

Research Article

Second Hankel Determinants for Some Subclasses of Biunivalent Functions Associated with Pseudo-Starlike Functions

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We introduce second Hankel determinant of biunivalent analytic functions associated with λ -pseudo-starlike function in the open unit disc Δ subordinate to a starlike univalent function whose range is symmetric with respect to the real axis.

1. Introduction

Let \mathcal{A} be the class of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

in the open unit disc $\Delta = \{z : |z| < 1\}$. Let \mathbb{S} be the subclass of \mathcal{A} consisting of univalent functions. Let \mathcal{P} be the family of analytic functions $p(z)$ in Δ such that $p(0) = 1$ and $\Re p(z) > 0$ ($z \in \Delta$) of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \quad (2)$$

For any two functions f and g analytic in Δ , we say that the function f is subordinate to g in Δ and we write it as $f(z) \prec g(z)$, if there exists an analytic function w , in Δ with $w(0) = 0$, $|w(z)| < 1$ ($z \in \Delta$) such that $f(z) = g(w(z))$. In view of Koebe 1/4 theorem, every function $f \in \mathbb{S}$ has an inverse f^{-1} , defined by

$$\begin{aligned} f^{-1}(f(z)) &= z, \quad (z \in \Delta), \\ f(f^{-1}(w)) &= w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right). \end{aligned} \quad (3)$$

In fact, the inverse function is given by

$$\begin{aligned} f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 - a_3) w^3 \\ &\quad - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots \end{aligned} \quad (4)$$

A function $f \in \mathcal{A}$ is said to be biunivalent in Δ if both f and f^{-1} are univalent in Δ . Let Σ denote the class of all biunivalent functions defined in the unit disc Δ . We notice that Σ is nonempty. The behavior of the coefficients is unpredictable when the biunivalence condition is imposed on the function $f \in \mathcal{A}$. In 1967, Lewin [1] introduced the class Σ of biunivalent functions and investigated second coefficient in Taylor-Maclaurin series expansion for every $f \in \Sigma$. Subsequently, in 1967, Brannan and Clunie [2] introduced bistarlike functions and biconvex functions similar to the familiar subclasses of univalent functions consisting of strongly starlike, strongly convex, starlike, and convex functions and so on and obtained estimates on the initial coefficients conjectured that $|a_2| \leq \sqrt{2}$ for bistarlike functions and $|a_2| \leq 1$ for biconvex functions. Only the last estimate is sharp; equality occurs only for $f(z) = z/(1-z)$ or its rotation. Since then, various subclasses of biunivalent functions class Σ were introduced and nonsharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in Taylor-Maclaurin series expansion were found in several investigations. The coefficient estimate problem for each of $|a_n|$ is still an open problem. In 1976, Noonan and

Thomas [3] defined q th Hankel determinants of f for $q \geq 1$ and $n \geq 1$ which is stated as follows:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & a_{n+2} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & a_{n+3} & \cdots & a_{n+q} \\ a_{n+2} & a_{n+3} & a_{n+4} & \cdots & a_{n+q+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & a_{n+q} & a_{n+q+1} & \cdots & a_{n+2q-2} \end{vmatrix}. \quad (5)$$

Easily one can observe that $H_2(1) = |a_3 - a_2^2|$ is a special case of the well known Fekete-Szegő functional $|a_3 - \mu a_2^2|$ where μ is real, for $\mu = 1$. Now for $q = 2, n = 2$, we get second Hankel determinant

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2|. \quad (6)$$

In particular, sharp upper bounds on $H_2(2)$ were obtained by the authors of articles [4–6] for various subclasses of analytic and univalent functions. In 2013, Babalola [7] determined the second Hankel determinant with Fekete-Szegő parameter $|a_2 a_4 - \lambda a_3^2|$ for some subclasses of analytic functions. Let ϕ be an analytic function with positive real part in Δ such that $\phi(0) = 1, \phi'(0) > 0$ which is symmetric with respect to the real axis. Such a function has a Maclaurin series expansion of the form $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ ($B_1 > 0$).

Researchers like Duren [8], Singh [9], and so on have studied various subclasses of usual known Bazilevič function of order α denoted by $B(\alpha)$ which satisfy the geometric condition $\operatorname{Re}(f(z)^{\alpha-1} f'(z)/z^{\alpha-1}) > 0$, where α is nonnegative real number, different ways of perspectives of convexity, radii of convexity and starlikeness, inclusion properties, and so on. The class $B(\alpha)$ reduces to the starlike function and bounded turning function whenever $\alpha = 0$ and $\alpha = 1$, respectively. This class is extended to $B(\alpha, \beta)$ which satisfy the geometric condition $\operatorname{Re}(f(z)^{\alpha-1} f'(z)/z^{\alpha-1}) > \beta$, where α is nonnegative real number and $0 \leq \beta < 1$. Recently, Babalola [7] defined new subclass λ -pseudo-starlike functions of order β ($0 \leq \beta < 1$) which satisfy the condition $\operatorname{Re}(z[f'(z)]^\lambda / f(z)) > \beta$, ($\lambda \geq 1 \in \mathfrak{R}, 0 \leq \beta < 1, z \in \Delta$) and is denoted by $\mathcal{L}_\lambda(\beta)$. Babalola [7] proved that all pseudo-starlike functions are Bazilevič of type $(1-1/\lambda)$, order $\beta^{(1/\lambda)}$, and univalent in the open unit disc Δ . For $\lambda = 2$ we note that functions in $\mathcal{L}_2(\beta)$ are defined by $\operatorname{Re} f'(z)(zf'(z)/f(z)) > \beta$ which is a product combination of geometric expressions for bounded turning and starlike functions. Note that the singleton subclass $\mathcal{L}_\infty(\beta)$ of S contains the identity map. In 2016, Joshi et al. [10] defined two new subclasses of biunivalent functions using pseudo-starlike functions, one is $\mathcal{LB}_\Sigma^\lambda(\alpha)$ class of strong λ -bi-pseudo-starlike functions of order α and other is $\mathcal{LB}_\Sigma(\lambda, \beta)$ λ -bi-pseudo-starlike functions of order β in the open unit disc. Many researchers [11–15] have estimated the second Hankel determinants for some subclasses of biunivalent functions. Motivated by the above-mentioned work, in this paper we have introduced λ -bi-pseudo-starlike functions subordinate

