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On Trotter–Kato type inductive limits in the category of C_0-semigroups

Christian Budde

University of the Free State, Department of Mathematics and Applied Mathematics, Faculty of Natural and Agriculture Sciences, PO Box 339, Bloemfontein 9300
E-mail: BuddeCJ@ufs.ac.za

Abstract

In this paper we will show that the category of C_0-semigroups possesses inductive limits under certain Trotter–Kato type conditions.
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INTRODUCTION

One-parameter semigroups of operators, or C_0-semigroups for short, have been studied extensively in the last decades as they provide solutions to so-called evolution equations. The latter one is an umbrella term for equations that can be interpreted as differential laws describing the development of a system or as a mathematical treatment of motion in time. A detailed overview on C_0-semigroups and evolution equations can for example be found in the monographs by Engel and Nagel [10], Pazy [17] or Goldstein [11]. Recently, the theory of C_0-semigroups firstly has been approached by A. Ng by means of category theory, cf. [16]. We want to jump on the bandwagon and continue the study of this approach. In particular, we want to study a specific construction the category theory of C_0-semigroups, the so-called inductive limits. We will, see that the typical Trotter–Kato approximation conditions appear naturally when constructing the desired limit. Evolution equations in their own right in connection with category already appeared earlier in the work of Liu [13]. Category theory in the framework of functional analysis appears in different areas and also with different perspectives, see for example [3, 15, 9, 24].

This paper is organized as follow: In the first section, we will recall the basic definitions of categories, inductive limits as well as C_0-semigroups. The second section consists of the definitions of categories we will work with and we will investigate the Banach inductive limit. In the last section, we will eventually prove the main result stating that under certain Trotter–Kato type conditions one has an inductive limit of C_0-semigroups.

1. PRELIMINARIES

1.1. Categories and inductive limits

Before we go deeper into the theory of this paper, let us recall the definition of a category, see for example the monographs [14] or [2].

Definition 1.1. A category $\mathcal{C}$ consists of *objects* $A, B, C, \ldots$ and *arrows* $f, g, h, \ldots$ (also called *morphisms*). For each arrow f there are given objects $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$ called the *domain* and *codomain* of f. We write $f : A \to B$ to indicate that $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$. Given arrows $f : A \to B$ and $g : B \to C$, that is, with $\mathrm{cod}(f) = \mathrm{dom}(g)$ there is given an arrow $g \circ f : A \to C$ called the *composite* of f and g. Furthermore, for each object A there is given an arrow $1_A : A \to A$ called the *identity arrow* of A. These arrows are required to satisfy the following axiomas:

(a) *Associativity*, i.e., for $f : A \to B$, $g : B \to C$ and $h : C \to D$ one has

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

(b) *Unit law*, i.e., for each $f : A \to B$ one has

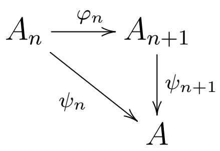
$$f \circ 1_A = f = 1_B \circ f.$$

As the title of the paper promises, we will investigate inductive limits. For the sake of completeness, we recall the definition of an inductive limit in a category. Notice that we simplified the original definition to sequences of objects instead of directed systems of objects, cf. [14, Chapter V, Sect. 1] or [2, Def. 5.17 & 5.18].

Definition 1.2. Let $\mathcal{C}$ be a category and $(A_n)_{n\in\mathbb{N}}$ a sequence of objects in $\mathcal{C}$ such that there exist maps $\varphi_n : A_n \to A_{n+1}$ for each $n \in \mathbb{N}$, i.e., we have the following diagram

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} A_4 \longrightarrow \cdots$$

We say that an objects A in $\mathcal{C}$ is the *inductive limit* of the sequence $(A_n)_{n\in\mathbb{N}}$ if for any $n \in \mathbb{N}$ there exists an arrow $\psi_n : A_n \to A$ such that $\psi_{n+1} \circ \varphi_n = \psi_n$ for each $n \in \mathbb{N}$, i.e., the following diagrams commute for each $n \in \mathbb{N}$:



Moreover, A is *universal* in the sense that if another object B such that there exist arrows $\vartheta_n : A_n \to B$ such that $\vartheta_{n+1} \circ \varphi_n = \vartheta_n$ for all $n \in \mathbb{N}$, then there exists a unique arrow $\alpha : A \to B$ such that $\alpha \circ \psi_n = \vartheta_n$ for each $n \in \mathbb{N}$.

