{\infty}$ to zero operator $\mathcal{O}$ then $W_{22}^k y(0) \otimes \mathcal{C}^k y'(0) \underset{k\to\infty}{\longrightarrow} 0$. The operator $W_{22}$ cannot converge to the zero operator since it is stochastic by our assumption. □

**Theorem 4.13** *(Sufficient condition) If the spectral radius* $\rho(\Lambda W \otimes \mathcal{C}) < 1$ *then the model (4.5) converges and its fix-point is*

$$x^\sharp = (I_n \otimes \mathcal{I} - \Lambda W \otimes \mathcal{C})^{-1}\big((I_n - \Lambda) \otimes \mathcal{I}\big)x(0).$$

***Proof*** It is easy to prove by induction that after iteration of recursive equation (4.5) one would get that

$$x(k+1) = \Big[(\Lambda W \otimes \mathcal{C})^k + \Big(\sum_{i=0}^{k-1}(\Lambda W \otimes \mathcal{C})^i\Big)\big((I_n - \Lambda) \otimes \mathcal{I}\big)\Big]x(0).$$

So from the assumption on spectral radius we know that these two operator sequences converge in norm:

(1) $\lim\limits_{k\to\infty} (\Lambda W \otimes \mathcal{C})^k = \mathcal{O}$,

(2) $\lim_{k\to\infty} \sum_{i=0}^{k-1} (\Lambda W \otimes \mathcal{C})^k = (I_n \otimes \mathcal{I} - \Lambda W \otimes \mathcal{C})^{-1}$.

Hence,

$$\lim_{k\to\infty} x(k+1) = (I_n \otimes \mathcal{I} - \Lambda W \otimes \mathcal{C})^{-1}\big((I_n - \Lambda) \otimes \mathcal{I}\big)x(0).$$

□

From Theorem 4.13 it is easy to conclude the following corollary.

**Corollary 4.14** *If $\rho(\mathcal{C}) < \rho(\Lambda W)^{-1}$ then the model converges.*

We will see that a proof of this statement is trivial having the following Lemma 4.15 and its Corollary 4.16.

Before we formulate the Lemma 4.15 we need to specify the setting.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces and $T$ a linear operator acting between them, i.e. $T : \mathcal{H}_1 \to \mathcal{H}_2$.

Let $\sigma(T)$ denote the spectrum of the operator $T$. Recall also that the spectral radius $\rho(T)$ of the operator $T$ is the supremum of the moduli of the (complex) numbers from the spectrum $\sigma(T)$ of the operator $T$.

The following Lemma 4.15 provides a fact of crucial importance for the proof of our theorems on opinion dynamics, where opinions can be elements of a separable Hilbert space.

**Lemma 4.15** *Let $T_1$ and $T_2$ be bounded linear operators $T_i : \mathcal{H}_1 \mapsto \mathcal{H}_2$, $i = 1, 2$, acting between separable Hilbert spaces $\mathcal{H}_i$, $i = 1, 2$. Then the following equation holds*

$$\sigma(T_1 \otimes T_2) = \sigma(T_1)\sigma(T_2) = \{xy \mid x \in \sigma(T_1),\ y \in \sigma(T_2)\}.$$

***Proof*** See the paper by Brown and Pearcy (1966). One can extract proof from the one given in Schechter (1969), as a special case for polynomial $P(z_1, z_2) = z_1 z_2$, $z_1, z_2 \in \mathbb{C}$. □

**Corollary 4.16** *Let the setting be as in Lemma 4.15. Then the spectral radii of $T_1$ and $T_2$ have the following property:*

$$\begin{aligned}
\rho(T_1 \otimes T_1) &= \sup\left\{|\lambda| \ : \ \lambda \in \sigma(T_1 \otimes T_1)\right\} \\
&= \sup\left\{|\lambda| \ : \ \lambda \in \sigma(T_1)\sigma(T_2)\right\} \\
&= \sup\left\{|\lambda_1||\lambda_2| \ : \ \lambda_1 \in \sigma(T_1),\ \lambda_2 \in \sigma(T_2)\right\} \\
&= \sup\left\{|\lambda| \ : \ \lambda \in \sigma(T_1)\right\} \sup\left\{|\lambda| \ : \ \lambda \in \sigma(T_2)\right\} \\
&= \rho(T_1)\rho(T_2).
\end{aligned}$$

Now let us prove Corollary 4.14.