to a starlike univalent function whose range is symmetric with respect to the real axis and estimated second Hankel determinants.

Definition 1. A function $f \in \Sigma$ is said to be in the class $\mathcal{LB}_\Sigma^\lambda(\phi), \lambda \geq 1$, if it satisfies the following conditions:

$$\begin{aligned} \frac{z[f'(z)]^\lambda}{f(z)} &< \phi(z), \quad z \in \Delta, \\ \frac{w[g'(w)]^\lambda}{g(w)} &< \phi(w), \quad w \in \Delta, \end{aligned} \quad (7)$$

where g is an extension of f^{-1} to Δ .

- (1) If $\phi(z) = (1+z)/(1-z)$, then the class $\mathcal{LB}_\Sigma^\lambda(\phi)$ reduces to the class $\mathcal{LB}_\Sigma^\lambda$ and satisfies the following conditions:

$$\begin{aligned} \operatorname{Re} \frac{z[f'(z)]^\lambda}{f(z)} &> 0, \quad z \in \Delta, \\ \operatorname{Re} \frac{w[g'(w)]^\lambda}{g(w)} &> 0, \quad w \in \Delta, \end{aligned} \quad (8)$$

where g is an extension of f^{-1} to Δ .

- (2) If $\phi(z) = (1+(1-2\alpha)z)/(1-z), 0 \leq \alpha < 1$, then the class $\mathcal{LB}_\Sigma^\lambda(\phi)$ reduces to the class $\mathcal{LB}_\Sigma^\lambda(\alpha)$ and satisfies the following conditions:

$$\begin{aligned} \operatorname{Re} \frac{z[f'(z)]^\lambda}{f(z)} &> \alpha, \quad z \in \Delta, \\ \operatorname{Re} \frac{w[g'(w)]^\lambda}{g(w)} &> \alpha, \quad w \in \Delta, \end{aligned} \quad (9)$$

where g is an extension of f^{-1} to Δ .

- (3) If $\phi(z) = ((1+z)/(1-z))^\beta$, then the class $\mathcal{LB}_\Sigma^\lambda(\phi)$ reduces to the class $\mathcal{LB}_\Sigma^\lambda(\beta), 0 < \beta \leq 1$ and satisfies following conditions:

$$\begin{aligned} \left| \arg \frac{z[f'(z)]^\lambda}{f(z)} \right| &< \frac{\beta\pi}{2}, \quad z \in \Delta, \\ \left| \arg \frac{w[g'(w)]^\lambda}{g(w)} \right| &< \frac{\beta\pi}{2}, \quad w \in \Delta, \end{aligned} \quad (10)$$

where g is an extension of f^{-1} to Δ .

(4) If $\lambda = 1$, then the class $\mathcal{L}B_{\Sigma}^{\lambda}(\phi)$ reduces to the class of bistarlike functions $ST_{\Sigma}(\phi)$ and satisfies the following conditions:

$$\begin{aligned} \frac{zf'(z)}{f(z)} &< \phi(z), \quad z \in \Delta, \\ \frac{wg'(w)}{g(w)} &< \phi(w), \quad w \in \Delta, \end{aligned} \tag{11}$$

where g is an extension of f^{-1} to Δ .

Several choices of ϕ would reduce the class $ST_{\Sigma}(\phi)$ to some well known subclasses of Σ .

(1) For the function ϕ given by $\phi(z) = (1 + (1 - 2\alpha)z)/(1 - z)$, $0 \leq \alpha < 1$, the class $ST_{\Sigma}(\phi)$ reduces to the class $ST_{\Sigma}(\alpha)$ and satisfies the following conditions:

$$\begin{aligned} \operatorname{Re} \frac{zf'(z)}{f(z)} &> \alpha, \quad z \in \Delta, \\ \operatorname{Re} \frac{wg'(w)}{g(w)} &> \alpha, \quad w \in \Delta, \end{aligned} \tag{12}$$

where g is an extension of f^{-1} to Δ and this class is called class of bistarlike function of order α .

(2) For the function ϕ given by $\phi(z) = (1 + z)/(1 - z)$, the class $ST_{\Sigma}(\phi)$ reduces to the class ST_{Σ} and satisfies the following conditions:

$$\begin{aligned} \operatorname{Re} \frac{zf'(z)}{f(z)} &> 0, \quad z \in \Delta, \\ \operatorname{Re} \frac{wg'(w)}{g(w)} &> 0, \quad w \in \Delta, \end{aligned} \tag{13}$$

where g is an extension of f^{-1} to Δ and this class is called class of bistarlike function.