1.2. C_0-semigroups

Before, we finally introduce the category of C_0-semigroups, we quickly recall the basic definitions, cf. [10, Chapter I, Def. 5.1].

Definition 1.3. A family of bounded linear operators $(T(t))_{t\geq 0}$ is called *strongly continuous one-parameter semigroup of linear operators*, or C_0-*semigroup* for short, if the following properties are satisfied:

(i) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$ and $T(0) = \mathrm{I}$.

(ii) $\lim_{t\to 0} \|T(t)x - x\| = 0$ for all $x \in X$.

Definition 1.4. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. The *(infinitesimal) generator* of $(T(t))_{t\geq 0}$ is the linear operator $(A, \mathrm{D}(A))$ on X defined by

$$Ax := \lim_{t\searrow 0} \frac{T(t)x - x}{t}, \quad \mathrm{D}(A) := \left\{ x \in X : \lim_{t\searrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}.$$

As mentioned above, C_0-semigroups yield solutions of evolution equations. Evolution equations can be studied by an operator theoretical approach by means of abstract Cauchy problems. These are Banach space valued initial value problems of the following form

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0 \\ u(0) = x \in X, \end{cases} \tag{ACP}$$

where $(A, \mathrm{D}(A))$ is a linear operator on a Banach space X. For the sake of completeness, we will outline how abstract Cauchy problems and operator semigroups work together. For this we need the notions of solutions and well-posedness of abstract Cauchy problems, cf. [10, Chapter II, Thm. 6.7].

Definition 1.5. A function $u : \mathbb{R}_+ \to X$ is called a *(classical) solution* of (ACP) if u is continuously differentiable with respect to X, $u(t) \in \mathrm{D}(A)$ for all $t \geq 0$ and (ACP) holds.

Definition 1.6. The abstract Cauchy problem (ACP) is called *well-posed* if for every $f \in \mathrm{D}(A)$, there exists a unique solution $u(\cdot, f)$ of (ACP), $\mathrm{D}(A)$ is dense in X and if for every sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{D}(A)$ with $\lim_{n\to\infty} f_n = 0$, one has $\lim_{n\to\infty} u(t, f_n) = 0$ uniformly in compact intervals.

We saw that by means of Definition 1.4 each C_0-semigroup yields a linear operator. The converse question, which operator $(A, \mathrm{D}(A))$ is the generator of a C_0-semigroups is more involving. As a matter of fact, this question is answered by the so-called Hille–Yosida theorem, cf. [10, Chapter II, Thm. 3.8] or [25]. The most important fact is, that the abstract Cauchy problem (ACP) is well-posed in the sense of Definition 1.6 if and only if the operator $(A, \mathrm{D}(A))$ is the generator of a C_0-semigroup, cf. [10, Chapter II, Cor. 6.9].

2. THE CATEGORIES $\mathbf{C_0SG_1}$ AND $\mathbf{BAN_1}$

Here, we will introduce the categories we are interested in. We start with the discussion on the category of Banach spaces. The most obvious definition of this category is the following.

Definition 2.1. Let **Ban** be the category of Banach spaces. The objects and the morphisms in this category are given by

$$\mathrm{ob}(\mathbf{Ban}) := \{X : \ X \text{ Banach space}\},$$

and

$$\mathrm{hom}(X, Y) := \mathscr{L}(X, Y).$$

This definition is of course straightforward, however, it is not useful when working in a category theoretical framework. For this reason, one also has the following related category

Definition 2.2. Let $\mathbf{Ban_1}$ be the category of Banach spaces. The objects and the morphisms in this category are given by

$$\mathrm{ob}(\mathbf{Ban_1}) := \{X : \ X \text{ Banach space}\},$$

and

$$\mathrm{hom}(X, Y) := \{T \in \mathscr{L}(X, Y) : \ \|T\| \leq 1\}.$$

In particular, the morphisms in the category $\mathbf{Ban_1}$ are linear contractions. As already mentioned in [15, Chapter 1] the big advantage is that $\mathbf{Ban_1}$ is both complete and co-complete, i.e., it contains all limits and co-limits of small spectral families. Moreover, by the work of Semadeni and Zidenberg [19] one even knows that even every diagram in $\mathbf{Ban_1}$ admits limits and co-limits. In this paper, we are explicitly interested in limits coming from diagrams of the form

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

In particular, we are interested in inductive limits as introduced in Definition 1.2.