***Proof of Corollary 4.14*** By Lemma 4.15 we get that the following two inequalities are equivalent

- $\rho(\Lambda W)\rho(\mathcal{C}) = \rho(\Lambda W \otimes \mathcal{C}) < 1$,
- $\rho(\mathcal{C}) < \rho(\Lambda W)^{-1}$.

□

***Remark 4.17*** Theorem 4.13 gives us sufficient condition for the convergence of the model (4.5). If the model has no oblivious agents it is also necessary. Indeed, without oblivious agents matrices $W_{12}$ and $W_{22}$ are absent. However, if there are oblivious agents condition of Theorem 4.13 is not necessary. We can construct an example of convergent model where condition from Theorem 4.13 is not met. Simply take $\rho(\Lambda W) = \rho(\mathcal{C}) = 1$ but also with $\rho(\Lambda_{11} W_{11}) < 1$ and semi-convergent matrix $W_{22}$. By Theorem 4.12 $c^n \to C^\sharp$ as $n$ tends to infinity. Then we have convergence to the limit $x^\sharp = (\overline{x}^\sharp\, \underline{x}^\sharp)^T$, where

$$\overline{x}^\sharp = (I_n \otimes \mathcal{I} - \Lambda W \otimes \mathcal{C})^{-1}\big((I_n - \Lambda) \otimes \mathcal{I}\overline{x}(0) + (\Lambda_{11} W_{12} W_{22}^\sharp) \otimes \mathcal{C}\mathcal{C}^\sharp \underline{x}(0)\big)$$

and

$$\underline{x}^\sharp = W_{22}^\sharp \otimes \mathcal{C}^\sharp \underline{x}(0).$$

A detailed proof follows easily from Parsegov et al. (2017).

We can easily generalize Theorem 4.13 to the situation of a model in which we have an infinite number of agents.

**Corollary 4.18** *Suppose that the infinite matrix $W$ is a bounded operator acting on the Hilbert space $\ell^2(\mathbb{N}) = \{(a_1, a_2, \ldots) : \sum_{n=1}^\infty a_n^2 < \infty\}$, i.e.*

$$W : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$$

*and*

$$\|W\|_{\ell^2(\mathbb{N}) \to \ell^2(N)} \leq M.$$

*Suppose also that the spectral radius $\rho$ of the bounded operator*

$$\Lambda W \otimes \mathcal{C} : \ell^2(\mathbb{N}) \otimes \mathcal{H} \to \ell^2(\mathbb{N}) \otimes \mathcal{H}$$

*satisfies*

$$\rho(\Lambda W \otimes \mathcal{C}) < 1$$

*then the model (4.5) converges and its fix-point is*

$$x^\sharp = (I_n \otimes \mathcal{I} - \Lambda W \otimes \mathcal{C})^{-1}\big((I_n - \Lambda) \otimes \mathcal{I}\big)x(0).$$

***Proof*** The same as the proof of Theorem 4.13. □

## 5 Dynamics in Conceptual Spaces

When we have the model (4.5) defined it is not difficult to see how to combine it with the theory of conceptual spaces.

Let $\mathcal{H}$ be a conceptual space. There are $n$ actors and every one of them has his $m$ prototypes. Denote them by $p_1^i, \ldots, p_m^i \in \mathcal{H}$, $1 \leqslant i \leqslant n$. We also have a matrix of social influences $W \in M_{n\times n}(\mathbb{R})$ and a diagonal matrix of prejudices[2] $\Lambda \in M_{n\times n}(\mathbb{R})$. The last missing piece is an operator $\mathcal{C}$ of correlation between dimensions.