2. Preliminary Lemmas

Let P denote the class of functions consisting of p , such that

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \dots = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{14}$$

which are analytic in the open unit disc Δ and satisfy $\Re\{p(z)\} > 0$ for any $z \in \Delta$.

Lemma 2 (see [8]). *If $p \in P$, then $|c_n| \leq 2$ for each $n \geq 1$ and the inequality is sharp for the function $(1 + z)/(1 - z)$.*

Lemma 3 (see [16]). *The power series for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ given in (14) converges in the open unit disc Δ to a function in P if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & c_3 & \dots & c_n \\ c_{-1} & 2 & c_1 & c_2 & \dots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & c_1 & \dots & c_{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_{-n} & c_{-n+1} & c_{-n+2} & c_{-n+3} & \dots & 2 \end{vmatrix}; \tag{15}$$

$\forall n \in \mathbb{N}$

and $c_{-k} = \overline{c_k}$ are all nonnegative. They are strictly positive except for $p(z) = \sum_{k=1}^m \rho_k p_o(\exp(it_k)z)$, $\rho_k > 0$, t_k real, and $t_k \neq t_j$, for $k \neq j$, where $p_o(z) = (1 + z)/(1 - z)$; in this case $D_n > 0$ for $n < (m - 1)$ and $D_n = 0$ for $n \geq m$.

We may assume without any restriction that $c_1 > 0$, on using Lemma 3 for $n = 2$ and $n = 3$, respectively, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c_1} & 2 & c_1 \\ \overline{c_2} & \overline{c_1} & 2 \end{vmatrix} = [8 + 2\Re\{c_1^2 c_2\} - 2|c_2|^2 - 4|c_1|^2] \tag{16}$$

≥ 0

which is equivalent to

$$2c_2 = c_1^2 + x(4 - c_1^2), \quad \text{for some } x, |x| \leq 1. \tag{17}$$

If we consider the determinant

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c_1} & 2 & c_1 & c_2 \\ \overline{c_2} & \overline{c_1} & 2 & c_1 \\ \overline{c_3} & \overline{c_2} & \overline{c_1} & 2 \end{vmatrix} \geq 0 \tag{18}$$

we get the following inequality:

$$\begin{aligned} & \left| (4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2 \right| \\ & \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \end{aligned} \tag{19}$$

From (17) and (19), it is obtained that

$$\begin{aligned} 4c_3 &= c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 \\ &+ 2c_1(4 - c_1^2)(1 - |x|^2)z \end{aligned} \tag{20}$$

for some $z, |z| \leq 1$.

Another required result is the optimal value of quadratic expression. Standard computations show that

$$\begin{aligned} & \max_{0 \leq t^2 \leq 4} (Pt^2 + Qt + R) \\ &= \begin{cases} R, & Q \leq 0, P \leq \frac{-Q}{4}, \\ 16P + 4Q + R, & Q \geq 0, P \geq \frac{-Q}{8} \text{ or } Q \leq 0, P \geq \frac{-Q}{4}, \\ \frac{4PR - Q^2}{4P}, & Q > 0, P \leq \frac{-Q}{8}. \end{cases} \end{aligned} \quad (21)$$

3. Main Results

Theorem 4. *If $f \in \mathcal{L}B_{\Sigma}^{\lambda}(\phi)$ and is of the form (1) then we have the following.*

- (1) $|a_2| \leq B_1/(2\lambda - 1)$.
- (2) $|a_3| \leq B_1^2/(2\lambda - 1)^2 + B_1/(3\lambda - 1)$.
- (3) $|a_4| \leq B_1^3|(-4\lambda^3 + 13\lambda - 3)|/3(2\lambda - 1)^3(4\lambda - 1) + 5B_1^2/2(2\lambda - 1)(3\lambda - 1) + 4B_1/(4\lambda - 1) + |B_3|/(4\lambda - 1)$.

Proof. Since $f \in \mathcal{L}B_{\Sigma}^{\lambda}(\phi)$, there exist two Schwartz functions $u(z), v(w)$ in Δ with $u(0) = 0, v(0) = 0$ and $|u(z)| \leq 1, |v(w)| \leq 1$ such that

$$\begin{aligned} \frac{z [f'(z)]^{\lambda}}{f(z)} &= \{\phi[u(z)]\}, \\ \frac{w [g'(w)]^{\lambda}}{g(w)} &= \{\phi[v(w)]\}. \end{aligned} \quad (22)$$

Define two functions $p(z), q(w)$ such that

$$\begin{aligned} p(z) &= \frac{1 + u(z)}{1 - u(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + \dots, \\ q(w) &= \frac{1 + v(w)}{1 - v(w)} = 1 + d_1w + d_2w^2 + d_3w^3 + \dots. \end{aligned} \quad (23)$$

Then

$$\begin{aligned} \phi\left(\frac{p(z) - 1}{p(z) + 1}\right) &= 1 + \frac{B_1c_1z}{2} + \left(\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}\right)z^2 \\ &+ \left(\frac{B_1}{2}\left(\frac{c_1^3}{4} - c_1c_2 + c_3\right) + \frac{B_2}{4}(2c_1c_2 - c_1^3) + \frac{B_3c_1^3}{8}\right)z^3 + \dots, \\ \phi\left(\frac{q(w) - 1}{q(w) + 1}\right) &= 1 + \frac{B_1d_1w}{2} + \left(\frac{B_1}{2}\left(d_2 - \frac{d_1^2}{2}\right) + \frac{B_2d_1^2}{4}\right)w^2 \\ &+ \left(\frac{B_1}{2}\left(\frac{d_1^3}{4} - d_1d_2 + d_3\right) + \frac{B_2}{4}(2d_1d_2 - d_1^3) + \frac{B_3d_1^3}{8}\right)w^3 + \dots. \end{aligned} \quad (24)$$