The category $\mathbf{C_0SG}$ of C_0-semigroups on Banach spaces has been introduced in [16, Def. 5]. The objects of this category are tuples (X, T) where X is a Banach space and $(T(t))_{t\geq 0}$ a C_0-semigroup on X. A morphism $(X, T) \to (Y, S)$ between two objects in $\mathbf{C_0SG}$ is a linear map $U \in \mathscr{L}(X, Y)$ such that $UT(t) = S(t)U$ for all $t \geq 0$. For our purposes, we will slightly modify the morphisms of the category which yields another category which we will denote by $\mathbf{C_0SG_1}$.

Definition 2.3. Let $\mathbf{C_0SG_1}$ be the category of C_0-semigroups of operators on Banach spaces. The objects and the morphisms in this category are given by

$$\mathrm{ob}(\mathbf{C_0SG_1}) := \{(X, T(\cdot)) : \ X \text{ Banach space}, \ (T(t))_{t\geq 0} \ C_0\text{-semigroup}\},$$

and

$$\hom((X, T(\cdot)), (Y, S(\cdot))) := \{U \in \mathscr{L}(X, Y) : \ \|U\| \leq 1 \text{ and } UT(t) = S(t)U \ \forall t \geq 0\}$$

Notice, that this modification is not too restrictive as every bounded operator becomes a contraction after suitable scaling. Moreover, for our purposes it is suitable to restrict to contraction maps as we mentioned earlier that the category $\mathbf{Ban_1}$ possesses both limits and co-limits.

2.1. Inductive limits in $\mathbf{Ban_1}$

In this section we recall the notion of Banach inductive limit. For every $n \in \mathbb{N}$ let $(X_n, \|\cdot\|_n)$ be a Banach space and $U_n \in \mathscr{L}(X_n, X_{n+1})$ satisfying $\|U_n\| \leq 1$, i.e., we have the following diagram

$$X_1 \xrightarrow{U_1} X_2 \xrightarrow{U_2} X_3 \xrightarrow{U_3} \cdots$$

In what follows, we will describe explicitly how the inductive limit looks like. For this, let

$$X := \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \exists N \in \mathbb{N} : U_n(x_n) = x_{n+1} \,\forall n \geq N \right\},$$

where $\prod_{n\in\mathbb{N}} X_n$ is the Banach space consisting of all sequences which are uniformly bounded, i.e., there exists $M \geq 0$ such that $\|x_n\|_n \leq M$ for all $n \in \mathbb{N}$. As a matter of fact, this space corresponds to the categorical product in the category $\mathbf{Ban_1}$. One equips X with the seminorm $\|(x_n)_{n\in\mathbb{N}}\| := \lim_{n\to\infty} \|x_n\|_n$ where $(x_n)_{n\in\mathbb{N}} \in X$. Let K denote the kernel of the seminorm, i.e., $K := \{(x_n)_{n\in\mathbb{N}} \in X : \|(x_n)_{n\in\mathbb{N}}\| = 0\}$ and take the quotient X/K. Then X/K becomes a normed space by construction. We denote the completion of X/K with respect to the norm by $\mathcal{X}$, which yields a Banach space. As a matter of fact, the space $\mathcal{X}$ is the desired inductive limit in the category $\mathbf{Ban_1}$, see for example [8, Sect. 4.2]. One also has inclusion maps $\iota_n : X_n \to \mathcal{X}$ that belong to the inductive limit $\mathcal{X}$ in a natural and universal way defined by

$$\iota_n(x) = \Big[(\underbrace{0, 0, \ldots, 0}_{n-1}, x, U_n x, U_{n+1}U_n x, \ldots)\Big], \quad x \in X_n. \tag{2.1}$$

Inductive limits of Banach spaces have been used for several purposes in the literature. They firstly appeared in [18] and were later used in [5, 6]. More on functors and category in general functional analysis can for example be found in [24].