Now we can run $m$ independent extended FJ-models and combine the results if the models are convergent. At any stage $k$ we can combine the set of resulting prototypes into $n$ conceptual spaces. One for each actor. In this way we can find out how dynamic of learning process changes world comprehension of all agents.

***Example 1*** Let us consider a simplified, easy to visualize, 2-dimensional example, where we can visualize this first state of a system as four conceptual spaces showed in Fig. 1.

$$p^1 = \begin{bmatrix} (8.5, 3) \\ (4.5, -4.5) \\ (10, -2) \\ (-3.5, 3) \\ (-9.7, 9.7) \end{bmatrix} p^2 = \begin{bmatrix} (0.11, 5) \\ (4.2, -4.2) \\ (-1.2, 2.8) \\ (-3, 3.1) \\ (-9.7, 0) \end{bmatrix} p^3 = \begin{bmatrix} (1.8, 5.3) \\ (4.7, -4.2) \\ (1.1, -1.8) \\ (-5.2, 0.1) \\ (-8, -1) \end{bmatrix} p^4 = \begin{bmatrix} (6, 4.2) \\ (4, -8) \\ (1.2, -2.5) \\ (-3.1, 2.2) \\ (-5, 4) \end{bmatrix}$$

Dependencies between dimensions can also be presented in form of a matrix $\mathcal{C}$ since the space in this example is finitely dimensional. Also for simplification we set a stubbornness matrix $\Lambda$ to be $I - W$. Thus

$$\mathcal{C} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

.

To show a difference between dynamics of the oblivious and non-oblivious agents we consider two cases. In the first one $W = W_1$ we have 3 oblivious agents and in $W = W_2$ we have none. The agents' dependencies $W$ are presented in Fig. 2.

For the first dependencies matrix the model almost converges after 8 steps and next iteration barely changes anything. As said, here we have 3 oblivious agents 1, 3, 4. This means that they are not bound to any point in space longer than one step. Hence they have a lot more freedom and they obey the DeGroot dynamic. In the picture one can see that the agents reached consensus or in other words they agreed upon some common vision of the world. Whereas the second stubborn agent agrees with them to some extent, nevertheless he sticks to his initial biases or beliefs Fig. 3 .

In the second case the model reaches equilibrium after a dozen or so steps. The result can be seen in Fig. 4. Now there is no oblivious agent. Reaching stable state takes more time, and the final equilibrium state is certainly not a consensus. It is close to the consensus, but since each agent considers its own stage $k = 0$ prototype in every

[2] If such data is difficult to obtain one can simply set $\Lambda = I_n - W$.