Then (22) becomes

$$\begin{aligned} \frac{z [f'(z)]^{\lambda}}{f(z)} &= \phi\left(\frac{1 + u(z)}{1 - u(z)}\right), \\ \frac{w [g'(w)]^{\lambda}}{g(w)} &= \phi\left(\frac{1 + v(w)}{1 - v(w)}\right). \end{aligned} \quad (25)$$

Now equating the coefficients in (25)

$$(2\lambda - 1)a_2 = \frac{B_1c_1}{2}, \quad (26)$$

$$\begin{aligned} (3\lambda - 1)a_3 - (-2\lambda^2 + 4\lambda - 1)a_2^2 &= \frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) \\ &+ \frac{B_2c_1^2}{4}, \end{aligned} \quad (27)$$

$$\begin{aligned} (4\lambda - 1)a_4 - (-6\lambda^2 + 11\lambda - 2)a_2a_3 \\ + \left[\frac{4\lambda(\lambda - 1)(\lambda - 2)}{3} + (-2\lambda^2 + 4\lambda - 1)\right]a_2^3 \\ = \frac{B_1}{2}\left(\frac{c_1^3}{4} - c_1c_2 + c_3\right) + \frac{B_2}{4}(2c_1c_2 - c_1^3) + \frac{B_3c_1^3}{8}, \end{aligned} \quad (28)$$

$$-(2\lambda - 1)a_2 = \frac{B_1d_1}{2}, \quad (29)$$

$$\begin{aligned} (2\lambda^2 + 2\lambda - 1)a_2^2 - (3\lambda - 1)a_3 &= \frac{B_1}{2}\left(d_2 - \frac{d_1^2}{2}\right) \\ &+ \frac{B_2d_1^2}{4}, \end{aligned} \quad (30)$$

$$\begin{aligned} -(4\lambda - 1)a_4 + (6\lambda^2 + 9\lambda - 3)a_2a_3 \\ - \left[\frac{4\lambda(\lambda - 1)(\lambda - 2)}{3} + (10\lambda^2 + 2\lambda - 2)\right]a_2^3 \\ = \frac{B_1}{2}\left(\frac{d_1^3}{4} - d_1d_2 + d_3\right) + \frac{B_2}{4}(2d_1d_2 - d_1^3) \\ + \frac{B_3d_1^3}{8}. \end{aligned} \quad (31)$$

Now from (26) and (29)

$$\begin{aligned} c_1 &= -d_1, \\ a_2 &= \frac{B_1c_1}{2(2\lambda - 1)}. \end{aligned} \quad (32)$$

Now from (27) and (30)

$$a_3 = \frac{B_1^2c_1^2}{4(2\lambda - 1)^2} + \frac{B_1(c_2 - d_2)}{4(3\lambda - 1)}. \quad (33)$$

Now from (28) and (31)

$$\begin{aligned}
 a_4 = & \frac{B_1^3 c_1^3 (-4\lambda^3 + 13\lambda - 3)}{24(2\lambda - 1)^3(4\lambda - 1)} + \frac{5B_1^2 c_1 (c_2 - d_2)}{16(2\lambda - 1)(3\lambda - 1)} \\
 & + \frac{B_2}{4(4\lambda - 1)} (c_1 (c_2 + d_2) - c_1^3) \\
 & + \frac{B_1}{4(4\lambda - 1)} \left[\frac{c_1^3}{2} - c_1 (c_2 + d_2) + (c_3 - d_3) \right] \\
 & + \frac{B_3 c_1^3}{8(4\lambda - 1)}
 \end{aligned} \tag{34}$$

and with the help of the above Lemma 2, we get the required results. \square

Theorem 5. If $\mathcal{L}B_\Sigma^\lambda(\phi)$ is of the form (1) then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1 |h(\mu)|}{3\lambda - 1}; & |h(\mu)| \geq 1, \\ \frac{B_1}{3\lambda - 1}; & 0 \leq |h(\mu)| \leq 1. \end{cases} \tag{35}$$

Proof. Now adding (27) and (30), we get that

$$(4\lambda^2 - 2\lambda) a_2^2 = \frac{B_1}{2} (c_2 + d_2) + (c_1^2 + d_1^2) \frac{(B_2 - B_1)}{4}. \tag{36}$$

Now from (26) and (29), we get that

$$a_2^2 = \frac{B_1 (c_1^2 + d_1^2)}{8(2\lambda - 1)^2}. \tag{37}$$

Now from (36) and (37)

$$\begin{aligned}
 |a_3 - \mu a_2^2| \\
 = \frac{B_1}{4(3\lambda - 1)} [c_2 (1 + h(\mu)) + d_2 (-1 + h(\mu))],
 \end{aligned} \tag{38}$$

where $h(\mu) = B_1^2(1-\mu)(3\lambda-1)/[\lambda(2\lambda-1)B_1^2+(B_1-B_2)(2\lambda-1)^2]$ which completes the proof of the theorem. \square

Theorem 6. If $\mathcal{L}B_\Sigma^\lambda(\phi)$ is of the form (1) then we have the following.