Remark 2.4. It is worth to emphasize, see also [8, Sect. 4.2], that in general the Banach inductive limit, i.e., the space $\mathcal{X}$ as constructed above, does not coincide with the inductive limit in the sense of locally convex spaces. In fact, locally convex inductive limits of Banach spaces are far away from being Banach or Frechet spaces. In functional analysis, there is a separate notion for topological vector spaces that are a locally convex inductive limit of a countable inductive system of Banach spaces, the so-called LB-spaces. Functional analytic research in the framework of category theory has attracted some interested in the past, see for example [4, 21, 22, 23].

3. TROTTER–KATO TYPE INDUCTIVE LIMITS IN $\mathbf{C_0SG_1}$

In this section we will show that the category $\mathbf{C_0SG_1}$ under certain Trotter–Kato type assumptions indeed possesses inductive limits.

Assume that for every $n \in \mathbb{N}$ there exists $(X_n, T_n) \in \mathrm{ob}(\mathbf{C_0SG_1})$ as well as $U_n \in \hom((X_n, T_n), (X_{n+1}, T_{n+1}))$, i.e., in $\mathbf{C_0SG_1}$ we consider the following diagram

$$(X_1, T_1) \xrightarrow{U_1} (X_2, T_2) \xrightarrow{U_2} (X_3, T_3) \xrightarrow{U_3} \cdots$$

especially, we have $T_{n+1}(t)U_n = U_n T_n(t)$ for all $t \geq 0$ and for every $n \in \mathbb{N}$, i.e., we have the following commuting diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{U_1} & X_2 & \xrightarrow{U_2} & X_3 & \xrightarrow{U_3} & \cdots \\ {\scriptstyle T_1}\downarrow & & {\scriptstyle T_2}\downarrow & & {\scriptstyle T_3}\downarrow & & \\ X_1 & \xrightarrow[U_1]{} & X_2 & \xrightarrow[U_2]{} & X_3 & \xrightarrow[U_3]{} & \cdots \end{array}$$

We want to construct an inductive limit of the above diagram in the category $\mathbf{C_0SG_1}$. For this, we need two objects: a Banach space $\mathcal{X}$ and a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X}$. A choice for $\mathcal{X}$, which is not far from seeking, is the Banach inductive limit constructed in Section 2.1. Indeed, consider the forgetful functor $F : \mathbf{C_0SG_1} \to \mathbf{Ban_1}$ defined by $F((X,T)) := X$ and $F((X,T) \xrightarrow{U} (Y,S)) := \left(X \xrightarrow{U} Y\right)$. Then we are exactly in the situation described in Section 2.1, hence we are able to take the inductive limit $\mathcal{X}$ in $\mathbf{Ban_1}$ of the Banach space $F((X_n, T_n))$ subjected to the morphisms $F(U_n)$ for $n \in \mathbb{N}$.

The next steps are constructing a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X}$ and showing that $(\mathcal{X}, \mathcal{T})$ is the inductive limit by means of the universal property. Firstly, we define a family of operators $(T(t))_{t\geq 0}$ for each $(x_n)_{n\in\mathbb{N}} \in X$ by

$$T(t)(x_n)_{n\in\mathbb{N}} := (T_n(t)x_n)_{n\in\mathbb{N}}, \quad (x_n)_{n\in\mathbb{N}} \in X,\ t \geq 0. \tag{3.1}$$

In general, the family of operators defined by (3.1) will not yield a C_0-semigroup on $\mathcal{X}$. For this reason, we introduce the following assumption.

Assumption A. Let each $n \in \mathbb{N}$ let $(X_n, T_n) \in \mathrm{ob}(\mathbf{C_0SG_1})$. We say that the sequence satisfies the Trotter–Kato approximation property (TK) if the following are satisfied:

(i) There exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T_n(t)\| \leq M\mathrm{e}^{\omega t}$ for all $n \in \mathbb{N}$ and $t \geq 0$.