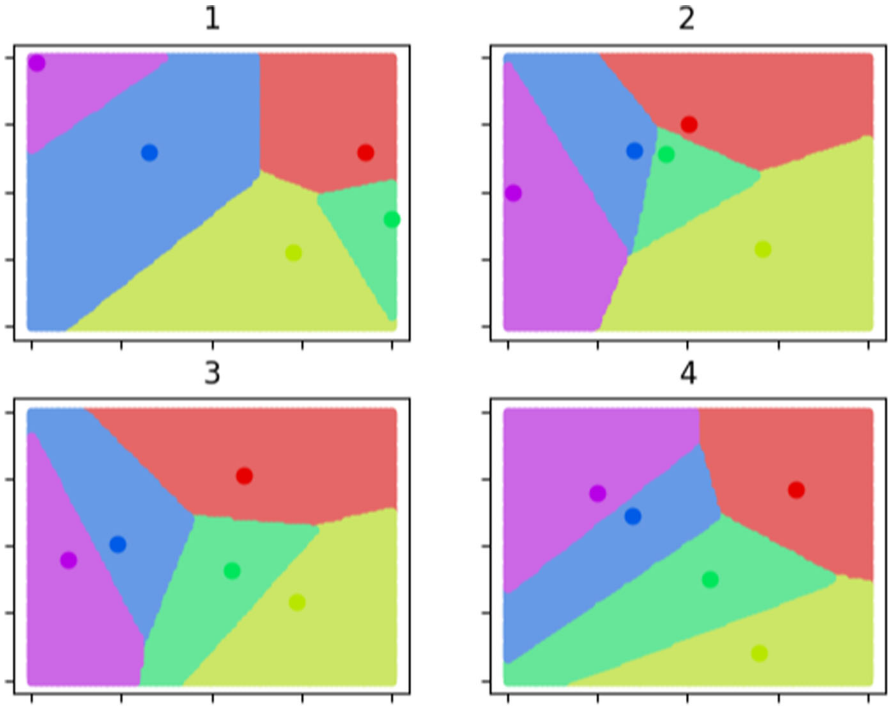


**Fig. 1** Conceptual spaces at $k = 0$.

**Fig. 2** Agent dependencies coefficients

$$W_1 = \begin{bmatrix} 0 & 0 & 0.5 & 0.5 \\ 0.147 & 0.215 & 0.344 & 0.294 \\ 1 & 0 & 0 & 0 \\ 0.66 & 0 & 0.33 & 0 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 0 & 0.25 & 0.25 & 0.5 \\ 0.147 & 0.215 & 0.344 & 0.294 \\ 0.9 & 0 & 0.1 & 0 \\ 0.25 & 0.2 & 0.25 & 0.3 \end{bmatrix}$$

iteration they are all influenced by their initial beliefs. Also they are all connected to each other. Therefore every initial condition has a small impact on every agent. At this point it is clear that we need to be more precise and some kind of a measure of such a "non-consensusness" is required. We consider this problem in the next section.

In the next example we will exploit the full potential that Hilbert space extension gives us. But since a visualization of, say, $\ell_2(\mathbb{N})$ space is not a trivial task we only describe the nature of the problem and we leave a numerical simulation as well as more detailed study for future research.

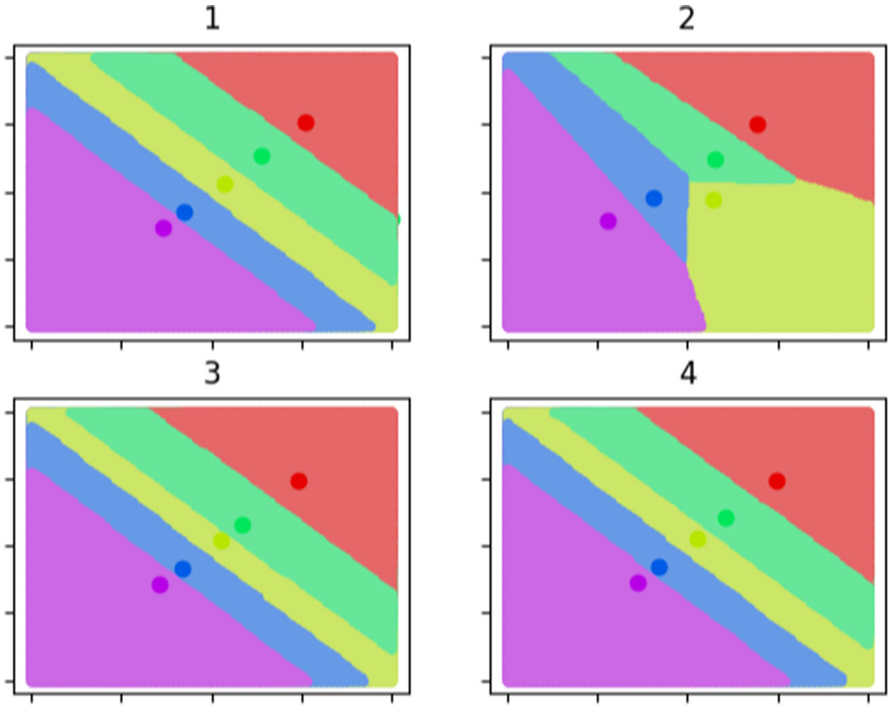


**Fig. 3** The model with $W_1$ after 8 steps of iteration

***Example 2*** The problem is to model a classification of what actors see in the picture. There can be many different ways to comprehend image but we only focus on an edge detection. For example we can process only black and white drawings of shapes or cards used in Rorschach test (Searls, 2017).