- (1) If $4\xi_1 \leq \xi_3, \xi_2 \leq B_1/2(3\lambda - 2)^2$ then $|a_2 a_4 - a_3^2| \leq B_1^2/(3\lambda - 1)^2$.
- (2) If $4\xi_1 \geq \xi_3, \xi_1 - \xi_2/2 - B_1(1/(2\lambda - 1)(4\lambda - 1) - 1/(3\lambda - 1)^2) \geq 0$ or $4\xi_1 \leq \xi_3, \xi_2 \geq B_1/2(3\lambda - 2)^2$ then $|a_2 a_4 - a_3^2| \leq 2B_1 \xi_2$.
- (3) $4\xi_1 > \xi_3, \xi_1 - \xi_2/2 - B_1(1/(2\lambda - 1)(4\lambda - 1) - 1/(3\lambda - 1)^2) \leq 0$; then

$$|a_2 a_4 - a_3^2| \leq \frac{(B_1/8) [8B_1 \xi_2 / (3\lambda - 1)^2 - 4B_1 \xi_1 / (2\lambda - 1)(4\lambda - 1) - B_1^2 / (2\lambda - 1)^2 (4\lambda - 1)^2 - 4\xi_1^2]}{\xi_2 - \xi_1 - (B_1/2) [1 / (2\lambda - 1)(4\lambda - 1) - 1 / (3\lambda - 1)^2]}. \tag{39}$$

where

$$\begin{aligned}
 \xi_1 = & \frac{B_1^2}{4(3\lambda - 1)(2\lambda - 1)^2} + \frac{|B_2|}{(2\lambda - 1)(4\lambda - 1)}, \\
 \xi_2 = & \frac{\lambda(2\lambda + 1)B_1^3}{6(2\lambda - 1)^3(4\lambda - 1)} + \frac{|B_3|}{2(2\lambda - 1)(4\lambda - 1)}, \\
 \xi_3 = & B_1 \left(\frac{4}{(3\lambda - 1)^2} - \frac{2}{(2\lambda - 1)(4\lambda - 1)} \right).
 \end{aligned} \tag{40}$$

Proof. Using the values of a_2, a_3, a_4 from the above theorem, one can obtain

$$\begin{aligned}
 |a_2 a_4 - a_3^2| = & \frac{B_1}{8} \left| \frac{-c_1^4}{2(2\lambda - 1)(4\lambda - 1)} \left[\frac{\lambda B_1^3 (4\lambda^2 - 1)}{3(2\lambda - 1)^3} \right. \right. \\
 & \left. \left. - B_1 - B_3 + 2B_2 \right] + c^2 \left[\frac{B_1^2 (c_2 - d_2)}{4(3\lambda - 1)(2\lambda - 1)^2} \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & \left. + \frac{(B_2 - B_1)(c_2 + d_2)}{(2\lambda - 1)(4\lambda - 1)} \right] + \frac{c_1 B_1 (c_3 - d_3)}{(2\lambda - 1)(4\lambda - 1)} \\
 & \left. - \frac{B_1 (c_2 - d_2)^2}{2(2\lambda - 1)^2} \right|.
 \end{aligned} \tag{41}$$

According to Lemma 3 we get that

$$\begin{aligned}
 2c_2 = & c_1^2 + x(4 - c_1^2), \\
 2d_2 = & d_1^2 + y(4 - d_1^2) \\
 \Downarrow \\
 (c_2 - d_2) = & \frac{(4 - c_1^2)(x - y)}{2}; \\
 (c_2 + d_2) = & c_1^2 + \frac{(4 - c_1^2)(x + y)}{2}, \\
 (c_3 - d_3) = & \frac{c_1^3}{2} + \frac{c_1(4 - c_1^2)(x + y)}{2}
 \end{aligned}$$

$$\begin{aligned}
& - \frac{c_1(4-c_1^2)(x^2+y^2)}{4} \\
& + \frac{(4-c_1^2)}{2} [(1-|x|^2)z - (1-|y|^2)w].
\end{aligned} \tag{42}$$

For some z, w with $|z| \leq 1, |w| \leq 1$. using (42), we have

$$\begin{aligned}
|a_2a_4 - a_3^2| & \leq \frac{B_1}{8} \left[c_1^4 \left(\frac{\lambda(2\lambda+1)B_1^3}{6(2\lambda-1)^3(4\lambda-1)} \right. \right. \\
& \left. \left. + \frac{|B_3|}{2(2\lambda-1)(4\lambda-1)} \right) + \frac{c_1B_1(4-c_1^2)}{(2\lambda-1)(4\lambda-1)} \right. \\
& \left. + \left(\frac{c_1^2B_1^2(4-c_1^2)}{8(3\lambda-1)(2\lambda-1)^2} + \frac{c_1^2(4-c_1^2)|B_2|}{2(2\lambda-1)(4\lambda-1)} \right) \right. \\
& \cdot (|x|+|y|) + \left. \left(\frac{B_1(c_1^2-2c)(4-c_1^2)}{4(4\lambda-1)(2\lambda-1)} \right) (|x|^2 \right. \\
& \left. + |y|^2) + \frac{B_1(4-c_1^2)^2}{8(3\lambda-1)^2} (|x|+|y|)^2 \right].
\end{aligned} \tag{43}$$