(ii) For every $(x_n)_{n\in\mathbb{N}} \in X$ one has $\lim_{n\to\infty} \|T_n(t)x_n - x_n\|_n = 0$ uniformly on compact intervals.

Remark 3.1. It is worth to mention that the conditions of Assumption A are not randomly. Similar assumptions naturally appear in the framework of operator semigroup approximation, cf. [20, 12, 1, 7]. We will see that Assumption A indeed yield the right requirements so that the family of operators defined by (3.1) is a C_0-semigroup on $\mathcal{X}$

Proposition 3.2. *If Assumption A is satisfied, then the family of operators* $(T(t))_{t\geq 0}$ *defined by* (3.1) *is a semigroup of bounded linear operators on* X.

Proof. That $(T(t))_{t\geq 0}$ also satisfies the semigroup law is obvious, as $(T_n(t))_{t\geq 0}$ is a C_0-semigroup for each $n \in \mathbb{N}$. Let $(x_n)_{n\in\mathbb{N}} \in X$. As $x_n \in X_n$ and $(T_n(t))_{t\geq 0}$ is a C_0-semigroup on X_n we conclude that $T_n(t)x_n \in X_n$ for each $n \in \mathbb{N}$. Moreover,

$$\begin{aligned}\|T(t)(x_n)_{n\in\mathbb{N}}\| &= \|(T_n(t)x_n)_{n\in\mathbb{N}}\| = \lim_{n\to\infty} \|T_n(t)x_n\|_n \\ &\leq \lim_{n\to\infty} M\mathrm{e}^{\omega t} \|x_n\|_n = M\mathrm{e}^{\omega t} \|(x_n)_{n\in\mathbb{N}}\|,\end{aligned}$$

whenever $(x_n)_{n\in\mathbb{N}} \in X$. Hence $T(t) \in \mathscr{L}(X)$ for each $t \geq 0$ and $(T(t))_{t\geq 0}$ is exponentially bounded. By the properties of $U_n \in \hom((X_n, T_n(\cdot)), (X_{n+1}, T_{n+1}(\cdot)))$ we obtain that

$$U_nT_n(t)x_n = T_{n+1}(t)U_{n+1}x_n = T_{n+1}(t)x_{n+1},$$

for all $n \in \mathbb{N}$ and $t \geq 0$. This shows that indeed $T(t)(x_n)_{n\in\mathbb{N}} \in X$. □

Remark 3.3. First of all, we notice that for Proposition 3.2 the first condition of Assumption A is sufficient as we do not speak about continuity yet. In fact, Proposition 3.2 does not state that $(T(t))_{t\geq 0}$ is a C_0-semigroup. For this, we need the underlying space to be a Banach space. Notice that the family of operators $(T(t))_{t\geq 0}$ defined by (3.1) also defines a semigroup of linear operators on the quotient space X/K. Therefore, one has that $T(t) \in \mathscr{L}(X/K)$ for each $t \geq 0$. As $\mathcal{X}$ is the completion of X/K, see Section 2.1, $T(t) \in \mathscr{L}(\mathcal{X})$ by the density of X/K in $\mathcal{X}$ and $(T(t))_{t\geq 0}$ is a semigroup of linear operators on $\mathcal{X}$. For the sake of simplicity, we will denote by $(\mathcal{T}(t))_{t\geq 0}$ this semigroup of linear operators on $\mathcal{X}$.

Lemma 1. *If Assumption A is satisfied, then* $(\mathcal{X}, \mathcal{T}) \in \mathrm{ob}(\boldsymbol{C_0SG_1})$.

Proof. We already know, that $\mathcal{X}$ is a Banach space, cf. Section 2.1. It suffices to show that $(\mathcal{T}(t))_{t\geq 0}$ is a C_0-semigroup on $\mathcal{X}$. By Proposition 3.2 and Remark 3.3 we know already that $(\mathcal{T}(t))_{t\geq 0}$ is a semigroup of linear operators on $\mathcal{X}$. Hence, it suffices to prove that $(\mathcal{T}(t))_{t\geq 0}$ is strongly continuous. By [10, Chapter I, Lemma 5.2 & Prop. 5.3] it suffices to show strong continuity on a dense subset, in particular it suffices to show strong continuity on X/K. Without loss of generality, let $(x_n)_{n\in\mathbb{N}}$ in X, then