So consider a model with the conceptual space being $L_2([0, 1]^2)$. We identify an element of this space, i.e. $L_2$-function, with a 2-dimensional picture located in the $[0, 1] \times [0, 1]$-square. In the experiment actors looking at a picture (i.e. prototype of a concept in the conceptual space theory) detect edges using some form of Fourier transform,[3] which takes place in their brain processes, then place the result in form of a point in $L_2([0, 1]^2)$-space. Then picture is taken away and after a while agents discuss with other participants. There are three major aspects of the experiment that force the dynamics. First of all, agents have influence on each other. Secondly, comprehension of an image is certainly different for every agent. And finally, memories fade away in time. The prototype shown earlier should "lose its precision" in their consciousness. Unimportant parts are forgotten. Therefore, at the end of the experiment agents should have their prototypes changed since by discussion they can convince each other that some parts of their own view is the right one.

[3] Edge detection and is beyond the scope of this paper. Therefore, we only point to some (almost random) resources about this topic: (Tang et al., 2000; Jaffard et al., 2001; Bishop, 2006)

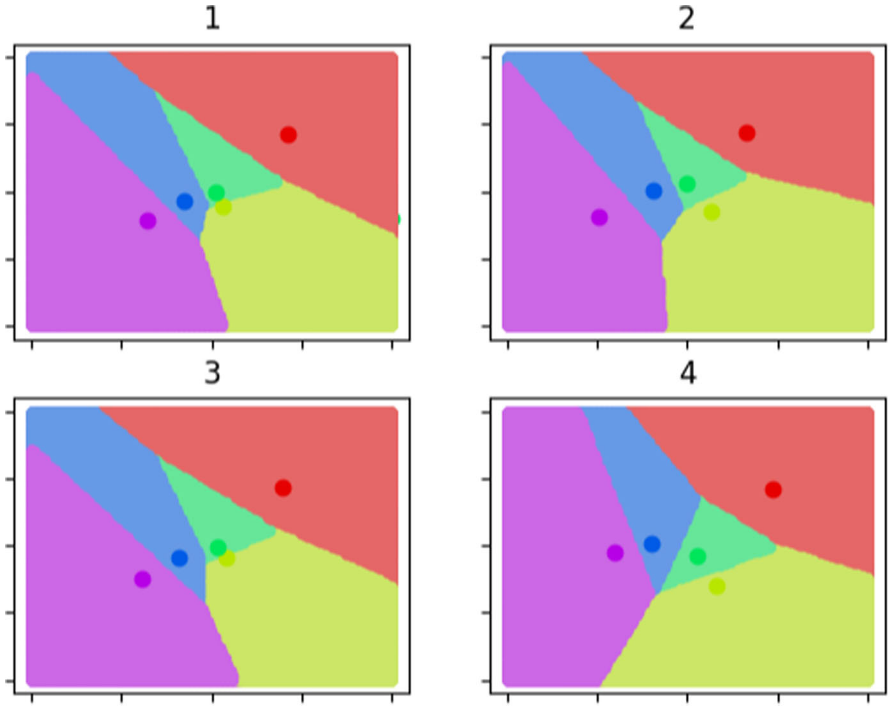


**Fig. 4** Model with $W_2$ after 13 steps of iteration

Now suppose that we want to model this experiment. If we assume that the picture was very simple and the agents are able to make a hand drawn picture, then we can assume that we have a given initial condition in terms of prototypes that are functions from $L_2([0, 1]^2)$. Finding the matrix $W$ should not be difficult since we have finitely many agents and this is no different from classical situations.