Since $p \in P, |c_1| \leq 2$. Letting $c_1 = c$ we may assume without any restriction that $c \in [0, 2]$. Thus for $\gamma_1 = |x| \leq 1$ and $\gamma_2 = |y| \leq 1$, we obtain

$$\begin{aligned}
|a_2a_4 - a_3^2| & \leq T_1 + T_2(\gamma_1 + \gamma_2) + T_3(\gamma_1^2 + \gamma_2^2) \\
& + T_4(\gamma_1 + \gamma_2)^2 = F(\gamma_1, \gamma_2),
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
T_1 & = \frac{B_1}{8} \left[c^4 \left(\frac{\lambda(2\lambda+1)B_1^3}{6(2\lambda-1)^3(4\lambda-1)} \right. \right. \\
& \left. \left. + \frac{|B_3|}{2(2\lambda-1)(4\lambda-1)} \right) + \frac{cB_1(4-c^2)}{(4\lambda-1)(2\lambda-1)} \right], \\
T_2 & = \frac{B_1}{8} \left(\frac{c^2B_1^2(4-c^2)}{8(3\lambda-1)(2\lambda-1)^2} \right. \\
& \left. + \frac{c^2(4-c^2)|B_2|}{2(4\lambda-1)(2\lambda-1)} \right), \\
T_3 & = \frac{B_1^2(c^2-2c)(4-c^2)}{32(4\lambda-1)(2\lambda-1)}, \\
T_4 & = \frac{B_1^2(4-c^2)^2}{64(3\lambda-1)^2}.
\end{aligned} \tag{45}$$

Now we need to maximize $F(\gamma_1, \gamma_2)$ in the closed square $S = [0, 1] \times [0, 1]$ for $c \in [0, 2]$. We must investigate the maximum

of $F(\gamma_1, \gamma_2)$ according to $c \in (0, 2), c = 2$, and $c = 0$ taking into account the sign of $F_{\gamma_1\gamma_1}F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2$.

First, let $c \in (0, 2)$. Since $T_3 < 0$ and $T_3 + 2T_4 > 0$, we conclude that $F_{\gamma_1\gamma_1}F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2 < 0$. Thus the function F cannot have a local maximum in the interior of the square S . Now, we investigate the maximum of F on the boundary of the square S .

For $\gamma_1 = 0$ and $0 \leq \gamma_2 \leq 1$ (similarly $\gamma_2 = 0$ and $0 \leq \gamma_1 \leq 1$), we obtain

$$F(0, \gamma_2) = G(\gamma_2) = T_1 + T_2\gamma_2 + (T_3 + T_4)\gamma_2^2. \tag{46}$$

(i) *The Case $T_3 + T_4 \geq 0$.* In this case $0 \leq \gamma_2 \leq 1$ and for any fixed c with $0 < c < 2$, it is clear that $G'(\gamma_2) = 2(T_3 + T_4)\gamma_2 + T_2 > 0$; that is, $G(\gamma_2)$ is an increasing function. Hence for any fixed $c \in (0, 2)$ the maximum of $G(\gamma_2)$ occurs at $\gamma_2 = 1$ and

$$\max G(\gamma_2) = G(1) = T_1 + T_2 + T_3 + T_4. \tag{47}$$

(ii) *The Case $T_3 + T_4 < 0$.* Since $2(T_3 + T_4) + T_2 \geq 0$ for $0 \leq \gamma_2 \leq 1$ and for any fixed c with $0 < c < 2$, it is clear that $2(T_3 + T_4) + T_2 < 2(T_3 + T_4)\gamma_2 + T_2 < T_2$ and so $G'(\gamma_2) > 0$. Hence for any fixed $c \in [0, 2)$ the maximum of $G(\gamma_2)$ occurs at $\gamma_2 = 1$. Also for $c = 2$ we obtain

$$\begin{aligned}
& F(\gamma_1, \gamma_2) \\
& = B_1 \left(\frac{\lambda(2\lambda+1)B_1^3}{3(2\lambda-1)^3(4\lambda-1)} + \frac{|B_3|}{(2\lambda-1)(4\lambda-1)} \right).
\end{aligned} \tag{48}$$

Taking into account the value of (48) and case (i) and case (ii), for $0 \leq \gamma_2 \leq 1$ and for any fixed c with $0 \leq c \leq 2$,

$$\max G(\gamma_2) = G(1) = T_1 + T_2 + T_3 + T_4. \tag{49}$$

For $\gamma_1 = 1$ and $0 \leq \gamma_2 \leq 1$ (similarly $\gamma_2 = 1$ and $0 \leq \gamma_1 \leq 1$), we obtain

$$\begin{aligned}
& F(1, \gamma_2) = H(\gamma_2) \\
& = (T_3 + T_4)\gamma_2^2 + (T_2 + 2T_4)\gamma_2 + T_1 + T_2 \\
& \quad + T_3 + T_4.
\end{aligned} \tag{50}$$

Similar to the above case of $T_3 + T_4$, we get that

$$\max H(\gamma_2) = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4. \tag{51}$$

Since $G(1) \leq H(1)$ for $c \in [0, 2]$, $\max F(\gamma_1, \gamma_2) = F(1, 1)$ on the boundary of the square S . Thus the maximum of F occurs at $\gamma_1 = 1$ and $\gamma_2 = 1$ in the closed square S .