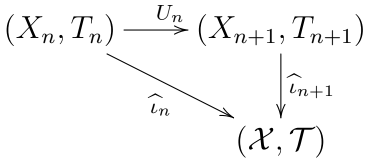
$$\lim_{t\to 0} \|T(t)(x_n)_{n\in\mathbb{N}} - (x_n)_{n\in\mathbb{N}}\| = \lim_{t\to 0} \|(T_n(t)x_n - x_n)_{n\in\mathbb{N}}\| = \lim_{t\to 0}\lim_{n\to\infty} \|T_n(t)x_n - x_n\|_n.$$

By Assumption A(ii) one has that $\lim_{n\to\infty} \|T_n(t)x_n - x_n\|_n = 0$ uniformly on compact intervals which shows that indeed the previous limit exists and is equal to 0. Hence $(\mathcal{T}(t))_{t\geq 0}$ is C_0-semigroup on $\mathcal{X}$ which concludes the proof. □

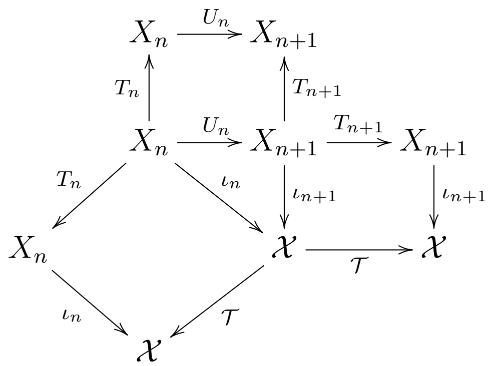
By Lemma 1 we actually know, that under Assumption A constructing $(\mathcal{X}, \mathcal{T})$ for a given sequence of objects in $\mathbf{C_0SG_1}$ leaves the category invariant. We will now show that $(\mathcal{X}, \mathcal{T})$ is indeed the desired inductive limit.

Theorem 3.4. *For every $n \in \mathbb{N}$ let $(X_n, T_n) \in \mathrm{ob}(\boldsymbol{C_0SG_1})$ and $U_n \in \mathrm{hom}((X_n, T_n), (X_{n+1}, T_{n+1})$. If Assumption A is satisfied, then $(\mathcal{X}, \mathcal{T})$ is the inductive limit.*

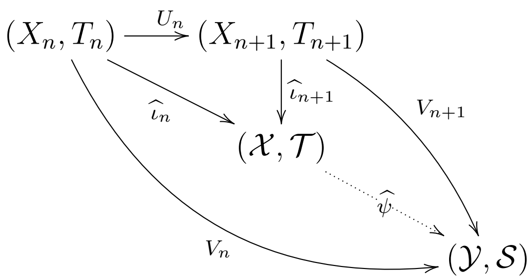
Proof. As mentioned before, Lemma 1 ensures that $(\mathcal{X}, \mathcal{T}) \in \mathrm{ob}(\mathbf{C_0SG_1})$. We have to check the conditions of Definition 1.2 to conclude the proof. In $\mathbf{C_0SG_1}$ we have the following diagram



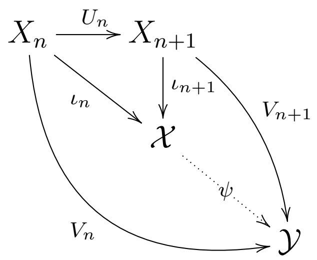
where $\widehat{\iota}_n$ is the morphism in $\mathbf{C_0SG_1}$ induced by the map $\iota_n : X_n \to \mathcal{X}$ as constructed by (2.1). For obvious reasons, one indeed has $\iota_n \circ T_n(t) = \mathcal{T}(t) \circ \iota_n$ for all $t \geq 0$ by construction, i.e., $\widehat{\iota}_n \in \mathrm{hom}((X_n, T_n), (\mathcal{X}, \mathcal{T}))$ for each $n \in \mathbb{N}$. In fact, one has the following commuting diagram