So suppose that we already have data in form of agents' conceptual spaces and we know the matrix $W$.

Another problem is to formulate how dimensions in $L_2([0, 1]^2)$ are correlated. Since $L_2([0, 1]^2)$ is separable be need to do that only on some countable basis. For example for $L_2([0, 1]^2)$ one could use standard trigonometric orthonormal basis $\{e^{2\pi i(nx+my)}\}_{n,m=-\infty}^{\infty}$. It seems that statistical methods can be used for this purpose and an infinite correlation matrix $\mathcal{C}$ can be estimated as was done in the finite dimensional case.

Now we can start iterating our algorithm that models our situation and compare the theoretical result with the experimental one.

### 5.1 Consensus and Measure of Agreement in Conceptual Spaces

As seen in previous section even though the model converges it doesn't mean that all agents have reached a state of consensus. They can no longer learn from each other but they still can have distinct perception or knowledge about some concept. So in order to capture this idea more formally let us introduce the following definition.

**Definition 5.1** *Measure of agreement* (MoA) is a function

$$\mu_d^j \ \mathcal{H}^n \to [0, \infty),$$

such that for any $x_1, \ldots, x_n \in \mathcal{H}$ the following condition holds

$$\mu_d^j(x_1, \ldots, x_n) = 0 \iff x_1 = x_2 = \ldots = x_n \tag{5.1}$$

This function is parameterized by metric $d$ from domain space and by $j$ to emphasize that all input prototypes $x_1, \ldots, x_n$ comes from common concept $j$. The simplest example of such a functions would be a sum of all distances between prototypes

$$\mu_d^j(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{k=i+1}^{n} d(x_i, x_k)$$

or smallest radius $r$ of an open ball $B$ containing all prototypes in natural topology generated by the metric $d$

$$\mu_d^j(x_1, \ldots, x_n) = \inf \left\{ r \;\middle|\; B(x; r) \supseteq \{x_1, \ldots, x_n\},\ x \in \mathcal{H} \right\}.$$

It is easy to check that for those functions condition 5.1 holds.
At this point we can finally delimit that to reach a consensus means to have a consensus degree equal to 0 for some family of MoAs $\{\mu_d^j\}_j$.

**Definition 5.2** A *consensus degree* $\varepsilon^*$ is a sum of all measures of agreement of concepts.

$$\varepsilon^* = \sum_{j=1}^{m} \mu_d^j(p_j^1, p_j^2, \ldots, p_j^n).$$

At the end we would like to notice that if iteration of some model $M$ over conceptual spaces has $\varepsilon^* = 0$ it does not imply that it has a limit. So in conclusion those two properties are independent and more research on this topic is needed.

## 6 Conclusions and Future Research

Our main contribution in this paper is the application of opinion dynamics theory to the study of conceptual spaces from a dynamic point of view.

The second major contribution is that we can consider conceptual spaces that are separable Hilbert spaces. This allows us to model complex concepts.

There are several potential directions for further research. In the context of Theorem 4.18, the first research problem should be to study the situation of infinitely many agents carefully. Besides convergence, what can be said for this situation? In particular, it is very interesting to ask about what kind of new information about conceptual spaces we can gain in this context.

In this work we consider only separable Hilbert spaces. Of course, one can also ask about analogous theorems for Banach spaces. It is interesting what insight we obtain in the theory of conceptual spaces by considering the transition to infinite dimensional Banach spaces.

Let us mention the possibility of extending the theorems obtained in this paper to the situation of non-synchronous models in opinion dynamics theory. By this we mean models in which in each step not all agents communicate with each other.

Last but not least, the consensus problem remains. Here, any kind of result that allows us to conclude that there is consensus, i.e. that all actors have the same opinion, is extremely important.

What has been written in this paper is only the first step in the study of conceptual spaces using methods of opinion dynamics theory. Further problems stand open and waiting to be solved.

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## Declarations

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**Ethical Approval** This article does not contain any studies with human participants or animals performed by any of the authors.



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