Letting $K : [0, 2] \rightarrow \mathfrak{R}$,

$$\begin{aligned}
K(c) & = \max F(\gamma_1, \gamma_2) = F(1, 1) \\
& = T_1 + 2T_2 + 2T_3 + 4T_4.
\end{aligned} \tag{52}$$

Substituting the values of T_1, T_2, T_3, T_4 in the above equation,

$$\begin{aligned}
 K(c) = & \frac{B_1}{8} \left[c^4 \left(\frac{\lambda(2\lambda+1)B_1^3}{6(2\lambda-1)^3(4\lambda-1)} \right. \right. \\
 & + \frac{|B_3|}{2(2\lambda-1)(4\lambda-1)} \\
 & - \left. \left(\frac{B_1^2}{4(3\lambda-1)(2\lambda-1)^2} + \frac{|B_2|}{(2\lambda-1)(4\lambda-1)} \right) \right. \\
 & - \frac{B_1}{2} \left. \left(\frac{1}{(2\lambda-1)(4\lambda-1)} - \frac{1}{(3\lambda-1)^2} \right) \right) \quad (53) \\
 & + c^2 \left(\frac{B_1^2}{(2\lambda-1)^2(3\lambda-1)} + \frac{4|B_2|}{(2\lambda-1)(4\lambda-1)} \right. \\
 & - \left. B_1 \left(\frac{4}{(3\lambda-1)^2} - \frac{2}{(2\lambda-1)(4\lambda-1)} \right) \right) \\
 & \left. + \frac{8B_1}{(3\lambda-1)^2} \right].
 \end{aligned}$$

Let

$$\begin{aligned}
 P = & \frac{B_1}{8} \left[\frac{\lambda(2\lambda+1)B_1^3}{6(2\lambda-1)^3(4\lambda-1)} + \frac{|B_3|}{2(2\lambda-1)(4\lambda-1)} \right. \\
 & - \left. \left(\frac{B_1^2}{4(3\lambda-1)(2\lambda-1)^2} + \frac{|B_2|}{(2\lambda-1)(4\lambda-1)} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{B_1}{2} \left(\frac{1}{(2\lambda-1)(4\lambda-1)} - \frac{1}{(3\lambda-1)^2} \right) \Bigg], \\
 Q = & \frac{B_1}{8} \left(\frac{B_1^2}{(2\lambda-1)^2(3\lambda-1)} + \frac{4|B_2|}{(2\lambda-1)(4\lambda-1)} \right. \\
 & - \left. B_1 \left(\frac{4}{(3\lambda-1)^2} - \frac{2}{(2\lambda-1)(4\lambda-1)} \right) \right), \\
 R = & \frac{B_1^2}{(3\lambda-1)^2}. \quad (54)
 \end{aligned}$$

Then $K(c) = Pt^2 + Qt + R$, where $t = c^2$.

Then with help of optimal value of quadratic expression, we get the required result. This completes the proof of the theorem. \square

Corollary 7. If $f \in \mathcal{LB}_\Sigma^\lambda$ and is of the form (1) then

$$\begin{aligned}
 & |a_2a_4 - a_3^2| \\
 & \leq 4 \left[\frac{4\lambda(2\lambda+1)}{3(2\lambda-1)^3(4\lambda-1)} + \frac{1}{(2\lambda-1)(4\lambda-1)} \right]. \quad (55)
 \end{aligned}$$

Corollary 8. If $f \in \mathcal{LB}_\Sigma^\lambda(\alpha)$ and is of the form (1) then

$$|a_2a_4 - a_3^2| \leq \begin{cases} 4(1-\alpha)^2 \left[\frac{4\lambda(1-\alpha)^2(2\lambda+1)}{3(2\lambda-1)^3(4\lambda-1)} + \frac{1}{(2\lambda-1)(4\lambda-1)} \right], & \alpha \in [0, \tau], \\ \frac{(1-\alpha)^2 [\rho_1(1-\alpha)^2 - 18(1-\alpha)\rho_2 + \rho_3]}{(2\lambda-1)(4\lambda-1) [\rho_4(1-\alpha)^2 - 3\rho_2(1-\alpha) + \rho_5]}, & \alpha \in [\tau, 1), \end{cases} \quad (56)$$

where

$$\begin{aligned}
 \rho_1 = & (4\lambda-1)[16\lambda(2\lambda+1)(2\lambda-1) - 3(4\lambda-1)], \\
 \rho_2 = & (2\lambda-1)(3\lambda-1)(4\lambda-1), \\
 \rho_3 = & 3(2\lambda-1)^2 [4(2\lambda-1)(4\lambda-1) - 9(3\lambda-1)^2], \\
 \rho_4 = & 4\lambda(2\lambda+1)(3\lambda-1)^2, \\
 \rho_5 = & 3(2\lambda-1)^3(4\lambda-1) - 6(2\lambda-1)^2(3\lambda-1)^2, \\
 \tau = & 1 - \frac{3(2\lambda-1)(4\lambda-1) + (2\lambda-1)\sqrt{9(4\lambda-1)^2 + 96(2\lambda+1)(3\lambda-1)^2}}{16\lambda(2\lambda+1)(3\lambda-1)}. \quad (57)
 \end{aligned}$$

Corollary 9. If $f \in \mathcal{L}B_{\Sigma}^{\lambda}(\beta)$ and is of the form (1) then