Now, we have to show that $(\mathcal{X}, \mathcal{T})$ has the universal property. To do so, assume that there exists $(\mathcal{Y}, \mathcal{S}) \in \mathrm{ob}(\mathbf{C_0SG_1})$ and $V_n \in \mathrm{hom}((X_n, T_n), (\mathcal{Y}, \mathcal{S}))$ such that $V_{n+1}U_n = V_n$ for all $n \in \mathbb{N}$, i.e., one has the following diagram



where $\widehat{\psi} \in \hom((\mathcal{X},\mathcal{T}),(\mathcal{Y},\mathcal{S})$ is the unique homomorphism we have to construct such that the diagram above is commutative in $\mathbf{C_0SG_1}$. Again, by making use of the forgetful functor $F : \mathbf{C_0SG_1} \to \mathbf{Ban_1}$ as mentioned previously, the diagram above yields a commutative diagram in the category of Banach spaces, i.e., one has



where $\psi \in \mathscr{L}(\mathcal{X},\mathcal{Y})$ satisfying $\|\psi\| \leq 1$ and making the above diagram commuting. Such an unique linear map ψ indeed exists as we already observed that $\mathcal{X}$ is the inductive limit in $\mathbf{Ban_1}$. We will show that this map ψ can be lifted to a morphism $\widehat{\psi} \in \hom((\mathcal{X},\mathcal{T}),(\mathcal{Y},\mathcal{S}))$. In fact, we will show that $\mathcal{S}(t)\circ\psi = \psi\circ\mathcal{T}(t)$ for all $t \geq 0$. As mentioned in [8, Sect. 4.2] one actually can identify X/K with $\bigcup_{n\in\mathbb{N}} \iota_n(X_n)$ and $\mathcal{X}$ with $\overline{\bigcup_{n\in\mathbb{N}} \iota_n(X_n)}$. Firstly, let $x \in \bigcup_{n\in\mathbb{N}} \iota_n(X_n) \subseteq \mathcal{X}$ be arbitrary. Then there exists $m \in \mathbb{N}$ and $y \in X_m$ such that $x = \iota_m(y)$. Hence

$$\begin{aligned}\mathcal{S}(t)\psi(x) &= \mathcal{S}(t)\psi(\iota_m(y)) = \mathcal{S}(t)V_m y = V_m T_m(t)y \\ &= \psi(\iota_m T_m(t)y) = \psi(\mathcal{T}(t)\iota_m(y)) = \psi(\mathcal{T}(t)x),\end{aligned}$$

for all $t \geq 0$. For arbitrary $x \in \mathcal{X}$ one can approximate x by a sequence of elements of the form $\iota_k(y)$, $k \in \mathbb{N}$, $y \in X_k$. By continuity of all morphism, we obtain that indeed $\widehat{\psi} \in \hom((\mathcal{X},\mathcal{T}),(\mathcal{Y},\mathcal{S}))$. The uniqueness of $\widehat{\psi}$ immediately follows from the uniqueness of ψ. □

Remark 3.5. We saw in Section 1.2 that each operator semigroup yields a generator, see Definition 1.4. In [16] also another related category has been introduced, namely $\mathbf{C_0SGG}$. The objects are tuples (X, A) where X is a Banach space and $A : \mathrm{D}(A) \to X$ is the generator of a C_0-semigroup. A morphism between two objects (X, A) and (Y, B) is a map $U \in \mathscr{L}(X, Y)$ with the properties that $Ux \in \mathrm{D}(B)$ and $UAx = BUx$ for each $x \in \mathrm{D}(A)$, cf. [16, Def. 6]. Moreover, there exists an isomorphism of categories $G : \mathbf{C_0SGG} \to \mathbf{C_0SG}$ converting C_0-semigroup generators into the corresponding C_0-semigroup. Let us denote by $\mathbf{C_0SGG}_1$ to be the image of $\mathbf{C_0SG}_1$ under G^{-1}, i.e., $\mathbf{C_0SG}_1$ and $\mathbf{C_0SGG}_1$ are isomorphic categories as well. It is expected that, according to the (second) Trotter–Kato approximation results, cf. cf. [20, 12, 1, 7], the main result Theorem 3.4 has a "translation" in the category $\mathbf{C_0SGG}_1$ by means of conditions concerning the convergence of resolvents of the semigroup generators.

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