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{4\beta^2}{(3\lambda - 1)^2}, & \beta\zeta_2 \leq \zeta_4, \beta^2\zeta_1 \leq \tau_5, \\ 4\beta \left[\beta^3\zeta_1 + \frac{\beta}{3(2\lambda - 1)(4\lambda - 1)} \right], & \beta\zeta_2 \geq \zeta_4; \zeta_1\beta^2 - \frac{\zeta_2\beta}{2} + \frac{1}{6(2\lambda - 1)(4\lambda - 1)} \geq 0 \text{ or } \beta\zeta_2 \leq \zeta_4, \beta^2\zeta_1 \geq \tau_5, \\ \frac{\beta^2[\beta^2(\tau_1 - \tau_2) - \beta\tau_3 + \tau_4]}{\beta^2\zeta_1 - \beta\zeta_2 + \zeta_3}, & \beta\zeta_2 > \zeta_4, \zeta_1\beta^2 - \frac{\zeta_2\beta}{2} + \frac{1}{6(2\lambda - 1)(4\lambda - 1)} \leq 0, \end{cases} \quad (58)$$

where

$$\begin{aligned} \tau_1 &= \frac{16\lambda(2\lambda + 1)}{3(2\lambda - 1)^3(3\lambda - 1)^2(4\lambda - 1)} \\ &\quad + \frac{8}{3(2\lambda - 1)(3\lambda - 1)^2(4\lambda - 1)}, \\ \tau_2 &= \frac{1}{(3\lambda - 1)^2(2\lambda - 1)^4} + \frac{4}{(2\lambda - 1)^2(4\lambda - 1)^2} \\ &\quad - \frac{4}{(2\lambda - 1)^3(3\lambda - 1)(4\lambda - 1)}, \\ \tau_3 &= \frac{2}{(2\lambda - 1)^3(3\lambda - 1)(4\lambda - 1)} \\ &\quad + \frac{4}{(2\lambda - 1)^2(4\lambda - 1)^2}, \\ \tau_4 &= \frac{4}{3(2\lambda - 1)(3\lambda - 1)^2(4\lambda - 1)} \\ &\quad - \frac{1}{(2\lambda - 1)^2(4\lambda - 1)^2}, \end{aligned}$$

$$\begin{aligned} \tau_5 &= \frac{1}{(3\lambda - 1)^2} - \frac{1}{3(2\lambda - 1)(4\lambda - 1)}, \\ \zeta_1 &= \frac{4\lambda(2\lambda + 1)}{3(2\lambda - 1)^3(4\lambda - 1)} + \frac{2}{3(2\lambda - 1)(4\lambda - 1)}, \\ \zeta_2 &= \frac{1}{(2\lambda - 1)^2(3\lambda - 1)} + \frac{2}{(2\lambda - 1)(4\lambda - 1)}, \\ \zeta_3 &= \frac{1}{(3\lambda - 1)^2} - \frac{2}{3(2\lambda - 1)(4\lambda - 1)}, \\ \zeta_4 &= \frac{2}{(3\lambda - 1)^2} - \frac{1}{(2\lambda - 1)(4\lambda - 1)}. \end{aligned} \quad (59)$$

Corollary 10. If $f \in ST_{\Sigma}(\phi)$ and is of the form (1) then we have the following.

- (1) If $4(B_1^2/8 + |B_2|/3) \leq B_1/3, (B_1^3 + |B_3|)/6 \leq B_1/8$ then $|a_2a_4 - a_3^2| \leq B_1^2/4$.
- (2) If $4(B_1^2/8 + |B_2|/3) \geq B_1/3, (B_1^3 + |B_3|)/6 - (1/2)(B_1^2/8 + |B_2|/3) - B_1/12 \geq 0$ or $4(B_1^2/8 + |B_2|/3) \leq B_1/3, (B_1^3 + |B_3|)/6 \geq B_1/8$ then $|a_2a_4 - a_3^2| \leq B_1(B_1^3 + |B_3|)/3$.
- (3) $4(B_1^2/8 + |B_2|/3) > B_1/3, (B_1^3 + |B_3|)/6 - (1/2)(B_1^2/8 + |B_2|/3) - B_1/12 \leq 0$; then

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{8} \left[\frac{(B_1/3)(B_1^3 + |B_3|) - (4B_1/3)(B_1^2/8 + |B_2|/3) - B_1^2/9 - 4(B_1^2/8 + |B_2|/3)^2}{(B_1^3 + |B_3|)/6 - (B_1^2/8 + |B_2|/3) - B_1/24} \right]. \quad (60)$$

The above result is obtained by taking $\lambda = 1$ in Theorem 6, which is the second Hankel determinant of bistarlike function.

Corollary 11. If $f \in ST_{\Sigma}(\alpha)$ and is of the form (1) then

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{4(1 - \alpha)^2}{3}(4\alpha^2 - 8\alpha + 5); & \text{if } \alpha \in \left[0, \frac{29 - \sqrt{137}}{32}\right], \\ \frac{(1 - \alpha)^2(13\alpha^2 - 14\alpha - 7)}{(16\alpha^2 - 26\alpha + 5)}; & \text{if } \alpha \in \left(\frac{29 - \sqrt{137}}{32}, 1\right). \end{cases} \quad (61)$$

Corollary 12. If $f \in ST_{\Sigma}$ and is of the form (1) then

$$|a_2a_4 - a_3^2| \leq \frac{20}{3}. \quad (62)$$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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